KNOT GROUPS AND SLICE CONDITIONS

JONATHAN A. HILLMAN

Abstract. We introduce the notions of "k-connected-slice" and "π_1-slice", interpolating between "homotopy ribbon" and "slice". We show that every high-dimensional knot group π is the group of an $(n-1)$-connected-slice n-knot for all $n \geq 3$. However if π is the group of an n-connected-slice n-knot the augmentation ideal $I(\pi)$ must have deficiency 1 as a module. If moreover $n = 2$ and π' is finitely generated then π' is free. In this case $\mathrm{def}(\pi) = 1$ also.

An n-knot is a locally flat embedding $K : S^n \to S^{n+2}$. Such a knot K is *homotopy ribbon* if it is a slice knot with a slice disc whose exterior W has a handlebody decomposition consisting of 0-, 1- and 2-handles. The dual decomposition of W relative to ∂W has only $(n+1)$-, $(n+2)$- and $(n+3)$-handles, and so the inclusion of ∂W into W is n-connected. More generally, we shall say that K is *k-connected-slice* if there is a slice disc with exterior W such that $(W, \partial W)$ is k-connected, and that K is *π_1-slice* if the inclusion of the knot exterior $X(K) = \overline{S^{n+2} - K(S^n) \times D^2}$ into the exterior of some slice disc induces an isomorphism on fundamental groups.

Every ribbon knot is homotopy ribbon [4], while if $n \geq 2$ "homotopy ribbon" $\Rightarrow$ "n-connected-slice" $\Rightarrow$ "π_1-slice" $\Rightarrow$ "slice". Nontrivial classical knots are never π_1-slice, since the longitude of a slice knot is nullhomotopic in the exterior of a slice disc. (A 1-knot is "homotopically ribbon" in the sense used in Problem 4.22 of [6] if and only if it is 1-connected-slice.) It is an open question whether every classical slice knot is ribbon. However in higher dimensions these notions are generally distinct. Every even-dimensional knot is slice, but a knot group is the group of a ribbon n-knot (for $n \geq 2$) if and only if it has a Wirtinger presentation of deficiency 1 [11]. (More generally, if W is homotopy equivalent to a finite 2-complex and $\chi(W) = 0$ then $\mathrm{def}(\pi_1(W)) \geq 1$.) There are n-knot groups with deficiency ≤ 0 for every $n \geq 2$.

In this note we shall show that every high-dimensional knot group π is the group of an $(n-1)$-connected-slice n-knot for all $n \geq 3$. However the groups of n-connected-slice n-knots satisfy constraints related to

1991 *Mathematics Subject Classification.* 57Q45.
Key words and phrases. deficiency, knot, ribbon, slice.

deficiency. We shall show that if π is the group of an n-connected-slice n-knot the augmentation ideal $I(\pi)$ must have deficiency 1 as a $\mathbb{Z}[\pi]$-module. In all known cases $\mathrm{def}(\pi) = 1$, and we shall show that the latter condition must hold if π is the group of a π_1-slice 2-knot and π' is finitely generated. (In fact the commutator subgroup π' is then free.)

1. $(n-1)$-Connected-slice n-knots

If K is an n-knot let $\pi K = \pi_1(X(K))$ and $M(K) = X(K) \cup D^{n+1} \times S^1$ denote the knot group and the closed $(n+2)$-manifold obtained by surgery on K, respectively. Then $M(K)$ has the homology of $S^{n+1} \times S^1$, $\chi(M(K)) = 0$ and $\pi_1(M(K)) \cong \pi K$.

The following result is a variation on Theorem 1.7 of [9].

