PD_3-COMPLEXES BOUND

JONATHAN A. HILLMAN

Abstract. We show that every PD_3-complex P bounds a PD_4-pair (Z, P). If P is orientable we may assume that $\pi_1(Z) = 1$. We show also that if P has a manifold 1-skeleton then it is homotopy equivalent to a closed 3-manifold.

It is well known that every closed connected 3-manifold bounds a compact smooth 4-manifold (which may be assumed orientable if the 3-manifold is orientable). This follows from the calculation of the bordism rings, but there are also *ad hoc* low-dimensional proofs [14]. There is an analogous notion of PD-bordism (as studied in [8]). Much of the published work on this topic (and related notions, such as PD-surgery and transversality) was driven by the needs and results of high-dimensional manifold topology, and we have not found an explicit treatment of the low-dimensional cases.

In the very lowest dimensions $n = 1$ or 2 every PD_n-complex X is homotopy equivalent to a closed n-manifold, and X bounds if and only if the corresponding manifold bounds. Our interest is in the case $n = 3$. In §1 we show that every PD_3-complex P is the range of a homology equivalence $f : M \to P$ with domain a closed 3-manifold. The union of the mapping cylinder of this map with a suitable 4-manifold bounded by M is the ambient space of a PD_4-pair with boundary P. Some argument is needed, since there are PD_3-complexes which are not homotopy equivalent to closed 3-manifolds. We use special features of the low-dimensional case, and leave aside the general problem of Poincaré duality bordism.

Every aspherical 3-manifold is the π_1-injective boundary of an aspherical 4-manifold [4], and in §2 we introduce "injective bordism" of PD_n-groups, to put the corresponding question for PD_3-groups in a wider context. In §3 we consider another aspect of the structure of PD_3-complexes: we show that if a PD_3-complex has a manifold 1-skeleton then it is homotopy equivalent to a closed 3-manifold.

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We have demoted to an appendix a section on constructions of injective null-bordisms for some 3-manifolds which fibre non-trivially.

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1. PD_3-complexes

If $f : M \to P$ is a degree-1 map from a 3-manifold M to a PD_3-complex P then surgery may be used to improve it to a $\mathbb{Z}[\pi]$-homology equivalence, provided an obstruction in $L_3(\mathbb{Z}[\pi])$ vanishes [11]. Here we need only a $\mathbb{Z}$-homology equivalence. The arguments of [2] probably apply to this situation, since $L_3(\mathbb{Z}) = 0$, but we shall use naive, unobstructed surgery below the middle dimension, as in Theorem 5.1 of [10]. (The issue of promoting a degree-1 map to a normal map does not arise, as the orientation characters determine the normal fibrations.)

Theorem 1. *Orientable PD_3-complexes bound orientably.*

Proof. Let P be a PD_3^+-complex. The fundamental class $[P]$ may be represented by a 3-cycle $\Sigma\psi_i$, where each summand ψ_i is a singular 3-simplex. Since $\partial\Sigma\psi_i = 0$, the faces of the summands must match in pairs. Choosing such a pairing gives a map $f : E \to P$ with domain a finite 3-complex which is an orientable 3-manifold away from its vertices. The finitely many non-manifold points have links which are aspherical orientable surfaces. After replacing conical neighbourhoods of such vertices by handlebodies, we obtain a degree-1 map $g : M \to P$ with domain a closed orientable 3-manifold. (The existence of such a map also follows from the Atiyah-Hirzebruch spectral sequence for oriented bordism.)

Since g is a degree-1 map, $\pi_1(G)$ is an epimorphism, and the Hurewicz homomorphism maps $\mathrm{Ker}(\pi_1(g)$ onto the "homology surgery kernel" $K_1 = \mathrm{Ker}(H_1(g;\mathbb{Z}))$. Moreover, K_1 is a direct summand of $H_1(M;\mathbb{Z})$. If K_1 is infinite we may choose a knot $L : S^1 \to M$ which represents an infinite direct summand of K_1, and such that $g \circ L$ is null-homotopic in P. Since the homology class of L generates an infinite direct summand of $H_1(M;\mathbb{Z})$ there is a closed orientable surface S in M which meets the image of L transversely in one point, by Poincaré duality. Let $N \cong S^1 \times D^2$ be a regular neighbourhood of L. We may assume that $S \cap N \cong D^2$, and so $H_1(\partial N;\mathbb{Z})$ has a longitude-meridian basis $\{\lambda, \mu\}$, where λ is freely homotopic in M to L and $\mu = \partial S \cap N$ bounds a transverse disc. (This property characterizes the meridian, up to sign. There is no canonical choice of longitude, but this shall not affect our argument here.) Hence μ is null-homologous in $\overline{M \setminus N}$, since it bounds $\overline{S \setminus N}$. Let $\phi : \partial D^2 \times S^1 \to \partial\overline{M \setminus N}$ be a homeomorphism which maps $\partial D^2 \times 1$ to λ, and let $M' = \overline{M \setminus N} \cup_\phi D^2 \times S^1$. Since $g \circ L$ is null-homotopic, so are $g|_N$ and $g|_{\partial N}$. Therefore $g|_{\overline{M \setminus N}}$ extends to a degree-1 map $g' : M' \to P$, and λ and μ are each null-homologous in M'. Hence $H_1(M'; \mathbb{Z}) \cong H_1(M; \mathbb{Z})/\langle\lambda\rangle$. Proceeding in this way, we may arrange that K_1 is finite.

