Finiteness conditions and PD_r-group covers of PD_n-complexes

J. A. Hillman, D.H.Kochloukova

School of Mathematics and Statistics,
University of Sydney, NSW 2006, AUSTRALIA,
UNICAMP-IMECC, Cx.P.6005, 13083-970 Campinas, SP, BRAZIL

Email: `jonh@maths.usyd.edu.au, desi@ime.unicamp.br`

Abstract We show that an infinite cyclic covering space M' of a PD_n-complex M is a PD_{n-1}-complex if and only if $\chi(M) = 0$ and M' is homotopy equivalent to a complex with finite $[(n-1)/2]$-skeleton and $\pi_1(M')$ is finitely presentable. This is best possible in terms of minimal finiteness assumptions on the covering space. We give also a corresponding result for covering spaces M_ν with covering group a PD_r-group under a slightly stricter finiteness condition.

AMS Classification 57P10, 55R99

Keywords coinduced module, finiteness condition, finite domination, infinite cyclic cover, Novikov ring, Poincaré duality

If $p : M \to B$ is a fibration of a PD_n-complex M over a PD_r-complex B the homotopy fibre of p is a PD_{n-r}-complex if and only if it is finitely dominated, by a theorem of Gottlieb and Quinn. (The paper [11] gives a very nice proof for the case when M, B and the homotopy fibre are finite complexes. The general case follows on taking products with copies of S^1 to reduce to the finite case and using the Künneth theorem). When B is aspherical and $p_* = \pi_1(p)$ is an epimorphism the homotopy fibre is the covering space corresponding to $\mathrm{Ker}(p_*)$. We shall show that in this case we may use duality to relax the hypothesis that the fibre be finitely dominated, to requiring merely that it be homotopy equivalent to a complex with finite $[n/2]$-skeleton. In the simplest nontrivial case, when the base is S^1, we can improve this slightly, and our result is then best possible. (Our argument shall be entirely homological, rather than homotopy-theoretic as in [11]).

The first section introduces some notation and terminology. In §2 we use the finiteness criterion of Brown and extend a duality argument of Barge to show that a covering space of a PD_n-complex with covering group a PD_r-group is a PD_{n-r}-complex if it is homotopy equivalent to a complex with finite $[n/2]$-skeleton and has finitely presentable fundamental group (Theorem 4). In §3 we provide some algebraic background relating to Novikov rings and the finiteness criterion of Ranicki. (In particular, we consider explicitly the twisted case). This is used in §4 together with the main result of [16] to show that if M' is an infinite cyclic covering space of a finite PD_n-complex M then M' satisfies Poincaré duality of formal dimension $n-1$ if $\chi(M)=0$ and M' is homotopy equivalent to a complex with finite $[(n-1)/2]$-skeleton (Theorem 7). Knot theory provides examples with $\pi=\pi_1(M)\cong Z$ and infinite cyclic covering space $[(n-3)/2]$-connected but not finitely dominated, so this finiteness hypothesis is best possible in general. (See the paragraph following Theorem 7 below). If $n\neq 4$ then M' must in fact be a PD_{n-1}-complex; this is not known when $n=4$. In the aspherical case if a PD_n-group π is a semidirect product $\pi\cong\nu\rtimes Z$ then ν is a PD_{n-1}-group if and only if $\chi(\pi)=0$ and ν is $FP_{[(n-1)/2]}$. We do not know whether the finiteness assumption on ν is best possible in this case.

1 Notation

If X is a space let $C_*(X)$ be its singular chain complex, $\widetilde{X}$ its universal covering space, and X_ν the covering space associated to a subgroup $\nu\leq\pi_1(X)$.

Since we wish to minimize finiteness hypotheses, we shall make the following distinctions. A *PD_n-space* is a connected space X with an orientation character $w:\pi_1(X)\to\mathbb{Z}^\times$ and a class $[X]\in H_n(X;\mathbb{Z}^w)$ which satisfies formal Poincaré duality of dimension n with w-twisted local coefficients. A *PD_n-complex* is a PD_n-space which is homotopy equivalent to a finitely dominated cell complex. It is *finite* if it is homotopy equivalent to a finite cell complex. A cell complex X is finitely dominated if and only if $X\times S^1$ is finite, by Theorem 1 of [19].

Let R be a ring. An R-chain complex has *finite k-skeleton* if it is chain homotopy equivalent to a projective complex P_* with P_j finitely generated for $j\leq k$. If $i:R\to S$ is an inclusion of R as a subring of a ring S and C is a S-module let $i^!C$ be the R-module obtained by restriction of coefficients. An S-chain complex C_* is *R-finitely dominated* if $i^!C_*$ is chain homotopy equivalent to a finite projective R-chain complex. If X is a PD_n-space with fundamental group π then $C_*(\widetilde{X})$ is $\mathbb{Z}[\pi]$-finitely dominated, so π is FP_2, and X is finitely dominated if and only if π is finitely presentable [7].

If G is a group and A is a left $\mathbb{Z}[G]$-module let $|A|$ be the $\mathbb{Z}[G]$-module with the same underlying group and trivial G-action, and let $A^G=Hom_{\mathbb{Z}}(\mathbb{Z}[G],A)$ be the module of functions $\alpha:G\to A$ with G-action given by $(g\alpha)(h)=g.\alpha(hg)$ for all $g,h\in G$. Then $|A|^G$ is coinduced from a module over the trivial group.

The *conjugate* of A with respect to an orientation character $w : G \to Z/2Z$ is the right $\mathbb{Z}[G]$-module $D_w A$ with the same underlying group and G-action given by $a.g = (-1)^{w(g)} g^{-1}.a$ for all $a \in A$ and $g \in G$. (Note that the conjugate of a free left $\mathbb{Z}[G]$-module is again free. In particular, $D_w(\mathbb{Z}[G]) \cong \mathbb{Z}[G]$).

A group G is a *weak* PD_r-group if $H^q(G;\mathbb{Z}[G]) \cong Z$ if $q = r$ and is 0 otherwise [1]. If $r \leq 2$ a group is a weak PD_r-group if and only if it is virtually a PD_r-group. This is easy for $r \leq 1$ and is due to Bowditch when $r = 2$ [6].

2 Brown's criterion and duality

In this section we shall combine the finiteness criterion of Brown with an extension of work of Barge to establish our first main result.

