Representation theory of towers of recollement: theory, notes, and examples

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Abstract

We give an axiomatic framework for studying the representation theory of towers of algebras. We introduce a new class of algebras, contour algebras, generalising (and interpolating between) blob algebras and cyclotomic Temperley-Lieb algebras. We demonstrate the utility of our formalism by applying it to this class.

Introduction

Let A be a finite-dimensional algebra and $e \in A$ be an idempotent. The category eAe-mod is fully embedded in A-mod and the remaining simples L for A are characterised by $eL = 0$. In particular, we have an exact 'localisation' functor

$$\begin{array}{ccc} F : A\text{-mod} & \to & eAe\text{-mod} \\ M & \longmapsto & eM \end{array}$$

which takes simples to simples or zero. Indeed, every simple eAe-module arises in this way:

Theorem (Green [Gre80]). *Let $\{L(\lambda) : \lambda \in \Lambda\}$ be a full set of simple A-modules, and set $\Lambda^e = \{\lambda \in \Lambda : eL(\lambda) \neq 0\}$. Then $\{eL(\lambda) : \lambda \in \Lambda^e\}$ is a full set of simple eAe-modules. Further, the simple modules $L(\lambda)$ with $\lambda \in \Lambda \backslash \Lambda^e$ are a full set of simple A/AeA-modules.*

We define the globalisation functor by

$$G : N \longmapsto Ae \otimes_{eAe} N$$

and note that $FG(N) \cong N$ and G is a full embedding. Cline, Parshall and Scott [CPS88] use this idea to provide examples of recollement [BBD82] in the context of quasi-heredity and highest weight categories. Following an application to the Temperley-Lieb algebra in [Mar91], the second author and Saleur then used it for the tower $b_1 \subset b_2 \subset \ldots$ of blob algebras [MS94], for which there exist idempotents $e_n \in b_n$ such that $e_n b_n e_n \cong b_{n-2}$, to recursively analyse the representation theory of the entire tower.

There are in fact a significant number of interesting towers of algebras with such idempotents, particularly among algebras equipped with a diagram calculus and algebras arising in invariant theory [Bra37, Bro55, Mar91, Mar94, Jon97, Gre98, Maz98, Maz02, Blo03, ME, RX]. In Section 1 we abstract and formalise aspects of the common procedure used to analyse such towers of algebras in [MS94, Mar94] (while largely avoiding the explicit construction of bases). In Sections 2 and 3 we demonstrate the utility of this formalism by applying it to a new class of diagram algebras, the contour algebras. This is a collection of towers of algebras which includes as special cases the Temperley-Lieb algebras and blob algebras, and the cyclotomic Temperley-Lieb algebras recently defined by Rui and the last author [RX]. The formalism allows us to index simple modules very

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easily, to construct standard modules, and to locate many standard module morphisms efficiently. In Section 4 we carry out the algebra-specific calculations required by our formalism. Finally in Section 5 we return to a discussion of our axiom scheme. We explore the consequences of modifying our axioms at various points, and the relationship between them and other such exercises in the literature.

Our notion of a tower of recollement combines certain ideas from the tower formalism in [GHJ89] (but relaxing the emphasis on semisimplicity) with the notion of recollement in [CPS88]. (The latter is a special case of the general notion of recollement in [BBD82].) We only make explicit one of the two defining functors in a recollement diagram; the other is implicit in this approach (see [CPS88, Section 2]) but not needed in this paper.

Although we will make no use of it in what follows, it is worth remarking on the physics that originally drove this approach. These algebras (over $\mathbb{C}$) are transfer matrix algebras in the sense of [Mar91]. The physical context naturally brings two properties into play. First that the algebras arise as a tower (corresponding to different physical system sizes), and second that their module categories embed in each other (corresponding to the thermodynamic limit). It is the interplay between these two ways of passing through the tower that lies at the heart of our axiomatisation.

1. Towers of recollement

Let A_n (with $n \geqslant 0$) be a family of finite-dimensional algebras, with idempotents e_n in A_n. For simplicity we shall assume that A_n is defined over an algebraically closed field k. We will impose a series of restrictions on such algebras sufficient for an analysis of their representation theory along the lines of that carried out in [MS94]. The rationale for introducing axioms (A1–6), which now follow, is that they allow us to inductively classify the simple A_n-modules, and to determine which of the algebras in the family are semisimple (along with lots of homological data when they are not), with only a minimum of calculations.

We first assume

(A1) *For each $n \geqslant 2$ we have an isomorphism*

$$\Phi_n : A_{n-2} \longrightarrow e_n A_n e_n.$$

With this assumption we define a pair of families of functors $F_n : A_n\text{-mod} \to A_{n-2}\text{-mod}$ and $G_n : A_n\text{-mod} \to A_{n+2}\text{-mod}$ as in the introduction. That is, $F_n(M) = e_n M$ and $G_{n-2}(N) = A_n e_n \otimes_{e_n A_n e_n} N$ (where in each case we are using the isomorphism in (A1)). Note that the right inverse to F_n is G_{n-2}.

Denote the indexing set for the simple A_n-modules by Λ_n, and that for the simple $A_n/A_n e_n A_n$-modules by Λ^n. Then by (A1) and the Theorem in the introduction we have

$$\Lambda_n = \Lambda^n \sqcup \Lambda_{n-2} \tag{1}$$

and hence, provided that Λ_0, Λ_1 and Λ^n are known, this immediately allows the simple modules for each A_n to be classified by induction. We will illustrate this by providing a very short proof of the classification of simple modules for the contour algebras in Corollary 2.8.

By (1) we may regard Λ_n as a subset of Λ_{n+2}, and set $\Lambda = (\lim_n \Lambda_{2n}) \sqcup (\lim_n \Lambda_{2n+1})$. We call elements of Λ *weights*. For $m, n \in \mathbb{N}$ with $m - n$ even we set $\Lambda^n_m = \Lambda^n$ *regarded as a subset of* Λ_m if $m \geqslant n$, and $\Lambda^n_m = \emptyset$ otherwise.

