3-MANIFOLDS WITH MORE THAN ONE ABELIAN EMBEDDING

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Abstract. We construct examples of 3-manifolds M which have at least two inequivalent embeddings in S^4 such that in each case the complementary regions have abelian fundamental groups.

A TOP locally flat embedding of a closed connected 3-manifold M in S^4 is *abelian* if each of the fundamental groups π_X and π_Y of the two complementary regions X and Y is abelian. If M has such an embedding then either $\beta = \beta_1(M) = 1, 3, 4$ or 6 and $H_1(M) = H_1(M;\mathbb{Z}) \cong \mathbb{Z}^\beta$, or $H_1(M) \cong C_n^2$ or $\mathbb{Z}^2 \oplus C_n^2$, for some $n > 0$ [5, Theorem 8.1]. In all cases the abelian groups π_X and π_Y have balanced presentations. If M is a homology 3-sphere then it has an essentially unique abelian embedding (and the complementary regions are then contractible), while if $H_1(M) \cong \mathbb{Z}$ then it has at most one such embedding [5, Theorem 8.9].

We shall show that there are examples with more than one abelian embedding. Our strategy is to find a link L which has several distinct partitions into a pair of sublinks, each of which is trivial (or split $Ap1$, as defined below), and to consider the associated embeddings of the manifold $M = M(L)$ obtained by 0-framed surgery on L. For appropriate choices of L the embeddings are abelian, and we can use the essential uniqueness of the JSJ decomposition of M to show that the embeddings are distinct. One example with $\beta = 6$ has (at least) 5 abelian embeddings.

At the end we attach a short section outlining how surgery may be applied when $\beta = 0$ and the complementary regions have fundamental group C_n, for some $n > 0$.

1. the examples

Embeddings j and $\tilde{j}$ of a 3-manifold M in S^4 are *equivalent* if there are self-homeomorphisms ϕ of M and ψ of S^4 such that $\psi j = \tilde{j}\phi$. Let $j_X : M \to X$ and $j_Y : M \to Y$ be the inclusions of M into each of the complementary regions for the embedding j (and similarly for j'). In particular, if the image of the complementary regions X and Y under ψ are X' and Y' then $H_1(\phi)$ maps the kernel of $H_1(j_X)$ onto the kernel of $H_1(j_{X'})$ and the kernel of $H_1(j_Y)$ onto the kernel of $H_1(j_{Y'})$. Thus in order to show that two embeddings of M are not equivalent it shall suffice to show that there is no such automorphism of $H_1(M)$.

Our examples shall all be variations on the Borromean rings link Bo. All the proper sublinks of Bo are trivial links, and the exterior $X(Bo)$ is hyperbolic [8, Exercise 3.3.10]. We shall say that a knot K in S^3 is $Ap1$ if it has Alexander polynomial $\Delta_K = 1$. Every $Ap1$ knot bounds a TOP locally flat disc in D^4 with complement having fundamental group $\mathbb{Z}$ [3, Theorem 11.7B]. A link in S^3 is *split* $Ap1$ if it is a split link and each component is an $Ap1$ knot. Such links are slice links,

Key words and phrases. abelian, embedding, 3-manifold.

and have a set of slice discs with complement having free fundamental group. We shall also arrange that the nontrivial components have hyperbolic exterior, as this may simplify the invocation of JSJ arguments. There are infinitely many hyperbolic $Ap1$ knots [6].

The simplest nontrivial $Ap1$ knot is the Kinoshita-Teresaka knot $K = 11_{42n}$, which is an 11 crossing knot with $\Delta_K = 1$. It is hyperbolic, and bounds a smooth disc D_K in D^4 such that $\pi_1(D^4 \setminus D_K) \cong \mathbb{Z}$. See [4, Figure 1.4]. (If we could find 4 other such hyperbolic $Ap1$ knots we could avoid any appeal to TOP surgery.)

Let L be the 3-component link obtained from Bo by replacing the third component Bo_3 by a nontrivial $Ap1$ knot K. (See Figure 1, in which $K_o \subset D^3$ is the tangle obtained by deleting a small ball around a point on $K \subset S^3$.) The link L has two distinct partitions into a pair of sublinks, each of which is a split $Ap1$ link: $\mathcal{P} = \{\{Bo_1, Bo_2\}, K\}$ and $\mathcal{P}' = \{\{Bo_1, K\}, K_2\}$.