Lemma 1 *Let G be a group and A a left G-module. Then $A^G \cong |A|^G$.*

Proof If $\alpha : G \to A$ let $|\alpha| : G \to |A|$ be the corresponding element of $|A|^G$, and let $\Theta(\alpha)(h) = h.\alpha(h)$ for all $h \in G$. Then $\Theta(g\alpha) = g|\Theta(\alpha)|$, since $\Theta(g\alpha)(h) = h.(g\alpha)(h) = hg.\alpha(hg) = \Theta(\alpha)(hg)$ for all $g, h \in G$. Thus Θ defines an isomorphism of left $\mathbb{Z}[G]$-modules from A^G to $|A|^G$. □

Theorem 2 *Let M be a PD_n-space and $p : \pi = \pi_1(M) \to G$ an epimorphism with G a PD_r-group, and let $\nu = \mathrm{Ker}(p)$. Let $i : \mathbb{Z}[\nu] \to \mathbb{Z}[\pi]$ be the natural inclusion. If $i^! C_*(\widetilde{M})$ has finite $[n/2]$-skeleton then $C_*(\widetilde{M})$ is $\mathbb{Z}[\nu]$-finitely dominated and $H^s(M_\nu;\mathbb{Z}[\nu]) \cong H_{n-r-s}(M_\nu;\mathbb{Z}[\nu])$ for all s.*

Proof Let $v = w_1(G)$ and $w = w_1(M)$. It is sufficient to show that the functors $H^s(M_\nu;-) = H^s(i^! C_*(\widetilde{M});-)$ from $\mathbb{Z}[\nu]$-modules to abelian groups commute with direct limit for all $s \leq n$, for then $i^! C_*(\widetilde{M})$ is finitely dominated, by Brown's finiteness criterion [8]. We may assume that $s > n/2$, since $i^! C_*(\widetilde{M})$ has finite $[n/2]$-skeleton . If A is a $\mathbb{Z}[\nu]$-module and $W = Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi], A)$ then $H^s(M_\nu; A) \cong H^s(M; W) \cong H_{n-s}(M; D_w W)$, by Shapiro's Lemma and Poincaré duality.

Let $A_q = H_q(M_\nu; D_w(A))$. As a $\mathbb{Z}[\nu]$-module $D_w(W)$ is the direct product of $|G|$ copies of $D_w(A)$. Hence $H_q(M_\nu; D_w(W)) \cong A_q^G$, for $0 \leq q \leq [n/2]$, since M_ν has finite $[n/2]$-skeleton. (Note that theses are *left* $\mathbb{Z}[G]$-modules). We shall apply the Cartan-Leray spectral sequence

$$E^2_{pq} = H_p(G; D_v(H_q(M_\nu; D_w(W)))) \Rightarrow H_{p+q}(M; D_w(W)).$$

Poincaré duality for G and another application of Shapiro's Lemma now give $H_p(G;D_v(A_q^G)) \cong H^{r-p}(G;A_q^G) \cong H^{r-p}(1;A_q)$, since A_q^G is coinduced from a module over the trivial group, by Lemma 1. If $s > [n/2]$ and $p+q=n-s$ then $q \leq [n/2]$ and so $H_p(G;A_q^G) \cong A_q$ if $p=r$ and is 0 otherwise. Thus the spectral sequence collapses to give $H_{n-s}(M;D_w(W)) \cong H_{n-r-s}(M_\nu;D_w(A))$. Since homology commutes with direct limits the result now follows easily. □

Corollary 2.1. *If π is a PD_n-group and ν is a normal subgroup of type $FP_{[n/2]}$ such that π/ν is a PD_r-group then ν is a PD_{n-r}-group.*

Proof Let $M=K(\pi,1)$. Then M is a PD_n-space and $C_*(\widetilde{M})$ is a resolution of the augmentation module $\mathbb{Z}$. As $C_*(\widetilde{M})$ is $\mathbb{Z}[\nu]$-finitely dominated ν is FP. Hence it is a PD_{n-r}-group, by Theorem 9.11 of [2]. □

The finiteness condition in this corollary cannot be relaxed further when $r=2$ and $n=4$. For Kapovich has given an example of a pair $\nu < \pi$ with π a PD_4-group, π/ν a PD_2-group and ν finitely generated but not FP_2 [13].

Corollary 2.2. *Under the same hypotheses on M and π, if either $r=n-1$ or $r=n-2$ and ν is infinite or $r=n-3$ and ν has one end then M is aspherical.*

Proof Since $H_q(\widetilde{M};\mathbb{Z}) = H_q(M_\nu;\mathbb{Z}[\nu]) \cong H^{n-r-q}(M_\nu;\mathbb{Z}[\nu])$, by the theorem, $H_q(\widetilde{M};\mathbb{Z})=0$ if $q>n-r$, $H_{n-r}(\widetilde{M};\mathbb{Z}) \cong H^0(M_\nu;\mathbb{Z}[\nu]) \cong H^0(\nu;\mathbb{Z}[\nu])$ and $H_{n-r-1}(\widetilde{M};\mathbb{Z}) \cong H^1(M_\nu;\mathbb{Z}[\nu]) \cong H^1(\nu;\mathbb{Z}[\nu])$. In all cases the hypotheses imply that $\widetilde{M}$ is contractible and so M is aspherical. □

In the non-aspherical case it is not immediately obvious that there are isomorphisms from $H^s(M_\nu;A)$ to $H_{n-r-s}(M_\nu;D_w(A))$ which are induced by cap product with a class in $H_{n-r}(M_\nu;\mathbb{Z}^w)$. If ν is finitely presentable then M_ν is finitely dominated; if moreover M is a PD_n-complex we could apply the Gottlieb-Quinn Theorem to conclude that M_ν is a PD_{n-r}-complex.

We shall give instead a purely homological argument which does not require π or ν to be finitely presentable, and so applies to PD_n-spaces. If G is a weak PD_r-group and M_ν is a PD_{n-r}-complex then M_ν has fundamental class $[M_\nu] = \eta_G \cap [M]$, where $\eta_G \in H^r(M;\mathbb{Z}[G])$ is the image of a generator of $H^r(G;\mathbb{Z}[G])$. Barge has given a simple homological argument to show that cap product with $[M_\nu]$ induces isomorphisms with simple coefficients [1]. We shall extend his argument to the case of arbitrary local coefficients. (See also Chapter 4 of [12] for the case $G=Z$ and $n=4$).

All tensor products $N\otimes P$ in the following theorem are taken over $\mathbb{Z}$.