Set $e_{n,0} = 1$ in A_n, and for $1 \leqslant i \leqslant \frac{n}{2}$ define new idempotents in A_n by setting $e_{n,i} = \Phi_n(e_{n-2,i-1})$. To these elements we associate corresponding quotients of A_n by setting $A_{n,i} = A_n/(A_n e_{n,i+1} A_n)$.

It will be convenient to have the machinery of quasi-heredity at our disposal. For this reason we next assume

(A2) *(i) The algebra $A_n/A_ne_nA_n$ is semisimple.*
(ii) For each $n \geqslant 0$ and $0 \leqslant i \leqslant \frac{n}{2}$, setting $e = e_{n,i}$ and $A = A_{n,i}$, the surjective multiplication map $Ae \otimes_{eAe} eA \to AeA$ is a bijection.

Note that condition (i) (with (A1)) implies that $e_{n,i}A_{n,i}e_{n,i}$ is semisimple for all $n \geqslant 0$ and $0 \leqslant i \leqslant \frac{n}{2}$. We have chosen to state (A2) in the form above to emphasise the elementary nature of the condition (and because this is the form in which it will be verified, which is an entirely routine matter in specific algebras, as we will exemplify in Proposition 2.9). However, by [DR89, Statement 7] (or [Mar93, Definition 3.3.1 and the remarks following]), it is straightforward to verify that we could replace (A2) by

(A2′) *For each $n \geqslant 0$ the algebra A_n is quasi-hereditary, with heredity chain of the form*

$$0 \subset \cdots \subset A_ne_{n,i}A_n \subset \cdots \subset A_ne_{n,0}A_n = A_n.$$

As A_n is quasi-hereditary, there is for each $\lambda \in \Lambda_n$ a standard module $\Delta_n(\lambda)$, with simple head $L_n(\lambda)$. If we set $\lambda \leqslant \mu$ if either $\lambda = \mu$ or $\lambda \in \Lambda_n^r$ and $\mu \in \Lambda_n^s$ with $r > s$, then all other composition factors of $\Delta_n(\lambda)$ are labelled by weights μ with $\mu < \lambda$. Note that for $\lambda \in \Lambda_n^n$, we have that $\Delta_n(\lambda) \cong L_n(\lambda)$, and that this is just the lift of a simple module for the quotient algebra $A_n/A_ne_nA_n$. Arguing as in [MRH, Proposition 3] we see that

$$G_n(\Delta_n(\lambda)) \cong \Delta_{n+2}(\lambda). \tag{2}$$

Similarly (see for example [Don98, A1]) we have

$$F_n(\Delta_n(\lambda)) \cong \begin{cases} \Delta_{n-2}(\lambda) & \text{if } \lambda \in \Lambda_{n-2} \\ 0 & \text{if } \lambda \in \Lambda^n. \end{cases} \tag{3}$$

Crucially we impose a *second* way of passing through the family of algebras:

(A3) *For each $n \geqslant 0$ the algebra A_n can be identified with a subalgebra of A_{n+1}.*

The other main tool we wish to use, then, is Frobenius reciprocity. For this we will need to have certain controls over induction and restriction for our families of modules. Essentially, we want these to have a *local behaviour* and be *compatible* with globalisation, in a sense we now describe.

If a module M in A_n-mod has a Δ_n-filtration (i.e. a filtration with successive quotients isomorphic to some $\Delta_n(\lambda_i)$'s) we define the *support* of M, denoted $\mathrm{supp}_n(M)$, to be the set of labels λ for which $\Delta(\lambda)$ occurs in this filtration. (As standard modules form a basis for the Grothendieck group of a quasi-hereditary algebra, this is well-defined.) We will also need to consider the restriction functor $\mathrm{res}_n : A_n\text{-mod} \to A_{n-1}\text{-mod}$ and the induction functor $\mathrm{ind}_n : A_n\text{-mod} \to A_{n+1}\text{-mod}$ given by $\mathrm{ind}_n(M) = A_{n+1} \otimes_{A_n} M$. We will omit suffixes from supp_n, ind_n, res_n and Δ_n-filtration whenever this is unambiguous. Our next three assumptions ensure that induction and restriction behave well in this setting.

(A4) *For all $n \geqslant 1$ we have that $A_ne_n \cong A_{n-1}$ as a left A_{n-1}–, right A_{n-2}–bimodule.*

We can immediately deduce from (A4) that for each $\lambda \in \Lambda_n$ we have that

$$\mathrm{res}(G_n(\Delta_n(\lambda))) \cong \mathrm{ind}\Delta_n(\lambda). \tag{4}$$

(A5) *For each* $\lambda \in \Lambda_n^m$ *we have that* $\mathrm{res}(\Delta_n(\lambda))$ *has a* Δ*-filtration and*

$$\mathrm{supp}(\mathrm{res}(\Delta_n(\lambda)) \subseteq \Lambda_{n-1}^{m-1} \sqcup \Lambda_{n-1}^{m+1}.$$

Equation (4) now implies the analogue of (A5) for induction. Using (2) we deduce from (A5) and (4) that for each $\lambda \in \Lambda_n^m$ the module $\mathrm{ind}(\Delta_n(\lambda))$ has a Δ-filtration, and

$$\mathrm{supp}(\mathrm{ind}(\Delta_n(\lambda)) \subseteq \Lambda_{n+1}^{m-1} \sqcup \Lambda_{n+1}^{m+1}. \tag{5}$$

(A6) *For each* $\lambda \in \Lambda_n^m$ *there exists* $\mu \in \Lambda_{n-1}^{m-1}$ *such that* $\lambda \in \mathrm{supp}(\mathrm{ind}\Delta_{n-1}(\mu))$.

