SOME REMARKS ON THE ISOPERIMETRIC PROBLEM FOR THE HIGHER EIGENVALUES OF THE ROBIN AND WENTZELL LAPLACIANS

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Abstract. We consider the problem of minimising the kth eigenvalue, $k \geq 2$, of the (p-)Laplacian with Robin boundary conditions with respect to all domains in $\mathbb{R}^N$ of given volume M. When $k = 2$, we prove that the second eigenvalue of the p-Laplacian is minimised by the domain consisting of the disjoint union of two balls of equal volume, and that this is the unique domain with this property. For $p = 2$ and $k \geq 3$, we prove that in many cases a minimiser cannot be independent of the value of the constant α in the boundary condition, or equivalently of the volume M. We obtain similar results for the Laplacian with generalised Wentzell boundary conditions $\Delta u + \beta \frac{\partial u}{\partial \nu} + \gamma u = 0$.

1. Introduction

We are interested in the eigenvalue problem

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= \lambda |u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \alpha |u|^{p-2}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded, Lipschitz domain, $1 < p < \infty$, $\alpha > 0$, and ν is the outward pointing unit normal to Ω. Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u and the boundary conditions in (1.1) are of Robin type.

It is known that if Ω is connected, then analogous to the case of Dirichlet boundary conditions there is an isolated simple first eigenvalue $\lambda_1 = \lambda_1(\Omega, \alpha) > 0$ such that only eigenfunctions associated with λ_1 do not change sign. Moreover, there is a well-defined second eigenvalue $\lambda_2 > \lambda_1$ at the base of the rest of the spectrum obtainable by the L-S principle (see [18, Section 5.5]). If $p = 2$, then we recover the usual sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$ exhausting the

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spectrum (see for example [7]). For not necessarily connected domains Ω, we wish to study minimisation problems of the form

$$\min\{\lambda_k(\Omega,\alpha):\Omega\subset\mathbb{R}^N \text{ is bounded, Lipschitz, } |\Omega|=M\} \tag{1.2}$$

where $M>0$ and $\alpha>0$ are fixed, $k\geq 2$ if $p=2$ and $k=2$ otherwise, and $|\,.\,|$ is N-dimensional Lebesgue measure. Note that we list repeated eigenvalues according to their multiplicities. Such problems are often called isoperimetric problems as they depend on the geometry of the underlying domain.

When $k=1$ the Faber-Krahn inequality asserts that the unique solution to (1.2) is a ball B with $|B|=M$ (see [2,5]). When $k=2$ and $p=2$ it was proved in [16] that a solution to (1.2), which we shall call D_2, is disjoint union of two equal balls of volume $M/2$.

For $k=2$, it was proved in [15] that the domain which we shall call D_2, consisting of the disjoint union of two equal balls of volume $M/2$, is a solution to (1.2) when $p=2$. Our first goal here is to generalise this result to all $1<p<\infty$, and at the same time prove uniqueness of this minimiser (that is, sharpness of the associated inequality). This is done in Section 2 (see Theorem 2.1).

We consider the problem (1.2) for $k\geq 3$ in Section 3. Here we restrict our attention to the case $p=2$ because the spectrum of the p-Laplacian is not well understood otherwise. In particular, it is not known if the L-S sequence exhausts the spectrum, although we expect our observations to generalise easily if this is the case. We prove that for many values of N and k there cannot be a solution (1.2) independent of $\alpha>0$ in (1.1), or equivalently, of the volume $M>0$. (See Theorem 3.1.) Note that actually proving the existence of a solution to (1.2) in general is an extremely difficult problem – this has not even yet been proved in the easier Dirichlet case (see [3,14]), and the Robin problem lacks many of the properties of the Dirichlet problem (see Remark 3.2).

In Section 4, we consider the Laplacian with generalised Wentzell boundary conditions

$$\begin{aligned} -\Delta u &= \Lambda u && \text{in } \Omega,\\ \Delta u+\beta\frac{\partial u}{\partial\nu}+\gamma u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $\beta,\gamma>0$. Here too there exists a sequence of eigenvalues $0<\Lambda_1(\Omega)\leq\Lambda_2(\Omega)\leq\ldots$ exhausting the spectrum. Moreover, the first eigenvalue Λ_1 satisfies the (sharp) Faber-Krahn inequality $\Lambda_1(\Omega)\geq\Lambda_1(B)$ for all bounded, Lipschitz $\Omega\subset\mathbb{R}^N$ as the solution for $k=1$ to the analogue of (1.2) (see [15]). This is a similar problem to (1.1), and we prove analogues of our results for the Robin problem in this case (see Theorem 4.1). Here we only consider the case $p=2$; it appears no work has yet been done on developing a theory of the p-Laplacian with boundary conditions $\Delta_p u+\beta|\nabla u|^{p-2}\frac{\partial u}{\partial\nu}+\gamma|u|^{p-2}u=0$ on $\partial\Omega$.

Before we proceed, we have a few general remarks.

Remark 1.1. (i) We will only consider bounded, Lipschitz domains of fixed volume $M > 0$ unless otherwise specified, since this is in some sense the "natural" setting for problems such as (1.1) and (1.3), although a solution to (1.2) could be unbounded or non-Lipschitz.

(ii) We allow our domains to be disconnected. Any disconnected Lipschitz domain Ω will consist of countably many separated connected components (c.c.s for short), each having Lipschitz boundary. In such a case the eigenvalues of Ω (for any operator or boundary condition) can be found by collecting and reordering the eigenvalues of the c.c.s.

(iii) For such domains U, V, in a slight abuse of notation we will say $U = V$ iff their c.c.s are in bijective correspondence and for each pair $\widetilde{U}, \widetilde{V}$ of c.c.s, there exists a rigid transformation τ such that $\tau(\widetilde{U}) = \widetilde{V}$. (Thus their spectra will coincide.)

(iv) We will always use $\lambda = \lambda_k(\Omega, \alpha)$ to stand for an eigenvalue of (1.1), $\Lambda = \Lambda_k(\Omega, \beta, \gamma)$ for (1.3), although we will drop one or more arguments if there is no danger of confusion, and we will denote by $\mu_k = \mu_k(\Omega)$ the kth eigenvalue of the Dirichlet p-Laplacian on Ω. We collect some elementary properties of these eigenvalues in the appendix.

2. The second eigenvalue of the Robin p-Laplacian

Choose $1 < p < \infty$, $\alpha > 0$ and $M > 0$, which will all be fixed for this section. Let $\lambda_2(\Omega)$ be the second eigenvalue of (1.1) on Ω, and let D_2 be the disjoint union of two balls of volume $M/2$ each.