Theorem 3 *Let M be a PD_n-space and $p:\pi=\pi_1(M)\to G$ an epimorphism with G a weak PD_r-group, and let $\nu=\mathrm{Ker}(p)$. If $C_*(\widetilde{M})$ is $\mathbb{Z}[\nu]$-finitely dominated then there are isomorphisms $H^p(M_\nu;\mathbb{Z}[\nu])\cong H^{p+r}(M;\mathbb{Z}[\pi])$, induced by cup product with η_G.*

Proof Let C_* be a finitely generated projective $\mathbb{Z}[\pi]$-chain complex which is chain homotopy equivalent to $C_*(\widetilde{M})$. Since $C_*(\widetilde{M})$ is $\mathbb{Z}[\nu]$-finitely dominated there is a finitely generated projective $\mathbb{Z}[\nu]$-chain complex E_* and a pair of $\mathbb{Z}[\nu]$-linear chain homomorphisms $\theta:E_*\to i^!C_*$ and $\phi:i^!C_*\to E_*$ such that $\theta\phi\sim I_{C_*}$ and $\phi\theta\sim I_{E_*}$. Let $C^q=Hom_{\mathbb{Z}[\pi]}(C_q,\mathbb{Z}[\pi])$ and $E^q=Hom_{\mathbb{Z}[\nu]}(E_q,\mathbb{Z}[\nu])$, and let $\widehat{\mathbb{Z}[\pi]}=Hom_{\mathbb{Z}[\nu]}(i^!\mathbb{Z}[\pi],\mathbb{Z}[\nu])$ be the module coinduced from $\mathbb{Z}[\nu]$. (The left π-action on $\widehat{\mathbb{Z}[\pi]}$ is given by $(g\alpha)(h)=\alpha(hg)$ for all $g,h\in\pi$.) Then there are isomorphisms $\Psi:H^q(E^*)\cong H^q(C_*;\widehat{\mathbb{Z}[\pi]})$, determined by θ and Shapiro's Lemma.

The complex $\mathbb{Z}[G]\otimes_{\mathbb{Z}[\pi]}C_*$ is an augmented complex of finitely generated projective $\mathbb{Z}[G]$-modules with finitely generated integral homology. Therefore G is of type FP_∞, by Theorem 3.1 of [22]. Hence the augmentation $\mathbb{Z}[G]$-module $\mathbb{Z}$ has a resolution A_* by finitely generated projective $\mathbb{Z}[G]$-modules. Let $A^q=Hom_{\mathbb{Z}[G]}(A_q,\mathbb{Z}[G])$ and let $\eta\in H^r(A^*)=H^r(G;\mathbb{Z}[G])$ be a generator. Let $\varepsilon_C:C_*\to A_*$ be a chain map corresponding to the projection of p onto G, and let $\eta_G=\varepsilon_C^*\eta\in H^r(C_*;\mathbb{Z}[G])$. The augmentation $A_*\to\mathbb{Z}$ determines a chain homotopy equivalence $p:C_*\otimes A_*\to C_*\otimes\mathbb{Z}=C_*$. Let $\sigma:G\to\pi$ be a set-theoretic section.

We may define cup-products relating the cohomology of M_ν and M as follows. Let $e:\widehat{\mathbb{Z}[\pi]}\otimes\mathbb{Z}[G]\to\mathbb{Z}[\pi]$ be the pairing given by $e(\alpha\otimes g)=\sigma(g).\alpha(\sigma(g)^{-1})$ for all $\alpha:\mathbb{Z}[\pi]\to\mathbb{Z}[\nu]$ and $g\in G$. Then e is independent of the choice of section σ and is $\mathbb{Z}[\pi]$-linear with respect to the diagonal left π-action on $\widehat{\mathbb{Z}[\pi]}\otimes\mathbb{Z}[G]$. Let $d:C_*\to C_*\otimes C_*$ be a π-equivariant diagonal, with respect to the diagonal left π-action on $C_*\otimes C_*$, and let $j=(1\otimes\varepsilon_C)d:C_*\to C_*\otimes A_*$. Then $pj=Id_{C_*}$ and so j is a chain homotopy equivalence. We define the cup-product $[f]\cup\eta_G$ in $H^{p+r}(C^*)=H^{p+r}(M;\mathbb{Z}[\pi])$ by $[f]\cup\eta_G=e_\# d^*(\Psi([f])\times\eta_G)=e_\# j^*(\Psi([f])\times\eta)$ for all $[f]\in H^p(E^*)=H^p(M_\nu;\mathbb{Z}[\nu])$.

If C is a left $\mathbb{Z}[\pi]$-module let $D=Hom_{\mathbb{Z}[\nu]}(i^!C,\mathbb{Z}[\pi])$ have the left G-action determined by $(g\lambda)(c)=\sigma(g)\lambda(\sigma(g)^{-1}c)$ for all $c\in C$ and $g\in G$. If C is free with basis $\{c_i | 1 \leq i \leq n\}$ there is an isomorphism of left $\mathbb{Z}[G]$-modules $\Theta : D \cong (|\mathbb{Z}[\pi]|^G)^n$ given by $\Theta(\lambda)(g) = (\sigma(g).\lambda(\sigma(g)^{-1}c_1), \ldots, \sigma(g).\lambda(\sigma(g)^{-1}c_n))$ for all $\lambda \in D$ and $g \in G$, and so D is coinduced from a module over the trivial group.

Let $D^q = Hom_{\mathbb{Z}[\nu]}(i^!C_q, \mathbb{Z}[\pi])$ and let $\rho : E^* \otimes \mathbb{Z}[G] \to D^*$ be the $\mathbb{Z}$-linear cochain homomorphism defined by $\rho(f \otimes g)(c) = \sigma(g) f \phi(\sigma(g)^{-1}c)$ for all $c \in C_q$, $\lambda \in D^q$, $f \in E^q$, $g \in G$ and all q. Then the G-action on D^q and ρ are independent of the choice of section σ, and ρ is $\mathbb{Z}[G]$-linear if $E^q \otimes \mathbb{Z}[G]$ has the left G-action given by $g(f \otimes g') = f \otimes gg'$ for all $g, g' \in G$ and $f \in E^q$.