In the presence of (A5) this is equivalent to

(A6′) *For each* $\lambda \in \Lambda_n^m$ *there exists* $\mu \in \Lambda_{n+1}^{m-1}$ *such that* $\lambda \in \mathrm{supp}(\mathrm{res}\Delta_{n+1}(\mu))$.

For a quasi-hereditary algebra we have that $\mathrm{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0$ implies that $\lambda < \mu$. Therefore (A6) is equivalent to the requirement that for each $\lambda \in \Lambda_n$ there exists $\mu \in \Lambda_{n-1}$ such that there is a surjection

$$\mathrm{ind}\Delta_{n-1}(\mu) \to \Delta_n(\lambda) \to 0. \tag{6}$$

We shall call a family of algebras satisfying (A1–6) a *tower of recollement*, since it broadly combines ideas from [GHJ89] and [CPS88] as discussed in the Introduction.

The axiomatic framework introduced so far is sufficient to reduce the study of various general homological problems to certain explicit calculations, as illustrated by

Theorem 1.1. *(i) For all pairs of weights* $\lambda \in \Lambda_n^m$ *and* $\mu \in \Lambda_n^l$ *we have*

$$\mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) \cong \begin{cases} \mathrm{Hom}(\Delta_m(\lambda), \Delta_m(\mu)) & \text{if } l \leqslant m \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Suppose that for all $n \geqslant 0$ *and pairs of weights* $\lambda \in \Lambda_n^n$ *and* $\mu \in \Lambda_n^{n-2}$ *we have*

$$\mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) = 0.$$

Then each of the algebras A_n *is semisimple.*

Proof. For (i) we first note that quasi-heredity implies that for any non-zero Hom-space between standard modules as above we must have $\lambda \leqslant \mu$, and hence we may assume that $l \leqslant m$. As each G_n is a full embedding, any non-zero homomorphism between standard modules $\Delta_n(\lambda)$ and $\Delta_n(\mu)$ corresponds to a morphism between some pair of standards $\Delta_m(\lambda)$ and $\Delta_m(\mu)$ with $\lambda \in \Lambda_m^m$.

For (ii) we will proceed by induction on n. Recall that in a quasi-hereditary algebra, the standard module $\Delta(\lambda)$ is defined to be the largest quotient of the projective cover $P(\lambda)$ of $L(\lambda)$ with the property that all of its composition factors $L(\mu)$ satisfy $\mu \leqslant \lambda$. For semisimplicity it is enough to show that all the $P(\lambda)$ are simple. For any finite dimensional module M we have $\dim \mathrm{Hom}(P(\lambda), M) = [M : L(\lambda)]$, the multiplicity of $L(\lambda)$ as a composition factor of M. Hence it is enough to show that $\mathrm{Hom}(P(\lambda), P(\mu)) = 0$ for $\mu \neq \lambda$. As $P(\lambda)$ has a filtration by standard modules, it is enough to show that $\mathrm{Hom}(\Delta(\lambda), \Delta(\mu)) = 0$ for $\mu \neq \lambda$.

Suppose that λ and μ are such that $\mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) \neq 0$. Then in the order induced by quasi-heredity we must have $\lambda \leqslant \mu$; i.e. either $\lambda \in \Lambda_n^r$ and $\mu \in \Lambda_n^s$ with $r > s$, or $\lambda = \mu$. In the latter case quasi-heredity implies that $\mathrm{End}(\Delta_n(\lambda)) \cong k$, and so we may assume that $r > s$.

If $r < n$ then $F_n\Delta_n(\lambda) \cong \Delta_{n-2}(\lambda)$ and $F_n\Delta_n(\mu) \cong \Delta_{n-2}(\mu)$. Further, as $\Delta_n(\lambda)$ has simple head $L_n(\lambda)$ which is not killed by F_n, any non-zero homomorphism from $\Delta_n(\lambda)$ to $\Delta_n(\mu)$ survives under F_n. Hence, as A_{n-2} is semisimple, there are no non-zero morphisms between $\Delta_n(\lambda)$ and $\Delta_n(\mu)$.

Thus we may assume that $r = n$ and $s < n$. Then by (6) there exists a weight $\tau \in \Lambda_{n-1}$ such that $\mathrm{ind}\Delta_{n-1}(\tau) \to \Delta_n(\lambda) \to 0$, and by (5) we have that $\tau \in \Lambda_{n-1}^{n-1}$. Now we have an injection

$$0 \to \mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) \to \mathrm{Hom}(\mathrm{ind}\Delta_{n-1}(\tau), \Delta_n(\mu))$$

and by Frobenius reciprocity we have

$$\mathrm{Hom}(\mathrm{ind}\Delta_{n-1}(\tau), \Delta_n(\mu)) \cong \mathrm{Hom}(\Delta_{n-1}(\tau), \mathrm{res}\Delta_n(\mu)).$$

By (A3) and the semisimplicity of A_{n-2} we have that

$$\mathrm{res}\Delta_n(\mu) \cong (\oplus_i \Delta_{n-1}(\nu_i)) \oplus (\oplus_j \Delta(\nu_j))$$

for some $\nu_i \in \Lambda_{n-1}^{s-1}$ and $\nu_j \in \Lambda_{n-1}^{s+1}$, and hence

$$\mathrm{Hom}(\Delta_{n-1}(\tau), \mathrm{res}\Delta_n(\mu)) \cong \mathrm{Hom}(\Delta_{n-1}(\tau), (\oplus_i \Delta_{n-1}(\nu_i)) \oplus (\oplus_j \Delta(\nu_j))).$$

By semisimplicity, this Hom-space is zero unless $s + 1 = n - 1$, i.e. unless $s = n - 2$. Thus we have reduced to considering the case $r = n$ and $s = n - 2$ as required. □

Note that the test for semisimplicity in the second part of this Theorem is typically a tractable algebra-specific calculation. This is because for any A_n satisfying (A2) (with λ and μ as above) both $\Delta(\lambda)$ and $\Delta(\mu)$ have few composition factors (indeed the former is a simple module). Thus the determination of homomorphisms between them will in many cases be a tractable algebra-specific calculation.