Theorem 2.1. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain of volume M. Then $\lambda_2(\Omega) \geq \lambda_2(D_2)$ with equality if and only if $\Omega = D_2$ in the sense of Remark 1.1(iii).*

To prove Theorem 2.1 we cannot directly apply the method used in the Dirichlet case (see for example [14, Section 4] and also [16, Section 2] for when $p = 2$; the arguments are the same when $p \neq 2$) since the nodal domains may not be smooth enough to apply the Faber-Krahn inequality, which is only known for Lipschitz domains (see [2]). The proof we give is a refinement of that in [16], which for $p = 2$ constructs an appropriate sequence of approximations to the nodal domain. A significant additional argument is needed to prove uniqueness of the minimiser.

Remark 2.2. When $p = 2$, Theorem 2.1 combined with [16, Example 2.2] shows that there is no minimiser of λ_2 amongst all *connected* domains of given volume, since we can find a sequence of connected Ω_n with $\lambda_2(\Omega_n) \to \lambda_2(D_2)$. A similar construct should work when $p \neq 2$, but we do not know of domain approximation results akin to those in [6] for this case.

Before we proceed with the proof of Theorem 2.1, we recall some properties of the eigenvalues and eigenfunctions of the problem (1.1).

Here for simplicity we will assume Ω is connected. We understand an eigenvalue $\lambda \in \mathbb{R}$ of (1.1) with eigenfunction $\psi \in W^{1,p}(\Omega)$ in the weak sense, as a solution of

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\varphi\,dx + \int_{\partial\Omega}\alpha|\psi|^{p-2}\psi\varphi\,d\sigma = \lambda\int_\Omega |\psi|^{p-2}\psi\varphi\,dx \tag{2.1}$$

for all $\varphi \in W^{1,p}(\Omega)$.

Proposition 2.3. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded, connected Lipschitz domain. Then*

(i) *there exists a sequence of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ of* (1.1)*, obtainable by the Ljusternik-Schnirelman (L-S) principle, of the form $0 < \lambda_1 < \lambda_2 \leq \ldots$;*

(ii) *the second L-S eigenvalue satisfies*
$$\lambda_2 = \inf\{\lambda > \lambda_1 : \lambda \text{ is an eigenvalue of } (1.1)\};$$

(iii) *the first eigenvalue $\lambda_1 > 0$ is simple and every eigenfunction ψ associated with λ_1 satisfies $\psi > 0$ or $\psi < 0$ in Ω;*

(iv) *only eigenfunctions associated with λ_1 do not change sign in Ω;*

(v) *every eigenfunction ψ of* (1.1) *lies in $W^{1,p}(\Omega)\cap C^{1,\eta}(\Omega)\cap C(\overline{\Omega})$ for some $0 < \eta < 1$.*

Proof. Parts (i)-(iv) are essentially contained in [18]. Although C^1 regularity of Ω is assumed there in order to derive (i) and $C^{1,\theta}$, $0 < \theta < 1$, is assumed for (ii)-(iv), a careful analysis of the proofs shows that only Lipschitz continuity of $\partial\Omega$ is needed, since all background results, including those in the appendices, are valid for Lipschitz domains. (The extra regularity of $\partial\Omega$ is needed only to prove extra boundary regularity of the eigenfunctions.) For (v), first note that by [8, Theorem 2.7], every eigenfunction $\psi \in L^\infty(\Omega)$ (see also Section 4 there). But now, as noted in [2, Section 2], the arguments in [17, pp. 466-7] imply that ψ is Hölder continuous on $\overline{\Omega}$. Also, by [21], $\nabla\psi$ is Hölder continuous inside Ω. □

To prove Theorem 2.1, we first reduce to the case that Ω is connected. For, suppose Theorem 2.1 holds for connected domains, and that $\Omega \neq D_2$ is not connected. There are two possibilities: either $\lambda_2(\Omega) = \lambda_2(\widetilde{\Omega})$ for some c.c. $\widetilde{\Omega}$ of Ω, or else there exist c.c.s Ω', Ω'' such that $\lambda_1(\Omega) = \lambda_1(\Omega')$, $\lambda_2(\Omega) = \lambda_1(\Omega')$. In the former case, if we let $\widetilde{D}_2$ be a scaled down version of D_2 with $|\widetilde{D}_2| = |\widetilde{\Omega}|$, since $\widetilde{\Omega}$ is connected we may apply Theorem 2.1 to get $\lambda_2(\widetilde{\Omega}) > \lambda_2(\widetilde{D}_2) \geq \lambda_2(D_2)$, where for the last step we have used Lemma A.3. In the latter case, let B', B'' be balls having the same volume as Ω', Ω'', respectively. Then by the Faber-Krahn inequality [2, Theorem 1.1], $\lambda_2(\Omega) \geq \max\{\lambda_1(\Omega'), \lambda_1(\Omega'')\} \geq \max\{\lambda_1(B'), \lambda_1(B'')\}$, and the latter maximum is minimised when $\lambda_1(B') = \lambda_1(B'') = \lambda_2(D_2)$. Finally, if $\lambda_2(\Omega) = \lambda_2(D_2)$ then equality everywhere in the above argument implies $|\Omega'| = |\Omega''| = M/2$ (also using strict monotonicity in Lemma A.3) and sharpness of the Faber-Krahn inequality [2, Theorem 1.1] implies $\Omega' = B'$, $\Omega'' = B''$; that is, $\Omega = D_2$.

So now suppose Ω is connected, and let $\psi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ be any eigenfunction associated with $\lambda_2(\Omega)$. Since ψ must change sign in Ω, the nodal domains $\Omega^+ := \{x \in \Omega : \psi(x) > 0\}$ and $\Omega^- := \{x \in \Omega : \psi(x) < 0\}$ are both nonempty and open. Set $\psi^+ := \max\{\psi, 0\}$, $\psi^- := \max\{-\psi, 0\}$; then we have $\psi^+, \psi^- \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, and

$$\nabla\psi^+ = \begin{cases} \nabla\psi & \text{if } \psi > 0 \\ 0 & \text{if } \psi \leq 0, \end{cases}$$

with an analogous formula for $\nabla\psi^-$ (see [11, Lemma 7.6]).

Let B^+, B^- be balls having the same volume as Ω^+, Ω^- respectively. We will show that $\lambda_2(\Omega) > \max\{\lambda_1(B^+), \lambda_1(B^-)\}$. By Lemma A.3 this maximum is minimal when $B^+ = B^-$ and $\lambda_1(B^+) = \lambda_1(B^-) = \lambda_2(D_2)$. Without loss of generality we only consider Ω^+. Let $\partial_e\Omega^+ := \partial\Omega^+ \cap \partial\Omega$ and $\partial_i\Omega^+ := \partial\Omega^+ \cap \Omega = \partial\Omega^+ \setminus \partial_e\Omega^+$ denote the exterior and interior parts of the boundary of Ω^+, respectively (note that $\partial_i\Omega^+$ will not be closed). We first show that a piece of $\partial_i\Omega^+$ must be smooth.