If L is a knot which represents a finite direct summand of K_1 then the image of the meridian μ in $H_1(\overline{M \setminus N}; \mathbb{Z})$ has infinite order. (For if the image of μ has finite order s in $H_1(\overline{M \setminus N}; \mathbb{Z})$ then $s\mu$ would bound a surface $F \in \overline{M \setminus N}$, and so L would have non-zero intersection number $\pm s$ with the closed surface $\widehat{F} = F \cup sD^2$. Hence the image of L has infinite order in $H_1(M; \mathbb{Z})$.) It is an easy consequence of Poincaré duality that if X is an orientable $(2k+1)$-manifold and F is a field then the image of $H_k(\partial X; F)$ in $H_k(X; F)$ has rank $\frac{1}{2}\beta_k(\partial X; F)$. Since $\partial\overline{M \setminus N} = \partial N$ is a torus, the image of $H_1(\partial N; \mathbb{Q})$ in $H_1(\overline{M \setminus N}; \mathbb{Q})$ has rank 1, and so $H_1(\partial N; \mathbb{Z})$ has a longitude-meridian basis $\{\lambda, \mu\}$, where the image of λ in $H_1(\overline{M \setminus N}; \mathbb{Z})$ has finite order. Surgery on L with respect to this choice of framing replaces (M, g) by a pair (M'', g'') such that $H_1(M''; \mathbb{Z})/\mathrm{Im}(\langle\mu\rangle) \cong H_1(M; \mathbb{Z})/\mathrm{Im}(\langle\lambda\rangle)$. (Compare [10, Lemma 5.6].) This reduces the torsion subgroup of the homology surgery kernel, at the cost of increasing the rank. We then apply the earlier argument, to reduce the rank without further changing the torsion subgroup. After several iterations of these steps, we reduce the homology surgery kernel to 0.

Thus we may assume that $H_1(g; \mathbb{Z})$ is an isomorphism. Hence g induces isomorphisms on homology in all degrees. Let W be a compact orientable 4-manifold with boundary M. After elementary surgeries on a basis for $\pi_1(W)$, we may assume also that $\pi_1(W) = 1$. Let $MCyl(g)$ be the mapping cylinder of g and let $Z = W \cup_M MCyl(g)$. Then $\pi_1(Z) = 1$, since $\pi_1(g)$ is an epimorphism, and the inclusions $j : W \to Z$ and $J : (W, M) \to (Z, MCyl(g)) \simeq (Z, P)$ induce isomorphisms on homology. Since $J_*(j^*\xi \cap [W, M]) = \xi \cap J_*[W, M]$ for all $\xi \in H^i(Z; \mathbb{Z})$ and $i \geq 0$ and since (W, M) is an orientable 4-manifold pair, it follows that (Z, P) is an orientable PD_4-pair with boundary P. $\square$

In general, $(MCyl(g), M \sqcup P)$ need not be a PD_4-pair, even if g is an integral homology equivalence. Let M be the flat 3-manifold with holonomy of order 6. Then there is an integral homology equivalence $g : M \to P = S^2 \times S^1$. The mapping cylinder $MCyl(g)$ fibres over S^1, with fibre $MCyl(g')$ the mapping cylinder of the degree-1 collapse g' :

$T \to S^2$. If $(MCyl(g), M \sqcup P)$ were a PD_4-pair then $(MCyl(g'), T \sqcup S^2)$ would be a 1-connected PD_3-pair. But this is clearly not the case.