It will be convenient to note the following property of algebras satisfying (A1). Let $m < n$ with $m - n = 2i$ for some $i \in \mathbb{N}$. Then by the remarks after (A1) we have that $A_m \cong e_{n,i} A_n e_{n,i}$. There is a corresponding globalisation functor, which we denote G_m^n, given by $G_m^n(N) = A_n e_{n,i} \otimes_{e_{n,i} A_n e_{n,i}} (N)$ for all A_m-modules N. It is now an elementary exercise to verify that

$$G_m^n(N) \cong G_{n-2} G_{n-4} \dots G_m(N) \tag{7}$$

for all A_m-modules N.

The value of this axiom scheme hangs on there being a large number of concrete algebras to which it applies. We will illustrate the utility of the theory by applying it to the contour algebra in Section 2. First though we briefly sketch some other examples of its usefulness from the literature.

Examples 1.2. (i) The Temperley-Lieb algebra. See [Mar91] and [CGM03] for details. In this case the indexing set is $\Lambda_n = \{n, n-2, n-4 \dots, 0 \text{ or } 1\}$ and $\Lambda^n = \{n\}$. We have a short exact sequence

$$0 \to \Delta_{n-1}(i-1) \to \mathrm{res}\Delta_n(i) \to \Delta_{n-1}(i+1) \to 0$$

for $0 \leqslant i < n$, and $\mathrm{res}\Delta_n(n) \cong \Delta_{n-1}(n-1)$, and similar sequences for $\mathrm{ind}\Delta_n(i)$.

(ii) The blob algebra b_n was introduced in [MS94], and an analysis of the form described above first carried out (in characteristic zero) in [MW00]. These results were later generalised to positive characteristic in [CGM03]. In particular (A1) is proved in [MS94, Proposition 3], (A2) in [MW00, (3.2)], (A3) is obvious, (A4) in [MS94, Proposition 2], (A5) and (A6) in [MW00, (3.4) Proposition and (8.2) Theorem] (see also [MRH, Proposition 3]).

In this case the indexing set $\Lambda_n = \{n, n-2, n-4, \dots, 2-n, -n\}$ with $\Lambda^n = \{\pm n\}$. We have a short exact sequence

$$0 \to \Delta_{n-1}(i \mp 1) \to \mathrm{res}\Delta_n(i) \to \Delta_{n-1}(i \pm 1) \to 0$$

for $0 \leqslant i < n$ respectively $-n < i < 0$, and $\mathrm{res}\Delta_n(\pm n) \cong \Delta_{n-1}(\pm n \mp 1)$. There are similar sequences for $\mathrm{ind}\Delta_n(i)$.

(iii) The partition algebra was introduced in [Mar94]. In this case the application of the theory in this section is a little more involved, as the tower of algebras interleaves partition algebras with auxilliary intermediates. Details can be found in [Mar00].

(iv) Certain planar algebras — for example planar algebras on 1-boxes (see [Jon97, Section 2.2]). Planar algebras were introduced by Jones in [Jon97] formalising and generalising the treatment of the Temperley-Lieb algebra suggested in [Mar91, Section 6.2] (and implemented in [Mar94] in the non-planar setting). The verification of the axioms in this case is left as an exercise (but see below).

2. Contour algebras

In this section we define a new class of algebras, the contour algebras $X^d_{n,m}$, over a general ring R. We then apply the general theory developed in the preceding section. As we will need to consider several different algebras, in this section we will denote the index set for the simple modules for an algebra A by $\Lambda(A)$.

We will be interested in two classes of decorated Temperley-Lieb diagrams: arrow diagrams and bead diagrams. By an *arrow diagram* we mean a rectangular box containing non-intersecting line segments, possibly with one or more arrows on each line (see Figure 1). A *bead diagram* is similar but with unoriented beads instead of arrows.

It will be convenient to recall some standard terminology for ordinary (undecorated) Temperley-Lieb diagrams which will also be needed here. We refer to the dotted boundary of a diagram as its *frame* and the interior line-segments as *lines*. Lines in a diagram are called *propagating lines* if they connect the northern and southern edges of the frame, and *northern* (respectively *southern*) *arcs* if they meet only the northern (respectively southern) edge of the frame. The endpoints of lines are called *nodes*. We identify two diagrams if they differ by an (edgewise) frame-preserving ambient isotopy. If the number of southern nodes in A equals the number of northern nodes in B then we define the product AB to be the concatenation of the diagram A above the diagram B. (In the product of two diagrams AB we assume that the southern nodes of A are identified with the corresponding northern nodes of B, and ignore the dotted line segment formed by their frames across the centre of the new diagram. Then AB is another diagram.)

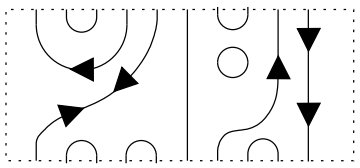

Figure 1.

We say that a line in a diagram is *of depth* 1 (or *exposed*) if the diagram can be deformed ambient isotopically such that the line touches the eastern edge of the frame. We now define the depth of a general line inductively by saying that a line is of *depth* d if it is not of depth less than d but can be deformed ambient isotopically to touch a line of depth $d-1$. We say that a diagram is *decorated to depth* d if all decorated lines in the diagram are of depth at most d. For example, the diagram illustrated in Figure 1 is decorated to depth 5, and indeed to depth d for any $d > 5$.