Lemma 2.4. *There exist $x_0 \in \Omega$ and $r > 0$ such that $\psi(x_0) = 0$, $B(x_0, r) \subset\subset \Omega$, $\nabla\psi(x) \neq 0$ for all $x \in B(x_0, r)$, and $\{x \in B(x_0, r) : \psi(x) = 0\}$ is a surface of class C^∞.*

Proof. We first show we can find $x_0 \in \partial_i\Omega^+$ with $\nabla\psi(x) \neq 0$ in a neighbourhood of x_0. Choose any $x \in \Omega^+$ close to $\partial_i\Omega^+$ and let $\delta_0 := \inf\{\delta > 0 : \partial B(x, \delta) \cap \partial_i\Omega^+ \neq \emptyset\}$. Then $B(x, \delta_0) \subset \Omega^+$ but there exists $x_0 \in \partial B(x, \delta_0) \cap \partial_i\Omega^+$.

We now apply a version of Hopf's Lemma for the p-Laplacian due to Vázquez. Since $\psi(x_0) = 0$, $\psi(x) > 0$ in $B(x, \delta_0)$ and $\psi \in C^1(\overline{B(x, \delta_0)})$, by [22, Theorem 5] we have $\frac{\partial\psi}{\partial\nu_B}(x_0) < 0$, where ν_B is the outer unit normal to $B(x, \delta_0)$. Hence $\nabla\psi(x_0) \neq 0$, and so by continuity of $\nabla\psi$ there exists a neighbourhood V_0 of x_0 and $m > 0$ such that $|\nabla\psi(x)| \geq m$ for all $x \in V_0$. In particular, inside V_0 we may write $-\Delta_p\psi = -\operatorname{div}(a(x)\nabla\psi)$, where $a(x) = |\nabla\psi(x)|^{p-2} \geq m^{p-2} > 0$. Since $\psi \in C^1(\overline{V_0})$ is an eigenfuction of the operator $-\operatorname{div}(a(x)\nabla u)$, a standard bootstrapping argument using elliptic regularity theory yields $\psi \in C^\infty(V_0)$. By the implicit function theorem it follows that the level surface $\{\psi = 0\}$ is locally the graph of a C^∞ function inside V_0. $\square$

Fix x_0 and r as in the lemma and set $\Gamma := \partial_i\Omega^+ \cap B(x_0, r/2)$ smooth; then the surface measure $\sigma(\Gamma) > 0$. We will impose Robin boundary conditions on Γ, strictly lowering the first eigenvalue of a suitable variational problem on Ω^+. To that end set $V_0 := \{\varphi \in W^{1,p}(\Omega^+) \cap C(\overline{\Omega^+}) : \varphi = 0 \text{ on } \partial_i\Omega^+ \setminus \Gamma\}$, for $\varphi \in V_0$ set

$$Q_p(\varphi) := \frac{\int_{\Omega^+} |\nabla\varphi|^p \, dx + \int_{\partial_e\Omega^+ \cup \Gamma} \alpha|\varphi|^p \, d\sigma}{\int_{\Omega^+} |\varphi|^p \, dx} \tag{2.2}$$

and let
$$\kappa(\Omega^+) := \inf_{\varphi \in V_0} Q_p(\varphi) \tag{2.3}$$
We may characterise $\lambda_2(\Omega)$ as follows. In an abuse of notation we will not distinguish between ψ^+ on Ω and $\psi^+|_{\Omega^+}$.

Lemma 2.5. *We have* $\psi^+ \in V_0$ *and*
$$\lambda_2(\Omega) = Q_p(\psi) = Q_p(\psi^+) \equiv \frac{\int_{\Omega^+} |\nabla \psi^+|^p \, dx + \int_{\partial_e \Omega^+} \alpha |\psi^+|^p \, d\sigma}{\int_{\Omega^+} |\psi^+|^p \, dx}. \tag{2.4}$$

Proof. We already know $\psi^+ \in V_0$, since $\psi^+ \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ is zero on $\partial_i \Omega^+$. To obtain (2.4), choose ψ^+ as a test function in the characterisation (2.1) of $\lambda_2(\Omega)$. Then $|\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \psi^+ = |\nabla \psi^+|^p$ in Ω and $|\psi|^{p-2}\psi\, \psi^+ = |\psi^+|^p$ pointwise in $\overline{\Omega}$. Since $\|\psi^+\|_p^p \neq 0$,
$$\lambda_2(\Omega) = \frac{\int_\Omega |\nabla \psi^+|^p \, dx + \int_{\partial\Omega} \alpha |\psi^+|^p \, d\sigma}{\int_\Omega |\psi^+|^p \, dx}. \tag{2.5}$$
Now (2.4) follows since $\{x \in \Omega : \psi^+(x) \neq 0\}$, $\{x \in \Omega : \nabla \psi^+(x) \neq 0\} \subset \Omega^+$, and the boundary integrand $\alpha|\psi^+|^p$ in (2.5) is nonzero only on $\partial_e \Omega^+$. Finally, $Q_p(\psi) = Q_p(\psi^+)$ is obvious since $\psi \equiv \psi^+$ on $\Omega^+ \cup \partial_e \Omega^+$. □

Lemma 2.6. $\lambda_2(\Omega) > \kappa(\Omega^+)$.

Proof. It is immediate from Lemma 2.5 and (2.3) that $\lambda_2(\Omega) \geq \kappa(\Omega^+)$. Suppose for a contradiction that we have equality. Then since $\lambda_2(\Omega)$ and ψ satisfy (2.3), we may also characterise them by
$$\int_{\Omega^+} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \varphi \, dx + \int_{\partial_e \Omega^+ \cup \Gamma} \alpha |\psi|^{p-2} \psi \varphi \, d\sigma = \lambda_2(\Omega) \int_{\Omega^+} |\psi|^{p-2} \psi \varphi \, dx$$
for all $\varphi \in V_0$. (This can be seen, for example, by solving
$$\frac{d}{dt}\Big(\frac{\int_{\Omega^+} |\nabla (\psi - t\varphi)|^p \, dx + \int_{\partial_e \Omega^+ \cup \Gamma} \alpha |\psi - t\varphi|^p \, d\sigma}{\int_{\Omega^+} |\psi - t\varphi|^p \, dx} \Big)\Big|_{t=0} = 0,$$
where $t \in \mathbb{R}$ and $\varphi \in V_0$.)

