STRONGLY MINIMAL PD_4-COMPLEXES

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Abstract. We consider the homotopy types of PD_4-complexes X with fundamental group π such that $c.d.\pi = 2$ and π has one end. Our main result is that (modulo two technical conditions) the orbits of the k-invariants determining "strongly minimal" complexes (i.e., those with homotopy intersection pairing λ_X trivial) correspond to elements of a subset of $H^2(\pi;\mathbb{F}_2)$. This group has order 2 when π is a PD_2-group. In general, the homotopy type of a PD_4-complex X with π a PD_2-group is determined by π, $w_1(X)$, λ_X and the v_2-type of X. Our result also implies that Fox's 2-knot with metabelian group is determined up to homeomorphism by its group.

It remains an open problem to give a homotopy classification of closed 4-manifolds, or more generally PD_4-complexes, in terms of standard invariants such as the fundamental group, characteristic classes and homotopy intersection pairings. The class of groups of cohomological dimension at most 2 seems to be both tractable and of direct interest to geometric topology, as it includes all surface groups, knot groups and the groups of many other bounded 3-manifolds. In our earlier papers we have shown that this case can largely be reduced to the study of "strongly minimal" PD_4-complexes Z with trivial intersection pairing on $\pi_2(Z)$. If X is a PD_4-complex with fundamental group π, $k_1(X) = 0$ and there is a 2-connected degree-1 map $p : X \to Z$, where Z is strongly minimal then the homotopy type of X is determined by Z and the intersection pairing λ_X on the "surgery kernel" $K_2(p) = \mathrm{Ker}(\pi_2(p))$, which is a finitely generated projective left $\mathbb{Z}[\pi]$-module [16].

The first two sections review material about generalized Eilenberg-Mac Lane spaces and cohomology with twisted coefficients, the Whitehead quadratic functor and PD_4-complexes. We assume thereafter that X is a PD_4-complex, $\pi = \pi_1(X)$ and $c.d.\pi = 2$. Such complexes have strongly minimal models $p : X \to Z$. In §3 we show that the homotopy type of X is determined by its first three homotopy groups and the second k-invariant $k_2(X) \in H^4(L_\pi(\pi_2(X), 2); \pi_3(X))$.

The key special cases in which the possible strongly minimal models are known are reviewed in §4. These are when

(1) $\pi \cong F(r)$ is a finitely generated free group;
(2) $\pi = F(r) \rtimes Z$; or
(3) π is a PD_2-group.

We review the first two cases briefly in §4, and in §5 we outline an argument for the case of PD_2-groups, which involves cup product in integral cohomology. (This is a model for our later work in Theorem 13.) In Theorem 8 we show that the homotopy type of a PD_4-complex X with π a PD_2-group is determined by π, $w_1(X)$, λ_X and the v_2-type of X. (The corresponding result was already known for π free and in

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the Spin case when π is a PD_2-group.) In §6 we assume further that π has one end, and give a partial realization theorem for k-invariants (Theorem 9); we do not know whether the 4-complexes we construct all satisfy Poincaré duality. In §7 and §8 we extend the cup product argument sketched in §5 to a situation involving local coefficient systems, to establish our main result (Theorem 13). Here we show that the number of homotopy types of minimal PD_4-complexes is bounded by the order of $H^2(\pi;\mathbb{F}_2)$, provided that $\mathbb{Z}^w \otimes_\pi \Gamma_W(\pi_2(X))$ has no 2-torsion. (However we do not have an explicit invariant.) This hypothesis fails for π a PD_2-group and $w_1(\pi)$ or w nontrivial, and thus our result is far from ideal. Nevertheless it holds in other interesting cases, notably when $\pi = Z*_m$ (with m even) and $w = 1$. (See §9.) In the final sections we consider other groups arising in low-dimensional topology. In particular, if π is the group of a fibred ribbon 2-knot K the knot manifold $M(K)$ is determined up to TOP s-cobordism by π, while Example 10 of Fox's "Quick Trip Through Knot Theory" [11] is determined up to homeomorphism by its group.

1. GENERALITIES

Let X be a topological space with fundamental group π and universal covering space $\widetilde{X}$, and let $f_{X,k} : X \to P_k(X)$ be the k^{th} stage of the Postnikov tower for X. We may construct $P_k(X)$ by adjoining cells of dimension at least $k+2$ to kill the higher homotopy groups of X. The map $f_{X,k}$ is then given by the inclusion of X into $P_k(X)$, and is a $(k+1)$-connected map. In particular, $P_1(X) \simeq K = K(\pi,1)$ and $c_X = f_{X,1}$ is the classifying map for the fundamental group $\pi = \pi_1(X)$.

Let $[X;Y]_K$ be the set of homotopy classes over K of maps $f : X \to Y$ such that $c_X = c_Y f$. If M is a left $\mathbb{Z}[\pi]$-module let $L_\pi(M,n)$ be the generalized Eilenberg-Mac Lane space over K realizing the given action of π on M. Thus the classifying map for $L = L_\pi(M,n)$ is a principal $K(M,n)$-fibration with a section $\sigma : K \to L$. We may view L as the ex-K loop space $\overline{\Omega}L_\pi(M,n+1)$, with section σ and projection c_L. Let $\mu : L \times_K L \to L$ be the (fibrewise) loop multiplication. Then $\mu(id_L, \sigma c_L) = \mu(\sigma c_L, id_L) = id_L$ in $[L;L]_K$. Let $\iota_{M,n} \in H^n(L;M)$ be the characteristic element. The function $\theta : [X,L]_K \to H^n(X;M)$ given by $\theta(f) = f^*\iota_{M,n}$ is a isomorphism with respect to the addition on $[X,L]_K$ determined by μ. Thus $\theta(id_L) = \iota_{M,n}$, $\theta(\sigma c_X) = 0$ and $\theta(\mu(f,f')) = \theta(f) + \theta(f')$. (See Definition III.6.5 of [3].)

Let Γ_W be the quadratic functor of J.H.C.Whitehead and let $\gamma_A : A \to \Gamma_W(A)$ be the universal quadratic function, for A an abelian group. The natural epimorphism from A onto $A/2A = \mathbb{F}_2 \otimes A$ is quadratic, and so induces a canonical epimorphism from $\Gamma_W(A)$ to $A/2A$. The kernel of this epimorphism is the image of the symmetric square $A \odot A$. If A is a $\mathbb{Z}$-torsion-free left $\mathbb{Z}[\pi]$-module the sequence

$$0 \to A \odot A \to \Gamma_W(A) \to A/2A \to 0$$

is an exact sequence of left $\mathbb{Z}[\pi]$-modules, when $A \odot A$ and $\Gamma_W(A)$ have the diagonal left π-action.

There is a natural exact sequence of $\mathbb{Z}[\pi]$-modules

$$(1) \qquad \pi_4(X) \xrightarrow{hwz_4} H_4(\widetilde{X};\mathbb{Z}) \to \Gamma_W(\Pi) \to \pi_3(X) \xrightarrow{hwz_3} H_3(\widetilde{X};\mathbb{Z}) \to 0,$$

where hwz_q is the Hurewicz homomorphism in dimension q. (See Chapter 1 of [4].)

Let $w : \pi \to \{\pm 1\}$ be a homomorphism, and let $\varepsilon_w : \mathbb{Z}[\pi] \to \mathbb{Z}^w$ be the w-twisted augmentation, given by w on elements of π. Let $I_w = \mathrm{Ker}(\varepsilon_w)$. If N is a right $\mathbb{Z}[\pi]$-module let $\overline{N}$ denote the conjugate left module determined by $g.n = w(g)n.g^{-1}$

for all $g \in \pi$ and $n \in N$. If M is a left $\mathbb{Z}[\pi]$-module let $M^\dagger = \overline{Hom_\pi(M, \mathbb{Z}[\pi])}$. The higher extension modules are naturally right modules, and we set $E^iM = \overline{Ext^i_{\mathbb{Z}[\pi]}(M, \mathbb{Z}[\pi])}$. In particular, $E^0M = M^\dagger$ and $E^i\mathbb{Z} = \overline{H^i(\pi; \mathbb{Z}[\pi])}$.

Lemma 1. *Let M be a $\mathbb{Z}[\pi]$-module with a finite resolution of length n and such that $E^iM = 0$ for $i < n$. Then $Aut_\pi(M) \cong Aut_\pi(E^nM)$.*

Proof. Since $c.d.\pi \leq 2$ and $E^iM = 0$ for $i < n$ the dual of a finite resolution for M is a finite resolution for E^nM. Taking duals again recovers the original resolution, and so $E^nE^nM \cong M$. If $f \in Aut(M)$ it extends to an endomorphism of the resolution inducing an automorphism E^nf of E^nM. Taking duals again gives $E^nE^nf = f$. Thus $f \mapsto E^nf$ determines an isomorphism $Aut_\pi(M) \cong Aut_\pi(E^nM)$. □

In particular, if π is a duality group of dimension n over $\mathbb{Z}$ and $D = H^n(G; \mathbb{Z}[G])$ is the dualizing module then $\overline{D} = E^n\mathbb{Z}$ and $Aut_\pi(\overline{D}) = \{\pm 1\}$. Free groups are duality groups of dimension 1, while if $c.d.\pi = 2$ then π is a duality group of dimension 2 if and only if it has one end and $E^2\mathbb{Z}$ is $\mathbb{Z}$-torsion-free. (See Chapter III of [5].)