Let $M = M(L)$ and let j and $j' : M \to S^4$ be the embeddings determined by these partitions, together with the obvious slice discs (as in [5, Chapter 2]). It is easy to see that in each case $\pi_X \cong \mathbb{Z}^2$ and $\pi_Y \cong \mathbb{Z}$, and so j and j' are abelian embeddings. (In each case $X \simeq T^2$ and $Y \simeq S^1 \vee 2S^2$ [5, Theorem 8.17].)

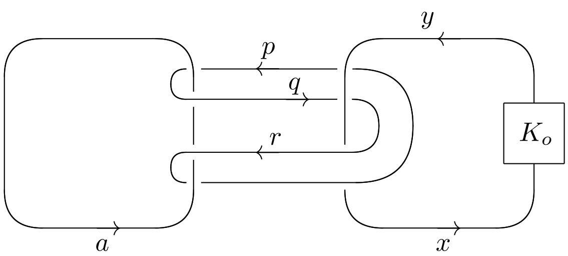

Figure 1.

The group $H_1(M)$ is freely generated by the images of the meridians $\{a, p, x\}$. The 3-manifold M has a JSJ decomposition of which one piece is homeomorphic to $D_{oo} \times S^1$, where D_{oo} is the "pair of pants", i.e., the twice punctured disc. Two of the boundary components of this piece are identified, to give a copy of $T_o \times S^1$ in M. Homology considerations show that the JSJ decomposition has no other such piece, and so self-homeomorphisms of M must leave this piece invariant, up to isotopy. In particular, there is no self-homeomorphism carrying the image of x in $H_1(M)$ into the subgroup generated by the other meridians. Since $j_{Y*}(x)$ generates $H_1(Y)$ and $j'_{Y*}(x) = 0$, it follows that j and j' are not equivalent.

If we tie distinct hyperbolic $Ap1$ knots K_1, K_2 and K_3 in each component of Bo then the three partitions $\mathcal{P}_{12} = \{\{K_1, K_2\}, K_3\}$, $\mathcal{P}_{13} = \{\{K_1, K_3\}, K_2\}$. and $\mathcal{P}_{23} = \{K_1, \{K_2, K_3\}\}$ are distinct. The JSJ decomposition of $M(L)$ has 4 pieces: $X(K_1)$, $X(K_2)$, $X(K_3)$ and $X(Bo)$. These are distinct, by Lemma 1 below, and so M has three inequivalent abelian embeddings.

Lemma 1. *Let K be a knot in S^3 and let μ_K be a meridian loop for K. Then the 3-manifold N with boundary T obtained by 0-framed surgery on K in the exterior $X(\mu_K) \cong S^1 \times D^2$ is homeomorphic to $X(K)$.*

Proof. The cocore of the surgery on S^3 giving $M(K)$ is isotopic to the image of μ_K in $X(K) \subset M(K)$. Deleting a regular neighbourhood of this core from $M(K)$ gives back $X(K)$. □

For the other cases we shall need links with at least 4 components.

If $\beta = 2$ then $\pi_X \cong \pi_Y \cong \mathbb{Z} \oplus C_n$, for some $n \geqslant 1$, and we start by replacing Bo_3 by its $(2, 2n)$-cable. If $\beta = 4$ then $\pi_X \cong \pi_Y \cong \mathbb{Z}^2$, and we replace Bo_3 by two parallel unlinked components. Call the resulting link $Bo(+)$.

In each case we then insert nontrivial $Ap1$ knots into the second and fourth components of $Bo(+)$. The new 4-component link L has two partitions into split sublinks which are also slice links:

$$\mathcal{P} = \{\{L_1, L_3\}, \{L_2, L_4\}\} \quad \text{and} \quad \mathcal{P}' = \{\{L_1, L_4\}, \{L_2, L_3\}\}.$$

The associated embeddings are abelian, and the JSJ argument again goes through (provided $n \neq 1$).

If $n = 0$ (so $\beta = 4$) then each of the 2-component sublinks of $Bo(+)$ is trivial, but the embedding associated to the partition $\mathcal{P} = \{\{L_1, L_2\}, \{L_3, L_4\}\}$ is not abelian. We may modify the second component of $Bo(+)$, as in Figure 3, so that it represents the commutator of the meridians of the third and fourth components. Each of the 2-component sublinks of the resulting link remains trivial. Now tie distinct hyperbolic $Ap1$ knots in each of the second, third and fourth components. Then the 3 partitions of the resulting L into pairs of disjoint 2-component sublinks each give rise to an abelian embedding of $M(L)$, and once again these embeddings are distinct.