An arrow assigns an orientation to a line. We say that two arrows on the same line are *opposing* if they assign opposite orientations to the line. An arrow on a northern or southern arc is called *easterly* (respectively *westerly*) if it point towards the eastern (respectively western) end of the line. Similarly arrows on propagating lines are either *northerly* or *southerly*.

Let $\bar{D}^l_n$ be the set of bead diagrams with l northern and n southern nodes, and $\bar{D}_n = \bar{D}^n_n$. The corresponding subsets of diagrams decorated to depth d will be denoted $\bar{D}^l_n[d]$ and $\bar{D}_n[d]$ respectively. Note that in the composition of any two diagrams we may expose new line segments but cannot produce new unexposed lines. Clearly similar remarks hold for lines of depth at most d, and hence

we have

LEMMA 2.1. *The diagram product gives a map from* $\bar{D}_n^l[d] \times \bar{D}_m^n[d]$ *to* $\bar{D}_m^l[d]$.

Another way to think of this is that the lines in a diagram are contours (or isobars) and that under composition non-closed lines can be combined to become closed contours. Fixing the eastern edge at sea-level, the maximum physical height a contour can realise on closure is its diagram depth. Thus depth cannot be increased by composition.

Fix m, and choose elements $\delta_0, \ldots, \delta_{m-1}$ in R. By Lemma 2.1 we may define the *contour algebra* $\bar{X}_{n,m}^d = \bar{X}_{n,m}^d(\delta_0 \ldots, \delta_{m-1})$ to be the algebra obtained from $R\bar{D}_n[d]$ under concatenation with the following additional relations:

(i) A diagram with m beads on the same line is identified with the same diagram with the beads omitted.

(ii) A diagram with an excess (modulo m) of k beads on a given closed loop is identified with δ_k times the same diagram with the closed loop omitted.

It is evident that $\bar{X}_{n,m}^d$ is associative, unital, and free as an R-module.

We denote by $\bar{X}_{n,m}^\infty$ the case where we allow decorated lines of arbitrary depth. Clearly we have that $\bar{X}_{n,m}^\infty \cong \bar{X}_{n,m}^n$, and for general d that $\bar{X}_{n,m}^d \subseteq \bar{X}_{n,m}^{d+1}$.

There is another presentation of these algebras in terms of arrow diagrams. Let D_l^n be the set of arrow diagrams with l northern and n southern nodes, and define sets D_n, $D_n^l[d]$, and $D_n[d]$ as in the corresponding bead cases. Now we define the algebra $X_{n,m}^d$ $(= X_{n,m}^d(\delta_0, \ldots, \delta_{m-1}))$ to be the algebra obtained from $RD_n[d]$ under concatenation with the following additional relations:

(i) A diagram with two opposing arrows on the same line is identified with the same diagram with the two arrows omitted.

(ii) A diagram with m non-opposing arrows on the same line is identified with the same diagram with the arrows omitted.

(iii) A diagram with an excess (modulo m) of k anti-clockwise arrows over clockwise arrows on a given closed loop is identified with δ_k times the same diagram with the closed loop omitted.

These three sets of relations are illustrated schematically in Figure 2.

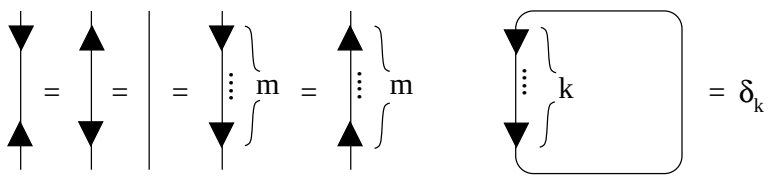

FIGURE 2.

It will be convenient to have names for certain diagrams. It is clear that the algebra $X_{n,m}^d$ is generated by the elements $E_n(i)$ (for $1 \leqslant i \leqslant n-1$) and $T_n(i)$ (for $\max(1, n+1-d) \leqslant i \leqslant n$) illustrated in Figure 3. Note that $E_n(i)^2 = \delta_0 E_n(i)$. The analogue of $T_n(i)$ with a bead instead of an arrow will be denoted $\bar{T}_n(i)$.

PROPOSITION 2.2. *The map*

$$E_n(i) \longmapsto E_n(i) \quad \text{and} \quad T_n(i) \longmapsto \begin{cases} \bar{T}_n(i) & \text{if } i \text{ odd} \\ (\bar{T}_n(i))^{m-1} & \text{if } i \text{ even} \end{cases}$$

induces an algebra isomorphism from $X_{n,m}^d$ *to* $\bar{X}_{n,m}^d$.

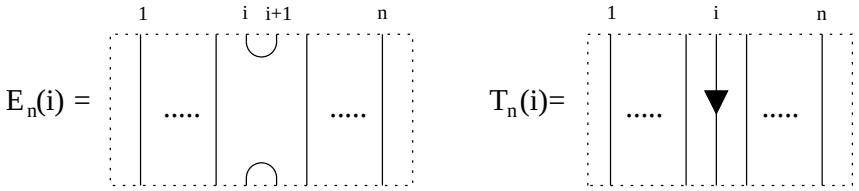

Figure 3.

Proof. This is an easy exercise. For example, any pair of opposing (respectively non-opposing) arrows must arise (from these generators) from a pair of elements $T_n(i)$ and $T_n(j)$ with $i-j$ odd (respectively even). □

Because of Proposition 2.2 we will henceforth also refer to $X^d_{n,m}$ as the contour algebra.

Remark 2.3. The algebra $X^0_{n,m}$ coincides with the Temperley-Lieb algebra (for any d), while $X^1_{n,2}$ is isomorphic to the blob algebra and $X^1_{n,m}$ to the coloured blob algebra introduced in [MWL00]. By comparing the arrow definition with that in [RX, Definition 3.3] it is easy to show that $X^\infty_{n,m}$ is isomorphic to the cyclotomic Temperley-Lieb algebra $\tilde{TL}_{n,m}$ introduced by Rui and Xi (which are planar algebras on 1-boxes). The algebras $X^d_{n,m}$ with $1 < d < n$ are new.