2. PD_4-complexes

We assume henceforth that X is a PD_4-complex. Then π is finitely presentable and X is homotopy equivalent to $X_o \cup_\phi e^4$, where X_o is a complex of dimension at most 3 and $\phi \in \pi_3(X_o)$ [24]. In [14] and [15] we used such cellular decompositions to study the homotopy types of PD_4-complexes. Here we shall follow [16] instead and rely more consistently on the dual Postnikov approach.

Lemma 2. *If π is infinite the homotopy type of X is determined by $P_3(X)$.*

Proof. If X and Y are two such PD_4-complexes and $h : P_3(X) \to P_3(Y)$ is a homotopy equivalence then $hf_{X,3}$ is homotopic to a map $g : X \to Y$. Since π is infinite $H_4(\widetilde{X}; \mathbb{Z}) = H_4(\widetilde{Y}; \mathbb{Z}) = 0$. Since g is 4-connected any lift to a map $\tilde{g} : \widetilde{X} \to \widetilde{Y}$ is a homotopy equivalence, by Whitehead's Theorem, and so g is a homotopy equivalence. □

Let $\Pi = \pi_2(X)$, with the natural left $\mathbb{Z}[\pi]$-module structure. In Theorem 11 of [16] we showed that two PD_4-complexes X and Y with the same strongly minimal model and with trivial first k-invariants ($k_1(X) = k_1(Y) = 0$ in $H^3(\pi; \Pi)$) are homotopy equivalent if and only if $\lambda_X \cong \lambda_Y$. The appeal to [20] there is inadequate. Instead we may use the following lemma (which requires no hypothesis on $k_1(X)$).

Lemma 3. *Let $P = P_2(X)$ and $Q = P_2(Z)$, and let $f, g : P \to Q$ be 2-connected maps such that $\pi_i(f) = \pi_i(g)$ for $i = 1, 2$. Then there is a homotopy equivalence $h : P \to P$ such that $gh \sim f$.*

Proof. This follows from Corollaries 2.6 and 2.7 of Chapter VIII of [3]. □

Lemma 4. *Let Z be a PD_4-complex with a finite covering space Z_ρ. Then Z is strongly minimal if and only if Z_ρ is strongly minimal.*

Proof. Let $\pi = \pi_1(Z)$, $\rho = \pi_1(Z_\rho)$ and $\Pi = \pi_2(Z)$. Then $\pi_2(Z_\rho) \cong \Pi|_\rho$, and so the lemma follows from the observations that since $[\pi : \rho]$ is finite $H^2(\pi; \mathbb{Z}[\pi])|_\rho \cong H^2(\rho; \mathbb{Z}[\rho])$ and $Hom_{\mathbb{Z}[\pi]}(\Pi, \mathbb{Z}[\pi])|_\rho \cong Hom_{\mathbb{Z}[\rho]}(\Pi|_\rho, \mathbb{Z}[\rho])$, as right $\mathbb{Z}[\rho]$-modules. □

In particular, if $v.c.d\pi \leq 2$ and ρ is a torsion-free subgroup of finite index then $c.d.\rho \leq 2$, and so $\chi(Z_\rho) = 2\chi(\rho)$, by Theorem 13 of [16]. Hence $[\pi : \rho]$ divides $2\chi(\rho)$, thus bounding the order of torsion subgroups of π.

The next theorem gives a much stronger restriction, under another hypothesis.

Theorem 5. *Let Z be a strongly minimal PD_4-complex and $\pi = \pi_1(Z)$. Suppose that π has one end and $E^2\mathbb{Z}$ is free abelian. If π has nontrivial torsion then it is a semidirect product $\kappa \rtimes (Z/2Z)$, where κ is a PD_2-group.*

Proof. Let κ be the kernel of the natural action of π on $\Pi = \pi_2(Z) \cong E^2\mathbb{Z}$. Then $[\pi : \kappa] \leq 2$, since $Aut(\Pi) = \{\pm 1\}$, by Lemma 1. Suppose that $g \in \pi$ has prime order $p > 1$. Then $H_{s+3}(Z/pZ; \mathbb{Z}) \cong H_s(Z/pZ; \Pi)$ for $s > 1$, by Lemma 2.10 of [13]. If g acts trivially on Π then $Z/pZ = H_3(Z/pZ; \Pi) \leq \Pi$, which is impossible, since Π is torsion-free. Therefore g acts via multiplication by -1 and $p = 2$. In particular, $\pi \cong \kappa \rtimes (Z/2Z)$, where κ is torsion-free. Moreover $H_2(Z/pZ; \Pi) = \Pi/2\Pi \cong Z/2Z$, and so the free abelian group $E^2\mathbb{Z} \cong \Pi$ must in fact be infinite cyclic. Hence κ is a PD_2-group [6]. □

This result settles the question on page 67 of [13].

Corollary 6. *Let X be a PD_4-complex with $\pi_1(X) = \pi \cong Z *_m \rtimes Z/2Z$. Then $\chi(X) > 0$.*

Proof. Let $\rho = Z*_m$. Then $\chi(X) = \frac{1}{2}\chi(X_\rho)$. Hence $\chi(X) \geq 0$, with equality if and only if X_ρ is strongly minimal, by Theorem 13 of [16]. In that case X would be strongly minimal, by Lemma 4. Since π is solvable $E^2\mathbb{Z}$ is free abelian [18]. Therefore X is not strongly minimal and so $\chi(X) > 0$. □

3. $c.d.\pi \leq 2$

We now assume that $c.d.\pi \leq 2$. Let $w = w_1(X)$ be the orientation character. In this case the following three notions of minimality are equivalent, by Theorem 13 of [16]:

(1) X is strongly minimal;
(2) X is minimal with respect to the partial order determined by 2-connected degree-1 maps;
(3) $\chi(X) = 2\chi(\pi) \leq \chi(Y)$ for Y any PD_4-complex with $(\pi_1(Y), w_1(Y)) \cong (\pi, w)$.

Thus we may drop the qualification "strongly" when $c.d.\pi \leq 2$.

We have $\Pi \cong E^2\mathbb{Z} \oplus P$, where P is a finitely generated projective left $\mathbb{Z}[\pi]$-module, and X is minimal if and only if $P = 0$. The first k-invariant is trivial, since $H^3(\pi; \Pi) = 0$, and so $P_2(X) \simeq L = L_\pi(\Pi, 2)$. Let σ be a section for c_L. The group $E_\pi(L)$ of based homotopy classes of based self-homotopy equivalences of L which induce the identity on π is the group of units of $[L, L]_K$ with respect to composition, and is isomorphic to a semidirect product $H^2(\pi; \Pi) \rtimes Aut_\pi(\Pi)$. (See Corollary 8.2.7 of [3].)

Lemma 7. *The homotopy type of X is determined by π, Π, $\pi_3(X)$ and the orbit of $k_2(X) \in H^4(L; \pi_3(X))$ under the actions of $E_\pi(L)$ and $Aut_\pi(\pi_3(X))$.*

Proof. Since these invariants determine $P_3(X)$ this follows from Lemma 1. □

It follows from the Whitehead sequence (1) that $H_3(\widetilde{L};\mathbb{Z}) = 0$ and $H_4(\widetilde{L};\mathbb{Z}) \cong \Gamma_W(\Pi)$, since $\widetilde{L} \simeq K(\Pi, 2)$. Hence the spectral sequence for the universal covering $p_L : \widetilde{L} \to L$ gives exact sequences

$$0 \to Ext^2_{\mathbb{Z}[\pi]}(\mathbb{Z}, \Pi) = H^2(\pi;\Pi) \to H^2(L;\Pi) \to Hom_{\mathbb{Z}[\pi]}(\Pi,\Pi) = End_\pi(\Pi) \to 0,$$

which is split by $H^2(\sigma;\Pi)$, and

(2)

$$0 \to Ext^2_{\mathbb{Z}[\pi]}(\Pi, \pi_3(X)) \to H^4(L;\pi_3(X)) \xrightarrow{p_L^*} Hom_{\mathbb{Z}[\pi]}(\Gamma_W(\Pi), \pi_3(X)) \to 0,$$

since $c.d.\pi \leq 2$. The right hand homomorphisms are the homomorphisms induced by p_L, in each case. (There are similar exact sequences with coefficients any left $\mathbb{Z}[\pi]$-module $\mathcal{A}$.) The image of $k_2(X)$ in $Hom(\Gamma_W(\Pi), \pi_3(X))$ is a representative for $k_2(\widetilde{X})$, and determines the middle homomorphism in the Whitehead sequence (1). If $p_L^* k_2(X)$ is an isomorphism its orbit under the action of $Aut_\pi(\pi_3(X))$ is unique. If π has one end the spectral sequence for $p_X : \widetilde{X} \to X$ gives isomorphisms $Ext^2_{\mathbb{Z}[\pi]}(\Pi, \mathcal{A})) \cong H^4(X;\mathcal{A})$ for any left $\mathbb{Z}[\pi]$-module $\mathcal{A}$, and so $f_{X,2}$ induces splittings $H^4(L;\mathcal{A}) \cong H^4(X;\mathcal{A}) \oplus H^4(\Pi, 2;\mathcal{A})^\pi$.