Now choose an open set $U \subset \Omega^+$ Lipschitz with $U \subset\subset \Omega$ and such that $\Gamma \subset \partial U$. (Recall $\partial_i \Omega^+$ is smooth in an open neighbourhood of $\overline{\Gamma}$.) For any $\varphi \in C_c^\infty(U \cup \Gamma) \subset V_0$,
$$\int_U |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \varphi \, dx + \int_\Gamma \alpha |\psi|^{p-2} \psi \varphi \, d\sigma = \lambda_2(\Omega) \int_U |\psi|^{p-2} \psi \varphi \, dx.$$
Also, since $-\Delta_p \psi = \lambda_2(\Omega) |\psi|^{p-2} \psi$ pointwise in U, a simple calculation gives
$$\int_U |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \varphi - \operatorname{div}(|\nabla \psi|^{p-2} \varphi \psi) \, dx = \lambda_2(\Omega) \int_U |\psi|^{p-2} \psi \varphi \, dx$$

for all $\varphi \in C_c^\infty(U \cup \Gamma)$. Applying the divergence theorem on U (see for example [10, Section 5.8]) and comparing the above identities,

$$\int_\Gamma \alpha|\psi|^{p-2}\psi\varphi\, d\sigma = -\int_\Gamma |\nabla\psi|^{p-2}\frac{\partial\psi}{\partial\nu}\varphi\, d\sigma$$

for all $\varphi \in C_c^\infty(U \cup \Gamma)$, where ν is the outward pointing unit normal to U (equivalently, Ω^+) on Γ. Since $C_c^\infty(U \cup \Gamma)$ is dense in $L^2(\Gamma)$, it follows that $\psi \in C^1(\overline{U})$ satisfies the boundary condition $|\nabla\psi|^{p-2}\frac{\partial\psi}{\partial\nu} + \alpha|\psi|^{p-2}\psi = 0$ pointwise in Γ. But we know $\psi = 0$ on Γ, while by Hopf's Lemma [22, Theorem 5] applied to U and $\psi \in C^1(\overline{U})$, we have $\frac{\partial\psi}{\partial\nu} > 0$ (and $|\nabla\psi| > 0$) on Γ, a contradiction. □

We will now construct a sequence of smooth domains U_n approximating Ω^+ from the outside, in order to overcome the possible lack of overall smoothness of $\partial\Omega^+$. As in [16, Section 3], we attach a "strip" near $\partial\Omega$ to Ω^+ to avoid the points where $\partial_e\Omega^+$ and $\partial_i\Omega^+$ meet. So fix $n \geq 1$ and set $S_n := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$, where $\delta = \delta(n)$ is chosen such that $|S_n| < 1/(2n)$. By [9, Theorem V.20] we can approximate $\Omega^+ \cup S_n$ from the outside by a sequence of smooth domains U_n as follows. Let $\Omega \supset U_n \supset \Omega^+ \cup S_n$ be such that $\partial U_n = \partial\Omega \cup \Gamma_n$, where $\Gamma_n \subset\subset \Omega$ is C^∞ and $|U_n \setminus (\Omega^+ \cup S_n)| < 1/(2n)$. We also impose the condition that $\Gamma \subset \Gamma_n$, which we can do since $\partial_i\Omega^+$ is C^∞ in an open neighbourhood $B(x_0, r) \subset \Omega$ containing $\overline{\Gamma}$. Then for any $n \geq 1$, U_n is Lipschitz, $|U_n \setminus \Omega^+| < 1/n$, and since $B(x_0, r) \subset\subset \Omega$, without loss of generality $\operatorname{dist}(\overline{U}_n \setminus \Omega^+, \Gamma) > 0$ as well. (See Figure 1.)

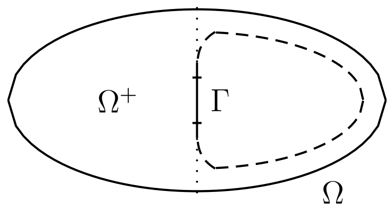

FIGURE 1. Ω^+ and U_n. The dotted line represents $\partial_i\Omega^+$ and the dashed line $\Gamma_n = \partial U_n \cap \Omega$.

In order to use the U_n, we need the following modification of the standard result that if $U \subset \mathbb{R}^N$ is open, arbitrary then functions in $W^{1,p}(U)$ vanishing continuously on ∂U lie in $W_0^{1,p}(U)$ (cf. [11, Section 7.5]).

Lemma 2.7. *Let $\varphi \in V_0$ and fix $n \geq 1$. The function $\tilde{\varphi} : U_n \to \mathbb{R}$ given by $\tilde{\varphi} = \varphi$ in Ω^+, $\tilde{\varphi} = 0$ in $U_n \setminus \Omega^+$ lies in $W^{1,p}(U_n)$.*

Proof. Let $\varphi \in V_0$ and $\tilde{\varphi}$ be as in the statement of the lemma. Using the lattice properties of V_0 and $W^{1,p}(U_n)$ (cf. [11, Lemma 7.6]) we may assume that $\varphi \geq 0$ in Ω^+. For $\xi > 0$ let $\varphi_\xi := (\varphi - \xi)^+ \in V_0$. Then by continuity of φ, there exists an open neighbourhood $U = U(\varphi, \xi)$ of $\partial_i\Omega^+ \setminus \Gamma$ such that $\varphi_\xi \equiv 0$ on $U \cap \overline{\Omega^+}$. Since the intersection of $U_n \setminus \Omega^+$

with $\overline{\Omega^+}$ is contained in $\partial_i\Omega^+ \setminus \Gamma$, we may certainly extend φ_ξ by 0 in $U_n \setminus \Omega^+$ to obtain a function $\tilde{\varphi}_\xi \in W^{1,p}(U_n)$. Since $\tilde{\varphi}_\xi \nearrow \tilde{\varphi}$ and

$$\nabla\tilde{\varphi}_\xi(x) \nearrow g(x) := \begin{cases} \nabla\varphi(x) & \text{if } x \in \Omega^+ \\ 0 & \text{if } x \in U_n \setminus \Omega^+ \end{cases}$$

pointwise monotonically in U_n as $\xi \to 0$, it follows easily that $g = \nabla\tilde{\varphi}$ and $\tilde{\varphi} \in W^{1,p}(U_n)$. □

For any $n \geq 1$ and $\varphi \in V_0$, using the extension $\tilde{\varphi} \in W^{1,p}(U_n)$ of φ in the representation

$$\lambda_1(U_n) = \inf_{\varphi \in W^{1,p}(U_n)} \frac{\int_{U_n} |\nabla\varphi|^p \, dx + \int_{\partial U_n} \alpha|\varphi|^p \, d\sigma}{\int_{U_n} |\varphi|^p \, dx}$$

we see $Q_p(\varphi) \geq \lambda_1(U_n)$. Hence $\kappa(\Omega^+) \geq \lambda_1(U_n)$ by (2.3). Now let B_n be a ball with $|B_n| = |U_n|$. By the Faber-Krahn inequality [2, Theorem 1.1], $\lambda_1(U_n) \geq \lambda_1(B_n)$. As $n \to \infty$, $|U_n| \to |\Omega^+|$ and so $\lambda_1(B_n) \to \lambda_1(B^+)$ by Lemma A.3. We conclude that $\lambda_2(\Omega) > \kappa(\Omega^+) \geq \limsup_{n\to\infty} \lambda_1(U_n) \geq \lambda_1(B^+)$, which in light of our earlier comments completes the proof.