Theorem 1. *Let π be a high dimensional knot group and $n \geq 3$. Then there is an $(n-1)$-connected-slice n-knot K with group $\pi K \cong \pi$.*

Proof. Let $\mathcal{P}$ be a finite presentation for π, and let X be the corresponding finite 2-complex. Then $H_2(X;\mathbb{Z})$ is a finitely generated free abelian group. The Hurewicz homomorphism in degree 2 is surjective, since $H_2(\pi;\mathbb{Z}) = 0$, and so we may attach 3-cells along representatives for a basis for $H_2(X;\mathbb{Z})$ to obtain a finite 3-complex Y with $\pi_1(Y) \cong \pi$ and $H_q(Y) = 0$ for $q \geq 2$. If $n \geq 3$ we may construct an $(n+3)$-dimensional handlebody $N \simeq Y$ with no handles of index > 3. Thus N may be obtained by adding handles of index at least n to a collar neighbourhood of $M = \partial N$, and so the inclusion of M into N is $(n-1)$-connected. Let Δ be the $(n+3)$-manifold obtained by adjoining a further 2-handle with attaching map representing a normal generator for π. Then Δ is contractible and $\partial\Delta$ is 1-connected, and so $\Delta \cong D^{n+3}$. The corecore of the final 2-handle is a slice disc for an n-knot $K : S^n \to \partial\Delta$, and K is easily seen to be $(n-1)$-connected-slice. ∎

In particular, π is the group of a π_1-slice n-knot, for all $n \geq 3$.

Note that if $\mathrm{def}(\pi) = 1$ then $H_2(X;\mathbb{Z}) = 0$ and so the argument gives a homotopy ribbon n-knot with group π for any $n \geq 2$.

2. n-Connected-slice n-knots

If (W,V) is a k-connected $(n+3)$-manifold pair and $k \leq n-1$ then W has a handlebody decomposition consisting only of handles of index $< n+3-k$ [10]. Thus $(n-1)$-connected-slice n-knots have slice discs with handlebody decompositions consisting of handles of index ≤ 3 only. If this "homotopy connectivity implies geometric connectivity" result held also when $k = n$ it would follow that every n-connected slice n-knot K is homotopy ribbon, and hence that $\mathrm{def}(\pi K) = 1$. Here we shall show that the linear analogue of this condition must hold.

If R is a ring and M is a finitely presentable R-module let

$$\mathrm{def}_R(M) = \sup\{g - r \mid \exists \textit{ exact sequence } R^r \to R^g \to M \to 0\}.$$

It is easy to see that if R maps nontrivially to a field then $\mathrm{def}_R(M)$ is finite.

Lemma 2. *Let G be a finitely presentable group and $I(G)$ be the augmentation ideal of $\mathbb{Z}[G]$. Then* $\mathrm{def}(G) \leq \mathrm{def}_{\mathbb{Z}[G]} I(G) \leq \beta_1(G) - \beta_2(G)$.

Proof. Let X is the finite 2-complex with one 0-cell, g 1-cells and r 2-cells associated to a presentation of G and let $C_*(\widetilde{X})$ be the equivariant cellular chain complex of the universal covering $\widetilde{X}$. Then $\chi(X) = 1 - g + r$ and $C_* = C_*(\widetilde{X})$ is a partial resolution of the augmentation $\mathbb{Z}[G]$-module $\mathbb{Z}$. Therefore $\partial_2 : C_2 \to C_1$ is a presentation for $I(\pi)$. The first inequality follow easily since C_1 and C_2 are free $\mathbb{Z}[G]$-modules of rank g and r, respectively. The second inequality follows on applying $\mathbb{Z} \otimes_{\mathbb{Z}[G]} -$ to a presentation of $I(G)$ and observing that $H_{i+1}(G; \mathbb{Z}) = Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, I(G))$ for $i \geq 0$. □

If every partial resolution of length 2 of the augmentation $\mathbb{Z}[G]$-module $\mathbb{Z}$ is chain homotopy equivalent to such a complex $C_*(\widetilde{X})$ then $\mathrm{def}(G) = \mathrm{def}_{\mathbb{Z}[G]} I(G)$. It is not known whether this "Realization Theorem for algebraic 2-complexes" holds for all groups G. (See [5].)

Theorem 3. *Let K be an n-connected-slice n-knot with group $\pi = \pi K$. Then* $\mathrm{def}_{\mathbb{Z}[\pi]} I(\pi) = 1$.

