Cramér-type large deviations for samples from a £nite population

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Abstract. Cramér-type large deviations for means of samples from a £nite population are established under weak conditions. The results are comparable to results for the so-called self-nomalized large deviation for independent random variables. Cramér-type large deviations for £nite population Student t -statistic are also investigated.

Key Words and Phrases: Cramér large deviation, moderate deviation, £nite population.

Short title: Cramér-type large deviation for £nite populations.

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1 Introduction and results

Let $X_1, X_2, ..., X_n$ be a simple random sample drawn without replacement from a £nite population $\{a\}_N = \{a_1, \cdots, a_N\}$, where $n < N$. Denote $\mu = EX_1,\ \sigma^2 = \text{var}(X_1)$,

$$S_n = \sum_{k=1}^{n} X_k, \qquad p = n/N, \qquad q = 1-p, \qquad \omega_N^2 = Npq.$$

Under appropriate conditions, the £nite central limit theorem [see Erdös and Rényi (1959)] states that $P\big(S_n - n\mu \geq x\sigma\omega_N\big)$ may be approximated by $1-\Phi(x)$, where $\Phi(x)$ is the distribution function of a standard normal variate. The absolute error of this normal approximation, via Berry-Esseen bounds and Edgeworth expansions, has been widely investigated in the literature. We only refer to Bikelis (1969) and Höglund (1978) for the rates in the Erdös and Rényi central limit theorem; Robinson (1978), Bickel and van Zwet (1978), Babu and Bai (1996) as well as Bloznelis (2000a, b) for the Edgeworth expansions. Extensions to U-statistics and, more generally, symmetric statistics can be found in Nandi and Sen (1963), Zhao and Chen (1987, 1990), Kokic and Weber (1990) as well as Bloznelis and Götze (2000, 2001).

In this paper we shall be concerned with the relative error of $P\big(S_n - n\mu \geq x\sigma\omega_N\big)$ to $1 - \Phi(x)$. In this direction, Robinson (1977) derived a large deviation result that is similar to the type for sums of independent random variables in Petrov (1975, Chapter VIII). However, to make the main results in Robinson (1977) applicable, it essentially requires the assumption that $0 < p_1 \leq p \leq p_2 < 1$. This kind of condition not only takes away a major dif£culty in proving large deviation results but also limits its potential applications. The aim of this paper is to establish a Cramér-type large deviation for samples from a £nite population under weak conditions. In a reasonably wide range for x, we show that the relative error of $P\big(S_n - n\mu \geq x\sigma\omega_N\big)$ to $1 - \Phi(x)$ is only related to $E|X_1 - \mu|^3/\sigma^3$ with an absolute constant. We also obtain a similar result for the so-called £nite population Student t-statistic de£ned by

$$t_n = \sqrt{n}(\bar{X} - \mu)/(\hat{\sigma}\sqrt{q}),$$

where $\bar{X} = S_n/n$ and $\hat{\sigma}^2 = \sum_{j=1}^{n}(X_j - \bar{X})^2/(n-1)$. It is interesting to note that the results for both £nite population standardized mean and Student t-statistic are comparable to the so-called self-nomalized large deviation for independent random variables, which has been recently developed by Jing, Shao and Wang (2003). Indeed, Theorems 1.1 and 1.3 below can be considered as analogous to Theorem 2.1 by Jing, Shao and Wang (2003) in the independent case. The Berry-Esseen bounds and Edgeworth expansions for the Student t-statistic have been investigated in Babu and Singh (1985), Rao and Zhao (1994) and Bloznelis (1999, 2003).

We now state our main £ndings.

Theorem 1.1. *There is an absolute constant $A > 0$ such that*

$$\exp\big\{-A(1+x)^3\beta_{3N}/\omega_N\big\} \leq \frac{P\big(S_n - n\mu \geq x\sigma\omega_N\big)}{1 - \Phi(x)} \leq \exp\big\{A(1+x)^3\beta_{3N}/\omega_N\big\}, \tag{1}$$

for $0 \leq x \leq (1/A)\omega_N\sigma/\max_k |a_k - \mu|$, *where* $\beta_{3N} = \sigma^{-3}E|X_1 - \mu|^3$.

The following result is a direct consequence of Theorem 1.1, and provides a Cramér-type large deviation result for samples from a £nite population.

Theorem 1.2. *There exists an absolute constant $A > 0$ such that*

$$\frac{P\big(S_n - n\mu \geq x\sigma\omega_N\big)}{1 - \Phi(x)} = 1 + O(1)(1+x)^3\beta_{3N}/\omega_N, \tag{2}$$

and

$$\frac{P\big(S_n - n\mu \le -x\sigma\omega_N\big)}{\Phi(-x)} \quad = \quad 1 + O(1)(1+x)^3\beta_{3N}/\omega_N, \tag{3}$$

for $0 \le x \le (1/A)\min\big\{\omega_N\sigma/\max_k|a_k-\mu|, (\omega_N/\beta_{3N})^{1/3}\big\}$, *where* $O(1)$ *is bounded by an absolute constant. In particular, if* $\omega_N/\beta_{3N} \to \infty$, *then, for any* $0 < \eta_N \to 0$,

$$\frac{P\big(S_n - n\mu \ge x\sigma\omega_N\big)}{1-\Phi(x)} \to 1, \quad \frac{P\big(S_n - n\mu \le -x\sigma\omega_N\big)}{\Phi(-x)} \to 1, \tag{4}$$

uniformly in $0 \le x \le \eta_N \min\big\{\omega_N\sigma/\max_k|a_k-\mu|, (\omega_N/\beta_{3N})^{1/3}\big\}$.

Results (2) and (3) are useful because they provide not only the relative error but also a Berry-Esseen rate of convergence. Indeed, by the fact that $1-\Phi(x) \le 2e^{-x^2/2}/(1+x)$ for $x \ge 0$, we may obtain

$$\big|P\big(S_n - n\mu \le x\sigma\omega_N\big) - \Phi(x)\big| \le A(1+|x|)^2 e^{-x^2/2}\beta_{3N}/\omega_N,$$

for $|x| \le (1/A)\min\big\{\omega_N\sigma/\max_k|a_k-\mu|, (\omega_N/\beta_{3N})^{1/3}\big\}$. This provides an exponential non-uniform Berry-Esseen bound for samples from a £nite population.

Remark 1.1. *We do not have any restriction on the* $\{a\}_N$ *in Theorems 1.1 and 1.2. Indeed, for any* $\{a\}_N$,

$$\mu = \frac{1}{N}\sum_{k=1}^N a_k, \qquad \sigma^2 = \frac{1}{N}\sum_{k=1}^N (a_k-\mu)^2, \qquad E|X_1-\mu|^3 = \frac{1}{N}\sum_{k=1}^N |a_k-\mu|^3.$$

Removing the trivial case that all a_k *are the same, we always have* $\max_k|a_k-\mu| > 0$, $\sigma^2 > 0$ *and* $E|X_1-\mu|^3 < \infty$.

Remark 1.2. *Hájek (1960) proved that if* $0 < p_1 \le p \le p_2 < 1$, *then* $(S_n - n\mu)/\sigma\omega_N \to_{\mathcal{D}} N(0,1)$ *if and only if* $\omega_N\sigma/\max_k|a_k-\mu| \to \infty$. *Theorems 1.1 and 1.2 therefore provide reasonably wide ranges for* x *to make the results hold true. To be more precise, as an example, consider* $a_k = k^\alpha$, *where* $\alpha > -1/3$. *In this special case, simple calculations show that*

$$\min\big\{\omega_N\sigma/\max_k|a_k-\mu|, (\omega_N/\beta_{3N})^{1/3}\big\} \quad \asymp \quad (Npq)^{1/6},$$

which implies that Theorem 1.2 holds true for x *being in the best range* $\big(0, o[(Npq)^{1/6}]\big)$.

The following Theorem 1.3 provides a relative error $P\big(t_n \ge x\big)$ to $1-\Phi(x)$, which is only related to $E|X_1-\mu|^3/\sigma^3$ with an absolute constant as in Theorem 1.1. Cramér-type large deviation results for the Student t-statistic may be obtained accordingly as in Theorem 1.2. We omit the details.

Theorem 1.3. *There is an absolute constant $A > 0$ such that*

$$\exp\left\{-A(1+x)^3\beta_{3N}/\omega_N\right\} \le \frac{P(t_n \ge x)}{1-\Phi(x)} \le \exp\left\{A(1+x)^3\beta_{3N}/\omega_N\right\}, \tag{5}$$

for all $0 \le x \le (1/A)\omega_N\sigma/\max_k |a_k - \mu|$, *where* β_{3N} *is de£ned as in Theorem 1.1.*

This paper is organized as follows. Major steps of the proofs of Theorems 1.1-1.3 are given in Section 2. As a preliminary, in a general setting, Section 3 provides a Berry-Esseen bound for the associated distribution of $P(S_n - n\mu \le x)$ related to the conjugate method. Proofs of three propositions used in the main proofs are offered in Sections 4-6. Throughout the paper we shall use $A, A_1, A_2, ...$ to denote absolute constants whose values may differ at each occurrence. We also write $b = x/\omega_N$, $V_n^2 = \sum_{k=1}^n X_k^2$,

$$V_{1n} = V_n^2 - n \quad \text{and} \quad V_{2n} = \sum_{k=1}^n \left[(X_k^2-1)^2 - E(X_k^2-1)^2\right],$$

and, when no confusion arises, $\sum$ denotes $\sum_{k=1}^N$, and $\prod$ denotes $\prod_{k=1}^N$. The symbol i will be used exclusively for $\sqrt{-1}$.

2 Proofs of theorems

Without loss of generality, we assume $\mu = 0$ and $\sigma^2 = 1$. Otherwise, it suf£ces to consider that $\{X_1, X_2, ..., X_n\}$ is a simple random sample drawn without replacement from a £nite population $\{a'\}_N = \{(a_1-\mu)/\sigma, \cdots, (a_N-\mu)/\sigma\}$, where $n < N$.

Proof of Theorem 1.1. When $0 \le x \le 2$, property (1) follows from the Berry-Esseen bound for samples from a £nite population (see, Höglund (1978), for example):

$$|P(S_n \ge x\omega_N) - (1-\Phi(x))| \le A\,\beta_{3N}/\omega_N.$$

When $2 \le x \le (1/A)\,\omega_N/\max_k |a_k|$, property (1) follows from the following Proposition 2.1 with $\xi = 0$, $\xi_1 = 0$ and $h = 0$. Proposition 2.1 will be proved in Section 4. □

Proposition 2.1. *There exists an absolute constant $A > 0$ such that, for all $0 \le \xi \le 1/2$, $|\xi_1| \le 36$ and $2 \le x \le (1/A)\,\omega_N/\max_k |a_k|$,*

$$\frac{P\Big(bS_n - \xi\, b^2 q V_{1n} + \xi_1 b^4 q^2 V_{2n} \ge x^2\Big)}{1-\Phi(x)} \quad \ge \quad \exp\left\{-Ax^3\beta_{3N}/\omega_N\right\}, \tag{6}$$

and

$$\frac{P\Big(bS_n - \xi\, b^2 q V_{1n} + \xi_1 b^4 q^2 V_{2n} \ge x^2 + h\Big)}{1-\Phi(x)} \le \big[1 + 9|h|x^{-2}\big] \exp\left\{-h + Ax^3\beta_{3N}/\omega_N\right\}, \tag{7}$$

where h is an arbitrary constant (which may depend on x) with $|h| \le x^2/5$.

Remark 2.1. *The restrictions for ξ and ξ_1 in proposition 2.1 may be reduced to more general $0 \le \xi \le A_0$ and $|\xi_1| \le A_1$, where A_0 and A_1 are two absolute constants.*

Proof of Theorem 1.2. This follows immediately from Theorem 1.1. □

Proof of Theorem 1.3. When $0 \le x \le 4$, property (5) follows from the Berry-Esseen bound for £nite population Student t-statistic. See, Bloznelis (1999), for example. Next, assume $4 \le x \le (1/A)\omega_N/\max_k |a_k|$. Without loss of generality, assume that $A \ge 8$ and $n \ge 4$. Note that $\max_k |a_k| \ge 1$ since $\sum a_k^2 = N$. It is readily seen that

$$\Big|\frac{x_0}{x} - 1\Big| = \Big|\big[1 + (x^2 q - 1)/n\big]^{-1/2} - 1\Big| \le 2x^2/n, \tag{8}$$

where $x_0 = x\, n^{1/2}/(n + x^2 q - 1)^{1/2}$. It follows from (8) that $2 \le x/2 \le x_0 \le 3x/2$ and $|x_0 - x| \le 2x^3\beta_{3N}/\omega_N^2$. Hence, by noting $1 - \Phi(x) \ge x\Phi'(x)/(1+x^2)$ for $x \ge 0$ (see, for example, Revuz and Yor(1999), p30), we have

$$\Big|\log\frac{1-\Phi(x_0)}{1-\Phi(x)}\Big| = \Big|\int_x^{x_0} \frac{\Phi'(t)}{1-\Phi(t)}dt\Big| \le \Big|\int_x^{x_0} \frac{1+t^2}{t}dt\Big| \le 2x|x-x_0| \le x^3\beta_{3N}/\omega_N,$$

which yields that

$$\exp\{-x^3\beta_{3N}/\omega_N\} \le \frac{1-\Phi(x_0)}{1-\Phi(x)} \le \exp\{x^3\beta_{3N}/\omega_N\}. \tag{9}$$

We are now ready to prove Theorem 1.3. As is well-known, for $x \ge 0$,

$$P(t_n \ge x) = P\big(S_n/V_n \ge x_0\sqrt{q}\big).$$

Note that $b_0\, x_0 \sqrt{q}\, V_n \le (x_0^2 + b_0^2 q V_n^2)/2 \le x_0^2 + b_0^2 q (V_n^2 - n)/2$, where $b_0 = x_0/\omega_N$. It follows from (6), (8) and (9) that, for $4 \le x \le (1/A)\omega_N/\max_k |a_k|$,

$$\begin{aligned} P(S_n \ge x_0\sqrt{q}V_n) &\ge P(b_0 S_n - b_0^2 q(V_n^2 - n)/2 \ge x_0^2) \\ &\ge \big(1-\Phi(x_0)\big)\, \exp\{-Ax_0^3\beta_{3N}/\omega_N\} \\ &\ge \big(1-\Phi(x)\big)\exp\left\{-A_1 x^3\beta_{3N}/\omega_N\right\}, \end{aligned}$$

which implies the £rst inequality of (5).

In view of the following Propositions 2.2 and 2.3, the second inequality of (5) may be obtained by a similar argument to that in the proof of (5.13) in Jing, Shao and Wang (2003), and the details are omitted. The proofs of Propositions 2.2 and 2.3 will be given in Section 5 and Section 6 respectively. □

Proposition 2.2. *There exists an absolute constant* $A > 0$ *such that*

$$P(S_n \geq x\sqrt{q}V_n) \leq (1 - \Phi(x)) \exp\{A\, x^3 \beta_{3N}/\omega_N\} + Ae^{-4x^2},$$

for $2 \leq x \leq (1/A)\, \omega_N / \max_k |a_k|$.

Proposition 2.3. *There exists an absolute constant* $A > 0$ *such that*

$$P(S_n \geq x\sqrt{q}V_n) \leq (1 - \Phi(x)) \exp\{Ax^3 \beta_{3N}/\omega_N\} + A\big(x\beta_{3N}/\omega_N\big)^{4/3},$$

for $2 \leq x \leq (1/A)\omega_N / \max_k |a_k|$.

3 Preliminaries

The main aim of this section is to derive a Berry-Esseen bound for the associated distribution of $P(S_n \leq x)$ related to the conjugate method. The result and several related lemmas are established in a general setting, and will be used in the proofs of the propositions.

For $z = x + iy$, de£ne,

$$K(z) = \log \beta(z) \qquad \text{with} \qquad \beta(z) = pe^{qz} + qe^{-pz}, \tag{10}$$

where $p, q > 0$ and $p + q = 1$. Consider a sequence of constants $\{b\}_N = \{b_1, \cdots, b_N\}$ with $\sum b_k = 0$, and let K_k, K_k' and K_k'' be the values of $K(x), K'(x)$ and $K''(x)$ evaluated at $x = u\, b_k + \alpha_N(u)$, where $\alpha_N(u)$ is the solution of the equation

$$\sum K'(u\, b_k + \alpha) = 0. \tag{11}$$

Throughout the section we assume that $C_0 > 0$ is a given constant and $|u| \leq C_0 / \max_k |b_k|$. Note that, for any real u with $|u| \leq C_0/\max_k |b_k|$, $\sum K'(ub_k + \alpha)$ is negative when $\alpha < -C_0$ and positive when $\alpha > C_0$, and it is strictly monotone in the range $-C_0 \leq \alpha \leq C_0$, by virtue of (13) and (14) below. It is readily seen that (11) has a unique solution $\alpha_N = \alpha_N(u)$, and $-C_0 \leq \alpha_N \leq C_0$.

We continue to assume that $X_1, X_2, ..., X_n$ is a random sample without replacement from $\{b\}_N$, where $n < N$, and continue to use the notation $S_n = \sum_{k=1}^n X_k$, p, q and $\omega_N^2 = Npq$ as in Section 1. De£ne

$$H_n(x; u) = Ee^{uS_n} I(S_n \le x)/Ee^{uS_n},$$

and assume $C > 0$ a constant depending only on C_0, which may differ at each occurrence.

The main result in this section is as follows.