We make no use of the fact that a PD_3-complex is finitely dominated, and much of the above argument extends to the non-orientable case. Here we must first find a $\mathbb{Z}[\mathbb{Z}^\times]$-homology equivalence. (We note also that $L_3(\mathbb{Z}[\mathbb{Z}^\times], w) = 0$ if w is nontrivial, but we do not use this fact.)

Addendum. *Every non-orientable PD_3-complex bounds.*

Proof. Let P be a PD_3-complex such that $w = w_1(P) \neq 0$. We could represent a generator of $H_3(P; \mathbb{Z}^w)$ by a geometric cycle $f : E \to P$ with singularities only at the vertices. However in this case it is not obvious that the links of the vertices have even Euler characteristic. Instead we appeal to the Atiyah-Hirzebruch spectral sequence for w-twisted bordism. This gives a 3-manifold M and a map $g : M \to P$ such that $g^*w = w_1(M)$ and $H_3(g; \mathbb{Z}^w)$ is an isomorphism. The surgery kernel $\mathrm{Ker}(\pi_1(g))$ is represented by knots with product neighbourhoods. Let M^+ be the orientable 2-fold covering space of M. After elementary surgeries as in the theorem, we may assume that the image of $\mathrm{Ker}(\pi_1(g))$ in $H_1(M^+; \mathbb{Z})$ is trivial, and so g is a $\mathbb{Z}[\mathbb{Z}^\times]$-homology equivalence. We may also assume that $M = \partial W$ where $w_1(W) : \pi_1(W) \to \mathbb{Z}^\times$ is an isomorphism. The rest of the argument is then as in the theorem. □

It is unlikely that such arguments extend much further. There are non-orientable PD_n-complexes and orientable PD_{n+1}-complexes whose Spivak normal bundle has no TOP reduction, for all $n \geq 4$ [6, 12]. Such complexes admit no degree-1 normal maps with domain a closed n-manifold.

However, M. Land has suggested the following argument for the case $n = 4$. If P is a PD_n-complex for some $n \geq 4$ then $P \simeq H \cup Y$, where H is a 1-handlebody, $(Y, \partial H)$ is a PD_n-pair and the inclusion induces an epimorphism from $\pi_1(H)$ onto $\pi_1(P)$ [17, Lemma 2.8]. We may perform elementary surgeries on a basis for $\pi_1(H)$ inside the manifold 1-skeleton H, and the trace W of the surgeries is a PD-bordism from P to a 1-connected PD_n-complex. If P is orientable then W is orientable. Every 1-connected PD_4-complex is homotopy equivalent to a TOP 4-manifold [5, §11.4]. Now $\Omega_4^{STOP} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, where the first summand is detected by the signature σ and the second by the Kirby-Siebenmann invariant KS. Let $*CP^2$ be the fake projective plane, with $KS(*CP^2) \neq 0$, and let $X = \overline{CP^2} \# *CP^2$. Then $\sigma(X) = 0$ and $KS(X) \neq 0$. The signature is an invariant of PD bordism, but X bounds as a PD_4-complex, since $*CP^2 \simeq CP^2$. Hence the signature defines an isomorphism $\Omega_4^{SPD} \cong \mathbb{Z}$.

In the non-orientable case we find that $\Omega_4^{PD} \cong (\mathbb{Z}/2\mathbb{Z})^2$, detected by SW numbers.

2. ASPHERICITY AND π_1-INJECTIVITY

Relative hyperbolization may be used to show that every closed orientable triangulable n-manifold is orientable cobordant to an aspherical n-manifold [3], and that every aspherical n-manifold which is the boundary of a triangulable $(n+1)$-manifold is in fact the π_1-injective boundary of an aspherical $(n+1)$-manifold. Similarly, every pair of aspherical n-manifolds which are cobordant together bound an aspherical $(n+1)$-manifold π_1-injectively [4]; see also [7, Theorem 5.1].

In the lowest dimensions $n = 1$ or 2 we may avoid hyperbolization (at least for the orientable cases). If $n = 1$ then S^1 is the boundary of the once-punctured torus T_o, and the inclusion of S^1 into T_o is π_1-injective.

If $n = 2$ then $T = \partial T_o \times S^1$, and the inclusion of T into $T_o \times S^1$ is π_1-injective. The Klein bottle bounds the mapping torus of an orientation-reversing involution of T_o. The exterior of the Θ-graph in S^3 depicted in [16, Figure 3.10] has a hyperbolic structure for which the boundary is totally geodesic (and hence incompressible) [16, Example 3.3.12]. The boundary has genus 2, and suitable finite cyclic covers are orientable hyperbolic 3-manifolds with connected totally geodesic boundary of arbitrary genus $g > 1$.