3. On the higher eigenvalues of the Robin problem

From now on we will assume $p = 2$ in (1.1). We will consider the problem (1.2) for $k \geq 3$ fixed. In contrast to the Dirichlet case, this is not one problem but a family depending on the parameter $\alpha > 0$. Here we will show that one cannot in general find a solution to (1.2) independent of α (alternatively, of the volume M). Roughly speaking, for large α we are close to the corresponding Dirichlet problem, while for α close to 0 (a Neumann problem), the domain D_k consisting of the disjoint union of k equal balls is in some sense a minimiser. We will denote by B_m a ball of volume m, so that D_k is the disjoint union of k copies of $B_{M/k}$, and $\lambda_k(D_k, \alpha) = \lambda_1(D_k, \alpha) = \lambda_1(B_{M/k}, \alpha)$.

Theorem 3.1. *Let $p = 2$ in* (1.1).

(i) *Given any bounded Lipschitz $\Omega \subset \mathbb{R}^N$ such that $\Omega \neq D_k$ in the sense of Remark 1.1(iii), there exists $\alpha_\Omega > 0$ possibly depending on Ω such that $\lambda_k(\Omega, \alpha) > \lambda_k(D_k, \alpha)$ for all $\alpha \in (0, \alpha_\Omega)$.*

(ii) *There exist $N \geq 2$ and $k \geq 3$ for which, given $M > 0$, there is no solution to* (1.2) *independent of α; equivalently, there is no domain D satisfying $\lambda_k(\Omega, \alpha) \geq \lambda_k(D, \alpha)$ for all $\alpha \in (0, \infty)$ and all Ω.*

(iii) *There exist $N \geq 2$ and $k \geq 3$ for which, given $\alpha > 0$, there is no solution to* (1.2) *independent of $M > 0$.*

Remark 3.2. (i) The conclusion of Theorem 3.1(ii) and (iii) holds whenever D_k does not minimise the kth Dirichlet eigenvalue μ_k. When

$N = 2$ this is true for all $k \geq 3$ (we prove this below) and when $N = 3$ at least for $k = 3$ (for the latter see [3, Section 3]).

(ii) It is easy to see (ii) and (iii) are equivalent assertions, since by making the homothety substitution $x \mapsto \alpha x$, (1.1) is equivalent to the problem $-\Delta u = (\lambda/\alpha^2)u$ in $\alpha\Omega = \{\alpha x : x \in \Omega\}$, $\frac{\partial u}{\partial \nu} + u = 0$ on $\partial(\alpha\Omega)$.

(iii) It is clear that any domain Ω with more than k connected components (c.c.s) cannot minimise λ_k for any value of α. However, the theorem makes a stronger statement than this and as a result the proof is somewhat more involved. Indeed, for some k, N, we can easily find a domain Ω_n with any $n \geq 1$ c.c.s and $\alpha_\Omega < \infty$. (Just take $N = k = 3$, so that for the ball B, $\alpha_B < \infty$. Shrink B slightly and add $n - 1$ disjoint tiny balls to get Ω_n.) Note that the Robin problem (1.1) lacks many useful properties that the corresponding Dirichlet problem satisfies. For example, the domain monotonicity property fails; that is, $U \subset V$ does not necessarily imply $\lambda_k(U, \alpha) \geq \lambda_k(V, \alpha)$ (see [20] or [12] for a counterexample). Similarly, if $\lambda_k(U, \alpha) > \lambda_k(V, \alpha)$ holds for some $\alpha > 0$, we cannot in general expect this for all $\alpha > 0$.

(iv) An examination of our proof shows that the conclusion of Theorem 3.1(i) holds for any domain Ω for which the Faber-Krahn inequality [2, Theorem 1.1] and Theorem 2.1 hold

Proof of Theorem 3.1(i). There are two cases to consider, depending on how many c.c.s Ω has.

(i) Suppose first that Ω has at most $k - 1$ c.c.s. If we set $\varepsilon := \min\{\lambda_2(\widetilde{\Omega}, 0) : \widetilde{\Omega}$ is a c.c. of $\Omega\}$, then $\varepsilon > 0$ by Lemma A.2. It follows from Lemma A.1(i) that there exists $\tilde{\alpha}_\Omega > 0$ such that

$$\max\{\lambda_1(\widetilde{\Omega}, \alpha) : \widetilde{\Omega} \text{ is a c.c. of } \Omega\} < \varepsilon$$

for all $\alpha \in (0, \tilde{\alpha}_\Omega)$. For all such α, by the pigeonhole principle at least one element of the set $\{\lambda_m(\widetilde{\Omega}, \alpha) : m \geq 2, \widetilde{\Omega}$ is a c.c. of $\Omega\}$ must be one of the first k eigenvalues of Ω (although precisely which m and c.c. may depend on α). In particular, using Lemma A.1(i),

$$\begin{aligned}\lambda_k(\Omega, \alpha) &\geq \inf\{\lambda_m(\widetilde{\Omega}, \alpha) : m \geq 2,\ \widetilde{\Omega} \text{ is a c.c. of } \Omega\} \\ &\geq \inf\{\lambda_2(\widetilde{\Omega}, 0) : \widetilde{\Omega} \text{ is a c.c. of } \Omega\} \geq \varepsilon\end{aligned}$$

for all $\alpha \in (0, \tilde{\alpha}_\Omega)$. Since $\lambda_k(D_k, \alpha) = \lambda_1(D_k, \alpha) \to 0$ as $\alpha \to 0$, there exists $\alpha_\Omega \leq \tilde{\alpha}_\Omega$ such that $\lambda_k(D_k, \alpha) < \varepsilon \leq \lambda_k(\Omega, \alpha)$ for all $\alpha \in (0, \alpha_\Omega)$.