If $\lambda \in D^q$ then $\lambda\theta_q(E_q)$ is a finitely generated $\mathbb{Z}[\nu]$-submodule of $\mathbb{Z}[\pi]$. Hence there is a family of homomorphisms $\{f_g \in E^q | g \in F\}$, where F is a finite subset of G, such that $\lambda\theta_q(e) = \Sigma_{g \in F} f_g(e)\sigma(g)$ for all $e \in E_q$. Let $\lambda_g(e) = \sigma(g)^{-1} f_g(\phi\sigma(g)\theta(e))\sigma(g)$ for all $e \in E_q$ and $g \in F$. Let $\Phi(\lambda) = \Sigma_{g \in F} \lambda_g \otimes g \in E^q \otimes \mathbb{Z}[G]$. Then Φ is a $\mathbb{Z}$-linear cochain homomorphism. Moreover $[\rho\Phi(\lambda)] = [\lambda]$ for all $[\lambda] \in H^q(D^*)$ and $[\Phi\rho(f \otimes g)] = [f \otimes g]$ for all $[f \otimes g] \in H^q(E^* \otimes \mathbb{Z}[G])$, and so ρ is a chain homotopy equivalence. (It is not clear that Φ is $\mathbb{Z}[G]$-linear on the cochain level, but we shall not need to know this).

We now compare the hypercohomology of G with coefficients in the cochain complexes $E^* \otimes \mathbb{Z}[G]$ and D^*. On one side we have $\mathbb{H}^n(G; E^* \otimes \mathbb{Z}[G]) = H^n_{tot}(Hom_{\mathbb{Z}[G]}(A_*, E^* \otimes \mathbb{Z}[G]))$, which may be identified with $H^n_{tot}(E^* \otimes A^*)$ since A_q is finitely generated for all $q \geq 0$. This is in turn isomorphic to $H^{n-r}(E^*) \otimes H^r(G; \mathbb{Z}[G]) \cong H^{n-r}(E^*)$, since G acts trivially on E^* and is a weak PD_r-group.

On the other side we have $\mathbb{H}^n(G; D^*) = H^n_{tot}(Hom_{\mathbb{Z}[G]}(A_*, D^*))$. The cochain homomorphism ρ induces a morphism of double complexes from $E^* \otimes A^*$ to $Hom_{\mathbb{Z}[G]}(A_*, D^*)$ by $\rho^{pq}(f \otimes \alpha)(a) = \rho(f \otimes \alpha(a)) \in D^p$ for all $f \in E^p$, $\alpha \in A^q$ and $a \in A_q$ and all $p, q \geq 0$. Let $\hat{\rho}^p([f]) = [\rho^{pr}(f \times \eta)] \in \mathbb{H}^{p+r}(G; D^*)$ for all $[f] \in H^p(E^*)$. Then $\hat{\rho}^p : H^p(E^*) \to \mathbb{H}^{p+r}(G; D^*)$ is an isomorphism, since $[f] \mapsto [f \times \eta]$ is an isomorphism and ρ is a chain homotopy equivalence. Since C_p is a finitely generated projective $\mathbb{Z}[\pi]$-module D^p is a direct summand of a coinduced module. Therefore $H^i(G; D^p) = 0$ for all $i > 0$, while $H^0(G; D^p) = Hom_{\mathbb{Z}[\pi]}(C_p, \mathbb{Z}[\pi])$, for all $p \geq 0$. Hence $\mathbb{H}^n(G; D^*) \cong H^n(C^*)$ for all n.

Let $f \in E^p$, $a \in A_r$ and $c \in C_p$, and suppose that $\eta(a) = \Sigma n_g g$. Since $\hat{\rho}^p([f])(a)(c) = \rho(f \otimes \eta(a))(c) = \Sigma n_g \sigma(g) f \phi(\sigma(g)^{-1}c) = ([f] \cup \eta)(c, a)$ it follows that the homomorphisms from $H^p(E^*)$ to $H^{p+r}(C^*)$ given by cup-product with η_G are isomorphisms for all p. ∎

Theorems 2 and 3 together give the following version of the Gottlieb-Quinn Theorem for covering spaces.

Theorem 4 *Let M be a PD_n-space and $p:\pi=\pi_1(M)\to G$ an epimorphism with G a PD_r-group, and let $\nu=\mathrm{Ker}(p)$. Then M_ν is a PD_{n-r}-space if and only if $i^!C_*(\widetilde{M})$ has finite $[n/2]$-skeleton.*

Proof The conditions are clearly necessary. Conversely, if M_ν has finite $[n/2]$-skeleton then C_* is $\mathbb{Z}[\nu]$-finitely dominated, by Theorem 2, and so cup product with η_G induces isomorphisms $H^p(M_\nu;\mathbb{Z}[\nu])\cong H^{p+r}(M;\mathbb{Z}[\pi])$, by Theorem 3. Let $[M]\in H_n(M;\mathbb{Z}^w)$ be a fundamental class for M, and let $[M_\nu]=\eta_G\cap[M]\in H_{n-r}(M;\mathbb{Z}^w\otimes\mathbb{Z}[G])=H_{n-r}(M_\nu;\mathbb{Z}^{w|_\nu})$. Then cap product with $[M_\nu]$ induces isomorphisms $H^p(M_\nu;\mathbb{Z}[\nu])\cong H_{n-r-p}(M_\nu;\mathbb{Z}[\nu])$ for all p, since $c\cap[M_\nu]=(c\cup\eta_G)\cap[M]$ in $H_{n-r-p}(M;\mathbb{Z}[\pi])=H_{n-r-p}(M_\nu;\mathbb{Z}[\nu])=H_{n-r-p}(\widetilde{M};\mathbb{Z})$ for $c\in H^p(M_\nu;\mathbb{Z}[\nu])$. Since $i^!C_*(\widetilde{M})$ is finitely dominated it follows that cap product with $[M_\nu]$ induces isomorphisms $H^p(M_\nu;\mathcal{F})\cong H_{n-r-p}(M_\nu;D_w(\mathcal{F}))$, for any free $\mathbb{Z}[\nu]$-module $\mathcal{F}$, and hence for arbitrary coefficient modules, by an easy 5-Lemma argument. □

Corollary 4.1. *Under the same hypotheses on M and π, the covering space M_ν is a PD_{n-r}-complex if and only if it is homotopy equivalent to a complex with finite $[n/2]$-skeleton and ν is finitely presentable.* □

Corollary 4.2. *If M is a PD_n-space and π is a PD_r-group then $\widetilde{M}$ is a PD_{n-r}-complex if and only if $H_q(\widetilde{M};\mathbb{Z})$ is finitely generated for all $q\leq[n/2]$.*

Proof The condition is clearly necessary. If it holds then $\widetilde{M}$ has finite $[n/2]$-skeleton [25], and so $\widetilde{M}$ is a PD_{n-r}-complex by Corollary 4.1. □

Stark used Theorem 3.1 of [22] with the Gottlieb-Quinn Theorem to deduce that if M is a PD_n-complex and $v.c.d.\pi/\nu<\infty$ then π/ν is of type vFP, and therefore is virtually a PD-group. Is there a purely algebraic argument to show that if M is a PD_n-space, ν is a normal subgroup of π and $C_*(\widetilde{M})$ is $\mathbb{Z}[\nu]$-finitely dominated then π/ν must be a weak PD-group?