Theorem 3.1. *We have*

$$\sup_x \Big| H_n(x; u) - \Phi\Big(\frac{x - m_N}{\sigma_N}\Big)\Big| \ \le\ C\,(pq)^{-1/2} \sum |b_k|^3 / \big(\sum b_k^2\big)^{3/2}, \tag{12}$$

where

$$m_N = \sum b_k K_k', \quad \sigma_N^2 = \sum b_k^2 K_k'' - \Big(\sum b_k K_k''\Big)^2 / \sum K_k''.$$

Theorem 3.1 provides an extension of the classical result for samples from a £nite population given by Höglund (1978). Its proof will be given after £ve lemmas.

Our £rst lemma summarizes some basic properties of $K(z)$.

Lemma 3.1. *We have* $K'(0) = 0$,

$$-pqe^{2t} \le K'(-x) < 0 < K'(x) \le pqe^{2t}, \qquad \textit{for } 0 < x \le t; \tag{13}$$

$$pq\, e^{-3t} < K''(x) < pq\, e^{3t}, \qquad \textit{for } |x| \le t; \tag{14}$$

$$|K'''(x + iy)| \le 2^{3/2} e^{5t} pq, \qquad \textit{for } |x| \le t \textit{ and } |y| \le \pi/2. \tag{15}$$

Furthermore, if $|x| \le 1/16$*, then*

$$\big|K(x)/\,pq\ - x^2/2\big| \ \le\ (1/2)\,|x|^3, \tag{16}$$

$$\big|K'(x)/\,pq\ - x\big| \ \le\ x^2, \tag{17}$$

$$\big|K''(x)/\,pq\ - 1 - (q - p)x\big| \ \le\ 8\,x^2. \tag{18}$$

Proof. The proof of Lemma 3.1 is straightforward and the details are omitted. □

To introduce the following lemmas, we write, for $1 \le k \le N$,

$$\eta_k = u\, b_k + \alpha_N \qquad \text{and} \qquad \xi_k = v\, b_k + y_0, \tag{19}$$

where ν and y_0 are two real variables speci£ed later. By using Lemma 3.1, it is readily seen that $e^{-2C_0} \le \beta(\eta_k) \le e^{2C_0}$,

$$|\eta_k| \le 2C_0, \quad |K_k'| \le pqe^{2C_0} \quad \text{and} \quad pqe^{-6C_0} \le K_k'' \le pqe^{6C_0}. \tag{20}$$

The property (20) will be used heavily in the lemmas below. In the remainder of this section, we also de£ne

$$\rho(u, v, y_0) = \prod \beta(\eta_k + i\xi_k).$$

Lemma 3.2. *There exist* $0 < \varepsilon_0 \le \pi/8$ *and* $\delta_0 > 0$ *depending only on* C_0*, such that, for* $|y_0| \le \varepsilon_0$ *and* $|v| < \delta_0 \sum b_k^2 / \sum |b_k|^3$,

$$\rho(u, v, y_0) \;=\; \exp\Big\{\sum \big(K_k + i\,\xi_k K_k' - \xi_k^2 K_k''/2\big)\Big\}\big(1 + R\big), \tag{21}$$

where $\beta(z)$ *is de£ned as in (10) and*

$$|R| \le C\; pq\; \sum |\xi_k|^3 \,\exp\Big(\frac{1}{4}\sum \xi_k^2 K_k''\Big).$$

Also, for $\varepsilon_0 \le |y_0| \le \pi$ *and* $|v| < \delta_0 \sum b_k^2 / \sum |b_k|^3$,

$$\big|\rho(u, v, y_0)\big| \le e^{2C_0} \prod_{k \ne k_0} \big|\beta(\eta_k + i\xi_k)\big| \le C \exp\Big\{\sum \big[K_k - \varepsilon_0^2 K_k''/4\big]\Big\}, \tag{22}$$

where $1 \le k_0 \le N$.

Proof. We £rst prove (21). De£ne

$$D_1 = \{k : |vb_k| \le \pi/4\} \quad \text{and} \quad D_2 = \{k : |vb_k| > \pi/4\}.$$

It suf£ces to show that there exist $0 < \varepsilon_0 \le \pi/8$ and $\gamma_1 > 0$ depending only on C_0 such that, if $|y_0| \le \varepsilon_0$ and $|v| < \gamma_1 \sum b_k^2 / \sum |b_k|^3$, then

$$\begin{aligned} I_{1N} \;&:=\; \prod_{k \in D_1} \beta(\eta_k + i\xi_k) \prod_{k \in D_2} \beta(\eta_k) \\ &=\; \exp\Big\{\sum \big(K_k + i\,\xi_k K_k' - \xi_k^2 K_k''/2\big)\Big\}\big(1 + R_1\big), \end{aligned} \tag{23}$$

where $|R_1| \le C\; pq\; \sum |\xi_k|^3 \,\exp\big(\frac{1}{4}\sum \xi_k^2 K_k''\big)$, and

$$\begin{aligned} |I_{2N}| \;&:=\; \Big|\prod_{k \in D_1} \beta(\eta_k + i\xi_k)\Big[\prod_{k \in D_2} \beta(\eta_k + i\xi_k) - \prod_{k \in D_2} \beta(\eta_k)\Big]\Big| \\ &\le\; C\; pq\; \sum |\xi_k|^3 \,\exp\Big\{\sum \big(K_k - \xi_k^2 K_k''/4\big)\Big\}. \end{aligned} \tag{24}$$

Indeed, it follows from (23)-(24) that

$$\rho(u, v, y_0) \;=\; \exp\Big\{\sum \big(K_k + i\,\xi_k K_k' - \xi_k^2 K_k''/2\big)\Big\}\big(1 + R\big),$$

where

$$\begin{aligned}|R| &\le |R_1| + |I_{2N}| \exp\Big\{\sum\big(-K_k + \xi_k^2 K_k''/2\big)\Big\} \\ &\le 2\,C\ pq\ \sum|\xi_k|^3\ \exp\big(\frac{1}{4}\sum \xi_k^2 K_k''\big),\end{aligned}$$

as required.

We next give the proofs of (23) and (24).

Recall we assume that $|\varepsilon_0| \le \pi/8$. If $k \in D_1$, then $|\xi_k| < \pi/2$ since $|y_0| \le \pi/8$. This fact, together with (15), (20) and Taylor's formula: for $x, y \in \mathcal{R}$,

$$K(x+iy) = K(x) + iyK'(x) - y^2K''(x)/2 - iy^3\int_0^1 (1-t)^2 K'''(x+ity)dt/2,$$

implies that, whenever $k \in D_1$,

$$\begin{aligned}&|K(\eta_k + i\xi_k) - K_k - i\xi_k K_k' + \xi_k^2 K_k''/2| \\ &\qquad \le |\xi_k|^3 \max_{\substack{|x|\le 2C_0\\ |y|<\pi/2}} |K'''(x+iy)|/6 \ \le\ e^{10C_0}\ pq\ |\xi_k|^3.\end{aligned}$$

Therefore,

$$\prod_{k\in D_1}\beta(\eta_k + i\xi_k) = \exp\Big\{\sum\nolimits_{k\in D_1}\big(K_k + i\,\xi_k K_k' - \xi_k^2 K_k''/2\big) + \mathcal{L}_{1N}\Big\}, \tag{25}$$

where $|\mathcal{L}_{1N}| \le e^{10C_0}\ pq\ \sum_{k\in D_1}|\xi_k|^3$. On the other hand, if $k \in D_2$, then $|\xi_k| \ge \pi/8$ since $|y_0| \le \pi/8$. This, together with (20), yields that, whenever $k \in D_2$,

$$|i\xi_k K_k' - \xi_k^2 K_k''/2| \le [(8/\pi)^2 + 4/\pi]e^{6C_0}\, pq\ |\xi_k|^3,$$

and hence

$$\begin{aligned}\prod_{k\in D_2}\beta(\eta_k) &= \exp\Big\{\sum\nolimits_{k\in D_2} K_k\Big\}, \\ &= \exp\Big\{\sum\nolimits_{k\in D_2}\big(K_k + i\,\xi_k K_k' - \xi_k^2 K_k''/2\big) + \mathcal{L}_{2N}\Big\},\end{aligned} \tag{26}$$

where $|\mathcal{L}_{2N}| \le [(8/\pi)^2 + 4/\pi]e^{6C_0}\ pq\ \sum_{k\in D_2}|\xi_k|^3$.

Recalling $\sum b_k = 0$, if we choose ε_0 and δ_0 so small that $4C_1\max\{\varepsilon_0, \delta_0\}e^{6C_0} \le 1/4$, where $C_1 = \max\{(8/\pi)^2 + 4/\pi, e^{4C_0}\}e^{6C_0}$, then for $|y_0| \le \varepsilon_0$ and $|v| < \delta_0 \sum b_k^2 / \sum |b_k|^3$,

$$\begin{aligned}|\mathcal{L}_{1N}| + |\mathcal{L}_{2N}| &\le C_1\, pq \sum |\xi_k|^3 \\ &\le 4C_1\ pq\ \Big(N|y_0|^3 + |v|^3 \sum |b_k|^3\Big) \\ &\le 4C_1 \max\{\varepsilon_0, \gamma_1\}\ pq\ \Big(Ny_0^2 + |v|^2 \sum b_k^2\Big) \\ &\le 4C_1\max\{\varepsilon_0, \gamma_1\}e^{6C_0} \sum \xi_k^2 K_k'' \\ &\le (1/4) \sum \xi_k^2 K_k'',\end{aligned} \tag{27}$$

by using (20). Combining (25)-(27),

$$I_{1N} = \exp\Big\{\sum \big(K_k + i\,\xi_k K_k' - \xi_k^2 K_k''/2\big)\Big\}\big(1+R_1\big),$$

where

$$\begin{aligned} |R_1| &= |e^{\mathcal{L}_{1N}+\mathcal{L}_{2N}} - 1| \le (|\mathcal{L}_{1N}| + |\mathcal{L}_{2N}|)e^{|\mathcal{L}_{1N}|+|\mathcal{L}_{2N}|} \\ &\le C\,pq \sum |\xi_k|^3 \exp\Big(\frac{1}{4}\sum \xi_k^2 K_k''\Big), \end{aligned}$$

which yields (23).

As for (24), by noting from (25)-(27) that, for any $k_0 \in D_2$,

$$\begin{aligned} \Big|\prod_{k\in D_1}\beta(\eta_k + i\xi_k)\prod_{k\in D_2-\{k_0\}}\beta(\eta_k)\Big| &\le e^{2C_0}\Big|\prod_{k\in D_1}\beta(\eta_k+i\xi_k)\prod_{k\in D_2}\beta(\eta_k)\Big| \\ &\le e^{2C_0}\exp\Big\{\sum\big(K_k - \xi_k^2K_k''/2\big) + |\mathcal{L}_{1N}| + |\mathcal{L}_{2N}|\Big\} \\ &\le e^{2C_0}\exp\Big\{\sum\big(K_k - \xi_k^2 K_k''/4\big)\Big\}, \end{aligned} \tag{28}$$

since $e^{-2C_0} \le \beta(\eta_k) \le e^{2C_0}$, we have

$$\begin{aligned} |I_{2N}| &\le \sum_{j\in D_2}\big|\beta(\eta_j+i\xi_j) - \beta(\eta_j)\big|\Big|\prod_{k\in D_1}\beta(\eta_k+i\xi_k)\Big|\prod_{k\in D_2-\{j\}}|\beta(\eta_k)| \\ &\le e^{2C_0}\exp\Big\{\sum\big(K_k - \xi_k^2K_k''/4\big)\Big\}\sum_{j\in D_2}\big|\beta(\eta_j+i\xi_j)-\beta(\eta_j)\big|. \end{aligned} \tag{29}$$

Now (24) follows from (29) and

$$|\beta(\eta_k+i\xi_k) - \beta(\eta_k)| = \Big|i\xi_k\int_0^1 \beta'(\eta_k+it\xi_k)dt\Big| \le 2e^{2C_0}\,pq\,|\xi_k| \le C\,pq\,|\xi_k|^3,$$

for $k \in D_2$, where we have used the estimates: $|\xi_k| \ge \pi/8$ for $k \in D_2$, and for all $0 \le t \le 1$,

$$|\beta'(\eta_k + it\xi_k)| = pq|e^{q(\eta_k+it\xi_k)} - e^{-p(\eta_k+it\xi_k)}| \le 2e^{2C_0}pq. \tag{30}$$

This proves (24) and also completes the proof of (21).

We next prove (22). As in (27) of Robinson (1977), we obtain

$$\begin{aligned} |\beta(\eta_k+i\xi_k)|^2 &= e^{2K_k}[1 - 2K_k''(1-\cos\xi_k)] \\ &\le \exp\big(2K_k - 2K_k''(1-\cos\xi_k)\big) \\ &= \exp\big\{2K_k - 2K_k''[1-\cos(y_0) - \mathcal{L}_{1k}]\big\}, \end{aligned} \tag{31}$$

where $\mathcal{L}_{1k} = \cos(\xi_k) - \cos(y_0)$. Note that $|\mathcal{L}_{1k}| \le |1-\cos(vb_k)| + |\sin(vb_k)| \le v^2 b_k^2/2 + |vb_k|$. It follows from (20) that, for any given $\delta_0 > 0$, if $|v| < \delta_0 \sum b_k^2 / \sum |b_k|^3$, then

$$\begin{aligned}
\sum |\mathcal{L}_{1k}| K_k'' &\le pq\, e^{6C_0} \sum |\mathcal{L}_{1k}| \\
&\le pq\, e^{6C_0} \Big[\delta_0^2/2\big(\sum b_k^2\big)^3/(\sum |b_k|^3)^2 + \delta_0 \sum b_k^2 \sum |b_k| / \sum |b_k|^3\Big] \\
&\le N\, pq\, e^{6C_0}(\delta_0^2/2 + \delta_0) \le e^{12C_0}(\delta_0^2/2+\delta_0) \sum K_k'', \qquad (32)
\end{aligned}$$

where we have used the fact that, by Hölder's inequality,

$$\sum |b_k| \le N^{2/3}\Big(\sum |b_k|^3\Big)^{1/3} \quad \text{and} \quad \sum b_k^2 \le N^{1/3}\Big(\sum |b_k|^3\Big)^{2/3}. \qquad (33)$$

By taking $\delta_0 = \min\{\gamma_1, \gamma_2\}$, where γ_1 is de£ned as in the proofs of (23)-(24) and γ_2 is a constant satisfying $e^{12C_0}(\gamma_2^2/2 + \gamma_2) \le (1-\cos\varepsilon_0)/4$, it follows easily from (31)-(32), and $|K_{k_0} - K_{k_0}''(1-\cos\xi_{k_0})| \le C$ [recall (20)] for any $1 \le k_0 \le N$, that if $|v| < \delta_0 \sum b_k^2 / \sum |b_k|^3$ and $\varepsilon_0 \le |y_0| \le \pi$, then

$$\begin{aligned}
\prod_{k \ne k_0} |\beta(\eta_k + i\xi_k)| &\le \exp\Big\{\sum \big[K_k - K_k''(1-\cos\xi_k)\big] + |K_{k_0} - K_{k_0}''(1-\cos\xi_{k_0})|\Big\} \\
&\le C \exp\Big\{\sum \big[K_k - \varepsilon_0^2 K_k''/4\big]\Big\},
\end{aligned}$$

for any $1 \le k_0 \le N$, where we have used the well-known facts:

$$1-\cos(y_0) \ge 1-\cos(\varepsilon_0) \ge \varepsilon_0^2/2 - \varepsilon_0^4/24 \ge \varepsilon_0^2/3,$$

since $0 < \varepsilon_0 \le \pi/8$. This proves the second inequality of (22). The £rst inequality of (22) holds true since $|\beta(\eta_{k_0} + i\xi_{k_0})| \le e^{2C_0}$ for each $1 \le k_0 \le N$.

The proof of Lemma 3.2 is now complete. $\square$

Lemma 3.3. *Let ε_0 and δ_0 be de£ned as in Lemma 3.2. Suppose that $|v| < \delta_0 \sum b_k^2 / \sum |b_k|^3$. Then, for $|y_0| \le \varepsilon_0$,*

$$\begin{aligned}
\Big|\frac{d\,\rho(u,v,y_0)}{dv} &- \big(i \sum b_k K_k' - \sum b_k \xi_k K_k''\big)\, \rho(u,v,y_0)\Big| \\
&\le C\, pq \sum |b_k| \xi_k^2 \exp\Big\{\sum \big(K_k - \xi_k^2 K_k''/4\big)\Big\}; \qquad (34)
\end{aligned}$$

and for $\varepsilon_0 \le |y_0| \le \pi$,

$$\Big|\frac{d\,\rho(u,v,y_0)}{dv}\Big| \le C\, pq \sum |b_k| \exp\Big\{\sum \big(K_k - \varepsilon_0^2 K_k''/4\big)\Big\}. \qquad (35)$$

Proof. Note that

$$\frac{d\,\rho(u,v,y_0)}{dv} \;=\; i\sum_{j=1}^{N} b_j\,\beta'(\eta_j+i\xi_j)\prod_{k\neq j}\beta(\eta_k+i\xi_k),$$

where $i=\sqrt{-1}$. The property (35) follows immediately from (22) and (30).