Henceforth we take $R = k$, an algebraically closed field. We will show that, with some conditions on the characteristic of k and the parameters δ_i, the algebras $X^d_{n,m}$ satisfy (A1–A6).

Proposition 2.4. *For $\delta_0 \neq 0$ we have*

$$E_n(1)X^d_{n,m}E_n(1) \cong X^d_{n-2,m}.$$

Proof. Any diagram $E_n(1)DE_n(1)$ in $E_n(1)X^d_{n,m}E_n(1)$ is of the form shown on the lefthand side of Figure 4, and can be put into the form on the righthand side of the Figure for some diagram D' in $X^d_{n-2,m}$. As $\delta_0 \neq 0$, the set of diagrams of the form shown on the righthand side defines an algebra isomorphic to $X^d_{n-2,m}$, via the map which sends $E_n(1)DE_n(1)$ to $\delta_0 D'$. □

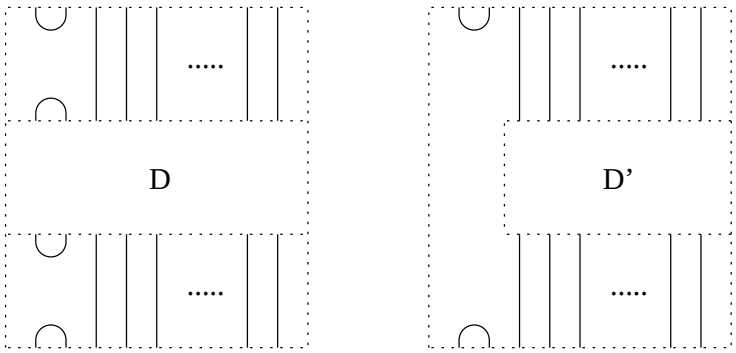

Figure 4.

This verifies (A1) when $\delta_0 \neq 0$. An analogous result can be obtained under the weaker assumption that there exists some j with $\delta_j \neq 0$. For this we argue as above, but replace every occurrence of $E_n(1)$ with the same diagram decorated with j westerly arrows on the southern arc. Henceforth we assume that there exists some $\delta_j \neq 0$, fix m, and denote $X^d_{n,m}$ by A_n. In proofs we will suppose that $\delta_0 \neq 0$ and denote $\delta_0^{-1}E_n(1)$ by e_n. The modifications for the general case are exactly as for Proposition 2.4 above.

We define the *propagating number* of a diagram D to be the number of propagating lines in D. Let $D_n[d;i]$ denote the subset of $D_n[d]$ consisting of diagrams with propagating number i. Note that there is a unique undecorated diagram with no closed loops in $D_n[d;n]$, which is the identity element in A_n. All other diagrams in $D_n[d;n]$ have the same underlying undecorated diagram, but with additional arrows and/or closed loops. The set $D_n[d;i]$ is not linearly independent, so we define $D_n^+[d;i]$ to be the subset of diagrams in $D_n[d;i]$ with no closed loops, no more than $m-1$ arrows on any single line, and all arrows either westerly or southerly. We set $D_n^+[d]$ to be the union of the $D_n^+[d;i]$. It is easy to see that such diagrams are linearly independent, and further that (after applying the defining relations) the composition of diagrams restricts to a map from $D^+[d] \times D^+[d]$ to $R \times D^+[d]$.

Let kC_m be the group algebra over k of the cyclic group of order m. As $T_n(i)^m = 1$, the element $T_n(i)$ generates a copy of kC_m.

Remark 2.5. It is a triviality to construct an enumerated basis of $X_{n,m}^d$ which coincides with the finite set $D_n^+[d]$, using the technique of [MS94, Proposition 2]. As in all the diagram algebras mentioned in Section 1, this construction exhibits bases for certain submodules of $RD_n^+[d]$ (regarded as the regular representation). It shows explicitly that the sum of squares of the ranks of these submodules is the rank of $X_{n,m}^d$. These modules coincide, in quasi-hereditary specialisations to be discussed shortly, with the standard modules considered in Section 3.

Suppose that $\delta_0 \neq 0$, and consider the filtration of A_n by two-sided ideals

$$\ldots \subset A_n E_n(1) E_n(3) A_n \subset A_n E_n(1) A_n \subset A_n. \tag{8}$$

We will denote the product $\prod_{j=1}^{i} E_n(2j-1)$ by $E_{n,i}$. As $\delta_0 \neq 0$ this is a preidempotent (i.e. a non-zero scalar multiple of an idempotent), and we define $e_{n,i}$ to be the corresponding idempotent $\delta_0^{(-i)} E_{n,i}$. The corresponding constructions for $\delta_j \neq 0$ are obvious.

PROPOSITION 2.6. *The i^{th} section*

$$A_n e_{n,i} A_n / A_n e_{n,i+1} A_n$$

in this filtration has basis $D_n^+[d; n-2i]$.

Proof. This is straightforward — confer [MS94, Corollary 1.1]. □

In particular we have

COROLLARY 2.7.

$$A_n / A_n e_n A_n \cong (kC_m)^{\min(n,d)}.$$

A parameterisation of the simple modules of A_n now follows immediately from (1):

COROLLARY 2.8. *Suppose that there exists some j with $\delta_j \neq 0$. Then for all $n \geqslant 0$ we have*

$$\Lambda(X_{n,m}^d) \;=\; \Lambda(X_{n-2,m}^d) \sqcup (\Lambda(kC_m))^{\min(n,d)} \;=\; \coprod_{i=n,n-2,..,1/0} (\Lambda(kC_m))^{\min(i,d)}.$$

The representation theory of $(kC_m)^n$ is well understood. For example, if k is a splitting field of $x^m - 1$ of characteristic p such that $p = 0$ or p does not divide m, then the set $\{1, 2, .., m\}$ may be taken as an index set $\Lambda(kC_m)$ for the simples of kC_m over k, and $\Lambda((kC_m)^n) = (\Lambda(kC_m))^n$. In the special case $d = \infty$ this provides a very short proof of [RX, Corollary 5.4].