We wish to classify the orbits of k-invariants for minimal PD_4-complexes. We shall first review the known cases, when π is a free group or a PD_2-group.

4. THE KNOWN CASES: FREE GROUPS AND SEMIDIRECT PRODUCTS

The cases with fundamental group a free group are well-understood. A minimal PD_4-complex X with $\pi \cong F(r)$ free of rank r is either $\#^r(S^1 \times S^3)$, if $w = 0$, or $\#^r(S^1 \tilde{\times} S^3)$, if $w \neq 0$. In [14] this is established by consideration of the chain complex $C_*(\widetilde{X})$, using the good homological properties of $\mathbb{Z}[F(r)]$. From the present point of view, if X is strongly minimal $\Pi = 0$, so $L = K(\pi, 1)$, $H^4(L;\pi_3(X)) = 0$ and $k_2(X)$ is trivial.

If X is not assumed to be minimal Π is a free $\mathbb{Z}[\pi]$-module of rank $\chi(X) + 2r - 2$ and the homotopy type of X is determined by the triple (π, w, λ_X) [14].

The second class of groups for which the minimal models are known are the extensions of Z by finitely generated free groups. If $\pi = F(s) \rtimes_\alpha Z$ the minimal models are mapping tori of based self-homeomorphisms of closed 3-manifolds $N = \#^s(S^1 \times S^2)$ (if $w|_\nu = 0$) or $\#^s(S^1 \tilde{\times} S^2)$ (if $w|_\nu \neq 0$). (See Chapter 4 of [13].) Two such mapping tori are orientation-preserving homeomorphic if the homotopy classes of the defining self-homeomorphisms are conjugate in the group of based self homotopy equivalences $E_0(N)$. There is a natural representation of $Aut(F(s))$ by isotopy classes of based homeomorphisms of N, and $E_0(N)$ is a semidirect product $D \rtimes Aut(F(s))$, where D is generated by Dehn twists about nonseparating 2-spheres [12]. We may identify D with $(Z/2Z)^s = H^1(F(s);\mathbb{F}_2)$, and then $E_0(N) = (Z/2Z)^s \rtimes Aut(F(s))$, with the natural action of $Aut(F(s))$.

Let f be a based self-homeomorphism of N, and let $M(f)$ be the mapping torus of f. If f has image (d, α) in $E_0(N)$ then $\pi = \pi_1(M(f)) \cong F(s) \rtimes_\alpha Z$. Let $\delta(f)$ be the image of d in $H^2(\pi;\mathbb{F}_2) = H^1(F(s);\mathbb{F}_2)/(\alpha - 1)H^1(F(s);\mathbb{F}_2)$. If g is another based self-homeomorphism of N with image (d', α) and $\delta(g) = \delta(f)$ then $d - d' = (\alpha - 1)(e)$ for some $e \in D$ and so (d, α) and (d', α) are conjugate. In fact this cohomology group parametrizes such homotopy types; see Theorem 13 for a more general result. However in this case we do not yet have explicit invariants enabling us to decide which are the possible minimal models for a given PD_4-complex. (It is a remarkable fact that if $\pi = F(s) \rtimes_\alpha Z$ and $\beta_1(\pi) \geq 2$ then π is such a semidirect product for infinitely many distinct values of s [7]. However this does not affect our present considerations.)

5. The known cases: PD_2-groups

The cases with fundamental group a PD_2-group are also well understood, from a different point of view. A minimal PD_4-complex X with π a PD_2-group is homotopy equivalent to the total space of a S^2-bundle over a closed aspherical surface. Thus there are two minimal models for each pair (π, w), distinguished by their second Wu classes. This follows easily from the fact that the inclusion of $O(3)$ into the monoid of self-homotopy equivalences $E(S^2)$ induces a bijection on components and an isomorphism on fundamental groups. (See Lemma 5.9 of [13].) However it is instructive to consider this case from the present point of view, in terms of k-invariants, as we shall extend the following argument to other groups in our main result.

When π is a PD_2-group and X is minimal Π and $\Gamma_W(\Pi)$ are infinite cyclic. The action $u : \pi \to Aut(\Pi)$ is given by $u(g) = w_1(\pi)(g)w(g)$ for all $g \in \pi$, by Lemma 10.3 of [13], while the induced action on $\Gamma_W(\Pi)$ is trivial.

Suppose first that π acts trivially on Π. Then $L \simeq K \times CP^\infty$. Fix generators t, x, η and z for $H^2(\pi; \mathbb{Z})$, Π, $\Gamma_W(\Pi)$ and $Hom(\Pi, \mathbb{Z}) = H^2(CP^\infty; \mathbb{Z})$, respectively, such that $z(x) = 1$ and $2\eta = [x, x]$ (the Whitehead product of x with itself). Let t, z denote also the generators of $H^2(L; \mathbb{Z})$ induced by the projections to K and CP^∞, respectively. Then $H^2(\pi; \Pi)$ is generated by $t \otimes x$, while $H^4(L; \Gamma_W(\Pi))$ is generated by $tz \otimes \eta$ and $z^2 \otimes \eta$. (Note that t has order 2 if $w_1(\pi) \neq 0$.)

The action of $[K, L]_K = [K, CP^\infty] \cong H^2(\pi; \mathbb{Z})$ on $H^2(L; \mathbb{Z})$ is generated by $t \mapsto t$ and $z \mapsto z + t$. The action on $H^4(L; \Gamma_W(\Pi))$ is then given by $tz \mapsto tz$ and $z^2 \mapsto z^2 + 2tz$. Hence $f_{t \otimes x}(z^2) = 2tz$ (which is the image of $2t \otimes x$). There are thus two possible $E_\pi(L)$-orbits of k-invariants, and each is in fact realized by the total space of an S^2-bundle over the surface K.

If the action u is nontrivial these calculations go through essentially unchanged with coefficients $\mathbb{F}_2$ instead of $\mathbb{Z}$. There are again two possible $E_\pi(L)$-orbits of k-invariants, and each is realized by an S^2-bundle space. (See §4 of [15] for another account.)

In all cases the orbits of k-invariants correspond to the elements of $H^2(\pi; \mathbb{F}_2) = Z/2Z$. In fact the k-invariant may be detected by the Wu class. Let $[c]_2$ denote the image of a cohomology class under reduction mod (2). Since $k_2(X) \pm (z^2 + mtz)$ has image 0 in $H^4(X; \mathbb{Z})$ it follows that $[z]_2^2 \equiv m[tz]_2$ in $H^4(X; \mathbb{F}_2)$. This holds also if π is nonorientable or the action u is nontrivial, and so $v_2(X) = m[z]_2$ and the orbit of $k_2(X)$ determine each other.

If X is not assumed to be minimal its minimal models may be determined from Theorem 7 of [15]. The enunciation of this theorem in [15] is not correct; an (implicit) quantifier over certain elements of $H^2(X; \mathbb{Z}^u)$ is misplaced and should be "there is" rather than "for all". More precisely, where it has

"*and let* $x \in H^2(X; \mathbb{Z}^u)$ *be such that* $(x \cup c_X^* \omega_F)[X] = 1$. *Then there is a 2-connected degree-1 map* $h : X \to E$ *such that* $c_E = c_X h$ *if and only if* $(c_X^*)^{-1} w_1(X) = (c_E^*)^{-1} w_1(E)$, $[x]_2^2 = 0$ *if* $v_2(E) = 0$ *and* $[x]_2^2 = [x]_2 \cup c_X^*[\omega_F]_2$ *otherwise*"

it should read

"*Then there is a 2-connected degree-1 map $h : X \to E$ such that $c_E = c_X h$ if and only if $(c_X^*)^{-1}w_1(X) = (c_E^*)^{-1}w_1(E)$ and there is an $x \in H^2(X;\mathbb{Z}^u)$ such that $(x \cup c_X^*\omega_F)[X] = 1$, with $[x]_2^2 = 0$ if $v_2(E) = 0$ and $[x]_2^2 = [x]_2 \cup c_X^*[\omega_F]_2$ otherwise*". The argument is otherwise correct. Thus if $v_2(\widetilde{X}) = 0$ the minimal model Z is uniquely determined by X; otherwise this is not so. Nevertheless we have the following result. It shall be useful to distinguish three "v_2-types" of PD_4-complexes:

(1) $v_2(\widetilde{X}) \neq 0$ (i.e., $v_2(X)$ is not in the image of $H^2(\pi;\mathbb{F}_2)$ under c_X^*);
(2) $v_2(X) = 0$;
(3) $v_2(X) \neq 0$ but $v_2(\widetilde{X}) = 0$ (i.e., $v_2(X)$ is in $c_X^*(H^2(\pi;\mathbb{F}_2)) - \{0\}$);

(This trichotomy is due to Kreck, who formulated it in terms of Stiefel-Whitney classes of the stable normal bundle of a closed 4-manifold.)