We next prove (34). Define D_1 and D_2 as in Lemma 3.2. We may write

$$\begin{aligned}\frac{d\,\rho(u,v,y_0)}{dv} \;=\;& i\sum_{k\in D_1} b_k\,K'(\eta_k+i\xi_k)\,\rho(u,v,y_0)\\ &+ i\sum_{k\in D_2} b_k\,\beta'(\eta_k+i\xi_k)\prod_{j\neq k}\beta(\eta_j+i\xi_j).\end{aligned}$$

By virtue of (28), it suffices to show that

$$\begin{aligned} II \;:=\;& \Big|\sum_{k\in D_1} i\,b_k\,K'(\eta_k+i\xi_k) - i\sum_{k\in D_1} b_k\,K'_k + \sum b_k\,\xi_k\,K''_k\Big|\\ \le\;& C\,(pq)\sum|b_k|\xi_k^2, \end{aligned} \tag{36}$$

and

$$\Big|\beta'(\eta_k+i\xi_k) - K'_k\,\beta(\eta_k+i\xi_k)\Big| \le C\,(pq)\,\xi_k^2, \qquad \text{for } k\in D_2. \tag{37}$$

In fact, as in the proof of Lemma 3.2, by using the Taylor's formula of $K'(x+iy)$,

$$\begin{aligned} II \;\le\;& \sum_{k\in D_1}|b_k|\,|K'(\eta_k+i\xi_k)-K'_k-i\xi_kK''_k| + \Big|\textstyle\sum_{k\in D_2} b_k\,\xi_k\,K''_k\Big|\\ \le\;& (1/2)\max_{\substack{|x|<2C_0\\ |y|<\pi/2}}|K'''(x+iy)|\sum_{k\in D_1}|b_k|\,|\xi_k|^2 + e^{6C_0}\,(pq)\textstyle\sum_{k\in D_2}|b_k|\,|\xi_k|\\ \le\;& C\,(pq)\sum|b_k|\xi_k^2, \end{aligned}$$

where we have used (15) and the fact that $|\xi_k|>\pi/8$ when $k\in D_2$. This proves (36). The property (37) follows from $|\xi_k|>\pi/8$ for $k\in D_2$, and hence

$$\begin{aligned} |\beta'(\eta_k+i\xi_k)-K'_k\,\beta(\eta_k+i\xi_k)| \;=\;& \frac{pq\,e^{q\eta_k}|e^{i\xi_k}-1|}{pe^{\eta_k}+q}\\ \le\;& e^{2C_0}\,(pq)\,|\xi_k| \;\le\; (8e^{2C_0}/\pi)\,(pq)\,\xi_k^2. \end{aligned}$$

The proof of Lemma 3.3 is complete. $\Box$

Lemma 3.4. *There exists a $\delta_1>0$ depending only on C_0, such that for $|v|<\delta_1\sum b_k^2/\sum|b_k|^3$,*

$$Ee^{(u+iv)S_n} = (G_n(p))^{-1}\Big(\sum K''_k\Big)^{-1/2}\exp\Big\{\sum K_k + ivm_N - \frac{1}{2}v^2\sigma_N^2\Big\}(1+R), \tag{38}$$

where $G_n(p) = \sqrt{2\pi}\binom{N}{n}p^n q^{N-n}$, m_N and σ_N^2 are de£ned as in Theorem 3.1 and

$$|R| \le C\Big(|v|^3\,(pq)\,\sum|b_k|^3 + 1/\omega_N\Big)e^{v^2\sigma_N^2/4}.$$

In particular, by letting $v = 0$ in (38),

$$Ee^{u\,S_n} = (G_n(p))^{-1}\Big(\sum K_k''\Big)^{-1/2}\exp\Big\{\sum K_k\Big\}(1 + O_1/\omega_N), \tag{39}$$

where $|O_1| \le C_1$ and C_1 is a constant depending only on C_0.

Proof. As in Erdos and Renyi(1959), for any α,

$$Ee^{(u+iv)S_n} = (\sqrt{2\pi}G_n(p))^{-1}\int_{-\pi}^{\pi}\prod_{k=1}^{N}(q + pe^{(u+iv)b_k+\alpha+i\theta})e^{-n(\alpha+i\theta)}d\theta.$$

Let α be the solution of (11), $y_0 = \psi/\omega_N$, and η_k and ξ_k as in (19). Some algebra shows that

$$\begin{aligned} Ee^{(u+iv)S_n} &= (\sqrt{2\pi}\omega_N G_n(p))^{-1}\Big(\int_{|\psi|\le\varepsilon_0\,\omega_N} + \int_{\varepsilon_0\,\omega_N<|\psi|<\pi\omega_N}\Big)\rho(u, v, \psi/\omega_N)d\psi \\ &= III_1 + III_2, \qquad \text{say}, \end{aligned} \tag{40}$$

where ε_0 is de£ned as in Lemma 3.2.

Let $\delta_1 = \min\{\delta_0, e^{-3C_0}\varepsilon_0/\sqrt{2}\}$, where ε_0 and δ_0 are de£ned as in Lemma 3.2. We will show that, for $|v| < \delta_1\sum b_k^2/\sum|b_k|^3$,

$$|III_2| \le \big(C/\omega_N\big)\,(G_n(p))^{-1}\,\Big(\sum K_k''\Big)^{-1/2}\exp\Big\{\sum K_k - \frac{1}{4}v^2\sigma_N^2\Big\}, \tag{41}$$

$$III_1 = (G_n(p))^{-1}\Big(\sum K_k''\Big)^{-1/2}\exp\Big\{\sum K_k + ivm_N - \frac{1}{2}v^2\sigma_N^2\Big\}(1 + R_1), \tag{42}$$

where $|R_1| \le C\big(|v|^3\,(pq)\,\sum|b_k|^3 + 1/\omega_N\big)e^{v^2\sigma_N^2/4}$. Then (38) follows easily from (40)-(42).

The proof of (41) is straightforward by (20) and Lemma 3.2. Indeed, it follows from (20) that

$$e^{-6C_0}\omega_N^2 \le \sum K_k'' \le e^{6C_0}\omega_N^2, \tag{43}$$

and hence for $|v| < \delta_1\sum b_k^2/\sum|b_k|^3$,

$$\begin{aligned} v^2\sigma_N^2 &\le v^2\sum b_k^2 K_k'' \le e^{6C_0}pqv^2\sum b_k^2 \\ &\le \delta_1^2 e^{6C_0}pq(\sum b_k^2)^3/(\sum|b_k|^3)^2 \le \delta_1^2 e^{6C_0}\omega_N^2 \le \varepsilon_0^2\sum K_k''/2, \end{aligned} \tag{44}$$

By (43)-(44) and Lemma 3.2, it is readily seen that

$$\begin{aligned} |III_2| &\le C\,(G_n(p))^{-1}\,\exp\Big[\sum K_k - \varepsilon_0^2\sum K_k''/4\Big] \\ &\le C\,(G_n(p))^{-1}\,\exp\Big[\sum K_k - \frac{1}{4}v^2\sigma_N^2 - \varepsilon_0^2\sum K_k''/8\Big] \\ &\le \big(C/\omega_N\big)\,(G_n(p))^{-1}\,\Big(\sum K_k''\Big)^{-1/2}\exp\Big\{\sum K_k - \frac{1}{4}v^2\sigma_N^2\Big\}, \end{aligned}$$

as required.

We next prove (42). Note that $\sum \xi_k K_k' = v \sum b_k K_k'$ since $\sum K_k' = 0$,

$$g(\psi, v) := \left\{ \psi + \frac{v\omega_N \sum b_k K_k''}{\sum K_k''} \right\}^2 \frac{\sum K_k''}{\omega_N^2} = \sum \xi_k^2 K_k'' - v^2 \sigma_N^2 \tag{45}$$

and $\int_{-\infty}^{\infty} e^{-g(\psi,v)/2} d\psi = (2\pi\omega_N^2 / \sum K_k'')^{1/2}$. It follows from (45) and Lemma 3.2 that, for $|v| < \delta_1 \sum b_k^2 / \sum |b_k|^3$,

$$III_1 = (G_n(p))^{-1} \Big(\sum K_k'' \Big)^{-1/2} \exp\Big\{ \sum K_k + ivm_N - \frac{1}{2} v^2 \sigma_N^2 \Big\} (1 + R_2), \tag{46}$$

where R is de£ned as in (21) and

$$|R_2| \le \int_{|\psi| \ge \varepsilon_0 \omega_N} e^{-g(\psi,v)/2} d\psi + e^{3C_0} \int_{|\psi| \le \varepsilon_0 \omega_N} |R| e^{-g(\psi,v)/2} d\psi := \mathcal{L}_{3N} + \mathcal{L}_{4N}.$$

By (20), (43) and Hölder's inequality,

$$\Big| \frac{\omega_N \sum b_k K_k''}{\sum K_k''} \Big| \le e^{3C_0} \Big(\sum b_k^2 K_k'' \Big)^{1/2} \le e^{6C_0} (pq)^{1/2} \Big(\sum b_k^2 \Big)^{1/2}. \tag{47}$$

It follows easily that

$$\int_{|\psi| \le \varepsilon_0 \omega_N} |\psi|^3 e^{-g(\psi,v)/4} d\psi \le C \Big(1 + \Big| \frac{v\, \omega_N \sum b_k K_k''}{\sum K_k''} \Big|^3 \Big) \le C \Big[1 + (pq)\, \omega_N\, |v|^3 \sum |b_k|^3 \Big].$$

This, together with the de£nitions of R and $g(\psi, v)$, implies that

$$\begin{aligned} \mathcal{L}_{4N} &\le C\,(pq)\, e^{v^2\sigma_N^2/4} \int_{|\psi| \le \varepsilon_0 \omega_N} \sum |\xi_k|^3 e^{-g(\psi,v)/4} d\psi \\ &\le 4C\,(pq)\, e^{v^2\sigma_N^2/4} \Big(|v|^3 \sum |b_k|^3 + N\omega_N^{-3} \int_{|\psi| \le \varepsilon_0 \omega_N} |\psi|^3 e^{-g(\psi,v)/4} d\psi \Big) \\ &\le C \Big(|v|^3\, (pq) \sum |b_k|^3 + 1/\omega_N \Big) e^{v^2\sigma_N^2/4}. \end{aligned} \tag{48}$$

As for $\mathcal{L}_{3N}$, by noting from (47) that, for $|v| < \delta_1 \sum b_k^2 / \sum |b_k|^3$,

$$\Big| \frac{v\omega_N \sum b_k K_k''}{\sum K_k''} \Big| \le \delta_1 e^{6C_0} (pq)^{1/2} \Big(\sum b_k^2 \Big)^{3/2} / \sum |b_k|^3 \le \varepsilon_0\, \omega_N / 2,$$

it is readily seen [recall (43)] that

$$\mathcal{L}_{3N} \le \int_{|\psi| \ge \varepsilon_0 \omega_N / 2} \exp\big(- e^{-6C_0} \psi^2 / 2 \big) d\psi \le C/\omega_N. \tag{49}$$

Taking the estimates (48) and (49) back into (46), we obtain the required (42).

The proof of Lemma 3.4 is now complete. □

Lemma 3.5. *If* $|v| < \min\{(pq\sum b_k^2)^{-1/2},\ \delta_1 \sum b_k^2/\sum|b_k|^3\}$, *then*

$$\begin{aligned}&\Big|\frac{d\left[e^{-ivm_N}Ee^{(u+iv)S_n}\right]}{dv} + v\sigma_N^2 e^{-\frac12 v^2\sigma_N^2}Ee^{uS_n}\Big| \\ &\qquad\le\ C\,\exp\Big\{\sum K_k\Big\}\sum|b_k|^3/\sum b_k^2, \qquad (50)\end{aligned}$$

where δ_1, m_N, σ_N *and* K_k *are de£ned as in Lemma 3.4.*

Proof. Let ε_0 be de£ned as in Lemma 3.2. By (40), we have

$$\begin{aligned}&\Big|\frac{d\left[e^{-ivm_N}Ee^{(u+iv)S_n}\right]}{dv} + v\sigma_N^2 e^{-\frac12 v^2\sigma_N^2}Ee^{uS_n}\Big| \\ &\qquad\le\ (\sqrt{2\pi}\omega_N G_n(p))^{-1}\Big(J_{1N}+J_{2N}+J_{3N}+J_{4N}\Big), \qquad (51)\end{aligned}$$

where

$$\begin{aligned}J_{1N} &= \int_{|\psi|\le\varepsilon_0\omega_N}\Big|\frac{d\rho(u,v,\psi/\omega_N)}{dv} - \Big(i\,m_N - \sum b_k\xi_k K_k''\Big)\,\rho(u,v,\psi/\omega_N)\Big|d\psi,\\ J_{2N} &= \Big|\int_{|\psi|\le\varepsilon_0\omega_N}\Big(\sum b_k\xi_k K_k'' - v\sigma_N^2\Big)\ \rho(u,v,\psi/\omega_N)\,e^{-ivm_N}d\psi\Big|,\\ J_{3N} &= |v|\sigma_N^2\Big|\int_{|\psi|\le\varepsilon_0\omega_N}\rho(u,v,\psi/\omega_N)e^{-ivm_N}d\psi - \sqrt{2\pi}\omega_N G_n(p)\,e^{-\frac12 v^2\sigma_N^2}Ee^{uS_n}\Big|,\\ J_{4N} &= \Big|\int_{\varepsilon_0\omega_N\le|\psi|\le\pi\omega_N}\frac{d\left[e^{-ivm_N}\rho(u,v,\psi/\omega_N)\right]}{dv}d\psi\Big|.\end{aligned}$$

De£ne $g(\psi,v)$ as in (45). Similarly to the proof of (48), it follows from Lemma 3.3 that

$$\begin{aligned}J_{1N} &\le C\,(pq)\,e^{\sum K_k}\int_{|\psi|\le\varepsilon_0\omega_N}\sum|b_k||\xi_k|^2e^{-g(\psi,v)/4}d\psi\\ &\le 2C\,(pq)\,e^{\sum K_k}\Big(|v|^2\sum|b_k|^3 + \omega_N^{-2}\sum|b_k|\int_{|\psi|\le\varepsilon_0\omega_N}|\psi|^2e^{-g(\psi,v)/4}d\psi\Big)\\ &\le 2C\,(pq)\,e^{\sum K_k}\Big(|v|^2\sum|b_k|^3 + C\omega_N^{-2}\sum|b_k|\left[1+v^2(pq)\sum b_k^2\right]\Big)\\ &\le 2C\,e^{\sum K_k}\sum|b_k|^3/\sum b_k^2, \qquad (52)\end{aligned}$$

since $|v| \le (pq\sum b_k^2)^{-1/2}$, where we have used the estimate: $\sum|b_k|\sum b_k^2 \le N\sum|b_k|^3$ by (33).

Also, by noting

$$\sum b_k\xi_k K_k'' = v\sigma_N^2 + g_1(\psi,v)\frac{\sum b_k K_k''}{\omega_N},$$

where $g_1(\psi,v) = \psi + \frac{v\omega_N\sum b_k K_k''}{\sum K_k''}$, it follows from (21) in Lemma 3.2 that

$$\begin{aligned}J_{2N} \le \frac{\left|\sum b_k K_k''\right|}{\omega_N}e^{\sum K_k}\Big(&\Big|\int_{|\psi|\le\varepsilon_0\omega_N}g_1(\psi,v)\,e^{-g(\psi,v)/2}d\psi\Big|\\ &+ C\,(pq)\int_{|\psi|\le\varepsilon_0\omega_N}\sum|\xi_k|^3|g_1(\psi,v)|\,e^{-g(\psi,v)/4}d\psi\Big). \qquad (53)\end{aligned}$$

Since $\int_{-\infty}^{\infty} g_1(\psi,v)\, e^{-g(\psi,v)/2}d\psi = 0$, and $|g_1(\psi,v)| \le e^{3C_0} g^{1/2}(\psi,v)$ by (43), as in the proof of (49), we have

$$\Big| \int_{|\psi|\le \varepsilon_0\omega_N} g_1(\psi,v)\, e^{-g(\psi,v)/2} d\psi \Big| \le \int_{|\psi|>\varepsilon_0\omega_N} |g_1(\psi,v)|\, e^{-g(\psi,v)/2} d\psi \le C/\omega_N.$$

On the other hand, as in the proof of (48),

$$\int_{|\psi|\le\varepsilon_0\omega_N} pq \sum |\xi_k|^3 |g_1(\psi,v)|\, e^{-g(\psi,v)/4} d\psi \le\ C\Big(|v|^3\,(pq)\ \sum |b_k|^3 + 1/\omega_N\Big).$$

Taking these estimates back into (53), and noting

$$\Big|\sum b_k K_k''\Big| \le e^{6C_0} pq \sum |b_k| \le e^{6C_0}\omega_N^2 \sum |b_k|^3 / \sum b_k^2,$$

and also $\Big|\sum b_k K_k''\Big| \le e^{6C_0}\, N^{1/2}\, pq\, (\sum b_k^2)^{1/2}$, by (20) and (33), we have that for $|v| \le (pq\sum b_k^2)^{-1/2}$,

$$\begin{aligned} J_{2N} &\le C\, \frac{\Big|\sum b_k K_k''\Big|}{\omega_N}\, e^{\sum K_k} \Big(|v|^3\,(pq)\ \sum |b_k|^3 + 1/\omega_N\Big) \\ &\le C\, e^{\sum K_k} \Big(|v|^3\,(pq)^{3/2}\,(\sum |b_k|^2)^{3/2} + 1\Big) \sum |b_k|^3 / \sum b_k^2 \\ &\le C\, e^{\sum K_k} \sum |b_k|^3 / \sum b_k^2. \end{aligned} \tag{54}$$

As for J_{3N}, by using (39) and (42), we obtain that for $|v| \le (pq\sum b_k^2)^{-1/2}$,

$$\begin{aligned} J_{3N} &\le C\,|v|\,\sigma_N^2\,\omega_N \Big(\sum K_k''\Big)^{-1/2} \big[|v|^3\,(pq)\ \sum |b_k|^3 + 1/\omega_N\big] e^{\sum K_k} \\ &\le C\, e^{\sum K_k} \sum |b_k|^3 / \sum b_k^2, \end{aligned} \tag{55}$$

where we have used (43), $\sigma_N^2 \le e^{6C_0}\,(pq)\ \sum b_k^2$ since (20), and some routine calculations.