(ii) Now suppose Ω has at least k c.c.s. We may write Ω as the disjoint union of Ω' and Ω'', where Ω' has $j < \infty$ c.c.s and $|\Omega''| < M/k$ (if $\Omega'' = \emptyset$, then we declare $\lambda_1(\Omega'', \alpha) = \infty$ for all $\alpha > 0$). Consider all possible open subdomains Ω_i of Ω', where Ω_i consists of $l_i \leq k-1$ c.c.s of Ω' (thus there are fewer than 2^j possible choices of Ω_i). For each i, let $D_{k,i}$ denote a scaled down version of D_k such that $|D_{k,i}| = |\Omega_i|$. Then by case (i) and Lemma A.3, there exists $\alpha_i := \alpha_{\Omega_i}$ such that

$$\lambda_k(\Omega_i, \alpha) > \lambda_k(D_{k,i}, \alpha) \geq \lambda_k(D_k, \alpha) \tag{3.1}$$

for all $\alpha \in (0, \alpha_i)$.

Set $\alpha_\Omega := \min_i \alpha_i > 0$ and fix $\alpha \in (0, \alpha_\Omega)$. We will show $\lambda_k(\Omega, \alpha) \geq \lambda_k(D_k, \alpha)$ with equality only if $\Omega = D_k$ in the sense of Remark 1.1(iii).

First suppose $\lambda_1(\Omega'', \alpha) \leq \lambda_k(\Omega, \alpha)$. Then by the Faber-Krahn inequality [2, Theorem 1.1] and Lemma A.3

$$\lambda_k(\Omega, \alpha) \geq \lambda_1(\Omega'', \alpha) \geq \lambda_1(B_{M/k}) = \lambda_k(D_k, \alpha). \tag{3.2}$$

Since $|\Omega''| < M/k$, Lemma A.3 implies that the second inequality in (3.2) must be strict.

So assume now that $\lambda_1(\Omega'', \alpha) > \lambda_k(\Omega, \alpha)$. There are two subcases to consider. First, if there are only $l < k$ c.c.s $\Omega_1, \ldots, \Omega_l$ of Ω' whose first eigenvalue is smaller than $\lambda_k(\Omega, \alpha)$, then setting $\widehat{\Omega}$ to be the disjoint union of $\Omega_1, \ldots, \Omega_l$, by (3.1) we have

$$\lambda_k(\Omega, \alpha) = \lambda_k(\widehat{\Omega}, \alpha) > \lambda_k(D_k, \alpha)$$

by choice of α_Ω and $\alpha < \alpha_\Omega$. Finally, suppose there are at least k c.c.s Ω_i of Ω' such that $\lambda_1(\Omega_i, \alpha) \leq \lambda_k(\Omega, \alpha)$ for all i. Then $\lambda_k(\Omega, \alpha) = \max_{1 \leq i \leq k} \lambda_1(\Omega_i, \alpha)$. For each i let B_i be a ball with $|B_i| = |\Omega_i|$. By the Faber-Krahn inequality $\lambda_1(\Omega_i, \alpha) \geq \lambda_1(B_i)$ for all i and thus

$$\lambda_k(\Omega, \alpha) \geq \max_i \lambda_1(B_i, \alpha) \geq \lambda_1(B_{M/k}, \alpha) = \lambda_k(D_k, \alpha), \tag{3.3}$$

where the second inequality in (3.3) follows easily from Lemma A.3 using $\sum_i |B_i| \leq |\Omega|$. If there is equality in (3.3), then for every $1 \leq i \leq k$, $\lambda_1(\Omega_i, \alpha) = \lambda_1(B_i, \alpha) = \lambda_1(B_{M/k}, \alpha)$ and so $\Omega_i = B_i = B_{M/k}$ using sharpness of the Faber-Krahn inequality [2, Theorem 1.1] and Lemma A.3, respectively. In this case $|\Omega_i| = M/k$ and so Ω must consist of k copies of $\Omega_i = B_{M/k}$, so $\Omega = D_k$. □

In order to complete the proof of the theorem and our claim in Remark 3.2(i), we will use the following lemma. Recall $\mu_k(\Omega)$ denotes the kth eigenvalue of the Dirichlet Laplacian (with $p = 2$) on Ω.

Lemma 3.3. *Let $N = 2$ and fix $k \geq 3$. The domain D_k does not minimise $\mu_k(\Omega)$ amongst all bounded Lipschitz domains in $\mathbb{R}^2$ of given volume.*

Proof. The proof is by an easy induction argument, using results from [23]. First note that D_k does not even minimise μ_k amongst all disjoint unions of balls if $3 \leq k \leq 17$ (see [23, Section 8]).

Now fix $k \geq 4$. We will show that if D_{k+1} minimises μ_{k+1}, then D_j must minimise μ_j for some $3 \leq j \leq k$. For, arguing as in [23, Theorem 8.1], D_{k+1} may be written as the disjoint union of open sets U and V, say, where U minimises μ_j and V minimises μ_{k-j+1} (both appropriately scaled) for some integer j between 1 and $k/2$. Now U and V must both be disjoint unions of equal balls, and since the minimiser of μ_j can have at most j c.c.s the only possibility is that $U = D_j$ and $V = D_{k-j+1}$ (both rescaled). Since $k \geq 4$, at least one of j, $k - j + 1$

must be at least 3. Noting that the Dirichlet minimiser is independent of the volume of the domain, our claim follows. □

Proof of Theorem 3.1(ii) and Remark 3.2(i). Suppose that D_k is not the minimiser of μ_k, which is true if $N = 2$ and $k \geq 3$ or $N = k = 3$. Then there exists a Lipschitz domain V such that $\mu_k(V) < \mu_k(D_k)$. By Lemma A.1(ii) and (iii), we have $\lambda_k(V, \alpha) < \mu_k(V)$ and $\lambda_k(D_k, \alpha) = \lambda_1(D_k, \alpha) \to \mu_1(D_k) = \mu_k(D_k)$ as $\alpha \to \infty$. Using continuity, it follows that for α sufficiently large, $\lambda_k(V, \alpha) < \mu_k(D_k, \alpha)$. Hence D_k does not minimise λ_k for all $\alpha \in (0, \infty)$. However, if $U \neq D_k$ is any (Lipschitz) domain which minimises λ_k for some $\tilde{\alpha} \in (0, \infty)$, then by part (i) $\lambda_k(U, \alpha) > \lambda_k(D_k, \alpha)$ for $\alpha < \tilde{\alpha}$ sufficiently small. Hence for such N and k no minimiser can exist for all $\alpha > 0$. □

4. On the higher eigenvalues of the Wentzell Laplacian

Here we will study the Laplacian with generalised Wentzell boundary conditions (1.3). This problem has been extensively studied in recent years; see for example [13, 19] and the references therein. We will denote by $\Lambda_k = \Lambda_k(\Omega, \beta, \gamma)$ the kth eigenvalue, with repeated eigenvalues counted according to their multiplicity. It was proved in [15] that if $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, then

$$\Lambda_1(\Omega, \beta, \gamma) \geq \Lambda_1(B, \beta, \gamma) \tag{4.1}$$

for all $\beta, \gamma > 0$. (As before B is a ball having the same volume M as Ω.) Moreover, the inequality is sharp if Ω is of class C^2. Note that combining the improved sharpness result in [2] for Robin problems with the method in [15], we immediately get sharpness of the Wentzell inequality (4.1) for all bounded Lipschitz domains. We will prove the following results which basically say that the minimisation problems for the Robin and Wentzell Laplacians are essentially the same.