Theorem 8. *If π is a PD_2-group the homotopy type of X is determined by the triple (π, w, λ_X) together with its v_2-type.*

Proof. Let t_2 generate $H^2(\pi;\mathbb{F}_2)$. Then $\tau = c_X^* t_2 \neq 0$. If $v_2(X) = m\tau$ and $p : X \to Z$ is a 2-connected degree-1 map then $v_2(Z) = mc_Z^* t_2$, and so there is an unique minimal model for X. Otherwise $v_2(X) \neq \tau$, and so there are elements $y, z \in H^2(X;\mathbb{F}_2)$ such that $y \cup \tau \neq y^2$ and $z \cup \tau \neq 0$. If $y \cup \tau = 0$ and $z^2 \neq 0$ then $(y + z) \cup \tau \neq 0$ and $(y + z)^2 = 0$. Taking $x = y, z$ or $y + z$ appropriately, we have $x \cup \tau \neq 0$ and $x^2 = 0$, so there is a minimal model Z with $v_2(Z) = 0$. In all cases the theorem now follows from the main result of [16]. □

In particular, if C is a smooth projective complex curve of genus ≥ 1 and $X = (C \times S^2)\#\overline{CP^2}$ is a blowup of the ruled surface $C \times CP^1 = C \times S^2$ each of the two orientable S^2-bundles over C is a minimal model for X. In this case they are also minimal models in the sense of complex surface theory. (See Chapter VI.§6 of [1].) Many of the other minimal complex surfaces in the Enriques-Kodaira classification are aspherical, and hence strongly minimal in our sense. However 1-connected complex surfaces are never minimal in our sense, since S^4 is the unique minimal 1-connected PD_4-complex and S^4 has no complex structure, by a classical result of Wu. (See Proposition IV.7.3 of [1].)

6. Realizing k-invariants

We assume now that π has one end. Then $c.d.\pi = 2$. If X is a PD_4-complex with $\pi_1(X) = \pi$ then $H_3(\widetilde{X};\mathbb{Z}) = H_4(\widetilde{X};\mathbb{Z}) = 0$. Hence $k_2(\widetilde{X}) : \Gamma_W(\Pi) \to \pi_3(X)$ is an isomorphism, by the Whitehead sequence (1), while $E_\pi(L) \cong H^2(\pi;\Pi) \rtimes \{\pm 1\}$, by Corollary 8.2.7 of [3] and Lemma 3. Thus if X is minimal its homotopy type is determined by π, w and the orbit of $k_2(X)$. We would like to find more explicit and accessible invariants that characterize such orbits. We would also like to know which k-invariants give rise to PD_4-complexes.

Let $P(k)$ denote the Postnikov 3-stage determined by $k \in H^4(L;\mathcal{A})$.

Theorem 9. *Let π be a finitely presentable group with c.d.$\pi = 2$ and one end, and let $w : \pi \to \{\pm 1\}$ be a homomorphism. Let $\Pi = E^2\mathbb{Z}$ and let $k \in H^4(L;\Gamma_W(\Pi))$. Then*

(1) There is a finitely dominated 4-complex Y with $H_3(\widetilde{Y};\mathbb{Z}) = H_4(\widetilde{Y};\mathbb{Z}) = 0$ and Postnikov 3-stage $P(k)$ if and only if $p_L^* k$ is an isomorphism and $P(k)$ has finite 3-skeleton. These conditions determine the homotopy type of Y.

(2) If π is of type FF we may assume that Y is a finite complex.
(3) $H_4(Y;\mathbb{Z}^w)\cong\mathbb{Z}$ and there are isomorphisms $\overline{H^p(Y;\mathbb{Z}[\pi])}\cong H_{4-p}(Y;\mathbb{Z}[\pi])$ induced by cap product with a generator $[Y]$, for $p\neq 2$.

Proof. Let Y be a finitely dominated 4-complex with $H_3(\widetilde{Y};\mathbb{Z})=H_4(\widetilde{Y};\mathbb{Z})=0$ and Postnikov 3-stage $P(k)$. Since Y is finitely dominated it is homotopy equivalent to a 4-complex with finite 3-skeleton, and since $P(k)\simeq Y\cup e^{q\geq 5}$ may be constructed by adjoining cells of dimension at least 5 we may assume that $P(k)$ has finite 3-skeleton. The homomorphism p_L^*k is an isomorphism, by the exactness of the Whitehead sequence (1).

Suppose now that p_L^*k is an isomorphism and $P(k)$ has finite 3-skeleton. Let $P=P(k)^{[4]}$ and let $C_*=C_*(\widetilde{P})$ be the equivariant cellular chain complex for $\widetilde{P}$. Then C_q is finitely generated for $q\leq 3$. Let $B_q\leq Z_q\leq C_q$ be the submodules of q-boundaries and q-cycles, respectively. Clearly $H_1(C_*)=0$ and $H_2(C_*)\cong\Pi$, while $H_3(C_*)=0$, since p_L^*k is an isomorphism. Hence there are exact sequences

$$0\to B_1\to C_1\to C_0\to\mathbb{Z}\to 0$$

and

$$0\to B_3\to C_3\to Z_2\to\Pi\to 0.$$

Schanuel's Lemma implies that B_1 is projective, since $c.d.\pi=2$. Hence $C_2\cong B_1\oplus Z_2$ and so Z_2 is finitely generated and projective. It then follows that B_3 is also finitely generated and projective, and so $C_4\cong B_3\oplus Z_4$. Thus $H_4(C_*)=Z_4$ is a projective direct summand of C_4.

After replacing P by $P\vee W$, where W is a wedge of copies of S^3, if necessary, we may assume that $Z_4=H_4(P;\mathbb{Z}[\pi])$ is free. Since $\Gamma_W(\Pi)\cong\pi_3(P)$ the Hurewicz homomorphism from $\pi_4(P)$ to $H_4(P;\mathbb{Z}[\pi])$ is onto. (See Chapter I§3 of [4].) We may then attach 5-cells along maps representing a basis to obtain a countable 5-complex Q with 3-skeleton $Q^{[3]}=P(k)^{[3]}$ and with $H_q(\widetilde{Q};\mathbb{Z})=0$ for $q\geq 3$. The inclusion of P into $P(k)$ extends to a 4-connected map from Q to $P(k)$. Now $C_*(\widetilde{Q})$ is chain homotopy equivalent to the complex obtained from C_* by replacing C_4 by B_3, which is a finite projective chain complex. It follows from the finiteness conditions of Wall that Q is homotopy equivalent to a finitely dominated complex Y of dimension ≤ 4 [23]. The homotopy type of Y is uniquely determined by the data, as in Lemma 1.

If π is of type FF then B_1 is stably free, by Schanuel's Lemma. Hence Z_2 is also stably free. Since dualizing a finite free resolution of $\mathbb{Z}$ gives a finite free resolution of $\Pi=E^2\mathbb{Z}$ we see in turn that B_3 must be stably free, and so $C_*(\widetilde{Y})$ is chain homotopy equivalent to a finite free complex. Hence Y is homotopy equivalent to a finite 4-complex [23].

Let D_* and E_* be the subcomplexes of C_* corresponding to the above projective resolutions of $\mathbb{Z}$ and Π. (Thus $D_0=C_0$, $D_1=C_1$, $D_2=B_1$ and $D_q=0$ for $q\neq 0,1,2$, while $E_2=Z_2$, $E_3=C_3$, $E_4=B_3$ and $E_r=0$ for $r\neq 2,3,4$.) Then $C_*(\widetilde{Y})\simeq D_*\oplus E_*$. (The splitting reflects the fact that c_Y is a retraction, since $k_1(Y)=0$.) Clearly $H^p(Y;\mathbb{Z}[\pi])=H_{4-p}(Y;\mathbb{Z}[\pi])=0$ if $p\neq 2$ or 4, while $H^4(Y;\mathbb{Z}[\pi])=E^2\Pi\cong\mathbb{Z}$ and $H_4(Y;\mathbb{Z}^w)=Tor_2(\mathbb{Z}^w;\Pi)\cong\mathbb{Z}^w\otimes_\pi\mathbb{Z}[\pi]\cong\mathbb{Z}$. The homomorphism $\varepsilon_{w\#}:H^4(Y;\mathbb{Z}[\pi])\to H^4(Y;\mathbb{Z}^w)$ induced by ε_w is surjective, since Y is 4-dimensional, and therefore is an isomorphism. Hence $-\cap[Y]$ induces isomorphisms in degrees other than 2. □

Since $\overline{H^2(Y;\mathbb{Z}[\pi])} \cong E^2\mathbb{Z}$, $H_2(Y;\mathbb{Z}[\pi]) = \Pi$ and $Hom_\pi(E^2\mathbb{Z},\Pi) \cong End_\pi(E^2\mathbb{Z}) = \mathbb{Z}$, cap product with $[Y]$ in degree 2 is determined by an integer, and Y is a PD_4-complex if and only if this integer is ±1. The obvious question is: what is this integer? Is it always ±1? The complex C_* is clearly chain homotopy equivalent to its dual, but is the chain homotopy equivalence given by slant product with $[Y]$?

There remains also the question of characterizing the k-invariants corresponding to Postnikov 3-stages with finite 3-skeleton.