Finally we estimate J_{4N}. In fact, by using (22), (35) and (43), and noting $|m_N| = |\sum b_k K_k'| \le pqe^{4C_0} \sum |b_k|$ since (20), we have

$$\begin{aligned} J_{4N} &\le \int_{\varepsilon_0\omega_N\le|\psi|\le\pi\omega_N} \Big(\Big|\frac{d\,\rho(u,v,\psi/\omega_N)}{dv}\Big| + |m_N|\,|\rho(u,v,\psi/\omega_N)|\Big)\, d\psi \\ &\le C\,(pq)\, e^{\sum K_k} \sum |b_k| \int_{\varepsilon_0\omega_N\le|\psi|\le\pi\omega_N} e^{-c\omega_N^2} d\psi \\ &\le C\,(pq)\, e^{\sum K_k} \sum |b_k|/\omega_N^2\ \le\ C\, e^{\sum K_k} \sum |b_k|^3 / \sum b_k^2. \end{aligned} \tag{56}$$

Combining (51)-(56) and noting [Lemma 1 in Höglund(1978)]

$$\sqrt{\pi}/2 \le \sqrt{2\pi}\omega_N G_n(p) < 1. \tag{57}$$

we obtain the required (50). The proof of Lemma 3.5 is complete. $\Box$.

We are now ready to prove Theorem 3.1.

Let $T = \delta \sum |b_k|^2 / \sum |b_k|^3$, where $\delta = \min\{\delta_0, \delta_1\}$ with that δ_0 and δ_1 are de£ned as in Lemmas 3.2 and 3.4. De£ne

$$f(v) = Ee^{(u+iv)S_n}/Ee^{uS_n} \quad \text{and} \quad g(v) = e^{ivm_N - v^2\sigma_N^2/2}.$$

Note that $f(v)$ and $g(v)$ are characteristic functions of the random variable with distribution function $H_n(x;u)$ and the normal random variable with mean m_N and variance σ_N^2, respectively. By Esseen's smoothing inequality,

$$\sup_x \Big| H_n(x;u) - \Phi\Big(\frac{x - m_N}{\sigma_N}\Big)\Big| \leq \int_{-T}^{T} |v|^{-1} |f(v) - g(v)| dv + 12/(T\sigma_N). \tag{58}$$

Recalling $\sum b_k = 0$ and (20), it is readily seen that

$$\begin{aligned} \sigma_N^2 &= \sum (b_k - \sum b_k K_k'' / \sum K_k'')^2 K_k'' \\ &> e^{-6C_0} pq \sum (b_k - \sum b_k K_k'' / \sum K_k'')^2 \geq e^{-6C_0} pq \sum |b_k|^2. \end{aligned} \tag{59}$$

This, together with (58), implies that (12) will follow if we prove

$$\int_{-T}^{T} |v|^{-1} |f(v) - g(v)| dv \leq C\,(pq)^{-1/2} \sum |b_k|^3 / \big(\sum b_k^2\big)^{3/2}. \tag{60}$$

Without loss of generality, we assume ω_N suf£ciently large so that $|O_1/\omega_N| \leq 1/2$, where O_1 is de£ned as in (39). Otherwise (60) is trivial by the fact $1/\sqrt{N} \leq \sum |b_k|^3 / (\sum b_k^2)^{3/2}$. For $|O_1/\omega_N| \leq 1/2$, it follows from Lemma 3.4 that

$$\begin{aligned} |f(v) - g(v)| &\leq \exp\big(-v^2\sigma_N^2/2\big) \frac{\big|R - O_1/\omega_N\big|}{\big|1 + O_1/\omega_N\big|} \\ &\leq C\Big(|v|^3\,(pq) \sum |b_k|^3 + 1/\omega_N\Big) e^{-v^2\sigma_N^2/4}. \end{aligned} \tag{61}$$

This, together with (59), implies that

$$\begin{aligned} \int_{T_1 \leq |v| \leq T} |v|^{-1} |f(v) - g(v)| dv &\leq C\,(pq)^{-1/2} \sum |b_k|^3 / (\sum |b_k|^2)^{3/2} + C/\omega_N \\ &\leq 2\,C\,(pq)^{-1/2} \sum |b_k|^3 / (\sum |b_k|^2)^{3/2}, \end{aligned} \tag{62}$$

where $T_1 = \min\{(pq \sum |b_k|^2)^{-1/2}, T\}$.

In the following, we let

$$f_1(v) = e^{-ivm_N} f(v) \quad \text{and} \quad g_1(v) = e^{-ivm_N} g(v) = \exp\{-\frac{1}{2} v^2 \sigma_N^2\}.$$

By (39) and Lemma 3.5, for $|v| \le T_1$,

$$\begin{aligned} |f_1'(v) - g_1'(v)| &= [Ee^{uS_n}]^{-1}\Big|\frac{d\big[e^{-ivm_N}Ee^{(u+iv)S_n}\big]}{dv} + v\sigma_N^2 e^{-\frac{1}{2}v^2\sigma_N^2}Ee^{uS_n}\Big| \\ &\le \frac{C\,G_n(p)\big(\sum K_k''\big)^{1/2}}{\big|1+O_1/\omega_N\big|}\sum |b_k|^3/\sum |b_k|^2 \\ &\le C\sum |b_k|^3/\sum |b_k|^2, \end{aligned}$$

where we have used $|O_1/\omega_N| \le 1/2$ and the fact that, by (43) and (57),

$$G_n(p)\big(\sum K_k''\big)^{1/2} \le e^{2C_0}G_n(p)\omega_N \le C.$$

This, together with the fact that $|f(v) - g(v)| = |f_1(v) - f_2(v)| \le |v|\sup_{0\le t\le v}|f_1'(t) - g_1'(t)|$, implies that

$$\int_{|v|\le T_1} |v|^{-1}|f(v) - g(v)|dv \;\le\; C(pq)^{-1/2}\sum |b_k|^3/(\sum b_k^2)^{3/2}. \tag{63}$$

Now (60) follows from (62) and (63). The proof of Theorem 3.1 is complete. □

4 Proof of Proposition 2.1

Roughly speaking, the proof of Proposition 2.1 is based on the conjugate method and an application of Theorem 3.1 to the b_k speci£ed in (64) below. We need some preliminaries £rst.

Let $0 < \lambda \le 2$, $0 \le \theta \le 1$ and $|\theta_1| \le 72$. De£ne, for $k = 1, \cdots, N$,

$$b_k = \lambda b a_k - \theta b^2 q\big(a_k^2 - 1\big) + \theta_1 b^4 q^2\Big[(a_k^2-1)^2 - \frac{1}{N}\sum (a_j^2 - 1)^2\Big]. \tag{64}$$

Since $\sum a_k = 0$ and $\sum a_k^2 = N$, it is readily seen that $\max_k |a_k| \ge 1$ and $\sum b_k = 0$. Also, when $b\,\max_k |a_k| \le 1/128$, we have that, $b\beta_{3N} \le 1/128$,

$$\begin{aligned} \max_k |b_k| &\le 1/32, & (65)\\ \Big|\sum b_k^2 - \lambda^2 b^2 N\Big| &\le 5Nb^3 q\beta_{3N}, & (66)\\ \sum |b_k|^3 &\le 9Nb^3\beta_{3N}. & (67) \end{aligned}$$

So, recalling $b = x/\omega_N$, (65)-(67) hold true if $0 \le x \le (1/128)\,\omega_N/\max_k |a_k|$.

De£ne $K(z)$ as in (10). We still use the notation K_k, K_k' and K_k'' denote the values of $K(z), K'(z)$ and $K''(z)$ evaluated at $z = b_k + \alpha_N$, where α_N is the solution of the equation

$$\sum K'(b_k + \alpha) = 0. \tag{68}$$

As shown in the solution of (11), if (65) holds true, then α_N is unique and $|\alpha_N| \le 1/32$.

We establish four lemmas before the proof of Proposition 2.1.

Lemma 4.1. *If* $0 \le x \le (1/128)\,\omega_N/\max_k |a_k|$, *then*

$$|\alpha_N| \;\le\; \min\Big\{1/32,\,(2/N)\sum b_k^2\Big\}, \qquad \alpha_N^2 \;\le\; (9/8)\,b^3\,\beta_{3N}. \tag{69}$$

Proof. The inequality that $|\alpha_N| \le 1/32$ has been proved above. By noting $|b_k| + |\alpha_N| \le 1/16$ by (65), it follows from (17), (68) and $\sum b_k = 0$ that

$$\begin{aligned} N|\alpha_N| &= \Big|\sum \big[K'(b_k+\alpha_N)/pq - (b_k+\alpha_N)\big]\Big| \\ &\le \sum (b_k+\alpha_N)^2 = \sum b_k^2 + N\alpha_N^2 \\ &\le \sum b_k^2 + N|\alpha_N|/2. \end{aligned}$$

This yields $|\alpha_N| \le (2/N)\sum b_k^2$, and hence the first result of (69) follows. Furthermore, by using Hölder's inequality, $|b_k| \le 1/32$ and (67),

$$\alpha_N^2 \;\le\; (4/N)\sum b_k^4 \;\le\; (9/8)b^3\beta_{3N},$$

which implies the second result of (69). The proof of Lemma 4.1 is complete. □

Lemma 4.2. *If* $0 \le x \le (1/128)\,\omega_N/\max_k |a_k|$, *then*

$$\Big|\sum K_k - \lambda^2 x^2/2\Big| \;\le\; 24\,x^3\beta_{3N}/\omega_N, \tag{70}$$

$$\Big|\sum b_k K_k' - \lambda^2 x^2\Big| \;\le\; 24\,x^3\beta_{3N}/\omega_N, \tag{71}$$

$$\Big|\sum K_k'' - \omega_N^2\Big| \;\le\; 41\,x^2, \tag{72}$$

$$\Big|\sum b_k K_k''\Big| \;\le\; 6\,x^2, \tag{73}$$

$$\Big|\sum b_k^2 K_k'' - \lambda^2 x^2\Big| \;\le\; 21\,x^3\beta_{3N}/\omega_N. \tag{74}$$

Proof. We prove (70). The others are similar and omitted. Applying (16) with $x = b_k + \alpha_N$ and using Hölder's inquality,

$$\Big|\sum \big[K_k - 2^{-1}pq\,(b_k+\alpha_N)^2\big]\Big| \le 2pq\big(\sum |b_k|^3 + N\alpha_N^3\big). \tag{75}$$

This, together with $\sum b_k = 0$, (66)-(67) and (69), implies that

$$\begin{aligned} \Big|\sum K_k - \lambda^2 x^2/2\Big| &\le \Big|\sum \Big[K_k - 2^{-1}pq\,(b_k+\alpha_N)^2\Big]\Big| \\ &\quad + 2^{-1}pq\Big|\sum b_k^2 - \lambda^2 b^2 N\Big| + 2^{-1}\omega_N^2\alpha_N^2 \\ &\le 24\,b^3\omega_N^2\beta_{3N} = 24x^3\beta_{3N}/\omega_N, \end{aligned}$$

as required. □

Let $Y_j, j = 1, 2, ..., n$ be a random sample of size n without replacement from $\{b_1, b_2, ..., b_N\}$ de£ned by (64), $T_n^* \equiv T_n(\lambda, \theta, \theta_1) = \sum_{k=1}^n Y_k$, $m_N^* \equiv m_N(\lambda, \theta, \theta_1) = \sum b_k K_k'$,

$$\sigma_N^{*\,2} \equiv \sigma_N^2(\lambda, \theta, \theta_1) = \sum b_k^2 K_k'' - (\sum b_k K_k'')^2 / \sum K_k'',$$

and $H_n^*(u) = E\exp(T_n^*) I(T_n^* \le u)/E\exp(T_n^*)$.

Lemma 4.3. *There exists an absolute constant* $\lambda_0 > 0$ *such that, for* $2 \le x \le \lambda_0\, \omega_N / \max_k |a_k|$,

$$\exp\{\lambda^2 x^2/2 - Ax^3\beta_{3N}/\omega_N\} \le E\exp(T_n^*) \le \exp\{\lambda^2 x^2/2 + Ax^3\beta_{3N}/\omega_N\}. \tag{76}$$

Proof. Without loss of generality, assume $\lambda_0 \le \min\{1/128, 1/(8C_1+4)\}$, where C_1 is de£ned as in (39). Recall that $\max_k |b_k| \le 1/32$ by (65). It follows from Lemma 3.4 with $C_0 = 1/32$, $u = 1$ and $v = 0$ that

$$E\exp(T_n^*) = (G_n(p))^{-1} (\sum K_k'')^{-1/2} \exp\{\sum_{j=1}^N K_k\}(1 + R^*), \tag{77}$$

where $G_n(p) = \sqrt{2\pi}\binom{N}{n} p^n q^{N-n}$ and $|R^*| \le C_1/\omega_N$. By Stirling's formula,

$$\binom{N}{n} p^n q^{N-n} = (2\pi\omega_N^2)^{-1/2}(1 + O_2\omega_N^{-2}),$$

where $|O_2| \le 1/6$. This, together with $\omega_N \ge x\, \max_k |a_k|/\lambda_0 \ge 128$ (recall $\max_k |a_k| \ge 1$), implies that

$$\omega_N^{-1} G_n(p)^{-1}(1 + R^*) = 1 + O_3\omega_N^{-1}, \tag{78}$$

where $|O_3| \le 2C_1 + 1$. On the other hand, it follows from (72) that

$$(\sum K_k'')^{-1/2}\omega_N = 1 + O_4 b^2, \tag{79}$$

where $|O_4| \le 82$. From (78)-(79), for $2 \le x \le \lambda_0\, \omega_N / \max_k |a_k|$,

$$\exp\{-2A_1 x^3\beta_{3N}/\omega_N\} \le (\sum K_k'')^{-1/2} G_n(p)^{-1}(1 + R^*) \le \exp\{A_1 x^3 \beta_{3N}/\omega_N\}, \tag{80}$$

where $A_1 = 2C_1 + 83$ and we have used the fact that $1/\omega_N + b^2 \le x^3\beta_{3N}/\omega_N$ since $b = x/\omega_N$ and $\beta_{3N} \ge 1$. Now (76) follows easily from (70), (77) and (80). The proof of Lemma 4.3 is complete. $\square$

Lemma 4.4. *There exists an absolute constant* $\lambda_1 > 0$ *such that, for* $2 \le x \le \lambda_1\, \omega_N / \max_k |a_k|$,

$$|m_N^* - \lambda^2 x^2| \;\le\; 24\, x^3 \beta_{3N}/\omega_N, \tag{81}$$

$$|\sigma_N^{*\,2} - \lambda^2 x^2| \;\le\; 22\, x^3\beta_{3N}/\omega_N, \tag{82}$$

If in addition $1 \leq \lambda \leq 2$, *then*

$$\Delta_N \ := \ \sup_y \Big| H_n^*(u(y)) - \Phi(y) \Big| \ \leq 12\, C\beta_{3N}/\omega_N \ \leq \ 1/4, \tag{83}$$

where $u(y) = y\,\sigma_N^* + m_N^*$ *and* C *is de£ned as in Theorem 3.1.*

Also, for all y *satisfying* $m_N^* \geq y + 2\sigma_N^*$,

$$P(T_n^* \geq y) \geq (1/2)\, \exp\{-m_N^* - 2\sigma_N^*\}\, E\exp(T_n^*). \tag{84}$$

Proof. Without loss of generality, assume $\lambda_1 \leq \min\{1/128, 1/(25C)\}$, where C is de£ned as in Theorem 3.1. Then (81) and (82) follow from (71)-(74) by a simple calculation.

If $1 \leq \lambda \leq 2$, by noting $\beta_{3N}/\omega_N \leq x\beta_{3N}/(2\omega_N) \leq \min\{1/128, 1/(50C)\}$ since $\beta_{3N} \leq \max_k |a_k|$, it follows easily from (65)-(67) that $pq \sum b_k^2 \geq 4x^2/5$ and

$$(pq)^{-1/2} \sum |b_k|^3 / (\sum b_k^2)^{3/2} \leq 12\beta_{3N}/\omega_N \leq 1/(4C). \tag{85}$$

By (85) and Theorem 3.1 with $C_0 = 1/32$ and $u = 1$ (recall $\max_k |b_k| \leq 1/32$),

$$\Delta_N \leq C(pq)^{-1/2} \sum |b_k|^3 / (\sum b_k^2)^{3/2} \ \leq 12\, C\beta_{3N}/\omega_N \ \leq \ 1/4,$$

which implies (83).

We next prove (84). In fact, by (83) and the conjugate method, for all y satisfying $m_N^* \geq y + 2\sigma_N^*$,

$$\begin{aligned} P(T_n^* \geq y) \Big/ E\exp(T_n^*) \ &= \ \int_y^\infty e^{-u} dH_n^*(u) \\ &= \ e^{-m_N^*} \int_{(y - m_N^*)/\sigma_N^*}^\infty e^{-x\sigma_N^*} dH_n^*(u(y)) \\ &\geq \ e^{-m_N^* - 2\sigma_N^*} \int_{-2}^2 dH_n^*(u(y)) \\ &\geq \ e^{-m_N^* - 2\sigma_N^*} (P(|N(0,1)| \leq 2) - \Delta_N) \\ &\geq \ (1/2)\exp\{-m_N^* - 2\sigma_N^*\}, \end{aligned}$$

where $N(0,1)$ is a standard normal random variable and we have used the fact that

$$P(|N(0,1)| \leq 2) > 3/4.$$

This proves (84) and also completes the proof of Lemma 4.4. $\square$

After these preliminaries, we are now ready to prove Proposition 2.1.