Theorem 4.1. *Let $\beta, \gamma > 0$ and $k \geq 2$ be fixed, let $D \subset \mathbb{R}^N$ be a bounded Lipschitz domain, and let $D_k \subset \mathbb{R}^N$ be as in Section 3.*

(i) *Suppose that for every bounded Lipschitz $\Omega \subset \mathbb{R}^N$ we have*

$$\lambda_k(D, \alpha) \leq \lambda_k(\Omega, \alpha) \tag{4.2}$$

for all $\alpha \in (0, \gamma/\beta)$. Then

$$\Lambda_k(D, \beta, \gamma) \leq \Lambda_k(\Omega, \beta, \gamma) \tag{4.3}$$

for all such Ω. Conversely, if (4.3) *holds, then* (4.2) *holds for some $\alpha \in (0, \gamma/\beta)$.*

(ii) *If* (4.2) *is sharp for all $\alpha \in (0, \gamma/\beta)$, then so is* (4.3) *for this β, γ. If* (4.3) *is sharp, then* (4.2) *holds and is sharp for some $\alpha \in (0, \gamma/\beta)$.*

(iii) *Suppose $\Omega \subset \mathbb{R}^N$ is bounded, Lipschitz. There exists $\alpha_\Omega > 0$ possibly depending Ω such that $\Lambda_k(\Omega, \beta, \gamma) > \Lambda_k(D_k, \beta, \gamma)$ for all β, γ with $\gamma/\beta < \alpha_\Omega$.*

(iv) *If for some k and N the conclusion of Theorem 3.1 holds, then there does not exist $D \subset \mathbb{R}^N$ bounded, Lipschitz such that $\Lambda_k(\Omega, \beta, \gamma) \geq \Lambda_k(D, \beta, \gamma)$ for all such Ω and all $\beta, \gamma > 0$.*

(v) *For any bounded, Lipschitz $\Omega \subset \mathbb{R}^N$ and any $\beta, \gamma > 0$, we have $\Lambda_2(\Omega, \beta, \gamma) \geq \Lambda_2(D_2, \beta, \gamma)$, with equality if and only if $\Omega = D_2$.*

In order to prove the theorem we will need some preliminary results. In what follows we will assume that $\beta, \gamma > 0$ and $k \geq 2$ are fixed, and $\Omega \subset \mathbb{R}^N$ is a fixed bounded Lipschitz domain. We start with an elementary identification which is the key to the approach.

Lemma 4.2. *Let $k \geq 1$ and $\alpha := (\gamma - \Lambda_k(\Omega, \beta, \gamma))/\beta \in \mathbb{R}$. Then*

$$\Lambda_k(\Omega, \beta, \gamma) = \lambda_k(\Omega, \alpha). \tag{4.4}$$

Proof. Consider the family of curves $g_n : \mathbb{R} \to \mathbb{R}$, $g_n(\alpha) := (\gamma - \lambda_n(\Omega, \alpha))/\beta$, $n \geq 1$, where we allow multiplicities in counting the λ_n (thus if $\lambda_n(\Omega, \tilde{\alpha}) = \lambda_{n+1}(\Omega, \tilde{\alpha})$ for some $\tilde{\alpha} \in \mathbb{R}$, then $g_n(\tilde{\alpha}) = g_{n+1}(\tilde{\alpha})$).

We know that the set of Wentzell eigenvalues $\{\Lambda_k : k \geq 1\}$ is in one-to-one correspondence with the set of fixed points $\{\alpha \in \mathbb{R} : g_n(\alpha) = \alpha \text{ for some } n\}$, via the identification as in [15, Proposition 3.3] (see also Remark 3.6(i) there). In particular, we know that $\Lambda_k(\Omega, \beta, \gamma) = \lambda_n(\Omega, \alpha)$ with $\alpha = (\gamma - \Lambda_k)/\beta$ for some $n \geq 1$; we have to show $n = k$.

Now by Lemma A.1(i) each curve g_n is a continuous and monotonically decreasing function of α. In particular for each n there will be exactly one fixed point $\alpha_n \in \mathbb{R}$ for which $g_n(\alpha_n) = \alpha_n$. Moreover, by definition $g_n(\alpha) \leq g_m(\alpha)$ whenever $n \geq m$ and hence $\alpha_n \leq \alpha_m$ if $n \geq m$. It follows inductively that $\lambda_n(\Omega, \alpha_n) = \gamma - \alpha_n\beta$ is the nth Wentzell eigenvalue $\Lambda_n(\Omega, \beta, \gamma)$ for all $n \geq 1$. □

Note that we have $0 < \Lambda_1(\Omega, \beta, \gamma) = \gamma - \alpha\beta$ for some $\alpha > 0$ (see [15, Remark 5.2]). In particular, we obtain the bound $\Lambda_1(\Omega, \beta, \gamma) < \gamma$ always, *independent of the volume of* Ω. This yields the following result, which obviously remains true if we replace D_k by any domain Ω having at least k c.c.s.

Lemma 4.3. *We have $\Lambda_k(D_k, \beta, \gamma) < \gamma$ for all $k \geq 1$.*

Proof. As in Section 3, we write D_k as the disjoint union of k balls $B_{M/k}$. Then $\Lambda_k(D_k, \beta, \gamma) = \Lambda_1(B_{M/k}, \beta, \gamma) < \gamma$. □

We are now in a position to give the proof of Theorem 4.1. Since β, γ are fixed we will write $\Lambda_k(\Omega, \beta, \gamma) = \Lambda_k(\Omega)$ if there is no danger of confusion. The following lemma contains the core of the argument.

Lemma 4.4. *Let $\beta, \gamma > 0$ be given and $U, V \subset \mathbb{R}^N$ bounded, Lipschitz.*

(i) *If $\Lambda_k(U) < \gamma$, then for $\alpha := (\gamma - \Lambda_k(U))/\beta$,*

$$\lambda_k(U, \alpha) \geq \lambda_k(V, \alpha) \tag{4.5}$$

implies

$$\Lambda_k(U) \geq \Lambda_k(V). \tag{4.6}$$

If the equality in (4.5) *is strict, then it is also strict in* (4.6).

(ii) *Suppose* $\Lambda_k(V) < \gamma$ *and let* $\alpha := (\gamma - \Lambda_k(V))/\beta$. *If* (4.6) *holds (resp. is strict), then* (4.5) *holds (resp. is strict) for this* α.