7. A LEMMA ON CUP PRODUCTS

In our main result (Theorem 12) we shall use a "cup-product" argument to relate cohomology in degrees 2 and 4. Let G be a group and let $\Gamma = \mathbb{Z}[G]$. Let C_* and D_* be chain complexes of left Γ-modules and $\mathcal{A}$ and $\mathcal{B}$ left Γ-modules. Using the diagonal homomorphism from G to $G \times G$ we may define *internal products*

$$H^p(Hom_\Gamma(C_*,\mathcal{A})) \otimes H^q(Hom_\Gamma(D_*,\mathcal{B})) \to H^{p+q}(Hom_\Gamma(C_* \otimes D_*,\mathcal{A}\otimes\mathcal{B}))$$

where the tensor products of Γ-modules are taken over $\mathbb{Z}$ and have the diagonal G-action. (See Chapter XI.§4 of [8].) If C_* and D_* are resolutions of $\mathcal{C}$ and $\mathcal{D}$, respectively, we get pairings

$$Ext^p_\Gamma(\mathcal{C},\mathcal{A}) \otimes Ext^q_\Gamma(\mathcal{D},\mathcal{B}) \to Ext^{p+q}_\Gamma(\mathcal{C}\otimes\mathcal{D},\mathcal{A}\otimes\mathcal{B}).$$

If instead $C_* = D_* = C_*(\widetilde{S})$ for some space S with $\pi_1(S) \cong G$ then composing with an equivariant diagonal approximation gives "cup product" pairings

$$H^p(S;\mathcal{A}) \otimes H^q(S;\mathcal{B}) \to H^{p+q}(S;\mathcal{A}\otimes\mathcal{B}).$$

These pairings are compatible with the universal coefficient spectral sequences $Ext^q_\Gamma(H_p(C_*),\mathcal{A}) \Rightarrow H^{p+q}(C^*;\mathcal{A}) = H^{p+q}(Hom_\Gamma(C_*,\mathcal{A}))$, etc. We shall use the symbol $\cup$ to express the values of these pairings.

We wish to show that if $c.d.\pi = 2$ and π has one end the homomorphism from $H^2(\pi;\Pi)$ to $Ext^2_{\mathbb{Z}[\pi]}(\Pi,\Pi\otimes\Pi)$ given by cup product with id_Π is an isomorphism. We state the next lemma in greater generality than we need, in order to clarify the hypotheses.

Lemma 10. *Let G be a group for which the augmentation (left) module $\mathbb{Z}$ has a finite projective resolution P_* of length n, and such that $H^j(G;\Gamma) = 0$ for $j < n$. Let $\mathcal{D} = H^n(G;\Gamma)$, $w : G \to \{\pm1\}$ be a homomorphism and $\mathcal{B}$ be a left Γ-module. Then there are natural isomorphisms*

(1) $h_\mathcal{B} : H^n(G;\mathcal{B}) \to \mathcal{D}\otimes_G \mathcal{B}$; and

(2) $e_\mathcal{B} : Ext^n_\Gamma(\overline{\mathcal{D}},\mathcal{B}) \to \mathbb{Z}^w \otimes_G \mathcal{B} = \mathcal{B}/I_w\mathcal{B}$.

Hence $\theta_\mathcal{B} = e^{-1}_{\overline{\mathcal{D}}\otimes\mathcal{B}} h_\mathcal{B} : H^n(G;\mathcal{B}) \cong Ext^n_\Gamma(\overline{\mathcal{D}},\overline{\mathcal{D}}\otimes\mathcal{B})$ is an isomorphism;

Proof. We may assume that $P_0 = \Gamma$. Let $Q_j = Hom_\Gamma(P_{n-j},\Gamma)$ and $\partial^Q_i = Hom_\Gamma(\partial^P_{n-j},\Gamma)$. This gives a resolution Q_* for $\mathcal{D}$ by finitely generated projective right modules, with $Q_n = \Gamma$. Let $\eta : Q_0 \to \mathcal{D}$ be the canonical epimorphism. Tensoring Q_* with $\mathcal{B}$ gives (1). Conjugating and applying $Hom_\Gamma(-,\mathcal{B})$ gives (2). Since we may identify $\mathcal{D}\otimes_G\mathcal{B}$ with $\mathbb{Z}^w \otimes_G (\overline{\mathcal{D}}\otimes\mathcal{B})$, composition gives an isomorphism $\theta_\mathcal{B} = e^{-1}_{\overline{\mathcal{D}}\otimes\mathcal{B}} h_\mathcal{B} : H^n(G;\mathcal{B}) \cong Ext^n_\Gamma(\overline{\mathcal{D}},\overline{\mathcal{D}}\otimes\mathcal{B})$. □

If $\mathcal{D}$ is $\mathbb{Z}$-torsion free then G is a duality group of dimension n, with dualizing module D. (See [5].) It is not known whether all the groups considered in the lemma are duality groups, even when $n = 2$.

Let $A : Q_0 \otimes_G \overline{\mathcal{D}} \to Hom_\Gamma(P_n, \overline{\mathcal{D}})$ be the homomorphism given by $A(q \otimes_G \delta)(p) = q(p)\delta$ for all $p \in P_n$, $q \in Q_0$ and $\delta \in \overline{\mathcal{D}}$, and let $[\xi] \in H^n(G; \overline{\mathcal{D}})$ be the image of $\xi \in Hom_\Gamma(P_n, \overline{\mathcal{D}})$. If $\xi = A(q \otimes_G \delta)$ then $h_{\overline{\mathcal{D}}}([\xi]) = \eta(q) \otimes \delta$ and $\xi \otimes \eta : P_n \otimes \overline{Q}_0 \to \overline{\mathcal{D}} \otimes \overline{\mathcal{D}}$ represents $[\xi] \cup id_{\overline{\mathcal{D}}}$ in $Ext^n_\Gamma(\overline{\mathcal{D}}, \overline{\mathcal{D}} \otimes \overline{\mathcal{D}})$. There is a chain homotopy equivalence $j_* : \overline{Q}_* \to P_* \otimes \overline{Q}_*$, since P_* is a resolution of $\mathbb{Z}$. Given such a chain homotopy equivalence, $e_{\overline{\mathcal{D}} \otimes \overline{\mathcal{D}}}([\xi] \cup id_{\overline{\mathcal{D}}})$ is the image of $(\xi \otimes \eta)(j_n(1^*))$, where 1^* is the canonical generator of $\overline{Q}_n$, defined by $1^*(1) = 1$.

Let τ be the ($\mathbb{Z}$-linear) involution of $H^n(G; \overline{\mathcal{D}})$ given by $\tau(h^{-1}_{\overline{\mathcal{D}}}(\rho \otimes_G \alpha)) = h^{-1}_{\overline{\mathcal{D}}}(\alpha \otimes_G \rho))$. If G is a PD_n-group then $H^n(G; \overline{\mathcal{D}}) \cong Z$ (if $w = w_1(\pi)$) or $Z/2Z$ (otherwise), and so τ is the identity. Let δ_{ij} be the Kronecker δ-function.

Examples. We shall give several simple examples (with w trivial and $n = 0, 1$ or 2), where we have managed to compute an explicit chain map j_*.

(1) Let $G = 1$. Then $\Gamma = \mathbb{Z}$, with trivial involution. In this case there are obvious isomorphisms of P_0, Q_0, $\mathcal{D}$, $\overline{\mathcal{D}}$, $H^0(G; \overline{\mathcal{D}})$ and $Ext^0_\Gamma(\overline{\mathcal{D}}, \overline{\mathcal{D}} \otimes \overline{\mathcal{D}})$ with $\mathbb{Z}$, and we may suppress $h_{\overline{\mathcal{D}}}$, $e_{\overline{\mathcal{D}} \otimes \overline{\mathcal{D}}}$ and $\theta_{\overline{\mathcal{D}}}$ from the notation. We see easily that $[\xi] \cup id_{\overline{\mathcal{D}}} = [\xi]$ for $\xi \in H^0(G; \overline{\mathcal{D}})$. (Note that G is a PD_0-group.)

(2) Let $G = F(r)$, with presentation $\langle x_i, 1 \le i \le r \mid \emptyset \rangle$. Then

$$P_* = \Gamma\langle p_i \rangle \to \Gamma, \quad \text{where} \quad \partial(p_i) = x_i - 1 \quad \text{for} \quad 1 \le i, j \le r.$$

Define a basis for Q_0 by $q_i(p_j) = \delta_{ij}$, for $1 \le i, j \le r$. Then

$$\overline{Q}_* = \Gamma 1^* \to \Gamma\langle q_i \rangle, \quad \text{where} \quad \partial(1^*) = \Sigma_{i=1}^{i=r}(x_i^{-1} - 1)q_i.$$

Let $j_0(q_i) = 1 \otimes q_i$, for $1 \le i \le r$, and $j_1(1^*) = 1 \otimes 1^* - \Sigma_{i=1}^{i=r} x_i^{-1}(p_i \otimes q_i)$. Then $(A(q_i \otimes_G \delta) \otimes \eta)(j_1(1^*)) = -x_i^{-1}(\delta \otimes \eta(q_i))$ for $1 \le i \le r$. Hence

$$e_{\overline{\mathcal{D}} \otimes \overline{\mathcal{D}}}([\xi] \cup id_{\overline{\mathcal{D}}}) = -h_{\overline{\mathcal{D}}}(\tau([\xi]))$$

(equivalently, $[\xi] \cup id_{\overline{\mathcal{D}}} = -\theta_{\overline{\mathcal{D}}}(\tau([\xi])))$ for $\xi \in H^1(G; \overline{\mathcal{D}})$.