3 Novikov rings and Ranicki's criterion

The results of the above section apply in particular when $G=Z$. In this case however we may use an alternative finiteness criterion of Ranicki to get a slightly stronger result, which we can show to be best possible. Here we shall outline the algebra relevant to our use of Ranicki's criterion in the next section.

Let R be a ring with an automorphism α, and let $S = R_\alpha[z, z^{-1}]$, $\widehat{S}_+ = R_\alpha((z))$ and $\widehat{S}_- = R_\alpha((z^{-1}))$ be the rings of twisted Laurent polynomials and series $\Sigma_{j\geq a} r_j z^{\pm j}$ with coefficients $r_j \in R$ and multiplication determined by $zr = \alpha(r)z$ for all $r \in R$.

An α-*twisted* endomorphism of an R-module E is an additive function $h : E \to E$ such that $h(re) = \alpha(r)h(e)$ for all $e \in E$ and $r \in R$, and h is an α-twisted automorphism if it is bijective. Such an endomorphism h extends to α-twisted endomorphisms of the modules $S \otimes_R E$, $\widehat{E}_+ = \widehat{S}_+ \otimes_R E$ and $\widehat{E}_- = \widehat{S}_- \otimes_R E$ by $h(s \otimes e) = zsz^{-1} \otimes h(e)$ for all $e \in E$ and $s \in S$, $\widehat{S}_+$ or $\widehat{S}_-$, respectively. In particular, left multiplication by z determines α-twisted automorphisms of $S \otimes_R E$, $\widehat{E}_+$ and $\widehat{E}_-$ which commute with h.

If E is finitely generated then $1 - z^{-1}h$ is an automorphism of $\widehat{E}_-$, with inverse given by a geometric series: $(1 - z^{-1}h)^{-1} = \Sigma_{k\geq 0} z^{-k}h^k$. (If E is not finitely generated this series may not give a function with values in $\widehat{E}_-$, and $z - h = z(1 - z^{-1}h)$ may not be surjective). Similarly, if k is an α^{-1}-twisted endomorphism of E then $1 - zk$ is an automorphism of $\widehat{E}_+$.

If P_* is a chain complex with an endomorphism $\beta : P_* \to P_*$ let $P_*[1]$ be the suspension and $\mathcal{C}(\beta)_*$ be the mapping cone. Thus $\mathcal{C}(\beta)_q = P_{q-1} \oplus P_q$, and $\partial_q(p, p') = (-\partial p, \beta(p) + \partial p')$, and there is a short exact sequence

$$0 \to P_* \to \mathcal{C}(\beta)_* \to P_*[1] \to 0.$$

The connecting homomorphisms in the associated long exact sequence of homology are induced by β. The *algebraic mapping torus* of an α-twisted self chain homotopy equivalence h of an R-chain complex E_* is the mapping cone $\mathcal{C}(1 - z^{-1}h)$ of the endomorphism $1 - z^{-1}h$ of the S-chain complex $S \otimes_R E_*$.

Lemma 5 *Let E_* be a projective chain complex over R which is finitely generated in degrees $\leq d$ and let $h : E_* \to E_*$ be an α-twisted chain homotopy equivalence. Then $H_q(\widehat{S}_- \otimes_S \mathcal{C}(1 - z^{-1}h)_*) = 0$ for $q \leq d$.*

Proof There is a short exact sequence

$$0 \to S \otimes_R E_* \to \mathcal{C}(1 - z^{-1}h)_* \to S \otimes_R E_*[1] \to 0.$$

Since E_* is a complex of projective R-modules the sequence

$$0 \to \widehat{E}_{*-} \to \widehat{S}_- \otimes_S \mathcal{C}(1 - z^{-1}h)_* \to \widehat{E}_{*-}[1] \to 0$$

obtained by extending coefficients is exact. The endomorphism $1 - z^{-1}h$ of $\widehat{E}_{*-}$ induces isomorphisms in degrees $\leq d$ and so induces isomorphisms on homology in degrees $< d$ and an epimorphism on homology in degree d. Therefore

$H_q(\widehat{S}_- \otimes_S \mathcal{C}(1-z^{-1}h)_*) = 0$ for $q \leq d$, by the long exact sequence for homology. $\square$

Theorem 6 *Let C_* be a finitely generated projective S-chain complex. Then $i^! C_*$ is chain homotopy equivalent (over R) to a projective complex E_* which is finitely generated in degrees $\leq d$ if and only if $H_q(\widehat{S}_\pm \otimes_S C_*) = 0$ for $q \leq d$.*

Proof We may assume without loss of generality that C_q is a finitely generated free S-module for all $q \leq d+1$, with basis $X_i = \{c_{q,i}\}_{i \in I(q)}$. We may also assume that $0 \notin \partial_i(X_i)$ for $i \leq d+1$, where $\partial_i : C_i \to C_{i-1}$ is the differential of the complex. Let $h_\pm$ be the $\alpha^{\pm 1}$-twisted automorphisms of $i^! C_*$ induced by multiplication by $z^{\pm 1}$ in C_*. Let $f_q(z^k r c_{q,i}) = (0, z^k \otimes r c_{q,i}) \in (S \otimes_R C_{q-1}) \oplus (S \otimes_R C_q)$. Then f_* defines S-chain homotopy equivalences from C_* to each of $\mathcal{C}(1-z^{-1}h_+)$ and $\mathcal{C}(1-zh_-)$.

Suppose first that $k_* : i^! C_* \to E_*$ and $g_* : E_* \to i^! C_*$ are chain homotopy equivalences, where E_* is a projective R-chain complex which is finitely generated in degrees $\leq d$. Then $\theta_\pm = k_* h_\pm g_*$ are $\alpha^{\pm 1}$-twisted self homotopy equivalences of E_*, and $\mathcal{C}(1-z^{-1}h_+)$ and $\mathcal{C}(1-zh_-)$ are chain homotopy equivalent to $\mathcal{C}(1-z^{-1}\theta_+)$ and $\mathcal{C}(1-z\theta_-)$, respectively. Therefore $H_q(\widehat{S}_- \otimes_S C_*) = H_q(\widehat{S}_- \otimes_S \mathcal{C}(1-z^{-1}\theta_+)) = 0$ and $H_q(\widehat{S}_+ \otimes_S C_*) = H_q(\widehat{S}_+ \otimes_S \mathcal{C}(1-z\theta_-)) = 0$ for $q \leq d$, by Lemma 5, applied twice.