The case $\beta = 6$ involves a little more effort. In [5, Example 8.3] we considered the links obtained as preimages of the Whitehead link Wh under 2- and 3-fold branched cyclic covers of S^3, branched over an unknotted axis. The associated manifolds $M(L)$ have abelian embeddings. However these links do not have distinct partitions leading to abelian embeddings. The 3-fold cover of Wh (with respect to this branching) is the link of Figure 2.

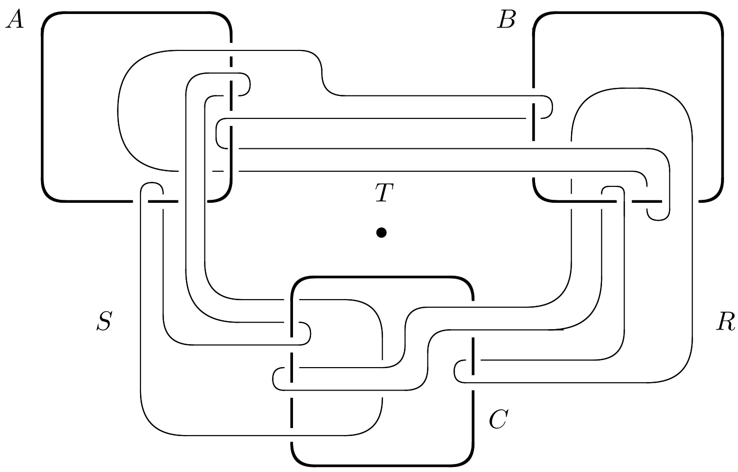

Figure 2. A 6-component link

We shall label the components of the preimage of one component of this link with A, B, C and the other components with R, S and T. Each of the six consecutive triples $\{A, T, B\}$, $\{T, B, R\}$, $\{B, R, C\}$, $\{R, C, S\}$, $\{C, S, A\}$ and $\{S, A, T\}$ is a nontrivial Brunnian link, while all 2-component sublinks and all the other 3-component sublinks are trivial. Each component represents the commutator of the meridians of its immediate neighbours (up to inversion). Thus the embedding determined by $\mathcal{P} = \{\{A, B, C\}, \{R, S, T\}\}$ and the obvious set of slice discs is abelian.

We may modify B, C, R and S to represent $[c, s], [a, r], [c, t]$ and $[b, r]$ (up to inversion), while the only nontrivial 3-component sublinks are $\{A, T, B\}$, $\{T, B, R\}$, $\{B, R, C\}$, $\{R, C, S\}$, $\{C, S, A\}$ and $\{S, A, T\}$, and $\{A, C, R\}, \{B, C, S\}, \{B, R, S\}$ and $\{C, R, S\}$. Figure 3 shows only the modification to R. (Note that the trivial link $\{C, R, T\}$ becomes a copy of Bo, if we ignore the other 3 components.)

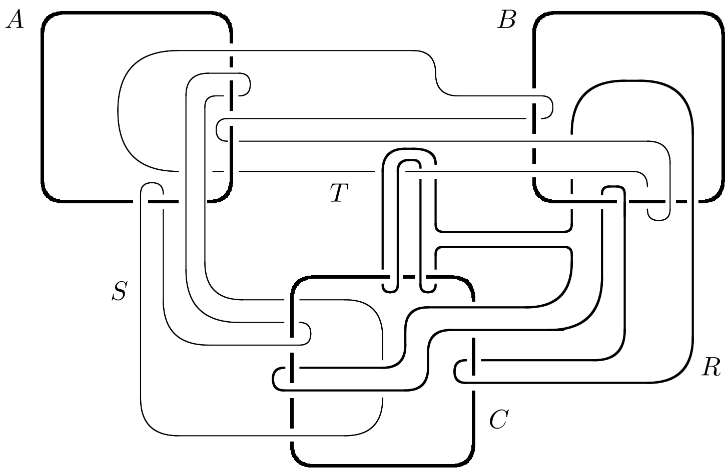

Figure 3. Modifying R so that it represents $[c, t]$.