Note that when $r = 1$ the group $G = F(1) = Z$ is a PD_1-group.

(3) Let $G = F(r) \times Z$, with presentation $\langle x_i, t \mid tx_i = x_i t \rangle$. We shall also write $x_0 = t$, for simplicity in the following formulae. Then

$$P_* = \Gamma\langle f_i \mid 1 \le i \le r \rangle \to \Gamma\langle e_i \mid 0 \le i \le r \rangle \to \Gamma,$$

where $\partial f_i = (t - 1)e_i + (1 - x)e_0$ and $\partial e_i = x_i - 1$.
Define bases for Q_0 and Q_1 by by $e_i^*(e_j) = \delta_{ij}$ and $f_k^*(f_l) = \delta_{kl}$. Then

$$\overline{Q}_* = \Gamma 1^* \to \Gamma\langle e_i^* \mid 0 \le i \le r \rangle \to \Gamma\langle f_i^* \mid 1 \le i \le r \rangle,$$

where $\partial 1^* = \Sigma_{i=0}^{i=r}(x_i^{-1} - 1)e_i^*$, $\partial e_0^* = \Sigma_{i=1}^{i=r}(1 - x_i^{-1})f_i^*$, and $\partial e_i^* = (t^{-1} - 1)f_i^*$ for $1 \le i \le r$.
Let $j_0(f_i^*) = 1 \otimes f_i^*$, and $j_1(e_i^*) = 1 \otimes e_i^* - t^{-1}(e_0 \otimes f_i^*)$, for $1 \le i \le r$,

$$j_1(e_0^*) = 1 \otimes e_0^* + \Sigma_{i=1}^{i=r} x_i^{-1}(e_i \otimes f_i^*) \quad \text{and}$$

$$j_2(1^*) = 1 \otimes 1^* - \Sigma_{i=0}^{i=r} x_i^{-1}(e_i \otimes e_i^*) - t^{-1}\Sigma_{i=1}^{i=r} x_i^{-1}(f_i \otimes f_i^*).$$

We again find that $[\xi] \cup id_{\overline{\mathcal{D}}} = -\theta_{\overline{\mathcal{D}}}(\tau([\xi]))$ for $\xi \in H^2(G; \overline{\mathcal{D}})$.

Note that when $r = 2$ the group $G = F(1) \times Z = Z^2$ is a PD_2-group.

(4) Let $G = Z*_m$ be the group with presentation $\langle a, t \mid ta = a^m t\rangle$ and let $\mu_k = \Sigma_{i=0}^{i=k-1} a^i$ and $\bar{\mu}_k = \Sigma_{i=0}^{i=k-1} a^{-i}$ for $0 \leq k \leq m$. Then

$$P_* = \Gamma p_2 \to \Gamma\langle p_a, p_t\rangle \to \Gamma,$$

where $\partial p_2 = (t - \mu_m)p_a + (1 - a^m)p_t$, $\partial p_a = a - 1$ and $\partial p_t = t - 1$. Define bases for Q_0 and Q_1 by $q(p_2) = 1$, $q_a(p_a) = q_t(p_t) = 1$ and $q_a(p_t) = q_t(p_a) = 0$. Then

$$\overline{Q}_* = \Gamma 1^* \to \Gamma\langle q_a, q_t\rangle \to \Gamma q,$$

where $\partial 1^* = (a^{-1} - 1)q_a + (t^{-1} - 1)q_t$, $\partial q_a = (t^{-1} - \bar{\mu}_m)q$ and $\partial q_t = (1 - a^{-m})q$.

Let $j_0(q) = 1 \otimes q$, $j_1(q_a) = 1 \otimes q_a + \Sigma_{i=1}^{i=m-1} a^{-i}(\mu_i p_a \otimes q) - t^{-1}(p_t \otimes q)$,

$$j_1(q_t) = 1 \otimes q_t + a^{-m}(\mu_m p_a \otimes q) \quad \text{and}$$

$$j_2(1^*) = 1 \otimes 1^* - a^{-1}(p_a \otimes q_a) - t^{-1}(p_t \otimes q_t) - a^{-1}t^{-1}(p_2 \otimes q).$$

Then it may be verified that $\partial j_2(1^*) - j_1(\partial 1^*) = 0$. (At one point we use the fact that $t^{-1}a^{-m} = a^{-1}t^{-1}$.)

We again find that $[\xi] \cup id_{\overline{\mathcal{D}}} = -\theta_{\overline{\mathcal{D}}}(\tau([\xi]))$ for $\xi \in H^2(G; \overline{\mathcal{D}})$.

Note that when $m = 1$ we have $Z*_1 = Z^2$, and our formulae agree with those of the previous example.

In each of cases (3) and (4) the group is an HNN extension with finitely generated free base and stable letter t. Do these formulae extend easily to other such groups, such as PD_2-groups or semidirect products $F(r) \rtimes Z$?

8. THE ACTION OF $E_\pi(L)$

In attempting to study the action of $E_\pi(L)$ on the set of possible k-invariants we shall switch between the algebraic and homotopical (obstruction-theoretic) interpretations of cohomology classes.

In this section we shall assume that X is minimal, and need to restrict further the pair (π, w). We believe this to be a shortcoming of our argument rather than a limitation of the result.

We shall use the following special case of a result of Tsukiyama [21]; we give only the part that we need below.

Lemma 11. *There is an exact sequence* $0 \to H^2(\pi; \Pi) \to E_\pi(L) \to Aut_\pi(\Pi) \to 0$.

Proof. Let $\theta : [K, L]_K \to H^2(\pi; \Pi)$ be the isomorphism given by $\theta(s) = s^*\iota_{\Pi,2}$, and let $\theta^{-1}(\phi) = s_\phi$ for $\phi \in H^2(\pi; \Pi)$. Then s_ϕ is a homotopy class of sections of c_L, $s_0 = \sigma$ and $s_{\phi+\psi} = \mu(s_\phi, s_\psi)$, while $\phi = s_\phi^*\iota_{\Pi,2}$. (Recall that $\mu : L \times_K L \to L$ is the fibrewise loop multiplication.)

Let $h_\phi = \mu(s_\phi c_L, id_L)$. Then $c_L h_\phi = c_L$ and so $h_\phi \in [L; L]_K$. Clearly $h_0 = \mu(\sigma c_L, id_L) = id_L$ and $h_\phi^*\iota_{\Pi,2} = \iota_{\Pi,2} + c_L^*\phi \in H^2(L; \Pi)$. We also see that

$$h_{\phi+\psi} = \mu(\mu(s_\phi, s_\psi)c_L, id_L) = \mu(\mu(s_\phi c_L, s_\psi c_L), id_L) = \mu(s_\phi c_L, \mu(s_\psi c_L, id_L))$$

(by homotopy associativity of μ) and so

$$h_{\phi+\psi} = \mu(s_\phi c_L, h_\psi) = \mu(s_\phi c_L h_\psi, h_\psi) = h_\phi h_\psi.$$

Therefore h_ϕ is a homotopy equivalence for all $\phi \in H^2(\pi; \Pi)$, and $\phi \mapsto h_\phi$ defines a homomorphism from $H^2(\pi; \Pi)$ to $E_\pi(L)$.

The lift of h_ϕ to the universal cover $\widetilde{L}$ is (non-equivariantly) homotopic to the identity, since the lift of c_L is (non-equivariantly) homotopic to a constant map. Therefore h_ϕ acts as the identity on Π. The homomorphism $h : \phi \mapsto h_\phi$ is in fact an isomorphism onto the kernel of the action of $E_\pi(L)$ on $\Pi = \pi_2(L)$ [21]. $\square$

Note also that we may view elements of $[K, L]_K$ (etc.) as π-equivariant homotopy classes of π-equivariant maps from $\widetilde{K}$ to $\widetilde{L}$. Let $c^2_\pi(\xi) = \xi \cup id_\Pi$ for all $\xi \in H^2(\pi;\Pi)$.

Theorem 12. *Let π be a finitely presentable group such that $c.d.\pi = 2$ and π has one end. Let $\Pi = E^2\mathbb{Z}$. Assume that c^2_π is surjective and $\mathbb{Z}^w \otimes_\pi \Gamma_W(\Pi)$ is 2-torsion-free. Then there is a bijection from the set of orbits of k-invariants of minimal PD_4-complexes with Postnikov 2-stage L under the actions of $E_\pi(L)$ and $Aut_\pi(\Gamma_W(\Pi))$ to a subset of $H^2(\pi;\mathbb{F}_2)$.*

Proof. Let $\phi \in H^2(\pi;\Pi)$ and let $s_\phi \in [K, L]_K$ and $h_\phi \in [L, L]_K$ be as defined in Lemma 9. Let $M = L_\pi(\Pi, 3)$ and let $\overline{\Omega} : [M, M]_K \to [L, L]_K$ be the loop map. Since $c.d.\pi = 2$ we have $[M, M]_K \cong H^3(M;\Pi) = End_\pi(\Pi)$. Let $g \in [M, M]_K$ have image $[g] = \pi_3(g) \in End_\pi(\Pi)$ and let $f = \overline{\Omega}g$. Then $\omega([g]) = f^*\iota_{\Pi,2}$ defines a homomorphism $\omega : End_\pi(\Pi) \to H^2(L;\Pi)$ such that $p^*_L\omega([g]) = [g]$ for all $[g] \in End_\pi(\Pi)$. Moreover $f\mu = \mu(f, f)$, since $f = \overline{\Omega}g$, and so $fh_\phi = \mu(fs_\phi c_L, f)$. Hence $h^*_\phi\xi = \xi + c^*_L s^*_\phi\xi$ for $\xi = \omega([g]) = f^*\iota_{\Pi,2}$. Naturality of the isomorphisms $H^2(X;\mathcal{A}) \cong [X, L_\pi(\mathcal{A}, 2)]_K$ for X a space over K and $\mathcal{A}$ a left $\mathbb{Z}[\pi]$-module implies that

$$s^*_\phi\omega([g]) = [g]_\# s^*_\phi\iota_{\Pi,2} = [g]_\#\phi$$

for all $\phi \in H^2(\pi;\Pi)$ and $g \in [M, M]_K$. (See Chapter 5.§4 of [2].)