Since every 3-manifold bounds, every aspherical 3-manifold is the π_1-injective boundary of an aspherical 4-manifold, by the result of [4]. On the other hand, for most values of $n \geq 4$ there are aspherical n-manifolds which do not bound at all (since $\Omega_n^{SO} \neq 0$ if $n \geq 8$). In particular, there are $\mathbb{H}^2(\mathbb{C})$-manifolds M with the rational cohomology of $\mathbb{CP}^2$ [15]. No such M can bound (even as an unoriented PD_4-complex), since $\chi(M) = 3$ is odd. Iterated products of such manifolds give counterexamples in all dimensions for which Ω_n^{SO} is infinite.

Definition. *A PD_n-group G bounds if it is a subgroup of a group π such that (π, G) is a PD_{n+1}-pair of groups.*

Two PD_n-groups G_1 and G_2 are injectively bordant if they are subgroups of a group π such that (π, G_1, G_2) is a PD_{n+1}-pair of groups.

Injective bordism of PD_n-groups is an equivalence relation. Reflexivity is displayed by the pair (G, G, G) with $\pi = G_1 = G_2 = G$, symmetry is obvious and transitivity follows from [1, Theorem 8.1].

It remains an open question whether every finitely presentable PD_n-group is the fundamental group of an aspherical closed n-manifold. However, we allow the possibility that G or the ambient group π of a pair (π, G) is of type FP, but not finitely presentable. For every $n \geq 4$

there are uncountable many PD_n-groups which are FP but not finitely presentable [13]. (The case $n = 3$ remains open.)

These observations suggest the following questions.

(1) Does every PD_3-group G bound?
(2) Is every PD_n-group G injectively bordant to a finitely presentable PD_n-group?

If every PD_3-group is a 3-manifold group then (1) has a positive answer. The second question is purely speculative.

3. manifold 1-skeleton

A PD_n-complex X has a *manifold 1-skeleton* if $X \simeq H \cup_\phi Y$, where H is obtained by attaching 1-handles to D^n, (Y, B) is a 1-connected PD_n-pair (i.e., the inclusion homomorphism from $\pi_1(B)$ to $\pi_1(Y)$ is onto) and $\phi : \partial H \to B$ is a homotopy equivalence. When $n = 3$ we shall call such a 3-manifold a *cube with handles.*

If $n \geq 4$ then every PD_n-complex has a manifold 1-skeleton [17, Lemma 2.8]. Here we shall show that if a PD_3-complex has a manifold 1-skeleton then it is essentially a 3-manifold.

Theorem 2. *Let X be a PD_3-complex with a manifold 1-skeleton. Then X is homotopy equivalent to a closed 3-manifold.*

Proof. We may assume that $X \simeq H \cup_\phi Y$, where H is a cube with handles and (Y, B) is a 1-connected PD_3-pair. We may clearly assume that B is non-empty.

Let $\Gamma = \mathbb{Z}[\pi]$. Since (Y, B) is 1-connected, Y may be obtained (up to homotopy equivalence) by adding cells of dimension > 1 to B. Therefore $H^i(Y, B; \Gamma) = H_i(Y, B; \Gamma) = 0$, for $i \leq 1$. Hence $H_j(Y; \Gamma) = 0$ for $j > 1$, by Poincaré duality, and so Y is aspherical. Similarly, if $\mathcal{M}$ is any left $\mathbb{Z}[G]$-module then $H^j(Y; \mathcal{M}) = 0$ for $j > 1$, and so $c.d.\pi \leq 1$. Thus π is a free group.

Since Y is aspherical and $\pi_1(Y)$ is a free group, there is a homotopy equivalence $f : (Y, B) \simeq (H', \partial H')$, where H' is a second cube with handles, by [9, Theorem 3.12] and its extension to the non-orientable case [9, page 38]. Since ∂H and $\partial H'$ are closed surfaces, the homotopy equivalence $f \circ \phi : \partial H \to \partial H'$ is homotopic to a homeomorphism Φ, by the Dehn-Nielsen Theorem [18, Theorem 5.6.1], and so X is homotopy equivalent to the 3-manifold $H \cup_\Phi H'$. □

There are finite PD_3-complexes which are not homotopy equivalent to 3-manifolds. The first and simplest such example has fundamental group S_3. (See [9, Chapter 5], and the references there.)

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APPENDIX: INJECTIVE NULL-BORDISMS FOR 3-MANIFOLDS

In this section we shall consider some further questions about explicit π_1-injective null-bordisms for aspherical 3-manifolds.