Conversely, suppose that $H_i(\widehat{S}_\pm \otimes_S C_*) = 0$ for all $i \leq k$. We can proceed as in [4] where the case of a partial free deleted resolution of a module over a group ring is considered (using a support function with values in the group). We shall define inductively a support function $supp_X$ for the elements λ of $\cup_{i \leq d+1} C_i$ with values finite subsets of $\{z^j\}_{j \in \mathbb{Z}}$ so that

(1) $supp_X(0) = \emptyset$

(2) if $x \in X_0$ then $supp_X(z^j x) = z^j$;

(3) if $x \in X_i$ for $1 \leq i \leq d+1$ then $supp_X(z^j x) = z^j . supp_X(\partial_i(x))$;

(4) if $s = \sum_j r_j z^j \in S$, where $r_j \in R$, $supp_X(sx) = \cup_{r_j \neq 0} supp_X(z^j x)$

(5) if $0 \leq i \leq d+1$ and $\lambda = \sum_{s_x \in S, x \in X_i} s_x x$ then $supp_X(\lambda) = \cup_{s_x \neq 0, x \in X_i} supp_X(s_x x)$

Then $supp_X(\partial_i(\lambda)) \subseteq supp_X(\lambda)$ for all $\lambda \in C_i$ and all $1 \leq i \leq d+1$.

Define two subcomplexes C^+ and C^- of C which are 0 in degrees $i \geq d+2$ as follows. Since $X = \cup_{i \leq d+1} X_i$ is finite there is a positive integer b such that $\cup_{x \in X_i, i \leq d+1} supp_X(x) \subseteq \{z^j\}_{-b \leq j \leq b}$.

(1) if $i \leq d+1$ an element $\lambda \in C_i$ is in C^+ if and only if $supp_X(\lambda) \subseteq \{z^j\}_{j\geq -b}$; and

(2) if $i \leq d+1$ an element $\lambda \in C_i$ is in C^- if and only if $supp_X(\lambda) \subseteq \{z^j\}_{j\leq b}$.

Then $\cup_{i\leq d+1} X_i \subseteq (C^+)^{[d+1]} \cap (C^-)^{[d+1]}$ and so $(C^+)^{[d+1]} \cup (C^-)^{[d+1]} = C^{[d+1]}$, where the upper index $*$ denotes the $*$-skeleton. Moreover $(C^+)^{[d+1]}$ is a complex of free finitely generated $R_\alpha[z]$-modules, $(C^-)^{[d+1]}$ is a complex of free finitely generated $R_\alpha[z^{-1}]$-modules, $(C^+)^{[d+1]} \cap (C^-)^{[d+1]}$ is a complex of free finitely generated R-modules and

$$C^{[d+1]} = S \otimes_{R_\alpha[z]} (C^+)^{[d+1]} = S \otimes_{R_\alpha[z^{-1}]} (C^-)^{[d+1]}.$$

Furthermore there is a Mayer-Vietoris exact sequence

$$0 \to (C^+)^{[d+1]} \cap (C^-)^{[d+1]} \to (C^+)^{[d+1]} \oplus (C^-)^{[d+1]} \to C^{[d+1]} \to 0.$$

Thus the $(d+1)$-skeletons of C, C^+ and C^- satisfy "algebraic transversality" in the sense of [21, Prop. 1].

Then to prove the theorem it suffices to show that C^+ and C^- are each chain homotopy equivalent over R to a complex of projective R-modules which is finitely generated in degrees $\leq d$. As in [21, p. 628] there is an exact sequence of $R_\alpha[z^{-1}]$-module chain complexes

$$0 \to (C^-)^{[d+1]} \to C^{[d+1]} \oplus R_\alpha[[z^{-1}]] \otimes_{R_\alpha[z^{-1}]} (C^-)^{[d+1]} \to \widehat{S}_- \otimes_S C^{[d+1]} \to 0.$$

Let $\tilde{i}$ denote the inclusion of $(C^-)^{[d+1]}$ into the central term. Inclusions on each component define a chain homomorphism

$$\tilde{j} : (C^+)^{[d+1]} \cap (C^-)^{[d+1]} \to (C^+)^{[d+1]} \oplus R_\alpha[[z^{-1}]] \otimes_{R_\alpha[z^{-1}]} (C^-)^{[d+1]}$$

such that the mapping cones of $\tilde{i}$ and $\tilde{j}$ are chain equivalent R-module chain complexes. The map induced by $\tilde{i}$ in homology is an epimorphism in degree d and an isomorphism in degree $< d$, since $H_i(\widehat{S}_- \otimes_S C^{[d+1]}) = 0$ for $i \leq d$. In particular all homologies in degrees $\leq d$ of the mapping cone of $\tilde{i}$ are 0. Hence all homologies of the mapping cone of $\tilde{j}$ are 0 in degrees $\leq d$. Then $(C^+)^{[d+1]}$ is homotopy equivalent over R to a chain complex of projectives over R whose d-skeleton is a summand of $(C^+)^{[d]} \cap (C^-)^{[d]}$. This completes the proof. □

If π is a group, $\rho : \pi \to Z$ is an epimorphism with kernel ν and $\rho(z) = 1$ then conjugation by z ($g \mapsto zgz^{-1}$) determines an automorphism α of $R = \mathbb{Z}[\nu]$. The corresponding twisted extensions S, $\widehat{S}_+$ and $\widehat{S}_-$ are the group ring $\mathbb{Z}[\pi]$ and the *Novikov rings* $\widehat{\mathbb{Z}[\pi]}_\rho$ and $\widehat{\mathbb{Z}[\pi]}_{-\rho}$. In [16] it is shown that if π is finitely generated the matrix rings $\mathbb{M}_n(\widehat{\mathbb{Z}[\pi]}_\rho)$ are von Neumann finite: i.e., if $A, B \in$

$\mathbb{M}_n(\widehat{\mathbb{Z}[\pi]}_\rho)$ and $AB = I$ then $BA = I$. Hence finitely generated stably free $\widehat{\mathbb{Z}[\pi]}_\rho$-modules have well defined ranks, and the rank is strictly positive if the module is nonzero. (In [12] rings satisfying the latter conditions are said to be *weakly finite*).