Let $\mathcal{A}$ be a left $\mathbb{Z}[\pi]$-module. If $u \in H^2(\pi;\mathcal{A})$ then $h^*_\phi c^*_L(u) = c^*_L(u)$, since $c_L h_\phi = c_L$. The homomorphism induced on the quotient $H^2(L;\mathcal{A})/c^*_L H^2(\pi;\mathcal{A}) \cong Hom_{\mathbb{Z}[\pi]}(\Pi, \mathcal{A})$ by h_ϕ is also the identity, since the lifts of h_ϕ are (non-equivariantly) homotopic to the identity in $\widetilde{L}$. Taking $\mathcal{A} = \Pi$ we obtain a homomorphism $\delta_\phi : End_\pi(\Pi) \to H^2(\pi;\Pi)$ such that $h^*_\phi(\xi) = \xi + \delta_\phi(p^*_L\xi)$ for all $\xi \in H^2(L;\Pi)$. Since $p^*_L\delta_\phi = 0$ and $h_{\phi+\psi} = h_\phi h_\psi$ it follows that δ_ϕ is additive as a function of ϕ. If $g \in [M, M]_K$ and $\phi = \rho \otimes_\pi \alpha \in H^2(\pi;\mathbb{Z}[\pi]) \otimes_\pi \Pi$ then

$$\delta_\phi([g]) = \delta_\phi(p^*_L\omega([g])) = c^*_L s^*_\phi\omega[g] = c^*_L(\rho \otimes_\pi [g](\alpha)).$$

The automorphism of $H^4(L;\mathcal{A})$ induced by h_ϕ preserves the subgroup $Ext^2_{\mathbb{Z}[\pi]}(\Pi, \mathcal{A})$ and induces the identity on the quotient $Hom_\pi(\Gamma_W(\Pi), \mathcal{A})$. Taking $\mathcal{A} = \Gamma_W(\Pi)$ we obtain a homomorphism $f_\phi = h^*_\phi - id$ from $H^4(L;\Gamma_W(\Pi))$ to $Ext^2_{\mathbb{Z}[\pi]}(\Pi, \Gamma_W(\Pi))$.

When $\mathcal{A} = \mathcal{D} = \Pi$, $\mathcal{C} = \mathbb{Z}$, $p = 2$ and $q = 0$ the pairing of Ext groups in §6 gives a pairing

$$H^2(\pi;\Pi) \otimes Hom_{\mathbb{Z}[\pi]}(\Pi, \mathcal{B}) \to Ext^2_{\mathbb{Z}[\pi]}(\Pi, \Pi \otimes \mathcal{B}).$$

Taking $S = L$, $\mathcal{A} = \mathcal{B} = \Pi$, $p = 2$ and $q = 0$ instead gives a pairing of $H^2(L;\Pi) \cong H^2(\pi;\Pi) \oplus End_\pi(\Pi)$ with itself with values in $H^4(L;\Pi \otimes \Pi)$. This restricts to the above pairing of $H^2(\pi;\Pi)$ with $End_\pi(\Pi)$ into $Ext^2_{\mathbb{Z}[\pi]}(\Pi, \Pi \otimes \Pi)$. We may also compose with the symmetrization homomorphism from $\Pi \otimes \Pi$ to $\Pi \odot \Pi < \Gamma_W(\Pi)$ to get pairings with values in $H^4(L;\Pi \odot \Pi)$ and $H^4(L;\Gamma_W(\Pi))$. Since $c.d.\pi = 2$ these pairings are trivial on the image of $H^2(\pi;\Pi) \otimes H^2(\pi;\Pi)$. On passing to $\widetilde{L}$ we find that

$$p^*_L(\xi \cup \xi')(\gamma_\Pi(x)) = p^*_L\xi(x) \odot p^*_L\xi'(x)$$

for all $\xi, \xi' \in H^2(L;\Pi)$ and $x \in \Pi$. Since $h_\phi^*(\xi \cup \xi') = h_\phi^*\xi \cup h_\phi^*\xi'$ we have also

$$f_\phi(\xi \cup \xi') = \delta_\phi(p_L^*\xi') \cup \xi + \delta_\phi(p_L^*\xi) \cup \xi'$$

for all $\xi, \xi' \in H^2(L;\Pi)$. In particular, if $p_L^*\xi = p_L^*\xi' = id_\Pi$ then $p_L^*(\xi \cup \xi') = 2id_{\Gamma_W(\Pi)}$ and $f_\phi(\xi \cup \xi') = 2\phi \cup id_\Pi = 2c_\pi^2(\phi)$.

If $k = k_2(X)$ for some minimal PD_4-complex X with $\pi_1(X) \cong \pi$ then p_L^*k is an isomorphism. After composition with an automorphism of $\Gamma_W(\Pi)$ we may assume that $p_L^*k = id_{\Gamma_W(\Pi)}$. We then have $2f_\phi(k) = 2c_\pi^2(\phi)$. Since $\mathbb{Z}^w \otimes_\pi \Gamma_W(\Pi)$ is 2-torsion-free $f_\phi(k) = c_\pi^2(\phi)$. Since c_π^2 is surjective the orbit of k under the action of $E_\pi(L)$ corresponds to an element of the quotient of $Ext^2_{\mathbb{Z}[\pi]}(\Pi, \Gamma_W(\Pi))$ by the image of $H^2(\pi;\Pi) \cong Ext^2_{\mathbb{Z}[\pi]}(\Pi, \Pi \otimes \Pi)$. Since $\Gamma_W(\Pi)/\Pi \odot \Pi \cong \Pi/2\Pi$ this quotient is $Ext^2_{\mathbb{Z}[\pi]}(\Pi, \Pi/2\Pi) \cong \Pi/(2, I_w)\Pi \cong H^2(\pi;\mathbb{F}_2)$. □

Minimality of X is used in the appeal to the work of §6. The hypotheses are satisfied if $\pi \cong Z^2$ or $Z*_2$ and $w = 1$. However if π is a PD_2-group and $w_1(\pi)$ or w is nontrivial then $\mathbb{Z}^w \otimes_\pi \Gamma_W(\Pi) \cong Z/2Z$.

We note that we do not yet have explicit invariants that might distinguish two such minimal PD_4-complexes.

Corollary 13. *If $H^2(\pi;\mathbb{F}_2) = 0$ and $\mathbb{Z}^w \otimes_\pi \Gamma_W(\Pi)$ is 2-torsion-free then there is an unique minimal PD_4-complex realizing (π, w). Hence two PD_4-complexes X and Y with $\pi_1(X) \cong \pi_1(Y) \cong \pi$ are homotopy equivalent if and only if there is an isomorphism $\theta : \pi_1(X) \to \pi_1(Y)$ such that $w_1(X) = w_1(Y) \circ \theta$ and an isometry of homotopy intersection pairings $\lambda_X \cong \theta^*\lambda_Y$.* □

Let $\varepsilon_2\Pi = H^2(\pi;\mathbb{F}_2)$. Then $\varepsilon_2\Pi \cong \Pi/(2, I_w)\Pi$ is the largest quotient of $\Pi/2\Pi$ on which π acts trivially. (Similarly, $Hom_{\mathbb{Z}[\pi]}(\Pi, \mathbb{F}_2) \cong H_2(\pi;\mathbb{F}_2)$.)

Let κ be the image of $k_2(X)$ in $H^4(L;\varepsilon_2\Pi)$ under the change of coefficients induced by the canonical epimorphism from $\Gamma_W(\Pi)$ to $\varepsilon_2\Pi$. To what extent does κ determine the cup product pairing on $H^2(X;\mathbb{F}_2)$?