In addition to the previous notation, we further let $T_{1n}=T_n(1,\xi,\xi_1)$,

$$m_{1N}=m_N(1,\xi,\xi_1),\quad \sigma_{1N}^2=\sigma_N^2(1,\xi,\xi_1),\quad \varepsilon_N=(x^2+h-m_{1N})/\sigma_{1N}$$

and $H_{1n}(u)=E\exp\{T_{1n}\}I(T_{1n}\le u)/E\exp\{T_{1n}\}$. Note that

$$bS_n-\xi\, b^2qV_{1n}+\xi_1\, b^4q^2V_{2n}=T_{1n}.$$

It follows from the conjugate method that,

$$\begin{aligned}
P\Big(bS_n-\xi\, b^2qV_{1n}+\xi_1\, b^4q^2V_{2n}\ge x^2+h\Big) &= P(T_{1n}\ge x^2+h)\\
&= E\exp\{T_{1n}\}\int_{x^2+h}^{\infty}e^{-t}dH_{1n}(t)\\
&= E\exp\{T_{1n}\}\,e^{-x^2-h}\int_0^{\infty}e^{-t\sigma_{1N}}dH_{1n}\Big[\sigma_{1N}(t+\varepsilon_N)+m_{1N}\Big]\\
&= E\exp\{T_{1n}\}\,e^{-x^2-h}\Big(\mathcal{L}_N+R_N\Big) \qquad (86)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}_N &= \int_0^{\infty}e^{-t\sigma_{1N}}d\Phi(t+\varepsilon_N),\\
R_N &= \int_0^{\infty}e^{-t\sigma_{1N}}d\Big\{H_{1n}\Big[\sigma_{1N}(t+\varepsilon_N)+m_{1N}\Big]-\Phi(t+\varepsilon_N)\Big\}.
\end{aligned}$$

We next estimate $E\exp\{T_{1n}\}$, $\mathcal{L}_N$ and R_N for $0\le\xi\le 1/2$, $|\xi_1|\le 36$, $|h|\le x^2/5$ and $2\le x\le \eta\omega_N/\max_k|a_k|$, where we assume η suffciently small such that $\eta\le\min\{1/128,\lambda_0,\lambda_1\}$, with λ_0 and λ_1 defined as in Lemmas 4.3 and 4.4. This η chosen guarantees that Lemmas 4.1-4.4 hold true, and since $\beta_{3N}\le\max_k|a_k|$,

$$\beta_{3N}/\omega_N\le x\beta_{3N}/(2\omega_N)\le\eta/2\le 1/256. \qquad (87)$$

Clearly, by Lemma 4.3,

$$\exp\big\{x^2/2-A\,x^3\beta_{3N}/\omega_N\big\}\le E\exp\{T_{1n}\}\le\exp\big\{x^2/2+A\,x^3\beta_{3N}/\omega_N\big\}. \qquad (88)$$

In order to estimate $\mathcal{L}_N$, we note that

$$\begin{aligned}
\mathcal{L}_N &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-\sigma_Nt-\frac12(t+\varepsilon_N)^2}dt\\
&= \frac{e^{-\varepsilon_N^2/2}}{\sqrt{2\pi}}\int_0^{\infty}e^{-(\varepsilon_N+\sigma_N)t-\frac12t^2}dt\\
&:= \frac{e^{-\varepsilon_N^2/2}}{\sqrt{2\pi}}\,\mathcal{L}_{1N}. \qquad (89)
\end{aligned}$$

Write $\psi(t) = \big\{1 - \Phi(t)\big\}/\Phi'(t) = e^{t^2/2}\int_t^\infty e^{-y^2/2}dy$. It is readily seen that,

$$3/4 \le t\psi(t) \le 1 \quad \text{for} \quad t \ge 2, \quad \text{and} \quad |\psi'(t)| = |t\psi(t) - 1| \le t^{-2} \quad \text{for} \quad t > 0. \tag{90}$$

On the other hand, $\psi\{\varepsilon_N + \sigma_N\} = \mathcal{L}_{1N}$, and by virtue of (81)-(82) and (87),

$$|\varepsilon_N - h/\sigma_N| \le 28x^2\beta_{3N}/\omega_N \tag{91}$$

and if in addition $|h| \le x^2/5$,

$$|\varepsilon_N + \sigma_N - x| \le 3|h|/(2x) + 41\, x^2\beta_{3N}/\omega_N \le 2x/3. \tag{92}$$

Using (90)-(92), it follows from Taylor's expansion that, for $|h| \le x^2/5$ and $2 \le x \le \eta\omega_N/\max_k |a_k|$,

$$\begin{aligned}
\mathcal{L}_{1N} &= \psi\{\varepsilon_N + \sigma_N\} \\
&= \psi(x) + \psi'(\theta)\big\{\varepsilon_N + \sigma_N - x\big\}, \qquad\qquad [\text{where } \theta \in (x/3, 5x/3)] \\
&= \psi(x)\big(1 + \tau + O_5\, x\, \beta_{3N}/\omega_N\big),
\end{aligned}$$

where $|\tau| \le 9|h|/x^2$ and $|O_5| \le 120$. Therefore, taking account of (89), we get for $|h| \le x^2/5$ and $2 \le x \le \eta\omega_N/\max_k |a_k|$,

$$\mathcal{L}_N \;=\; e^{x^2/2}\big\{1 - \Phi(x)\big\}\, e^{-\varepsilon_N^2/2}\big(1 + \tau + O_5\, x\, \beta_{3N}/\omega_N\big). \tag{93}$$

As for R_N, by (83) and integration by parts,

$$|R_N| \le 2\sup_t |H_{1n}\Big[\sigma_{1N}t + m_{1N}\Big] - \Phi(t)| \le 24C\beta_{3N}/\omega_N.$$

This, together with (90), implies that for $x \ge 2$,

$$R_N \;=\; O_6\, x\, \beta_{3N}/\omega_N\, e^{x^2/2}\big\{1 - \Phi(x)\big\}, \tag{94}$$

where $|O_6| \le 32\sqrt{2\pi}C$.

Combining (86), (88) and (93)-(94), it is readily seen that for any $|h| \le x^2/5$ and $2 \le x \le \eta\omega_N/\max_k |a_k|$,

$$\frac{P\Big(bS_n - \xi\, b^2 qV_{1n} + \xi_1\, b^4 q^2 V_{2n} \ge x^2 + h\Big)}{1 - \Phi(x)} \\
\le \big[1 + 9|h|x^{-2}\big]\exp\big\{-h + Ax^3\beta_{3N}/\omega_N\big\}.$$

This proves (7).

Similarly, by letting $h = 0$, it follows from (86), (88), (91) and (93)-(94) that if, in addition, $x^2 \le \omega_N/\beta_{3N}$, then

$$\begin{aligned}
&\frac{P\Big(bS_n - \xi\, b^2 q V_{1n} + \xi_1\, b^4 q^2 V_{2n} \ge x^2\Big)}{1-\Phi(x)} \\
&\ge\ \exp\left\{-Ax^3\beta_{3N}/\omega_N - \varepsilon_N^2/2\right\}\Big[1-\Big\{|O_5| + |O_6|e^{\varepsilon_N^2/2}\Big\}\, x\,\beta_{3N}/\omega_N\Big] \\
&\ge\ \exp\left\{-A_1x^3\beta_{3N}/\omega_N\right\}\Big[1 - A_2\, x\beta_{3N}/\omega_N\Big] \\
&\ge\ \exp\left\{-A_3\, x^3\beta_{3N}/\omega_N\right\}, \qquad (95)
\end{aligned}$$

by choosing η suf£ciently small. From (95), the property (6) will follow if we prove that, for $x^2 \ge \omega_N/\beta_{3N}$ and $2 \le x \le \eta\omega_N/\max_k |a_k|$,

$$\frac{P\Big(bS_n - \xi\, b^2 q V_{1n} + \xi_1\, b^4 q^2 V_{2n} \ge x^2\Big)}{1-\Phi(x)} \ge \exp\left\{-A\, x^3\beta_{3N}/\omega_N\right\}. \qquad (96)$$

We will prove (96) by using (84). Let $\lambda = 1 + 28x\beta_{3N}/\omega_N$, $\theta = \lambda\xi$ and $\theta_1 = \lambda\xi_1$. Note that, $1 \le \lambda \le 3/2$ by (87), $0 \le \theta \le 3/4$ since $0 \le \xi \le 1/2$ and $|\theta_1| \le 72$ since $|\xi_1| \le 36$. By virtue of (81)-(82), (87) and $x^2 \ge \omega_N/\beta_{3N}$, we have $m_N^* \le \lambda^2 x^2 + 24\, x^3\, \beta_{3N}/\omega_N$, $\sigma_N^* \le 2x \le 2x^3\, \beta_{3N}/\omega_N$ and

$$m_N^* - \lambda x^2 - 2\sigma_N^* \ \ \ge\ \ \lambda(\lambda-1)x^2 - 28x^3\beta_{3N}/\omega_N\ \ge\ 0.$$

Now, by (84) with $y = \lambda x^2$ and Lemma 4.3, for $x^2 \ge \omega_N/\beta_{3N}$ and $2 \le x \le \eta\omega_N/\max_k |a_k|$,

$$\begin{aligned}
P\Big(bS_n - \xi\, b^2 q V_{1n} + \xi_1\, b^4 q^2 V_{2n} \ge x^2\Big) &= P(T_n^* \ge \lambda x^2) \\
&\ge \frac{1}{2}\exp\{-m_N^* - 2\sigma_N^*\}E\exp\{T_n^*\} \\
&\ge \frac{1}{2}\exp\{-x^2/2 - 2x - Ax^3\beta_{3N}/\omega_N\} \\
&\ge (1-\Phi(x))\exp\{-A_1x^3\beta_{3N}/\omega_N\},
\end{aligned}$$

which implies (96). The proof of Proposition 2.1 is now complete. □

5 Proof of Proposition 2.2

By the inequality $(1+y)^{1/2} \geq 1 + y/2 - y^2$ for any $y \geq -1$,

$$
\begin{aligned}
P(S_n \geq x\sqrt{q} V_n) &= P\left(S_n \geq x\sqrt{nq}\left(1 + \frac{V_n^2 - n}{n}\right)^{1/2}\right) \\
&\leq P\left(S_n \geq x\sqrt{nq}\Big[1 + \frac{V_{1n}}{2n} - \frac{V_{1n}^2}{n^2}\Big]\right) \\
&\leq P\Big(V_{1n}^2 \geq 36x^2\Big(\sum_{k=1}^n (X_k^2-1)^2 + 5p\sum a_k^4\Big)\Big) \\
&\quad + P\Big(S_n \geq x\sqrt{nq}\Big(1 + \frac{V_{1n}}{2n} - \frac{36x^2}{n^2}\Big(\sum_{k=1}^n (X_k^2-1)^2 + 5p\sum a_k^4\Big)\Big)\Big) \\
&:= R_{1n} + R_{2n}, \quad \text{say.}
\end{aligned} \tag{97}
$$

Note that $R_{2n} = P\Big(bS_n - \frac{1}{2}b^2 q V_{1n} + 36\, b^4 q^2 V_{2n} \geq x^2 - h_0\Big)$, where, whenever $2 \leq x \leq (1/128)\,\omega_N/\max_k |a_k|$,

$$
h_0 = \frac{180p\, x^4 \sum a_k^4}{n^2} + \frac{36\, x^4 \sum_{k=1}^n E(X_k^2-1)^2}{n^2} \leq \frac{3x^3\beta_{3N}}{\omega_N},
$$

and also $0 \leq h_0 \leq x^2/5$. It follows from Proposition 2.1 with $\xi = 1/2$, $\xi_1 = 36$ and $h = h_0$ that there exists an absolute constant $A > 128$ such that, for all $2 \leq x \leq (1/A)\,\omega_N/\max_k |a_k|$,

$$
R_{2n} \;\leq\; (1 - \Phi(x))\exp\{Ax^3 \beta_{3N/\omega_N}\}. \tag{98}
$$

This, together with (97), implies that Proposition 2.2 will follow if we prove, for all $x > 0$,

$$
R_{1n} \;\leq\; 2\sqrt{2}\, e^{-4x^2}. \tag{99}
$$

Theorem 2.1 of de la Pena, Klass and Lai (2004) will be used to prove (99). To use the theorem, let $Y_i = X_i^2 - 1$, $\mathcal{A} = \sum_{k=1}^n Y_k$ and $\mathcal{B} = (2\sum_{k=1}^n Y_k^2 + 4p\sum a_k^4)^{1/2}$. It follows from

Theorem 4 of Hoeffding(1963) (also see Lemma 6.2 below) that, for any $\lambda \in R$,

$$
\begin{aligned}
& E \exp\Big\{\lambda\, \mathcal{A} - \frac{\lambda^2}{2}\mathcal{B}^2\Big\} \\
& = \exp\Big\{ - 2\lambda^2 p \sum a_k^4\Big\} E\exp\Big\{\sum_{k=1}^n (\lambda Y_k - \lambda^2 Y_k^2)\Big\} \\
& \le \exp\Big\{ - 2\lambda^2 p \sum a_k^4\Big\} \Big[E\exp\{\lambda Y_1 - \lambda^2 Y_1^2\}\Big]^n \\
& \le \exp\Big\{ - 2\lambda^2 p \sum a_k^4\Big\} \Big[1 + E(\lambda Y_1 I(\lambda Y_1 \ge -1/2))\Big]^n \\
& = \exp\Big\{ - 2\lambda^2 p \sum a_k^4\Big\} \Big[1 - E(\lambda Y_1 I(\lambda Y_1 \le -1/2))\Big]^n \\
& \le \exp\Big\{ - 2\lambda^2 p \sum a_k^4\Big\} \Big[1 + 2\lambda^2 E Y_1^2\Big]^n \\
& \le \exp\Big\{ - 2\lambda^2 p \sum a_k^4 + 2\lambda^2 n E Y_1^2\Big\} \\
& = \exp\Big\{ - 2\lambda^2 p \sum a_k^4 + 2\lambda^2 p \sum (a_k^2 - 1)^2\Big\} \le 1,
\end{aligned}
$$

where we have used the inequality $e^{x-x^2} \le 1 + xI(x \ge -1/2)$. This yields that two random variables $\mathcal{A}$ and $\mathcal{B} > 0$ satisfy the condition (1.4) in de la Pena, Klass and Lai (2004). Now, by noting $(E\mathcal{B})^2 \le E\mathcal{B}^2 \le 6p\sum a_k^4$ and applying Theorem 2.1 of de la Pena, Klass and Lai (2004), we have

$$
\begin{aligned}
& P\Big(V_{1n} \ge 6x\Big(\sum_{k=1}^n (X_k^2 - 1)^2 + 5p\sum a_k^4\Big)^{1/2}\Big) \\
& \le P\Big(\mathcal{A} \ge \frac{6x}{\sqrt{2}}\sqrt{\mathcal{B}^2 + (E\mathcal{B})^2}\Big) \\
& \le e^{-6xt/\sqrt{2}} E\exp\big(t\mathcal{A}/\sqrt{\mathcal{B}^2 + (E\mathcal{B})^2}\big) \\
& \le \sqrt{2} e^{-6xt/\sqrt{2} + t^2} \le \sqrt{2}\, e^{-4x^2},
\end{aligned}
\tag{100}
$$

by letting $t = \sqrt{2}x$. Similarly,

$$
P\Big(- V_{1n} \ge 6x\Big(\sum_{k=1}^n (X_k^2 - 1)^2 + 5p\sum a_k^4\Big)^{1/2}\Big) \le \sqrt{2}\, e^{-4x^2}. \tag{101}
$$

By virtue of (100) and (101), we obtain (99). The proof of Proposition 2.2 is now complete. □

6 Proof of Proposition 2.3

Throughout the section, let $\varepsilon_j, 1 \le j \le N$ be iid random variables with $P(\varepsilon_1 = 1) = 1 - P(\varepsilon_1 = 0) = p$, which are also independent of all other random variables, and $B_N = \sum_{j=1}^N (\varepsilon_j -$

p). By the inequality $(1+y)^{1/2} \geq 1 + y/2 - y^2$ for any $y \geq -1$ again, we have

$$
\begin{aligned}
P(S_n \geq x\sqrt{q}V_n) &= P\Big(S_n \geq x\sqrt{nq}\Big(1+\frac{V_n^2-n}{n}\Big)^{1/2}\Big) \\
&\leq P\Big(S_n \geq x\sqrt{nq}\Big(1+\frac{V_n^2-n}{2n}-\frac{(V_n^2-n)^2}{n^2}\Big)\Big) \\
&= P\Big(\sum \varepsilon_k a_k \geq x\sqrt{nq}\Big(1+\frac{\sum \varepsilon_k(a_k^2-1)}{2n}-\frac{(\sum \varepsilon_k(a_k^2-1))^2}{n^2}\Big)\ \Big| B_N = 0\Big) \\
&= P\Big(\sum(\varepsilon_k - p)g_k + \frac{x}{n^2}\sum_{1\leq k\neq j\leq N}\nu_k\nu_j \geq x - h\ \Big| B_N = 0\Big) \\
&= P(T_N + \Lambda_N \geq x - h\,|B_N = 0), \qquad (102)
\end{aligned}
$$

where $h = xpq\sum(a_k^2-1)^2/n^2$,

$$
T_N = \sum(\varepsilon_k - p)g_k, \qquad \Lambda_N = \frac{x}{n^2}\sum_{1\leq k\neq j\leq N}\nu_k\nu_j,
$$

where, for all $j = 1, \cdots, N,\ \nu_j = (\varepsilon_j - p)(a_j^2 - 1)$ and

$$
g_j = \frac{a_j}{\sqrt{nq}} - \frac{x(a_j^2-1)}{2n} + \frac{x(1-2p)}{n^2}\Big((a_j^2-1)^2 - \frac{1}{N}\sum(a_k^2-1)^2\Big),
$$

and where, in the proof of (102), we have used the fact that $\sum a_k = 0, \sum a_k^2 = N$ and

$$
(\varepsilon_k - p)^2 = \varepsilon_k(1-2p) + p^2 = (\varepsilon_k - p)(1-2p) + pq.
$$

We need the following lemmas before the proof of Proposition 2.3.