Proof. (i) Suppose (4.5) holds but (4.6) fails. Using Lemma 4.2 and (4.5) respectively,

$$\begin{aligned}\Lambda_k(U) &= \lambda_k(U, \frac{\gamma - \Lambda_k(U)}{\beta})\\ &\geq \lambda_k(V, \frac{\gamma - \Lambda_k(U)}{\beta}) \geq \lambda_k(V, \frac{\gamma - \Lambda_k(V)}{\beta}) = \Lambda_k(V),\end{aligned}$$

where the second inequality follows from Lemma A.1(i) since $\gamma - \Lambda_k(U) \geq \gamma - \Lambda_k(V)$ by the contradiction assumption. Hence $\Lambda_k(U) \geq \Lambda_k(V)$, contradicting the assumption that (4.6) fails. Now suppose (4.5) is strict and the contradiction assumption becomes $\Lambda_k(U) \leq \Lambda_k(V)$. Since the first inequality in the above line of reasoning is now strict, we still obtain a contradiction as nothing else changes. Hence we cannot have equality in (4.6).

(ii) Now suppose that (4.6) holds and that (4.5) fails. Interchanging the roles of U and V, we may argue essentially exactly as in (i) to obtain the desired conclusion (and do similarly for strictness). □

Proof of Theorem 4.1. (i) Suppose D satisfies (4.2). Let $(\Omega_m)_{m\in\mathbb{N}}$ be a minimising sequence for Λ_k. By Lemma 4.3, we may assume $\Lambda_k(\Omega_m) < \gamma$ for all m, so that $(\gamma - \Lambda_k(\Omega_m))/\beta \in (0, \gamma/\beta)$ and thus (4.2) holds for these values of α. Fixing $m \in \mathbb{N}$, we may apply Lemma 4.4(i) with Ω_m in place of U and D in place of V to conclude $\Lambda_k(\Omega_m) \geq \Lambda_k(D)$. Since $(\Omega_m)_{m\in\mathbb{N}}$ was a minimising sequence, D must minimise $\Lambda_k(\Omega)$. For the converse, suppose D satisfies (4.3). Since $\Lambda_k(D) < \gamma$ by Lemma 4.3, it follows directly from Lemma 4.4(ii) that D satisfies (4.2) for $\alpha = (\gamma - \Lambda_k(D))/\beta$.

(ii) Sharpness in both directions now follows immediately from strictness of the inequalities in Lemma 4.4.

(iii) Fix $\Omega \neq D_k$. By Theorem 3.1(i), there exists $\alpha_\Omega > 0$ such that $\lambda_k(\Omega, \alpha) > \lambda_k(D_k, \alpha)$ for all $\alpha \in (0, \alpha_\Omega)$. If β, γ are fixed with $\gamma/\beta < \alpha_\Omega$, then we have $\lambda_k(\Omega, \alpha) > \lambda_k(D_k, \alpha)$ for $\alpha = (\gamma - \Lambda_k(\Omega))/\beta$ in particular. Since also $\Lambda_k(D_k) < \gamma$ by Lemma 4.3, without loss of generality we may assume $\Lambda_k(\Omega) < \gamma$ (otherwise $\Lambda_k(\Omega) \geq \gamma > \Lambda_k(D)$ and we are done). But in this case it follows from Lemma 4.4(i) (with $\Omega = U$) that $\Lambda_k(\Omega) > \Lambda_k(D_k)$ anyway.

(iv) Let k and N be such that the conclusion of Theorem 3.1(ii) holds. By (iii) it suffices to show there exist $\beta, \gamma > 0$ and a domain Ω with $\Lambda_k(\Omega, \beta, \gamma) < \Lambda_k(D_k, \beta, \gamma)$. Choose Ω and $\alpha^* > 0$ such that $\lambda_k(\Omega, \alpha^*) < \lambda_k(D_k, \alpha^*)$. Now we may write $\Lambda_k(D_k, \beta, \gamma) = \Lambda_1(D_k, \beta, \gamma) = \gamma - \alpha\beta$, where α satisfies $(\gamma - \lambda_1(D_k, \alpha))\beta = \alpha$. Since $\lambda_1(D_k, \alpha)$ is continuous and monotonic with respect to α, an elementary argument shows that by fixing β and varying γ, we may obtain every $\alpha > 0$ as a solution to $(\gamma - \lambda_1(D_k, \alpha))\beta = \alpha$ for some $\beta, \gamma > 0$. Now choose β, γ such that $\Lambda_k(D_k, \beta, \gamma) = \gamma - \alpha^*\beta$. For this β, γ, we may apply Lemma 4.4(i) with $U = D_k$ and $V = \Omega$ to conclude $\Lambda_k(D_k, \beta, \gamma) > \Lambda_k(\Omega, \beta, \gamma)$.

(v) This follows immediately from (i) and (ii) combined with Theorem 2.1. □

Appendix A. Some basic eigenvalue properties

Here we collect some elementary but useful facts about the behaviour of the eigenvalues of the Robin and Neumann Laplacians.

Lemma A.1. *Suppose $\Omega \subset \mathbb{R}^N$ is a fixed Lipschitz domain and $p = 2$. Then the following assertions are true.*

(i) *Let $k \geq 1$. Then $\lambda_k(\Omega, \alpha)$ is continuous and monotonically increasing as a function of $\alpha \in \mathbb{R}$.*
(ii) *For any $\alpha \geq 0$ and $k \geq 1$, we have $\lambda_k(\Omega, \alpha) < \mu_k(\Omega)$.*
(iii) *$\lambda_1(\Omega, \alpha) \to \mu_1(\Omega)$ as $\alpha \to \infty$.*

Proof. Parts (i) and (ii) follow immediately from the minimax formula for the kth eigenvalue (see [4, Section VI.1]. Note that although [4] only deals with the case $N = 2$, none of the relevant arguments depend on the dimension of the space). For part (iii), see for example [12]. □

Our next lemma expresses in our notation the well-known fact that the first Neumann eigenvalue of a connected domain is simple (with the constant functions the only eigenfunctions). We omit the proof.

Lemma A.2. *Let $p = 2$. If Ω is connected, then $\lambda_2(\Omega, 0) > 0$.*

The following equally well-known result is true in general for the kth eigenvalue of (1.1) on any reasonably smooth domain, although we only need this for the first eigenvalue of a ball. A proof (for balls) can be found in [2, Lemma 4.1].

Lemma A.3. *Suppose $1 < p < \infty$. Let B_m denote the ball of volume m, centred at the origin. For $\alpha > 0$ fixed, $\lambda_1(B_m, \alpha)$ is a strictly decreasing, continuous function of $m > 0$.*

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