Note that the restriction rules for $(kC_m)^r$ to $(kC_m)^{r-1}$ are elementary. This will facilitate verification of (A5) shortly.

Before going on to consider quasi-heredity, we quickly note that (A3) and (A4) are both easily verified. For (A3) we can identify A_n as a subalgebra of A_{n+1} via the map which adds an undecorated

propagating line to the lefthand side of each diagram. For (A4), note that the left action of A_{n-1} is by concatenation from above on the rightmost $n-1$ strings, while the right action of A_{n-2} is by concatenation from below on the rightmost $n-2$ strings. We define a map from a diagram in $A_n e_n$ to a diagram in A_{n-1} by first deforming the original diagram ambient isotopically to move the leftmost northern node anticlockwise around the frame to become the leftmost southern node, and then removing the southern arc adjacent to this new node. An example of this is given in Figure 5, where the effect of the map on the lefthand diagram is illustrated on the right. (The shaded areas indicate the nodes acted on by the actions from above and below.) It is easy to verify that this map gives the desired left A_{n-1}-, right A_{n-2}- bimodule isomorphism.

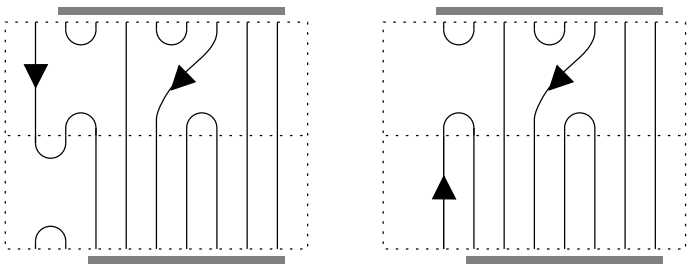

Figure 5.

We next verify (A2).

Proposition 2.9. *Suppose that there exists some j with $\delta_j \neq 0$, and that either $p = 0$ or p does not divide m. Then for all $n \geqslant 0$ the algebra A_n is quasi-hereditary, with heredity chain of the form given in (8).*

Proof. We consider the case $j = 0$, when the heredity chain will be precisely the chain in (8). For arbitrary j we must replace each $E_n(i)$ with the appropriately decorated analogue introduced after Proposition 2.4.

We wish to show that the filtration in (8) is a heredity chain for A_n; i.e. that each of the quotients $(A_n e_{n,i} A_n)/(A_n e_{n,i+1} A_n)$ is a heredity ideal of $A_{n,i} = (A_n)/(A_n e_{n,i+1} A_n)$. For this it is enough to show that the conditions (A2)(i) and (ii) both hold.

Condition (i) follows immediately from Corollary 2.7 and our assumptions on p. For (ii), we begin by noting that $A_{n,i} e_{n,i}$ has a basis represented by those diagrams with i non-nested southern arcs on the $2i$ westernmost vertices, and $n-2i$ propagating lines (possibly with decorations). We have a similar basis for $e_{n,i} A_{n,i}$ with northern instead of southern arcs. Thus the product D of such a diagram in $A_{n,i} e_{n,i}$ with such a diagram in $e_{n,i} A_{n,i}$ must have precisely $n-2i$ propagating lines, and it is clear that any pair of diagrams giving rise to D must be equivalent in $A_{n,i} e_{n,i} \otimes_{e_{n,i} A_{n,i} e_{n,i}} e_{n,i} A_{n,i}$. (To see this note that such pairs of diagrams can only differ in the distribution of decorations between them, which can be adjusted via an element of $e_{n,i} A_{n,i} e_{n,i}$.)

Thus we have verified (A2)(i) and (ii), and hence A_n is quasi-hereditary. □

3. Representations of contour algebras

Henceforth we will assume that A_n satisfies the conditions of Proposition 2.9. Then by the general theory in Section 1, every standard module $\Delta_n(\lambda)$ of A_n is the image under $G_{n-2} G_{n-4} \dots G_{n-2i}$ of some standard module for $(kC_m)^j$ lifted to A_j, for some $i, j \geqslant 0$ with $2i + j = n$. (We adopt the convention that $(kC_m)^0 = k$, with simple module labelled by $\emptyset$.) We call j the propagating number of λ. Thus we need to fix our convention for lifting modules from $(kC_m)^n$ to A_n.

We fix ν, a primitive mth root of unity, and define the element $\epsilon_n(i,j) = \sum_{t=0}^{m-1} \nu^{it} T_n(j)^t$ in A_n (where $T_n(j)^0 = 1_{A_n}$). Note that this element is a preidempotent: we have $(m^{-1}\epsilon_n(i,j))^2 = m^{-1}\epsilon_n(i,j)$. Graphically we represent $\epsilon_n(i,j)$ as shown in Figure 6 and refer to its decoration as $\bullet(i)$.

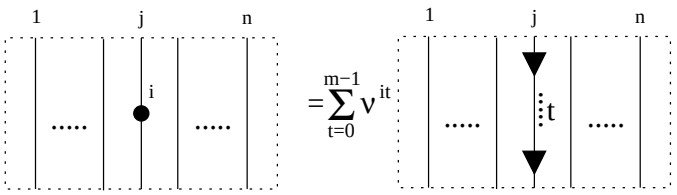

FIGURE 6.