After further modifications to B, C and S we insert distinct nontrivial hyperbolic $Ap1$ knots into each of the components R, S and T, to obtain a link with two partitions $\mathcal{P}$ and $\mathcal{P}' = \{\{A, B, R\}, \{C, S, T\}\}$ which each give rise to abelian embeddings j and j' of $M(L)$. The JSJ decomposition of $M(L)$ has 3 distinct hyperbolic components corresponding to R, S and T, by Lemma 1. It follows as before that $M(L)$ has no self-homeomorphisms which permute the basis of $H_1(M)$ in a manner necessary for an equivalence between j and j'.

A 6-component link $L = \{A, B, C, R, S, T\}$ has 10 partitions into pairs of 3-component sublinks. If each component of one sublink of a partition represents a commutator of meridians of two components of the other sublink then some of these partitions cannot represent abelian embeddings. Suppose for example that $\mathcal{P} = \{\{A, B, C\}, \{R, S, T\}\}$ is a partition such that A represents the commutator $[s, t]$ of the meridians for S and T in the exterior of $\{R, S, T\}$. Then $\{A, S, T\}$ cannot be a slice link, since the nilpotent completion of a slice link group is that of a free group – see [4, Chapter 12.7]. Consideration of the combinatorics of the problem then suggests that at most 5 of the partitions could give rise to abelian embeddings. We may start with the following partitions of a trivial 6-component link into pairs of trivial 3-component links:

$\mathcal{P}_1 = \{\{A, B, C\}, \{R, S, T\}\}$, $\mathcal{P}_2 = \{\{A, B, R\}, \{C, S, T\}\}$,

$\mathcal{P}_3 = \{\{A, B, S\}, \{C, S, T\}\}$, $\mathcal{P}_4 = \{\{A, C, R\}, \{B, S, T\}\}$ and
$\mathcal{P}_5 = \{\{A, C, T\}, \{B, R, S\}\}$.
We then modify each of the other ten 3-component sublinks as in Figure 3 to obtain copies of Bo, and tie distinct hyperbolic $Ap1$ knots in each of 5 of the components. Once again we may use the uniqueness of the JSJ decomposition to show that the 5 abelian embeddings corresponding to these partitions are inequivalent. We suspect that dropping the hypothesis on components representing commutators would not lead to more than 5 distinct embeddings, but have no proof for this.

Are there similar examples when $H_1(M) \cong C_n^2$ or $\mathbb{Z}^2$? Here arguments involving automorphisms of $H_1(M)$ do not seem to be adequate.

2. SOME REMARKS ON IDENTIFYING THE COMPLEMENTARY REGIONS WHEN $\beta = 0$

In [5, §8.6] we made some brief observations about the application of surgery to examples of abelian embeddings of 3-manifolds M with $H_1(M) \cong C_n^2$ in S^4, for the cases $n \leqslant 4$. We shall remove the latter restriction here. Let W be a complementary region of an embedding of a rational homology sphere M in S^4, such that $\pi_W = \pi_1 W \cong C_n$, for some $n \geqslant 1$. Then $W \simeq P_n = S^1 \cup_n e^2$ [5, Lemma 8.4], and homotopy equivalences between such pseudoprojective planes are simple [2]. Let $\mathcal{S}_{TOP}(W, M)$ be the simple-homotopy equivalence structure set. The normal invariants are detected by the signature, since $H^2(W, M; \mathbb{F}_2) = H_2(W; \mathbb{F}_2) = 0$. Hence exactness of the surgery sequence implies that $L_1^s(\mathbb{Z}[\pi_W])$ acts transitively on $\mathcal{S}_{TOP}(W, M)$. If moreover π_W has odd order then $L_1^s(\mathbb{Z}[\pi_W]) = 0$ [1], while if n is even $L_1^s(\mathbb{Z}[\pi_W])$ is a finite 2-group. Thus if n is odd the complementary regions of an abelian embedding of M are determined up to homeomorphism by M and the homotopy types of the inclusions of M as their boundaries.

The main difficulty in determining the abelian embeddings of such 3-manifolds is in computing the group $[M, P_n]_f$ of based homotopy classes of based maps inducing a given epimorphism $f : \pi_1 M \to C_n = \pi_1 P_n$.

Work of [7] implies that $S^3/Q(8)$ has an essentially unique abelian embedding.

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