Lemma 6.1. *For any random variable Z with $E|Z| < \infty$,*

$$
E\Big(Z|B_N = 0\Big) = \frac{1}{B_n(p)}\int_{-\pi\omega_N}^{\pi\omega_N} EZe^{itB_N/\omega_N}dt, \qquad (103)
$$

where $B_n(p) = 2\pi\omega_N P(B_N = 0)$ and

$$
1 \ \leq\ \sqrt{2\pi}/B_n(p)\ \leq\ 1 + \omega_N^{-2}. \qquad (104)
$$

Proof. Note that $B_N = \sum_{j=1}^N \varepsilon_j - n$ is an integer and for any integer k,

$$
\int_{-\pi}^{\pi} e^{ikt}dt = \begin{cases} 2\pi & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}
$$

The proof of (103) is now obvious. The estimate for $B_n(p)$ follows from $P(B_N = 0) = \binom{N}{n}p^n q^{N-n}$ and Stirling's formula. $\square$

Lemma 6.2. *Let the population $\{C\}_N$ consist of N values $c_1, \cdots, c_N$. Let $\tilde{X}_1, \cdots, \tilde{X}_n$ denote a random sample without replacement from $\{C\}_N$ and let $\tilde{Y}_1, \cdots, \tilde{Y}_n$ denote a random sample with replacement from $\{C\}_N$. Then for any continuous and convex function $f(x)$,*

$$Ef\Big(\sum_{k=1}^{n} \tilde{X}_k\Big) \;\le\; Ef\Big(\sum_{k=1}^{n} \tilde{Y}_k\Big). \tag{105}$$

$$Ef\Big(\frac{n-1}{N}\sum_{k=1}^{n} \tilde{X}_k^2 + \frac{N-1}{N}\sum_{1\le k\ne j\le n} \tilde{X}_k\,\tilde{X}_j\Big) \;\le\; Ef\Big(\sum_{1\le k\ne j\le n} \tilde{Y}_k\,\tilde{Y}_j\Big). \tag{106}$$

Proof. (105) is Theorem 4 of Hoeffding(1963). We next prove (106). As in the proof of Theorem 4 in Hoeffding(1963), for any function f, there exists a function $\bar{g}_f(x_1, \cdots, x_n)$ which is symmetric in $x_1, \cdots, x_n$ such that

$$Ef\Big(\sum_{1\le k\ne j\le n} \tilde{Y}_k\,\tilde{Y}_j\Big) = E\bar{g}_f(\tilde{X}_1, \cdots, \tilde{X}_n). \tag{107}$$

By noting

$$Ef\Big(\sum_{1\le k\ne j\le n} \tilde{Y}_k\,\tilde{Y}_j\Big) = \frac{1}{N^n}\sum_{k_1,\ldots,k_n=1}^{N} f\Big[\Big(\sum_{j=1}^{n} c_{k_j}\Big)^2 - \sum_{j=1}^{n} c_{k_j}^2\Big],$$

as in (6.6) of Hoeffding(1963), $\bar{g}_f$ can be written as

$$\bar{g}_f(x_1, \cdots, x_n) = {\sum}' p(k, i_1, \cdots, i_k, r_1, \cdots, r_k) f\Big[\Big(\sum_{j=1}^{k} r_j x_{i_j}\Big)^2 - \sum_{j=1}^{k} r_j x_{i_j}^2\Big], \tag{108}$$

where the sum ${\sum}'$ is taken over the positive integers $k, i_1, \cdots, i_k, r_1, \cdots, r_k$ such that $k = 1, 2, ..., n$, $\sum_{j=1}^{k} r_j = n$ and $i_1, ..., i_k$ are all different and do not exceed n. The coef£cients p are non-negative and do not depend on the function f. In particular, when $f(\cdot) = x$,

$$\bar{g}_x(x_1, \cdots, x_n) = K_0\sum_{k=1}^{n} x_k^2 + K_1\sum_{1\le k\ne j\le n} x_k\,x_j, \tag{109}$$

since $\bar{g}_f$ is symmetric on $(x_1, ..., x_n)$, where K_0 and K_1 are constants. Since

$$E\sum_{1\le k\ne j\le n} \tilde{Y}_k\,\tilde{Y}_j = K_0 E\sum_{k=1}^{n} \tilde{X}_k^2 + K_1 E\sum_{1\le k\ne j\le n} \tilde{X}_k\,\tilde{X}_j$$

by (107) and (109), we have that

$$\frac{n(n-1)}{N^2}\Big(\sum c_k\Big)^2 = \frac{K_0 n\sum c_k^2}{N} + \frac{K_1 n(n-1)}{N(N-1)}\Big(\Big(\sum c_k\Big)^2 - \sum c_k^2\Big)$$

holds true for any $c_1, \cdots, c_N \in \mathcal{R}$, and hence $K_0 = \frac{n-1}{N}$ and $K_1 = \frac{N-1}{N}$. On the other hand, by letting $f = 1$ in (107) and (108),

$$\sum{}' p(k, i_1, \cdots, i_k, r_1, \cdots, r_k) = 1. \tag{110}$$

By virtue of (107)-(110), it follows from the Jensen's inequality that, for any continuous and convex function $f(x)$,

$$
\begin{aligned}
Ef\Big(\frac{n-1}{N}\sum_{k=1}^{n}\tilde X_k^2+\frac{N-1}{N}\sum_{1\le k\ne j\le n}\tilde X_k\tilde X_j\Big) &= Ef\big(\bar g_x(\tilde X_1,\cdots,\tilde X_n)\big)\\
\le E\bar g_f(\tilde X_1,\cdots,\tilde X_n) &= Ef\Big(\sum_{1\le k\ne j\le n}\tilde Y_k\tilde Y_j\Big).
\end{aligned}
$$

This yields (106) and hence completes the proof of Lemma 6.2. $\Box$

Lemma 6.3. *(i). We have*

$$
\begin{aligned}
E\Big(\sum_{1\le k\ne j\le N}|\nu_k\nu_j|^{3/2}\Big|B_N=0\Big) &\le An^2\beta_{3N}^2, && (111)\\
E\Big(\sum_{k=1}^{N}\Big|\nu_k\sum_{j=1,\ne k}^{N}\nu_j\Big|^{3/2}\Big|B_N=0\Big) &\le An^2\beta_{3N}^2, && (112)\\
E\Big(\Big|\sum_{1\le k\ne j\le N}\nu_k\nu_j\Big|^{3/2}\Big|B_N=0\Big) &\le An^2\beta_{3N}^2. && (113)
\end{aligned}
$$

(ii). If $\eta_k, 1\le k\le N$, are iid random variables with

$$P(\eta_k=1)=1-P(\eta_k=0)=m(t),\quad 0\le m(t)\le 1,$$

independent of all other rv's, then

$$
\begin{aligned}
E\Big(\Big|\sum_{1\le k\ne j\le N}\eta_k\eta_j\nu_k\nu_j\Big|^{3/2}\Big|B_N=0\Big) &\le Am^2(t)n^2\beta_{3N}^2, && (114)\\
E\Big(\Big|\sum_{1\le k\ne j\le N}\eta_k(1-\eta_j)\nu_k\nu_j\Big|^{3/2}\Big|B_N=0\Big) &\le Am(t)n^2\beta_{3N}^2. && (115)
\end{aligned}
$$

Proof. We £rst prove (113). Note that

$$
\begin{aligned}
\sum_{1\le k\ne j\le N}\nu_k\nu_j &= \sum_{1\le k\ne j\le N}\varepsilon_j\varepsilon_k(a_j^2-1)(a_k^2-1)+2p\sum\varepsilon_k(a_k^2-1)^2\\
&\quad+p^2\sum_{1\le k\ne j\le N}(a_j^2-1)(a_k^2-1).
\end{aligned}
$$

By the c_r-inequality, we have

$$E\Big(\Big|\sum_{1\le k\ne j\le N}\nu_k\nu_j\Big|^{3/2}\Big|B_N=0\Big) \le 4(I_1+4I_2+I_3), \qquad (116)$$

where

$$I_1 = E\Big(\Big|\sum_{1\le k\ne j\le N} \varepsilon_j\varepsilon_k(a_j^2-1)(a_k^2-1)\Big|^{3/2}\Big|B_N=0\Big),$$
$$I_2 = p^{3/2}E\Big(\Big|\sum \varepsilon_k(a_k^2-1)^2\Big|^{3/2}\Big|B_N=0\Big),$$
$$I_3 = p^3\Big|\sum_{1\le k\ne j\le N}(a_j^2-1)(a_k^2-1)\Big|^{3/2}.$$

Since $\sum a_k^2 = N$,

$$I_3 \;\le\; p^3\Big|\sum(a_k^2-1)^2\Big|^{3/2} \le p^3\Big(\sum a_k^4\Big)^{3/2} \le p^3\Big(\sum |a_k|^3\Big)^2 \le n^2\beta_{3N}^2. \qquad (117)$$

Recall that $X_1, X_2, ..., X_n$ is a random sample without replacement from $\{a\}_N = \{a_1, \cdots, a_N\}$. Suppose that $Y_1, Y_2, ..., Y_n$ is a random sample with replacement from $\{a\}_N$. Note that Y_j are iid random variables with $Ef(Y_1) = \frac{1}{N}\sum f(a_k)$ for any $f(.)$. It follows from Lemma 6.2 and the classical results for iid random variables that

$$\begin{aligned}
I_2 \;&=\; p^{3/2}\,E\Big|\sum_{k=1}^n (X_k^2-1)^2\Big|^{3/2} \;\le\; p^{3/2}\,E\Big|\sum_{k=1}^n (Y_k^2-1)^2\Big|^{3/2} \\
&\le\; 2p^{3/2}\,E\Big|\sum_{k=1}^n\big((Y_k^2-1)^2 - E(Y_k^2-1)^2\big)\Big|^{3/2} + 2p^{3/2}\Big|nE(Y_1^2-1)^2\Big|^{3/2} \\
&\le\; 4p^{3/2}\sum_{k=1}^n E\Big|\big((Y_k^2-1)^2 - E(Y_k^2-1)^2\big)\Big|^{3/2} + 2p^3\Big|\sum(a_k^2-1)^2\Big|^{3/2} \\
&\le\; 16p^{5/2}\sum|a_k^2-1|^3 + 2p^3\Big|\sum(a_k^2-1)^2\Big|^{3/2} \\
&\le\; 18p^{5/2}\Big(\sum|a_k|^3\Big)^2 \le 18\,n^2\,\beta_{3N}^2. \qquad (118)
\end{aligned}$$

Similarly, it follows from Lemma 6.2 and the classical results for U-statistics that

$$\begin{aligned}
\Big(\frac{N-1}{N}\Big)^{3/2} I_1 \;&=\; \big(\frac{N-1}{N}\big)^{3/2}E\Big|\sum_{1\le k\ne j\le n}(X_j^2-1)(X_k^2-1)\Big|^{3/2} \\
&\le\; 2E\Big|\sum_{1\le k\ne j\le n}(Y_k^2-1)(Y_j^2-1)\Big|^{3/2} + 2p^{3/2}E\Big|\sum_{k=1}^n(X_k^2-1)^2\Big|^{3/2} \\
&\le\; A\,n^2\,(E|Y_1|^3)^2 + 36\,n^2\,\beta_{3N}^2 \le A_1\,n^2\,\beta_{3N}^2. \qquad (119)
\end{aligned}$$

Combining (116)-(119), we obtain the required (113).

We next prove (112). Note that, by $\sum a_k^2 = N$,

$$\begin{aligned}
\nu_k\sum_{j=1,\ne k}^N \nu_j \;&=\; \varepsilon_k(a_k^2-1)\sum_{j=1,\ne k}^N \varepsilon_j(a_j^2-1) \\
&\quad - p(a_k^2-1)\sum_{j=1}^N \varepsilon_j(a_j^2-1) \;+\; (2p\varepsilon_k - p^2)(a_k^2-1)^2.
\end{aligned}$$

By the c_r-inequality, we have

$$E\Big(\sum_{k=1}^{N}\big|\nu_k\sum_{j=1,\neq k}^{N}\nu_j\big|^{3/2}\Big|B_N=0\Big) \;\leq\; 4(I_4+I_5+I_6),$$

where, as in the proofs of (117)-(119),

$$\begin{aligned}
I_4 &= \sum_{k=1}^{N}E\Big|\varepsilon_k(a_k^2-1)\sum_{j=1,\neq k}^{N}\varepsilon_j(a_j^2-1)\Big|^{3/2}\Big|B_N=0\Big) \\
&= p\sum_{k=1}^{N}\big|a_k^2-1\big|^{3/2}E\Big(\Big|\sum_{j=1,\neq k}^{N}\varepsilon_j(a_j^2-1)\Big|^{3/2}\Big|\sum_{j=1,\neq k}^{N}\varepsilon_j=n-1\Big) \\
&\leq \frac{A\,n(n-1)}{N(N-1)}\sum_{k=1}^{N}\big|a_k^2-1\big|^{3/2}\sum_{j=1,\neq k}^{N}|a_j^2-1|^{3/2} \;\leq\; A\,n^2\,\beta_{3N}^2, \\
I_5 &= p\sum_{k=1}^{N}|a_k^2-1|^{3/2}E\Big(\Big|\sum_{j=1}^{N}\varepsilon_j(a_j^2-1)\Big|^{3/2}\Big|B_N=0\Big) \;\leq\; A\,n^2\,\beta_{3N}^2, \\
I_6 &= \sum_{k=1}^{N}|a_k^2-1|^3E\Big(\big|2p\varepsilon_k-p^2\big|^{3/2}\Big|B_N=0\Big) \;\leq\; A\,p^2\sum_{k=1}^{N}|a_k^6 \;\leq\; A\,n^2\,\beta_{3N}^2.
\end{aligned}$$

This yields (112).

The proof of (111) is simple. Indeed,

$$\begin{aligned}
\sum_{1\leq k\neq j\leq N}E\Big(|\nu_k\nu_j|^{3/2}\Big|B_N=0\Big) \;&\leq\; A\Big(\sum|a_k|^3\Big)^2E\Big(|(\varepsilon_1-p)(\varepsilon_2-p)|^{3/2}\Big|B_N=0\Big) \\
&\leq\; A_1\,p^2\Big(\sum|a_k|^3\Big)^2=A_1n^2\beta_{3N}^2.
\end{aligned}$$

We £nally prove (114) and (115). By (113) and the c_r inequality, it suf£ces to prove

$$E\Big(\Big|\sum_{1\leq k\neq j\leq N}(\eta_k-m(t))(\eta_j-m(t))\nu_k\nu_j\Big|^{3/2}\Big|B_N=0\Big) \;\leq\; Am^2(t)n^2\beta_{3N}^2, \quad (120)$$

$$E\Big(\Big|\sum_{1\leq k\neq j\leq N}(\eta_k-m(t))\nu_k\nu_j\Big|^{3/2}\Big|B_N=0\Big) \;\leq\; Am(t)n^2\beta_{3N}^2. \quad (121)$$

In fact, recalling that η_k are iid random variables with $E\eta_1=m(t)$, independent of all other random variables, it follow from conditional expectation arguments and moment results for degenerate U-statistics and (111) that

$$
\begin{aligned}
&E\Big(\Big|\sum_{1\le k\ne j\le N}(\eta_k-m(t))(\eta_j-m(t))\nu_k\nu_j\Big|^{3/2}\Big|B_N=0\Big)\\
&\le A\sum_{1\le k\ne j\le N}E\Big(|(\eta_k-m(t))(\eta_j-m(t))\nu_k\nu_j|^{3/2}\Big|B_N=0\Big)\\
&\le Am^2(t)\sum_{1\le k\ne j\le N}E\Big(|\nu_k\nu_j|^{3/2}\Big|B_N=0\Big)\\
&\le\ A\,m^2(t)\,n^2\,\beta_{3N}^2.
\end{aligned}
$$

This proves (120). Similarly, it follows from conditional expectation arguments and moment results for partial sums and (112) that

$$
\begin{aligned}
&E\Big(\Big|\sum_{1\le k\ne j\le N}(\eta_k-m(t))\nu_k\nu_j\Big|^{3/2}\Big|B_N=0\Big)\\
&\quad=\ E\Big(\Big|\sum_{k=1}^{N}(\eta_k-m(t))\nu_k\sum_{j=1,\ne k}^{N}\nu_j\Big|^{3/2}\Big|B_N=0\Big)\\
&\quad\le\ A\,m(t)\sum_{k=1}^{N}E\Big(\Big|\nu_k\sum_{j=1,\ne k}^{N}\nu_j\Big|^{3/2}\Big|B_N=0\Big)\\
&\quad\le\ A\,m(t)\,n^2\,\beta_{3N}^2,
\end{aligned}
$$

which implies (121). The proof of Lemma 6.3 is now complete. □

To introduce the following lemmas, we de£ne

$$
f(t)=E(e^{it(T_n+\Lambda_n)}|B_N=0),\ \ f_1(t)=E(e^{itT_n}|B_N=0),\ \ f_2(t)=E(\Lambda_ne^{itT_n}|B_N=0),
$$

and for $k=1,\cdots,N$,

$$
g_k(t,\psi)=E\exp\{i(\varepsilon_k-p)(tg_k+\psi/\omega_N)\}.
$$

We also use the notation $\Delta=x\beta_{3N}/\omega_N$.