4 Infinite cyclic coverings

One approach to duality when $G = \pi/\nu \cong Z$ might proceed as follows. Let $\Psi : H^q(M_\nu; \mathbb{Z}[\nu]) \to H^q(M; \widehat{\mathbb{Z}[\pi]})$ be the isomorphism determined by Shapiro's Lemma. The module $\widehat{\mathbb{Z}[\pi]}$ may be identified with the left $\mathbb{Z}[\pi]$-module of doubly infinite series $\Sigma_{n\in\mathbb{Z}} r_n z^n$ with coefficients in $\mathbb{Z}[\nu]$, and there is an exact sequence

$$\xi : \quad 0 \to \mathbb{Z}[\pi] \to A_+ \oplus A_- \to \widehat{\mathbb{Z}[\pi]} \to 0.$$

If $C_*(\widetilde{M})$ is $\mathbb{Z}[\nu]$-finitely dominated the Bockstein operation for ξ induces isomorphisms $\delta^\xi : H^q(M; \widehat{\mathbb{Z}[\pi]}) \to H^{q+1}(M; \mathbb{Z}[\pi])$. If we could show that $\delta^\xi\Psi(c) = \pm\Psi(c) \cup \eta_Z$ for all $c \in H^q(M_\nu; \mathbb{Z}[\nu])$ then we could conclude that M_ν is a PD_{n-1}-space, with fundamental class $\eta_Z \cap [M]$. However we have not managed to carry this through, and so we shall use Theorem 3 instead.

Theorem 7 *Let M be a PD_n-space with fundamental group π and let $p : \pi \to Z$ be an epimorphism with kernel ν. Then M_ν is a PD_{n-1}-space if and only if $\chi(M) = 0$ and $C_*(\widetilde{M_\nu}) = i^! C_*(\widetilde{M})$ has finite $[(n-1)/2]$-skeleton.*

Proof If M_ν is a PD_{n-1}-space then $C_*(\widetilde{M_\nu})$ is $\mathbb{Z}[\nu]$-finitely dominated [7]. In particular, $H_*(M; \mathbb{Z}[Z]) = H_*(M_\nu; \mathbb{Z})$ is finitely generated. Let $\Lambda = \mathbb{Z}[Z]$. The augmentation Λ-module $\mathbb{Z}$ has a short free resolution $0 \to \Lambda \to \Lambda \to \mathbb{Z} \to 0$, and it follows easily from the exact sequence of homology for this coefficient sequence that $\chi(M) = 0$ [20]. Thus the conditions are necessary.

Suppose that they hold. Let $A_\pm$ be the two Novikov rings corresponding to the two epimorphisms $\pm p : \pi \to Z$ with kernel ν. Then $H_j(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = 0$ for $j \leq [(n-1)/2]$, by Theorem 6. Hence $H_j(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = 0$ for $j \geq n - [(n-1)/2]$, by duality. If n is even there is one possible nonzero module, in degree $m = n/2$. But then $H_m(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*)$ is stably free, by Lemma 3.1 of [12]. Since $\chi(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = \chi(C_*) = \chi(M) = 0$ and the rings $A_\pm$ are weakly finite [16] these modules are 0. Thus $H_j(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = 0$ for all j, and so $i^! C_*$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\nu]$-complex, by Theorem 6. Thus the result follows from Theorem 3, as in Theorem 4. $\square$

When n is odd $[n/2] = [(n-1)/2]$, so the finiteness condition on M_ν agrees with that of Theorem 4 (for $G = Z$), but it is slightly weaker if n is even.

The infinite cyclic cover of the closed n-manifold $M(K)$ obtained by surgery on a simple $(n-2)$-knot K is $[(n-3)/2]$-connected. However there are examples for which $\pi_{[(n-1)/2]}(M(K))$ is not finitely generated as an abelian group [14, 17]. Thus the $FP_{[(n-1)/2]}$ condition is best possible, in general.

Corollary 7.1. *Let π be a PD_n-group and $p : \pi \to Z$ an epimorphism. Then $\nu = \mathrm{Ker}(p)$ is a PD_{n-1}-group if and only if $\chi(\pi) = 0$ and ν is $FP_{[(n-1)/2]}$.* ☐

The finiteness condition $FP_{[(n-1)/2]}$ is probably best possible, but we have no examples with $n > 4$ to confirm this. (This condition cannot be relaxed if $n \leq 4$. For let D be the closed 3-manifold obtained by doubling the exterior of a nontrivial knot with Alexander polynomial 1, and let $\pi = \pi_1(D)$. Then π is a PD_3-group with $\chi(\pi) = 0$, $\pi/\pi' \cong Z$ and $\nu = \pi'$ is not finitely generated. The products $\pi \times Z$ and $\nu = \pi' \times Z$ give a similar example for $n = 4$).

Corollary 7.2. *Under the same hypotheses on M and π, if $n \neq 4$ then M_ν is a PD_{n-1}-complex if and only if it is homotopy equivalent to a complex with finite $[(n-1)/2]$-skeleton.*

Proof If $n \leq 3$ every PD_{n-1}-space is a PD_{n-1}-complex, while if $n \geq 5$ then $[(n-1)/2] \geq 2$ and so ν is finitely presentable. ☐

If $n \leq 3$ we need only assume that M is a PD_n-space and ν is finitely *generated*. It remains an open question whether every PD_3-space is finitely dominated. The arguments of [24] and [9] on the factorization of PD_3-complexes into connected sums are essentially homological, and so every PD_3-space is a connected sum of aspherical PD_3-spaces and a PD_3-complex with virtually free fundamental group. (In particular, ν is FP_∞ and $v.c.d.\nu = 0$, 1 or 3). Thus this question reduces to whether every PD_3-group is finitely presentable. There are PD_4-groups which are not finitely presentable [10].

The case $n = 4$ was in fact the origin of this paper, and gives the following improvements to Theorems 4.1 and 5.18 of [12].