9. VERIFYING THE TORSION CONDITION FOR $Z*_{2m}$

Applying $\mathbb{Z} \otimes_\pi -$ to the exact sequence

$$0 \to \Pi \odot \Pi \to \Gamma_W(\Pi) \to \Pi/2\Pi \to 0$$

gives an exact sequence

$$Tor_1^{\mathbb{Z}[\pi]}(\mathbb{Z}, \Pi/2\Pi) \xrightarrow{\delta} \Pi \odot_\pi \Pi \to \mathbb{Z} \otimes_\pi \Gamma_W(\Pi) \to \mathbb{Z} \otimes_\pi \Pi/2\Pi \to 0.$$

Thus if $\mathbb{Z} \otimes_\pi \Pi/2\Pi = H^2(\pi;\mathbb{F}_2) = 0$ it is enough to show that $\Pi \odot_\pi \Pi$ is 2-torsion-free, for then the connecting homomorphism from the 2-torsion group $Tor_1^{\mathbb{Z}[\pi]}(\mathbb{Z}, \Pi/2\Pi)$ to $\Pi \odot_\pi \Pi$ is 0, and so $\Pi \odot_\pi \Pi \cong \mathbb{Z} \otimes_\pi \Gamma_W(\Pi)$.

Let $\pi = Z*_m$, which has a one-relator presentation $\langle a, t \mid ta = a^m t\rangle$ and is also a semidirect product $Z[\frac{1}{m}] \rtimes Z$. Let $R = \mathbb{Z}[\pi]$ and $D = \mathbb{Z}[a_n]/(a_{n+1} - a_n^m)$, where $a_n = t^n a t^{-n}$ for $n \in \mathbb{Z}$. Then $R = \oplus_{n \in \mathbb{Z}} t^n D$ is a twisted Laurent extension of the commutative domain D.

On dualizing the Fox-Lyndon resolution of the augmentation module we see that $H^2(\pi;\mathbb{Z}[\pi]) \cong R/(a^m - 1, t - \mu_m)R$ and so $\Pi \cong R/R(a^m - 1, t\mu_m - 1)$, where $\mu_m = \Sigma_{i=0}^{i=m-1} a^i$. Let $E = D/(a^m - 1)$. As an abelian group E is freely generated by $\{a_x \mid x = \frac{k}{m^n},\ 0 \le k < m^{n+1}\}$, where a_x is the image of a_{-n}^k, and D acts on E in an obvious way. Since $ta^k_{1-n} = a^k_{-n}t$ we have $\Pi \cong \oplus_{n\in\mathbb{Z}} t^n E/\sim$, where $t^m a_x \sim t^m a_x t\mu_m = t^{m+1}\mu_m a_{x/m}$.

Therefore $\Pi \odot \Pi \cong \oplus_{m\in\mathbb{Z}}(t^m E \odot t^m E)/\sim$, where

$$t^m a_x \odot t^m a_y \sim t^{m+1}\mu_m a_{x/m} \odot t^{m+1}\mu_m a_{y/m}.$$

Setting $z = y - x$ this gives

$$t^m a_x(1 \odot a_z) \sim t^{m+1}a_{x/m}(\mu_m \odot \mu_m a_{z/m}) = t^{m+1}a_{x/m}(\Sigma_{i,j=0}^{i,j=m-1} a^i(1 \odot a^{j-i}a_{z/m})).$$

On factoring out the action of π we see that

$$\Pi \odot_\pi \Pi \cong E/(a_z - m(\Sigma_{k=0}^{k=m-1} a^k a_{z/m})).$$

(In simplifying the double sum we may set $k = j - i$ for $j \geq i$ and $k = j + m - i$ otherwise, since $a^m a_{z/m} = a_{z/m}$ for all z.) Thus $\Pi \odot_\pi \Pi$ is a direct limit of free abelian groups and so is torsion-free. If m is even $H^2(Z*_m;\mathbb{F}_2) = 0$ and so $\mathbb{Z} \otimes_\pi \Gamma_W(\Pi)$ is torsion-free. This also holds for $Z*_1 = Z^2$; does it remain true for all m?

10. other groups

If $c.d.\pi = 2$ but π is not a PD_2-group then $E^2\mathbb{Z}$ is not finitely generated [10]. It remains an open question whether $E^2\mathbb{Z}$ is free abelian for all finitely presentable groups [18]. We shall verify that this is so for many of the groups of most direct interest to low dimensional topologists.

Lemma 14. *Let π have one end, and be either a semidirect product $F(s) \rtimes Z$, the fundamental group of an irreducible 3-manifold M with nonempty boundary or a torsion-free one-relator group. Then π is of type FF, $c.d.\pi = 2$ and $\Pi = E^2_\pi\mathbb{Z}$ is free abelian.*

Proof. If $\pi = \nu \rtimes Z$, where $\nu \cong F(s)$ is a nontrivial finitely generated free group, then $s \geq 1$, since π has one end. We may realize $K(\pi, 1)$ as a mapping torus of a self-map of $\vee^s S^1$. Hence π is of type FF and $c.d.\pi = 2$. An LHS spectral sequence argument shows that $\Pi|_\nu = E^2_\pi\mathbb{Z}|_\nu \cong E^1_\nu\mathbb{Z}$, which is free abelian.

If $\pi = \pi_1(M)$ for some irreducible 3-manifold M with nonempty boundary then M is aspherical and retracts onto a finite 2-complex. Hence π is of type FF and $c.d.\pi = 2$. Also $\Pi = H^2(\pi;\mathbb{Z}[\pi]) = H^2(M;\mathbb{Z}[\pi]) \cong H_1(\widetilde{M}, \partial\widetilde{M};\mathbb{Z})$, by Poincaré duality. This is free abelian since it is the kernel of the augmentation $H_0(\partial\widetilde{M};\mathbb{Z}) \to H_0(\widetilde{M};\mathbb{Z})$.

If π has a one-relator presentation $\langle X \mid r\rangle$ and is torsion-free the 2-complex associated to the presentation is aspherical, so $c.d.\pi = 2$ and π is of type FF. It is shown in [19] that one-relator groups are semistable at infinity and hence that Π is free abelian. □

In particular, all such groups are 2-dimensional duality groups. The class of groups covered by this lemma includes all PD_2-groups, classical knot groups and solvable HNN extensions $Z*_m$ other than Z. It remains an open question whether every finitely presentable group with one end and of cohomological dimension 2 is of type FF and semistable at infinity.

We note also that if π is either a semidirect product $F(s) \rtimes Z$ or the fundamental group of an irreducible 3-manifold M with nonempty boundary then $\widetilde{K}_0(\mathbb{Z}[\pi]) = 0$, i.e., projective $\mathbb{Z}[\pi]$-modules are stably free [22]. (This is not yet known for all torsion-free one relator groups.) In such cases finitely dominated complexes are homotopy finite.

Suppose that π is either the fundamental group of a finite graph of groups, with all vertex groups Z, or is square root closed accessible. (This includes all PD_2-groups, semidirect products $F(s) \rtimes Z$ and the solvable groups $Z*_m$.) Then $L_5(\pi, w)$ acts trivially on the s-cobordism structure set $S^s_{TOP}(M)$ and the surgery obstruction map $\sigma_4(M) : [M, G/TOP] \to L_4(\pi, w)$ is onto, for any closed 4-manifold M realizing (π, w). (See Lemma 6.9 of [13].) Thus there are finitely many s-cobordism classes within each homotopy type of such manifolds.

11. APPLICATIONS TO 2-KNOTS

Let π be a finitely presentable group with $c.d.\pi = 2$. If $H_1(\pi;\mathbb{Z}) = \pi/\pi' \cong Z$ and $H_2(\pi;\mathbb{Z}) = 0$ then $\mathrm{def}(\pi) = 1$, by Theorem 2.8 of [13] If moreover π is the normal closure of a single element then π is the group of a 2-knot $K : S^2 \to S^4$. (If the Whitehead Conjecture is true every knot group of deficiency 1 has cohomological dimension at most 2.) Since π is torsion-free it is indecomposable, by a theorem of Klyachko. Hence π has one end.

Let $M = M(K)$ be the closed 4-manifold obtained by surgery on the 2-knot K. Then $\pi_1(M) \cong \pi$ and $\chi(M(K)) = \chi(\pi) = 0$, and so M is a minimal model for π. If $\pi = F(s) \rtimes Z$ the homotopy type of M is determined by π, as explained in §4 above. This holds more generally for all knot groups for which $\mathbb{Z} \otimes_\pi \Gamma_W(\Pi)$ is torsion-free, by Corollary 13. (Does this condition hold for semidirect products $\pi = F(s) \rtimes Z$?) If moreover π is a classical knot group or $\pi = \pi\mathcal{G}$ for some graph of groups $\mathcal{G}$ with all vertex groups infinite cyclic then M is determined up to TOP s-cobordism by its homotopy type, by Theorem 17.8 of [13]. It follows that a fibred ribbon 2-knot is determined up to s-concordance and reflection by its fundamental group together with the conjugacy class of a meridian.

The group $S = Z*_m$ also has such a graph-of-groups structure, since it is an HNN extension with base Z. Solvable groups are "good" and so 5-dimensional s-cobordisms with such groups are TOP products. Thus if m is even the closed orientable 4-manifold M with $\pi_1(M) \cong Z*_m$ and $\chi(M) = 0$ is unique up to homeomorphism. If $m = 1$ there are two such homeomorphism types, distinguished by the second Wu class $v_2(M)$.

In particular, $Z*_2$ is the group of Fox's Example 10, which is a ribbon 2-knot [11]. Since metabelian knot groups have an unique conjugacy class of normal generators (up to inversion) and ribbon 2-knots are reflexive and -amphicheiral this is the unique 2-knot (up to homeomorphism) with this group.

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