Lemma 6.4. *If* $|t|\le(1/128)\Delta^{-1}$, *then for* $2\le x\le(1/128)\omega_N/\max_k|a_k|$ *and any* $0\le m(t)\le1$,

$$
\begin{aligned}
|f(t)|\ &\le\ A(1+|tx|)\Big[m^{-1/2}(t)\,e^{-m(t)t^2/4}+\omega_Ne^{-(1/40)m(t)\omega_N^2}\Big]\\
&\qquad+A\,|t|^{3/2}m(t)\Delta^2+A|t|m^{4/3}(t)\Delta^{4/3}.
\end{aligned}
\tag{122}
$$

Proof. Deﬁne $\{\eta_k, k = 1, \cdots, N\}$ as in Lemma 6.3 (ii). Furthermore, let

$$\begin{aligned}
T_{1N}^* &= \sum \eta_k(\varepsilon_k - p)g_k, & T_{2N}^* &= \sum (1-\eta_k)(\varepsilon_k - p)g_k, \\
\Lambda_{1N}^* &= \frac{x}{n^2}\sum_{1\le k\ne j\le N} \eta_k\eta_j\nu_k\nu_j, & \Lambda_{2N}^* &= \frac{x}{n^2}\sum_{1\le k\ne j\le N} \eta_k(1-\eta_j)\nu_k\nu_j, \\
\Lambda_{3N}^* &= \frac{x}{n^2}\sum_{1\le k\ne j\le N} (1-\eta_k)(1-\eta_j)\nu_k\nu_j.
\end{aligned}$$

Note that

$$T_N + \Lambda_N = T_{1N}^* + T_{2N}^* + \Lambda_{1N}^* + 2\Lambda_{2N}^* + \Lambda_{3N}^*. \tag{123}$$

It follows from (123), $|e^{it} - 1| \le |t|$ and $|e^{it} - 1 - it| \le 2|t|^{3/2}$, that

$$\begin{aligned}
|f(t)| &= \left|E(e^{it(T_{1N}^*+T_{2N}^*+\Lambda_{1N}^*+2\Lambda_{2N}^*+\Lambda_{3N}^*)}|B_N = 0)\right| \\
&\le \left|E(e^{it(T_{1N}^*+T_{2N}^*+2\Lambda_{2N}^*+\Lambda_{3N}^*)}|B_N = 0)\right| + |t|E(|\Lambda_{1N}^*||B_N = 0) \\
&\le \left|E(e^{it(T_{1N}^*+T_{2N}^*+\Lambda_{3N}^*)}|B_N = 0)\right| + 2|t|\left|E(\Lambda_{2N}^* e^{it(T_{1N}^*+T_{2N}^*+\Lambda_{3N}^*)}|B_N = 0)\right| \\
&\qquad +8|t|^{3/2}E(|\Lambda_{2N}^*|^{3/2}|B_N = 0) + |t|E(|\Lambda_{1N}^*||B_N = 0) \\
&:= \Xi_1(t,x) + \Xi_2(t,x) + \Xi_3(t,x) + \Xi_4(t,x).
\end{aligned} \tag{124}$$

We ﬁrst estimate $\Xi_3(t,x)$ and $\Xi_4(t,x)$. By Lemma 6.3 (ii), we obtain that,

$$E(|\Lambda_{2N}^*|^{3/2}|B_N = 0) \le Ax^{3/2}m(t)n^{-1}\beta_{3N}^2 \le Am(t)\Delta^2,$$

and, by Hölder's inequality,

$$E(|\Lambda_{1N}^*||B_N = 0) \le \left[E(|\Lambda_{1N}^*|^{3/2}|B_N = 0)\right]^{2/3} \le Am^{4/3}(t)\Delta^{4/3}.$$

These facts yield that

$$\Xi_3(t,x) + \Xi_4(t,x) \le A\,|t|^{3/2}m(t)\Delta^2 + A|t|m^{4/3}(t)\Delta^{4/3}. \tag{125}$$

Next we estimate $\Xi_1(t,x)$. Write $B_{1N}^* = \sum \eta_k(\varepsilon_k - p)$, $B_{2N}^* = \sum(1-\eta_k)(\varepsilon_k - p)$, and

$$B = \{k : \eta_k = 1\}, \qquad B^c = \{k : \eta_k = 0\}. \tag{126}$$

Note that, given $\eta_1, \cdots, \eta_N$,

$$T_{1N}^*,\ B_{1N}^* \in \sigma\{\varepsilon_k,\ k \in B\}, \qquad T_{2N}^*,\ \Lambda_{3N}^*\ B_{2N}^* \in \sigma\{\varepsilon_k,\ k \in B^c\},$$

and $B_N = B_{1N}^* + B_{2N}^*$. It follows that T_{1N}^* and B_{1N}^* are independent of $T_{2N}^*,\ \Lambda_{3N}^*,\ B_{2N}^*$, given $\eta_1, \cdots, \eta_N$, and hence by Lemma 6.1,

$$\begin{aligned}
\Xi_1(t,x) &= \frac{1}{B_n(p)} \int_{|\psi| \le \pi\omega_N} \Big| E \exp\{it(T_{1N}^* + T_{2N}^* + \Lambda_{3N}^*) + i\psi B_N/\omega_N\} \Big| d\psi \\
&\le 2 \int_{|\psi| \le \pi\omega_N} E \Big| E_\eta \exp\{itT_{1N}^* + i\psi B_{1N}^*/\omega_N\} \Big| d\psi \\
&= 2 \int_{|\psi| \le \pi\omega_N} \prod E \Big| E_\eta \exp\{i\eta_k(\varepsilon_k - p)(tg_k + \psi/\omega_N)\} \Big| d\psi, \qquad (127)
\end{aligned}$$

where E_η denotes the condition expectation given $\eta_k, k = 1, \cdots, N$.

Let ε_k^* be an independent copy of ε_k. Note that, by Taylor's expansion of e^{iz},

$$\begin{aligned}
&E \big| E_\eta \exp\{i\eta_k(\varepsilon_k - p)(tg_k + \psi/\omega_N)\} \big|^2 \\
&\quad = E\big(E_\eta \exp\{i\eta_k(\varepsilon_k - \varepsilon_k^*)(tg_k + \psi/\omega_N)\}\big) \\
&\quad = E \exp\{i\eta_k(\varepsilon_k - \varepsilon_k^*)(tg_k + \psi/\omega_N)\}\big) \\
&\quad \le 1 - (1/2)(tg_k + \psi/\omega_N)^2 E\eta_k^2 E(\varepsilon_k - \varepsilon_k^*)^2 + (1/6)|tg_k + \psi/\omega_N|^3 E\eta_k^3 E|\varepsilon_k - \varepsilon_k^*|^3 \\
&\quad \le 1 - pq\, m(t)\, (tg_k + \psi/\omega_N)^2 + (pq/3)\, m(t)\, |tg_k + \psi/\omega_N|^3.
\end{aligned}$$

This, together with that fact that $\sum g_k = 0$ and for $2 \le x \le (1/128)\omega_N/\max_k |a_k|$,

$$\Big| pq \sum g_k^2 - 1 \Big| \le 2x\beta_{3N}/\omega_N \quad \text{and} \quad \sum pq|g_k|^3 \le 5\beta_{3N}/\omega_N, \qquad (128)$$

yields that for $|t| < (1/128)\Delta^{-1},\ |\psi| < (3/8)\omega_N$ and $2 \le x \le (1/128)\omega_N/\max_k |a_k|$,

$$\begin{aligned}
J(t,\psi) &:= \prod E \Big| E_\eta \exp\{i\eta_k(\varepsilon_k - p)(tg_k + \psi/\omega_N)\} \Big| \\
&\le \Big(\prod E \big| E_\eta \exp\{i\eta_k(\varepsilon_k - p)(tg_k + \psi/\omega_N)\} \big|^2 \Big)^{1/2} \\
&\le \exp\Big\{ - (pq/2)m(t) \sum (tg_k + \psi/\omega_N)^2 + (pq/6)m(t) \sum |tg_k + \psi/\omega_N|^3 \Big\} \\
&\le \exp\Big\{ - (pq/2)m(t) \sum t^2 g_k^2 - m(t)\psi^2/2 \\
&\qquad + (2pq/3)m(t) \sum |tg_k|^3 + (2/3)m(t)|\psi|^3/\omega_N \Big\} \\
&\le \exp\Big\{ - (pq/2)m(t) \sum t^2 g_k^2 + (2pq/3)m(t) \sum |tg_k|^3 - m(t)\psi^2/4 \Big\} \\
&\le \exp\Big\{ - (1/2)m(t)t^2 \Big(1 - x\beta_{3N}/\omega_N - (5/3)|t|\beta_{3N}/\omega_N\Big) - m(t)\psi^2/4 \Big\} \\
&\le \exp\{-m(t)t^2/4 - m(t)\psi^2/4\}. \qquad (129)
\end{aligned}$$

To estimate $J(t,\psi)$ for $(3/8)\omega_N \le |\psi| \le \pi\omega_N$, we first note that

$$\begin{aligned}
&E \big| E_\eta \exp\{i\eta_k(\varepsilon_k - p)(tg_k + \psi/\omega_N)\} \big|^2 = E \exp\{i\eta_k(\varepsilon_k - \varepsilon_k^*)(tg_k + \psi/\omega_N)\}\big) \\
&\quad = 1 - 2pq + 2pq\, E \cos\big[\eta_k(tg_k + \psi/\omega_N)\big] \\
&\quad = 1 - 2pq\, m(t) + 2pq\, m(t)\, \cos\big(tg_k + \psi/\omega_N\big). \qquad (130)
\end{aligned}$$

De£ne $D = \{k : |g_k| \le 2\Delta\}$ and $D^c = \{k : |g_k| > 2\Delta\}$. It is readily seen that, for $k \in D$, $|t| < (1/128)\Delta^{-1}$ and $(3/8)\omega_N \le |\psi| \le \pi\omega_N$,

$$\frac{23}{64} \le tg_k + \psi/\omega_N \le \pi + \frac{1}{64} \quad \text{or} \quad -\frac{1}{64} - \pi \le tg_k + \psi/\omega_N \le -\frac{23}{64}$$

and hence $\cos\big(tg_k + \psi/\omega_N\big) \le \cos(23/64) < 0.95$. On the other hand, it follows from (128) that, for $2 \le x \le (1/128)\omega_N/\max_k |a_k|$,

$$4(Npq)^{-1}|D^c| \le \frac{4x^2\beta_{3N}^2}{\omega_N^2}|D^c| \le \sum_{k\in D^c} g_k^2 \le (pq)^{-1}(1 + 2x\beta_{3N}/\omega_N) \le 2(pq)^{-1},$$

where $|D^c|$ denotes the number of D^c. Thus $|D^c| \le N/2$ and $|D| = N - |D^c| \ge N/2$.

By virtue of (130) and all above facts, we obtain that for $|t| < (1/128)\Delta^{-1}$, $(3/8)\omega_N \le |\psi| \le \pi\omega_N$ and $2 \le x \le (1/128)\omega_N/\max_k |a_k|$,

$$\begin{aligned} J(t,\psi) &\le \Big(\prod_{k\in D} E\big|E_\eta \exp\{i\eta_k(\varepsilon_k - p)(tg_k + \psi/\omega_N)\}\big|^2\Big)^{1/2} \\ &\le \prod_{k\in D} \exp\Big\{-pq\, m(t)\,\big[1 - \cos\big(tg_k + \psi/\omega_N\big)\big]\Big\} \\ &\le \exp\big\{-(1/40)\, m(t)\, \omega_N^2\big\}. \end{aligned} \tag{131}$$

Combining (127), (129) and (131), it follows that, for $|t| < (1/128)\Delta^{-1}$ and $2 \le x \le (1/128)\omega_N/\max_k |a_k|$,

$$\Xi_1(t,x) \le Am(t)^{-1/2}e^{-m(t)t^2/4} + A\omega_N e^{-(1/40)m(t)\omega_N^2}. \tag{132}$$

Finally, we estimate $\Xi_2(t,x)$. Note that $\Lambda_{2N}^* = \frac{x}{n^2}\sum_{j\in B^c}\nu_j\sum_{k\in B}\nu_k$, where B and B^c is de£ned in (126). Similarly to (127),

$$\begin{aligned} \Xi_2(t,x) &= \frac{2|t|}{B_n(p)}\int_{|\psi|\le\pi\omega_N}\Big|E\Big(\Lambda_{2N}^* e^{it(T_{1N}^*+T_{2N}^*+\Lambda_{3N}^*)+i\psi B_N/\omega_N}\Big)\Big|d\psi \\ &\le \frac{4|t|x}{n^2}\int_{|\psi|\le\pi\omega_N} E\Big[\sum_{j\in B^c}\sum_{k\in B} E_\eta|\nu_j|\,\Big|E_\eta\Big(\nu_k \exp\{itT_{1N}^* + i\psi B_{1N}^*/\omega_N\}\Big)\Big|\Big]d\psi \\ &\le \frac{4|t|x}{n^2}\int_{|\psi|\le\pi\omega_N} E\Big[\sum_{1\le j\ne k\le N}(1-\eta_j)\eta_k E|\nu_j|\,E|\nu_k|\,\Omega_{jk}(t,\psi)\Big]d\psi \\ &\le \frac{4|t|x\, m(t)}{n^2}\sum_{1\le j\ne k\le N} E|\nu_j|\,E|\nu_k|\int_{|\psi|\le\pi\omega_N} E\Omega_{jk}(t,\psi)d\psi, \end{aligned} \tag{133}$$

where

$$\Omega_{jk}(t,\psi) = \prod_{l\ne j,k}\big|E_\eta \exp\{i\eta_l(\varepsilon_l - p)\,(tg_l + \psi/\omega_N)\}\big|.$$

As in the proof of (132) with minor modi£cations, we have that, for $|t| < (1/128)\Delta^{-1}$, $2 \le x \le (1/128)\omega_N/\max_k |a_k|$, and for all $1 \le j \ne k \le N$,

$$\int_{|\psi|\le\pi\omega_N} E\Omega_{jk}(t,\psi)d\psi \le Am(t)^{-1/2}e^{-m(t)t^2/4} + A\omega_N e^{-(1/40)m(t)\omega_N^2}.$$

This, together with (133) and the fact that

$$\sum_{1\le k\ne j\le N} E|\nu_j|E|\nu_k| \le \Big(2pq\sum(a_k^2+1)\Big)^2 = 16\omega_N^4,$$

yields that, for $2 \le x \le (1/128)\omega_N/\max_k |a_k|$ and $|t| < (1/128)\Delta^{-1}$,

$$\Xi_2(t,x) \le A|tx|(e^{-m(t)t^2/4} + \omega_N e^{-(1/40)m(t)\omega_N^2}). \tag{134}$$

Taking estimates (125), (132) and (134) into (124), we obtain (122). The proof of Lemma 6.4 is now complete. □

Lemma 6.5. *Suppose that* $2 \le x \le (1/128)\omega_N/\max_k |a_k|$.

(i). If $|t| \le (1/128)\Delta^{-1}$ *and* $|\psi| \le \pi\omega_N$, *then*

$$\prod_{l=1,\ne j,k}^{N} |g_l(t,\psi)| \;\le\; e^{-(t^2+\psi^2)/4} + e^{-(1/40)\omega_N^2}, \tag{135}$$

for all $1 \le k \ne j \le N$, *and*

$$\Big|\frac{d\prod g_k(t,\psi)}{dt}\Big| \;\le\; 4(|t|+|\psi|)\left(e^{-(t^2+\psi^2)/4} + e^{-(1/40)\omega_N^2}\right). \tag{136}$$

(ii). If $|t| \le (1/128)\Delta^{-1/3}$ *and* $|\psi| < (1/128)\Delta^{-1/3}$, *then*

$$\Big|\prod g_k(t,\psi) - g(t,\psi)\Big| \;\le\; A\Delta^{4/3}\, e^{-(t^2+\psi^2)/4}, \tag{137}$$

and if in addition $|t| \le 1/4$, *then*

$$\Big|\frac{d\prod g_k(t,\psi)}{dt} - \frac{d\,g(t,\psi)}{dt}\Big| \;\le\; A\,\Delta^{4/3}(1+\psi^6)e^{-\psi^2/4}, \tag{138}$$

where

$$g(t,\psi) = e^{-(t^2+\psi^2)/2}\Big\{1 + \sum(g_k(t,\psi)-1) + \frac{t^2+\psi^2}{2}\Big\}.$$

Proof. By letting $m(t) = 1$ in (129) and (131), together with minor modi£cations, we obtain (135). Note that, under the conditions of part (ii), $s := |t| + |\psi| \le (1/64)\Delta^{-1/3}$ and

$$\big|\sum(g_k(t,\psi)-1) + (t^2+\psi^2)/2\big| \;\le\; 2\big(s^2+s^3\big)\Delta, \tag{139}$$

by (128) and Taylor's expansion of e^{iz}. (137) follows easily from some routine calculations. See, for example, Lemma 10.1 of Jing, Shao and Wang (2003) with minor modi£cations.