Proof. Let W be the exterior of a slice disc for K such that $(W, \partial W)$ is n-connected, and let C_* be the equivariant cellular chain complex of the universal cover $\widetilde{W}$, which is a complex of finitely generated free left $\mathbb{Z}[\pi]$-modules. Then $H_p(W; \mathbb{Z}[\pi]) = H^{n+3-p}(W, \partial W; \mathbb{Z}[\pi]) = 0$ for $p \leq 2$ and $H^q(W; \mathcal{B}) = H_{n+3-q}(W, \partial W; \overline{\mathcal{B}}) = 0$ for any left $\mathbb{Z}[\pi]$-module $\mathcal{B}$ and $q \geq 3$, by Poincaré duality. (Here $\overline{\mathcal{B}}$ is the right $\mathbb{Z}[\pi]$-module obtained from $\mathcal{B}$ via the canonical involution of $\mathbb{Z}[\pi]$.) In particular, taking $\mathcal{B} = C_q$ we see that id_{C_q} is a cocycle, and so $id_{C_q} = \partial^q(f) = f_q \partial_q$ for some homomorphism $f_q : C_{q-1} \to C_q$, for $q = n+3, \ldots, 3$ (in descending order). Thus C_* splits as the sum of a contractible complex and a complex which is concentrated in degrees $0 \leq q \leq 2$. (Compare Lemma 2.3 of [9].) Since C_* is a finite free complex the direct summand $\mathrm{Im}(\partial_3)$ is stably free, and so C_* is chain homotopy equivalent to a finite free complex

$$0 \to D_2 \to D_1 \to D_0 \to 0$$

in which $D_0 \cong \mathbb{Z}[\pi]$. Since $\partial_2 : D_2 \to D_1$ is a presentation for $I(\pi)$ and $\chi(W) = 0$ we se that $\mathrm{def}_{\mathbb{Z}[\pi]} I(\pi) \geq 1$. On the other hand $\mathrm{def}_{\mathbb{Z}[\pi]} I(\pi) \leq \beta_1(\pi) - \beta_2(\pi) = 1$, by the lemma, and so $\mathrm{def}_{\mathbb{Z}[\pi]} I(\pi) = 1$. ☐

Is every high dimensional knot group π such that $\mathrm{def}_{\mathbb{Z}[\pi]} I(\pi) = 1$ realized by some n-connected-slice n-knot, for each $n \geq 2$?

If (D, Δ) is a k-connected ball pair of dimension $n + 3$ then the product with D^r gives a $(k + r)$-connected ball pair of dimension $n + r + 3$. Thus $n = 2$ is the case of greatest interest in attempting to realize knot groups by n-connected slice n-knots.

3. 2-knots

Although we do not yet know whether the result of Theorem 3 hold also for the groups of π_1-slice 2-knots, it is possible that all such groups may have deficiency 1, which is a stronger condition. In this section we shall give some evidence to support this possibility.

An n-knot K (with $n \geq 2$) is *fibred* if $M = M(K)$ fibres over S^1. The fibre F is then homotopy equivalent to the infinite cyclic covering space M', with fundamental group the commutator subgroup π' of $\pi = \pi K$. In [1] Cochran showed that if K is a fibred ribbon 2-knot with fibre F then the fundamental class $[F]$ has image 0 in $H_3(\pi'; \mathbb{Z})$, and so $F \simeq \#^r(S^1 \times S^2)$ for some $r \geq 0$. He also raised the question: "if a ribbon 2-knot has a minimal Seifert hypersurface V must $\pi_1(V)$ be free?". The argument of [1] applies equally well if the knot is π_1-slice, and extends to show that if a π_1-slice 2-knot K has a minimal Seifert hypersurface V and πK is an ascending HNN extension with base $\pi_1(V)$ then $\pi_1(V)$ is free. (Note however that there is a ribbon 2-knot whose group is not an HNN extension with free base [12].)

The following theorem provides another extension of this argument, under more algebraic hypotheses. (See also Theorem 17.10 of [2].)

Theorem 4. *Let π be the group of a π_1-slice 2-knot K. Then π' is finitely generated if and only if it is free. In that case* $\mathrm{def}(\pi) = 1$.