(3) if F is a non-orientable surface with $\chi(F) = 2k < 0$ is there a simple construction of an aspherical 3-manifold with π_1-injective boundary F?
(4) If N is an aspherical closed 3-manifold is there a simple construction of an aspherical 4-manifold with π_1-injective boundary N?
(5) If a self-homeomorphism θ of a closed orientable surface F is null-cobordant does it extend to a self-homeomorphism Θ of an aspherical 3-manifold N with π_1-injective boundary F?

For (3) it would suffice to construct an example with boundary $F = \#^4 RP^2$ (with $\chi(F) = -2$), since suitable finite cyclic covers would then realize the other non-orientable surfaces with $\chi = 2k < 0$ in this way. Does Example 3.3.12 of [16] have a 2-fold covering which admits an orientation-reversing free involution?

There is a partial answer to (4). An aspherical 3-manifold is either a graph manifold or is finitely covered by a mapping torus, by the Virtual Fibration Theorem. If N is the mapping torus of an orientation-preserving self-homeomorphism θ of an orientable surface F of genus $g > 2$ then the image of θ in the mapping class group $\mathcal{M}_g$ is a product of (say) r commutators, since $\mathcal{M}_g$ is perfect [Powell, J. Two theorems on the mapping class group of a surface, Proc. Amer. Math. Soc. 68 (1978), 347–350].

It follows easily that N bounds (π_1-injectively) the total space of an F-bundle over a once-punctured surface of genus $2r$. If F has genus 1 or 2 then $\mathcal{M}_1^{ab}$ and $\mathcal{M}_2^{ab}$ are cyclic, of orders 12 and 10 respectively, so a similar argument applies to a finite cover of $M(\theta)$.

We do not know of such simple constructions for the case of graph manifolds, even in the special case when N is the total spaces of a non-trivial S^1-bundle over an aspherical surface B. (Such S^1-bundle spaces bound the associated disc bundle spaces, but these boundaries are never π_1-injective.) If $N = \partial W$ where W fibres over a 3-manifold M with $\partial M = B$ then $N \cong B \times S^1$, since the restriction from $H^2(M;\mathbb{Z})$ to $H^2(\partial M;\mathbb{Z})$ is trivial. If N is not such a product, we may ask instead whether it bounds the total space of an F_o-bundle over B, where F_o is a once-punctured aspherical surface. (This is not possible when $N = T$, for any abelian subgroup of the mapping class group of F_o is generated by the images of self-diffeomorphisms with disjoint support [Birman,

J. S., Lubotzky, A. and McCarthy, J. Abelian and solvable subgroups of the mapping class groups, Duke Math. J. 50 (1983), 1107–1120].

The construction for (4) shows that 3-dimensional mapping tori usually bound surface bundles with the same fibre. Question (5) asks whether such mapping tori bound 4-dimensional mapping tori. This is a more stringent condition, since the group of bordism classes of self-diffeomorphisms of closed surfaces is a direct sum $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty$ [Bonahon, F. Cobordism of automorphisms of surfaces, Ann. Sc. Ec. Norm. Sup. 16 (1983), 237–270], and so we must assume that θ is null-cobordant.

Cusps of complete finite volume Riemannian 4-manifolds with one of the geometries $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$ or $\mathbb{H}^2 \times \mathbb{H}^2$ (and which are not virtually products, in the latter case) have cusps which are respectively $\mathbb{E}^3$-, $\mathbb{N}il^3$- or $\mathbb{S}ol^3$-manifolds. However not every such 3-manifold can be realized as the sole cusp of a geometric 4-manifold.

The flat 3-manifolds with holonomy of order 3 or 6 are a particular challenge. They are total spaces of T-bundles over S^1, but do not bound T-bundles over once-punctured surfaces, since in each case the image of θ in $SL(2,3)^{ab} \cong \mathbb{Z}/3\mathbb{Z}$ is non-trivial. Nor do they bound mapping tori, for a self homeomorphism θ of T bounds if and only if its image in $\theta \in SL(2,\mathbb{Z})$ has both eigenvalues ± 1, by the one half lives–one half dies principle. Finally they do not bound 4-manifolds whose interiors are complete $\mathbb{H}^4$-manifolds of finite volume [Long, D. D. and Reid, A. W. All flat 3-manifolds are cusps of hyperbolic orbifolds, Alg. Geom. Top. 2 (2002), 285–296].

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

Email address: jonathanhillman47@gmail.com