Corollary 7.3. *Let M be a PD_4-space with $\chi(M) = 0$ and $\pi = \pi_1(M) \cong \nu \rtimes Z$, where ν is finitely generated. Then M is aspherical if and only if ν has one end. In that case ν is a PD_3-group.*

Proof The space M_ν is a PD_3-space and ν is FP_2, by Theorem 7. If M is aspherical then so is M_ν. Hence ν is a PD_3-group, and so has one end. Conversely, if ν has one end $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by an LHS spectral sequence argument. Since ν is finitely generated $\beta_1^{(2)}(\pi) = 0$ [18]. Therefore M is aspherical, by Corollary 3.5.2 of [12]. $\square$

If $\pi \cong \nu \rtimes Z$ is a PD_4-group with ν finitely generated then $\chi(\pi) = 0$ if and only if ν is FP_2, by Corollary 2.1 and Theorem 7. However the latter conditions need not hold. Let F be the orientable surface of genus 2. Then $G = \pi_1(F)$ has a presentation $\langle a_1, a_2, b_1, b_2 \mid [a_1, b_1] = [a_2, b_2]\rangle$. The group $\pi = G \times G$ is a PD_4-group, and the subgroup $\nu \leq \pi$ generated by the images of (a_1, a_1) and the six elements $(x, 1)$ and $(1, x)$, for $x = a_2$, b_1 or b_2, is normal in π, with quotient $\pi/\nu \cong Z$. However $\chi(\pi) = 4 \neq 0$ and so ν cannot be FP_2.

Corollary 7.4. *Let M be a PD_4-space with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is an extension of Z^r by a finitely generated infinite normal subgroup ν, for some $r > 1$. Then M is aspherical and ν is a PD_{4-r}-group.* $\square$

Proof Let $\phi : \pi \to Z$ be an epimorphism which factors through π/ν. Then ν is a finitely generated infinite normal subgroup of $\mathrm{Ker}(\phi)$, and $\mathrm{Ker}(\phi)/\nu \cong Z^{r-1}$. Hence $\mathrm{Ker}(\phi)$ is finitely generated and has one end, and so the result follows from Corollaries 7.3 and 2.1. $\square$

A simple induction based on Theorem 7 shows that if M_ν is the covering space of a PD_n-complex M corresponding to an epimorphism $p : \pi_1(M) \to G$ and G is virtually poly-Z of Hirsch length r then M_ν is a PD_{n-r}-complex if $\chi(M) = 0$, $\mathrm{Ker}(\pi_1(p))$ is finitely presentable and M_ν is homotopy equivalent to a complex with finite $[(n-1)/2]$-skeleton.

However the methods and results described in this section break down for more general covering groups. Let S be an aspherical closed surface and let $G = \pi_1(S)$. Surface groups are left orderable, and a left order P on G determines a Novikov-like completion $R = \widehat{\mathbb{Z}[G]}_P$ of $\mathbb{Z}[G]$ in an obvious way. Since $\widetilde{S}$ is contractible the most straightforward extension of the Ranicki criterion would require that $H_*(R \otimes_{\mathbb{Z}[G]} C_*(\widetilde{S})) = 0$. If R were weakly finite this would imply that $\chi(G) = \chi(R \otimes_{\mathbb{Z}[G]} C_*(\widetilde{S})) = 0$. (A more geometric notion of Novikov completion is used in [3] to give criteria for the kernel ν of an epimorphism $f : \pi \to G$ to have a finitely dominated $K(\nu, 1)$-complex when there is a finite $K(\pi, 1)$ and G is a $CAT(0)$-group, such as a surface group).

References

[1] Barge, J. Dualité dans les revêtements galoisiens, Invent. Math. 58 (1980), 101-106.

[2] Bieri, R. *Homological Dimension of Discrete Groups*, Queen Mary College Mathematics Notes, London, 2nd ed. 1981

[3] Bieri, R. and Geogeghan, R. Kernels of actions on non-positively curved spaces, in *Geometry and Cohomology in Group Theory* (edited by P.H.Kropholler, G.A.Niblo and R.Stöhr), LMS Lecture Notes Series 252, Cambridge University Press, Cambridge - New York - Melbourne (1998), 24-38.

[4] R. Bieri, B. Renz, Valuations on free resolutions and higher geometric invariants of groups, Comment. Math. Helv. 63(1988), 464-497

[5] Bieri, R. and Strebel, R. *Geometric Invariants for Discrete groups*, manuscript-book, Frankfurt University

[6] Bowditch, B.H. Planar groups and the Seifert conjecture, J. Reine u. Angew. Math. 576 (2004), 11-62.

[7] Browder, W. Poincaré complexes, their normal fibrations and surgery, Invent. Math. 17 (1972), 191-202.

[8] K. S. Brown A homological criterion for finiteness, Commentarii Math. Helvetici 50 (1975), 129-135.

[9] Crisp, J.S. The decomposition of Poincaré duality complexes, Commentarii Math. Helvetici 75 (2000), 232-246.

[10] Davis, M. The cohomology of a Coxeter group with group ring coefficients, Duke Math. J. 91 (1998), 297-314.

[11] Gottlieb, D.H. Poincaré duality and fibrations, Proc. Amer. Math. Soc. 76 (1979), 148-150.

[12] Hillman, J.A. *Four-Manifolds, Geometries and Knots*, Geometry and Topology Monographs Vol. 5 (2002).

[13] Kapovich, M. On normal subgroups in the fundamental groups of complex surfaces, preprint, University of Utah (1998).

[14] Kearton, C. An algebraic classification of some even-dimensional knots, Topology 15 (1976), 363-373.

[15] Kochloukova, D.H. On a conjecture of E.Rapaport Strasser about knot-like groups and its pro-p version, J. Pure App. Algebra, to appear.

[16] Kochloukova, D.H. Some Novikov rings that are von Neumann finite and knot-like groups, preprint, UNICAMP, Brazil (2005).

[17] Levine, J.P. An algebraic classification of some knots of codimension two, Commentarii Math. Helvetici 45 (1970), 185-198.

[18] Lück, W. L^2-Betti numbers of mapping tori and groups, Topology 33 (1994), 203-214.

[19] Mather, M. Counting homotopy types of manifolds,
Topology 4 (1965), 93-94.

[20] Milnor, J.W. Infinite cyclic coverings,
in *Conference on the Topology of Manifolds* (edited by J.G.Hocking), Prindle, Weber and Schmidt, Boston (1968), 115-133.

[21] Ranicki, A.A. Finite domination and Novikov rings,
Topology 34 (1995), 619-632.

[22] Stark, C.W. Resolutions modeled on ternary trees,
Pacific J. Math. 173 (1996), 557-569.

[23] Strebel, R. A remark on subgroups of infinite index in Poincaré duality groups,
Comment. Math. Helv. 52 (1977), 317-324.

[24] Turaev, V.G. Three-dimensional Poincaré complexes: classification and splitting, Math. Sbornik 180 (1989), 809-830.

[25] Wall, C.T.C. Finiteness conditions for CW-complexes,
Ann. Math. 81 (1965), 56-69.