Proof. Let W be the exterior of a π_1-slice disc for K and $M = \partial W$. Then $M \cong M(K)$ and is a closed orientable 4-manifold with $\chi(M) = 0$ and $\pi_1(M) \cong \pi$. If π' is finitely generated the infinite cyclic cover M' is a PD_3-space, by Theorem 6 of [3]. Hence π' is FP_2 and the image of the fundamental class $[M']$ in $H_3(\pi'; \mathbb{Z})$ determines a projective homotopy equivalence of modules $C^2/\partial^1(C^1) \simeq I(\pi')$, by the argument of Theorem 4 of [8]. (The implication used here does not need π' to be finitely presentable.)

Since the classifying map $c_M : M \to K(\pi, 1)$ factors through W it follows from the exact sequence of homology for the pair (W, M) with coefficients $\mathbb{Z}[\pi/\pi']$ that $[M']$ has image 0 in $H_3(\pi'; \mathbb{Z})$. Hence $id_{I(\pi')} \sim 0$, so $I(\pi')$ is projective and $c.d.\pi' \leq 1$. Therefore π' is free.

The "knot module" $\pi'/\pi'' \cong H_1(M'; \mathbb{Z})$ is a finitely generated $\mathbb{Z}[\pi/\pi']$-torsion module, since $\mathbb{Z}[\pi/\pi'] \cong \mathbb{Z}[t, t^{-1}]$ is noetherian and $t - 1$ acts invertibly, by the Wang sequence for the covering $M' \to M$. Therefore if π' is free it must be finitely generated. Moreover since $\pi \cong \pi' \rtimes Z$ it is then clear that $\mathrm{def}(\pi) = 1$. □

In particular, the group of the 2-twist spin of the trefoil knot is not the group of a π_1-slice 2-knot, since it has commutator subgroup $Z/3Z$.

As observed in §1, every knot group of deficiency 1 is the group of some (homotopy ribbon) π_1-slice 2-knot. In [7] it is shown that if $G \cong N \rtimes Z$ has deficiency 1 then N is finitely generated if and only if it is free (The result from [3] used above depends on the "weak finiteness" of certain Novikov rings, proven in [7].)

References

[1] Cochran, T.D. Ribbon knots in S^4, J. London Math. Soc. 28 (1983), 563-576.

[2] Hillman, J.A. *Four-Manifolds, Geometries and Knots*, Geometry and Topology Monographs vol. 5 (2002).

[3] Hillman, J.A. and Kochloukova, D.H. Finiteness conditions and PD_r-group covers of PD_n-complexes, Math. Z., to appear.

[4] Hitt, L.R. Examples of higher dimensional slice knots which are not ribbon knots, Proc. Amer. Math. Soc. 77 (1979), 291-297.

[5] Johnson, F.E.A. *Stable Modules and the $D(2)$-Problem*, London Mathematical Society Lecture Note Series 301, Cambridge University Press, Cambridge - New York - Melbourne (2003).

[6] Kirby, R.C. Problems in low-dimensional topology, in *Geometric Topology* (editor W.H.Kazez) vol. 2, Studies in Advanced Mathematics 2.2, American Mathematical Society, Providence, R.I. (1997), 35-473.

[7] Kochloukova, D.H. Some Novikov rings that are von Neumann finite and knot-like groups, Comment. Math. Helv., to appear.

[8] Turaev, V.G. Three-dimensional Poincaré complexes: homotopy classification and splitting, Math. USSR Sbornik 67 (1990), 261-282.

[9] Wall, C.T.C. *Surgery on Compact Manifolds*, AAcademic Press, London - New York (1970).

[10] Wall, C.T.C. Geometric connectivity I, J. London Math. Soc. 3 (1971)), 597-604.

[11] Yajima, T. On a characterization of knot groups of some spheres in R^4, Osaka J. Math. 6 (1969), 435-446.

[12] Yoshikawa, K. A ribbon knot group which has no free base, Proc. Amer. Math. Soc. 102 (1988), 1065-1070.

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

E-mail address: `jonh@maths.usyd.edu.au`