We next prove (136) and (138). Note that

$$\frac{d\prod g_k(t,\psi)}{dt} = g^*(t,\psi)\prod g_k(t,\psi),$$

where $g^*(t,\psi) = \sum [g_k(t,\psi)]^{-1}\frac{dg_k(t,\psi)}{dt}$, and

$$\frac{dg(t,\psi)}{dt} = -tg(t,\psi) + \big(\sum \frac{dg_k(t,\psi)}{dt} + t\big)e^{-(t^2+\psi^2)/2}. \tag{140}$$

Simple calculations show that

$$\Big|\frac{d\prod g_k(t,\psi)}{dt} - \frac{d\,g(t,\psi)}{dt}\Big| \;\le\; \mathcal{J}_{1N} + (\mathcal{J}_{2N}+\mathcal{J}_{3N})e^{-(t^2+\psi^2)/2}, \tag{141}$$

where

$$\begin{aligned}
\mathcal{J}_{1N} &= |g^*(t,\psi)|\,\Big|\prod g_k(t,\psi) - g(t,\psi)\Big|,\\
\mathcal{J}_{2N} &= \big|g^*(t,\psi)+t\big|\Big|\sum (g_k(t,\psi)-1) + (t^2+\psi^2)/2\Big|,\\
\mathcal{J}_{3N} &= \Big|g^*(t,\psi) - \sum \frac{dg_k(t,\psi)}{dt}\Big|.
\end{aligned}$$

By the inequality $|e^{iz}-1-iz|\le z^2/2$, it is readily seen that, for any t and ψ,

$$|g_k(t,\psi)-1| \;\le\; (pq/2)(tg_k+\psi/\omega_N)^2, \tag{142}$$

and

$$\Big|\frac{dg_k(t,\psi)}{dt} + pq\,g_k\,(tg_k+\psi/\omega_N)\Big| \;\le\; (pq/2)\,|g_k|\,(tg_k+\psi/\omega_N)^2. \tag{143}$$

Since $pqg_k^2\le 2$ by (128), it follows from (142) that, if $|\psi| < (1/128)\Delta^{-1/3}$ and $|t|\le 1/4$, then $|g_k(t,\psi)-1|\le 1/4$ and hence

$$[g_k(t,\psi)]^{-1} \;=\; 1+\theta_1\,pq\,(tg_k+\psi/\omega_N)^2, \tag{144}$$

where $|\theta_1|<1$. In view of (143) and (144), it follows from (128) again that

$$\begin{aligned}
\mathcal{J}_{3N} &\le \Big|\sum\big([g_k(t,\psi)]^{-1}-1\big)\frac{dg_k(t,\psi)}{dt}\Big|\\
&\le 2(pq)^2\sum |g_k||tg_k+\psi/\omega_N|^3\\
&\le 8(pq)^2(1+|\psi|^3)\Big(\sum g_k^4 + \sum |g_k|/\omega_N^3\Big)\\
&\le 8(pq)^2(1+|\psi|^3)\Big(\Big(\sum |g_k|^3\Big)^{4/3} + \Big(\sum |g_k|^3\Big)^{1/3}N^{2/3}/\omega_N^3\Big)\\
&\le 8\,(1+|\psi|^3)\,\Delta^{4/3},
\end{aligned}$$

Similarly, by recalling $\sum g_k = 0$, we have

$$\begin{aligned} \big|g^*(t,\psi)+t\big| &\le \big|pq\sum g_k^2-1\big| + 2(pq)^2\sum |g_k||tg_k+\psi/\omega_N|^3 \\ &\le 10\,(1+|\psi|^3)\,\Delta. \end{aligned}$$

which, together with (137) and (139), implies that $\mathcal{J}_{1N} \le A\,(1+|\psi|^3)\,\Delta^{4/3}\,e^{-\psi^2/4}$ and $\mathcal{J}_{2N} \le A\,(1+|\psi|^6)\,\Delta^{4/3}$. Taking the estimates of $\mathcal{J}_{1N}, \mathcal{J}_{2N}$ and $\mathcal{J}_{3N}$ into (141), we obtain (138).

Similarly, by noting that

$$\begin{aligned} \sum\big|\frac{dg_k(t,\psi)}{dt}\big| &\le pq\sum|g_k|\,|tg_k+\psi/\omega_N| \\ &\le |t|pq\sum|g_k|^2 + |\psi|\left(pq\sum|g_k|^2\right)^{1/2} \le 4(|t|+|\psi|), \end{aligned} \tag{145}$$

it follows from (135) that

$$\begin{aligned} \Big|\frac{d\prod g_k(t,\psi)}{dt}\Big| &\le \sum\prod_{j=1,\ne k}^{N}|g_j(t,\psi)|\,\big|\frac{dg_k(t,\psi)}{dt}\big| \\ &\le 4(|t|+|\psi|)\left(e^{-(t^2+\psi^2)/4}+e^{-(1/40)\omega_N^2}\right), \end{aligned}$$

which implies (136). The proof of Lemma 6.5 is now complete. $\square$

Lemma 6.6. *Suppose that* $2\le x\le (1/128)\omega_N/\max_k|a_k|$. *Then, for* $|t|\le(1/128)\Delta^{-1/3}$,

$$\Big|f_1(t)-e^{-t^2/2}\Big| \le A\,\min\{|t|,1\}\Big(\Delta\,(1+t^6)e^{-t^2/4}+\omega_N^{-6}\Big), \tag{146}$$

and

$$\Big|f_1(t)-g(t,0)\Big| \le A\,\min\{|t|,1\}\Big(\Delta^{4/3}\,(1+t^6)e^{-t^2/4}+\omega_N^{-6}\Big), \tag{147}$$

where $g(t,\psi)$ is de£ned as in Lemma 6.5.

Proof. We only prove (147). (146) follows from (147) and (139) with $\psi=0$.

First assume $|t|\ge 1/4$. By Lemma 6.1, we have

$$f_1(t) = \frac{1}{B_n(p)}\int_{|\psi|\le\pi\omega_N}\prod g_k(t,\psi)d\psi = II_1(t)+II_2(t)+II_3(t)+II_4(t), \tag{148}$$

where

$$\begin{aligned} II_1(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t,\psi)d\psi, \\ II_2(t) &= \Big(\frac{1}{B_n(p)}-\frac{1}{\sqrt{2\pi}}\Big)\int_{-\infty}^{\infty}g(t,\psi)d\psi - \frac{1}{B_n(p)}\int_{|\psi|\ge(1/128)\Delta^{-1/3}}g(t,\psi)d\psi, \\ II_3(t) &= \frac{1}{B_n(p)}\int_{|\psi|\le(1/128)\Delta^{-1/3}}\Big(\prod g_k(t,\psi)-g(t,\psi)\Big)d\psi, \\ II_4(t) &= \frac{1}{B_n(p)}\int_{(1/128)\Delta^{-1/3}\le|\psi|\le\pi\omega_N}\prod g_k(t,\psi)d\psi. \end{aligned}$$

In view of (104), (135), (137) and (139), it is readily seen that

$$|II_2(t)| + |II_3(t)| + |II_4(t)| \;\leq\; A\Delta^{4/3}\,(1+t^6)e^{-t^2/4} + A\,\omega_N^{-6}. \tag{149}$$

In order to estimate $II_1(t)$, write $g_k^{(m)}(t,0) = E(\varepsilon_k - p)^m e^{itg_k(\varepsilon_k - p)}, m = 1,2,3$. We first note that, by Taylor's expansion of e^{iz},

$$\begin{aligned} g_k(t,\psi) &= g_k(t,0) + \frac{i\psi}{\omega_N} g_k^{(1)}(t,0) - \frac{\psi^2}{2\omega_N^2} g_k^{(2)}(t,0) \\ &\qquad + \frac{i^3\psi^3}{6\omega_N^3} g_k^{(3)}(t,0) + R_k(t,\psi), \end{aligned} \tag{150}$$

where $|R_k(t,\psi)| \leq (1/24)(\psi/\omega_N)^4 E|\varepsilon_k - p|^4 \leq (1/24)\,pq\,\psi^4/\omega_N^4$, and

$$g_k^{(2)}(t,0) \;=\; pq + itg_k E(\varepsilon_k - p)^3 + R_{1k}(t), \tag{151}$$

where $|R_{1k}(t)| \leq t^2 g_k^2 E|\varepsilon_k - p|^4/2 \leq pqt^2 g_k^2/2$. By virtue of (150)-(151) and the fact that $\sum g_k = 0$, $\int e^{-\psi^2/2} d\psi = \sqrt{2\pi}$ and $\int \psi^k e^{-\psi^2/2} = 0, k = 1,3$, we have

$$\begin{aligned} II_1(t) - g(t,0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \big(g(t,\psi) - g(t,0)e^{-\psi^2/2}\big) d\psi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big[\sum \big(g_k(t,\psi) - g_k(t,0)\big) + \psi^2/2\Big] e^{-(\psi^2+t^2)/2} d\psi \\ &= R(t), \end{aligned} \tag{152}$$

where

$$\begin{aligned} |R(t)| &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(\sum |R_k(t,\psi)| + \frac{\psi^2}{2\omega_N^2} \sum |R_{1k}(t)|\Big) e^{-(\psi^2+t^2)/2}\, d\psi \\ &\leq A\,\omega_N^{-2}\,\big(1 + t^2 pq \sum g_k^2\big) e^{-t^2/2} \leq A_1 \Delta^{4/3}\,(1+t^2)e^{-t^2/2}. \end{aligned}$$

Combining (148), (149) and (152), we obtain (147) for $|t| \geq 1/4$.

Next assume $|t| \leq 1/4$. Note that $f_1(t) - g(t,0) = \int_0^t (f_1'(s) - g'(s,0))ds$. It suffices to show that, for $|t| \leq 1/4$,

$$\big|f_1'(t) - g'(t,0)\big| \;\leq\; A\,\Delta^{4/3} + A\,\omega_N^{-6}. \tag{153}$$

We continue to use the decomposition of $f_1(t)$ in (148). In view of (136) and (138),

$$|II_3'(t)| + |II_4'(t)| \;\leq\; A\Delta^{4/3} + A\,\omega_N^{-6},$$

for $|t| \leq 1/4$. It follows easily from (140), (145) and (149) that,

$$|II_2'(t)| \leq A\Delta^{4/3} + A\,\omega_N^{-6},$$

for $|t| \le 1/4$. In order to estimate $II_1'(t)$, we first note that, as in (150)-(151),

$$\frac{dg_k(t,\psi)}{dt} - \frac{dg_k(t,0)}{dt} = \frac{i\psi}{\omega_N}\frac{dg_k^{(1)}(t,0)}{dt} - \frac{\psi^2}{2\omega_N^2}\frac{dg_k^{(2)}(t,0)}{dt} + R_k^*(t,\psi), \tag{154}$$

where $|R_k^*(t,\psi)| \le (1/6)(|\psi|/\omega_N)^3 |tg_k| E|\varepsilon_k - p|^4 \le (1/6)\, pq\, |g_k|\, |t|\, |\psi|^3/\omega_N^3$, and

$$\frac{dg_k^{(2)}(t,0)}{dt} = ig_k E(\varepsilon_k - p)^3 + R_{1k}^*(t), \tag{155}$$

where $|R_{1k}^*(t)| \le |t| g_k^2 E|\varepsilon_k - p|^4/2 \le pq\,|t|\, g_k^2/2$. It follows from (154)-(155), $\sum g_k = 0$, $pq \sum g_k^2 \le 2$ and $\int \psi e^{-\psi^2/2} = 0$ that, for $|t| \le 1/4$,

$$\begin{aligned} \Upsilon &:= \Big| \int_{-\infty}^{\infty} \Big(\frac{dg_k(t,\psi)}{dt} - \frac{dg_k(t,0)}{dt} \Big) e^{-\psi^2/2} d\psi \Big| \\ &\le A\omega_N^{-2} \sum |R_{1k}^*(t)| + A \int \sum |R_k^*(t,\psi)| e^{-\psi^2/2} d\psi \\ &\le A\omega_N^{-2} pq \sum g_k^2 + A\omega_N^{-3}\, pq \sum |g_k| \le A\,\Delta^2. \end{aligned} \tag{156}$$

Therefore, by (140), (152) and (156), we have that, for $|t| \le 1/4$,

$$\begin{aligned} \big| II_1'(t) - g'(t,0) \big| &= \frac{1}{\sqrt{2\pi}} \Big| \int_{-\infty}^{\infty} \Big(\frac{dg(t,\psi)}{dt} - \frac{dg(t,0)}{dt} e^{-\psi^2/2} \Big) d\psi \Big| \\ &\le |t|\, \big| II_1(t) - g(t,0) \big| + \frac{\Upsilon}{\sqrt{2\pi}} e^{-t^2/2} \\ &\le A\,\Delta^{4/3}. \end{aligned}$$

Combining (148) and all above estimates for $II_k'(t)$, $k = 1, 2, 3, 4$, we obtain (153).

The proof of Lemma 6.6 is now complete. $\Box$

Lemma 6.7. *Suppose that* $2 \le x \le (1/128)\omega_N / \max_k |a_k|$. *Then, for* $|t| \le (1/128)\, \Delta^{-1/3}$,

$$|f_2(t)| \le A(1+t^2)\, \Delta^2 \big(e^{-t^2/4} + \omega_N^{-6} \big), \tag{157}$$

$$|f(t) - f_1(t)| \le A\, \Delta^2\, |t|^{3/2} + A\, |t| (1+t^2)\, \Delta^2 \big(e^{-t^2/4} + \omega_N^{-6} \big). \tag{158}$$

Proof. We first prove (157). Write $\varepsilon_k^* = (\varepsilon_k - p)(tg_k + \psi/\omega_N)$. Note that, by (128), $E\nu_k = 0$, $\sum a_k^2 = N$ and Taylor's expansion of e^{iz},

$$\begin{aligned} \sum \big| E\big(\nu_k e^{i\varepsilon_k^*}\big) \big| &\le \sum |E\nu_k (e^{itg_k(\varepsilon_k - p)} - 1) e^{i(\varepsilon_k - p)\psi/\omega_N}| + \sum |E\nu_k (e^{i(\varepsilon_k - p)\psi/\omega_N} - 1)| \\ &\le \sum |tg_k| (a_k^2 + 1) E(\varepsilon_k - p)^2 + (|\psi|/\omega_N) \sum (a_k^2 + 1) E(\varepsilon_k - p)^2 \\ &\le 2|t| pq (\sum |g_k|^3)^{1/3} (\sum |a_k|^3)^{2/3} + 2|\psi| \omega_N \\ &\le 6|t|\, \beta_{3N}\, \omega_N + 2|\psi| \omega_N \le 6(|t| + |\psi|)\, \beta_{3N}\, \omega_N. \end{aligned}$$

This, together with Lemma 6.1, (135) and the independence of ε_k, implies that

$$\begin{aligned}
|f_2(t)| &= \frac{x}{n^2 B_n(p)} \Big| \int_{|\psi|\le \pi\omega_N} \sum_{1\le k\ne j\le N} E\big(\nu_k \nu_j e^{i\sum \varepsilon_l^*}\big) d\psi \Big| \\
&\le \frac{2x}{n^2} \int_{|\psi|\le \pi\omega_N} \sum_{1\le k\ne j\le N} \big|E\big(\nu_k e^{i\varepsilon_k^*}\big)\big| \, \big|E\big(\nu_j e^{i\varepsilon_j^*}\big)\big| \prod_{l=1,\ne j,k}^{N} |g_l(t,\psi)| \, d\psi \\
&\le A\, x n^{-2} \,(1+|t|)^2 \,\beta_{3N}^2 \,\omega_N^2 \big(e^{-t^2/4} + \omega_N^3 e^{-(1/40)\omega_N^2}\big) \\
&\le A(1+t^2)\Delta^2 \big(e^{-t^2/4} + \omega_N^{-6}\big),
\end{aligned}$$

which yields (157).

By virtue of (157) and (113), the proof of (158) is simple. Indeed, by (113), we have

$$\begin{aligned}
|f(t) - f_1(t) - it\, f_2(t)| &= \Big| E e^{itT_n} \big(e^{it\Lambda_n} - 1 - it\Lambda_n\big) \Big| B_N = 0\Big) \Big| \\
&\le 2|t|^{3/2} E\Big(|\Lambda_n|^{3/2} \Big| B_N = 0\Big) \le A\,|t|^{3/2}\, x^{3/2}\, \beta_{3N}^2/n \\
&\le A\,|t|^{3/2}\,\Delta^2,
\end{aligned}$$

and hence

$$\begin{aligned}
|f(t) - f_1(t)| &\le |f(t) - f_1(t) - it\, f_2(t)| + |t|\,|f_2(t)| \\
&\le A\,\Delta^2\,|t|^{3/2} + A\,|t|(1+t^2)\,\Delta^2 \big(e^{-t^2/4} + \omega_N^{-6}\big),
\end{aligned}$$

as required. The proof of Lemma 6.7 is now complete. □

Lemma 6.8. *Suppose that* $2 \le x \le (1/128)\omega_N / \max_k |a_k|$. *There exists an absolute constant* A *such that, for all* $|y| \le 4x$,

$$P(T_N + \Lambda_N \ge y \,|B_N = 0) \;\le\; (1 - \Phi(y)) + A\, x\, \Delta\, e^{-y^2/2} + A\,\Delta^{4/3}.$$

Proof. Note that Lemmas 6.4, 6.6 and 6.7 are similar to Lemmas 10.1-10.3 in Jing, Shao and Wang (2003). The proof of Lemma 6.8 is similar to Lemma 10.5 of Jing, Shao and Wang (2003) with some routine modi£cations. We omit the details. □

We are now ready to prove Proposition 2.3. Note that $\max |a_k| \le \omega_N$,

$$h = xpq \sum (a_k^2 - 1)^2/n^2 \le x\, \max |a_k|\, \beta_{3N}/n \le \Delta,$$

and $|x-h| \le 2x$. It follows from (102) and Lemma 6.8 that

$$\begin{aligned}
P(S_n \ge x\sqrt{q}V_n) &\le P(T_N + \Lambda_N \ge x - h\,|B_N = 0) \\
&\le (1-\Phi(x-h)) + Ax\Delta e^{-(x-h)^2/2} + A\Delta^{4/3} \\
&\le 1-\Phi(x) + A(1+x)\Delta e^{-x^2/2+x\Delta} + A\Delta^{4/3} \\
&\le (1-\Phi(x))(1 + Ax^2\Delta e^{x\Delta}) + A\Delta^{4/3} \\
&\le (1-\Phi(x))\exp\{Ax^3\beta_{3N}/\omega_N\} + A\left(x\beta_{3N}/\omega_N\right)^{4/3},
\end{aligned}$$

where we have used the result:

$$\Phi(x) - \Phi(x-h) \le h\Phi'(x-h) \le h\, e^{-(x-h)^2/2} \le \Delta e^{-x^2/2+x\Delta}.$$

This yields Proposition 2.3.

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