Now the simple module labelled by $(i_1, \ldots, i_n)$ for $(kC_m)^n$ can be realised as an A_n-module (via Corollary 2.7) as the module $A_n\epsilon_n(i_1,1)\ldots\epsilon_n(i_n,n)$, with the convention that we identify any diagram with fewer than n propagating lines with zero. There is an obvious extension of the graphical notation for $\epsilon_n(i,j)$, where we represent $\epsilon_n(i_1,1)\ldots\epsilon_n(i_n,n)$ by the corresponding product of the diagrams for each $\epsilon_n(i,j)$.

By the general theory in Section 1 we have for $n > l$ with $n - l$ even that

$$\Delta_n(i_1,\ldots,i_l) \cong G_l^n \Delta_l(i_1,\ldots,i_l) \cong A_n e_{n,t} \otimes_{e_{n,t}A_n e_{n,t}} A_l \epsilon_l(i_1,1)\ldots\epsilon_l(i_l,l)$$

where $t = \frac{n-l}{2}$. Let $D_l^n(i_1,\ldots,i_l)$ denote the set of diagrams with n northern and l southern nodes, l propagating lines and no closed loops, such that the jth propagating line is decorated with $\bullet(i_j)$. Let $\Delta'_n(i_1,\ldots,i_l)$ denote the A_n-module with basis $D_l^n(i_1,\ldots,i_l)$, where the action of A_n is by concatenation from above, such that any product of diagrams with fewer than l propagating lines is set to zero. It will be evident that a fixed distribution of southern arcs could be added to every diagram without changing the action, and hence we have

PROPOSITION 3.1. *The modules $\Delta_n(i_1,\ldots,i_l)$ and $\Delta'_n(i_1,\ldots,i_l)$ can be identified.*

We now consider (A5) and (A6). We first note that there is an A_{n-1}-submodule of $\Delta_n(i_1,\ldots,i_l)$ (as a diagram module) spanned by those diagrams with a propagating line from the most westerly northern node is isomorphic to $\Delta_{n-1}(\mu)$ where $\mu = (i_2,\ldots,i_l) \in \Lambda_{n-1}^{l-1}$. (This is clear, as A_{n-1} acts on all but the most westerly northern node.)

All remaining diagrams in $\Delta_n(i_1,\ldots,i_l)$ have a northern arc starting at the most westerly northern node. We consider a new basis for this set formed by taking linear combinations of diagrams such that this northern arc is decorated with a $\bullet(i)$ for some i, as illustrated in the left-hand diagram in Figure 7 (where the shaded region denotes some collection of lines whose precise configuration does not concern us). If we take the subset of such diagrams with fixed decoration $\bullet(i)$ then, modulo the submodule $\Delta_{n-1}(\mu)$ described above, there is an A_{n-1}-module isomorphism with $\Delta_{n-1}(\nu)$ (where $\nu = (i,i_1,\ldots,i_l) \in \Lambda_{n-1}^{l+1}$) given by the map which deforms the diagram ambient isotopically as shown in Figure 7.

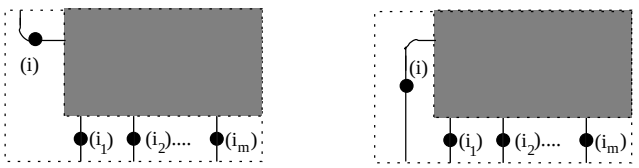

FIGURE 7.

This completes the verification of (A5); it is also clear from the above that (A6′) holds. Thus we may apply all the general theory from Section 1 to these algebras.

To apply Theorem 1.1 it only remains to calculate $\mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu))$ for all $\lambda \in \Lambda_n^n$ and $\mu \in \Lambda_n^{n-2}$. If there exists a $\mu \in \Lambda_n^{n-2}$ with $\Delta_n(\mu)$ non-simple, then at least one such Hom-space will be non-zero. Thus to prove that our algebras are semisimple it is enough, for example, to show that the Gram matrix for $\Delta_n(\mu)$ is non-degenerate for all $\mu \in \Lambda_n^{n-2}$.

4. Gram matrix results

We now consider the Gram matrix $G_n(\lambda)$ of inner products with respect to the diagram basis of $\Delta_n(\lambda)$ (confer [MS94]). Let $D^n_{l,p}(i_1, \ldots, i_p)$ be the mild generalisation of $D^n_p(i_1, \ldots, i_p)$ consisting of diagrams with n northern and l southern nodes, p propagating lines, and propagating line decorations as for $D^n_p(i_1, \ldots, i_p)$. Then $\hat{D}^p_n(i_1, \ldots, i_p) = D^p_{n,p}(i_1, \ldots, i_p)$ is the upside down version of $D^n_p(i_1, \ldots, i_p)$. Let $\epsilon(\lambda)$ denote the unique element of $D^p_p(\lambda)$. Consider the map

$$\begin{aligned} \hat{D}^p_n(\lambda) \times D^n_p(\lambda) &\to \mathbb{Z}[\delta_0, \ldots, \delta_{m-1}] \\ (a, b) &\mapsto \langle a|b\rangle \end{aligned}$$

where $\langle a|b\rangle$ is such that the diagram product $ab = \langle a|b\rangle\, \epsilon(\lambda)$ if ab lies in $\mathbb{Z}[\delta_0, \ldots, \delta_{m-1}]D^p_p(\lambda)$, and is zero otherwise. Note that $\langle -|-\rangle$ defines an inner product on $\Delta_n(\lambda)$.

We will first consider the case $m = 2$ and $d = \infty$ for the sake of definiteness. However, neither restriction is significant. In pictures we will denote $\bullet(1)$ just by $\bullet$. When $n = 2$ we then have

$$\Lambda_2 = \Lambda^2 \cup \Lambda_0 = \{(1,1), (1,2), (2,1), (2,2)\} \cup \{\emptyset\}$$

(using the index set introduced above Corollary 2.8).

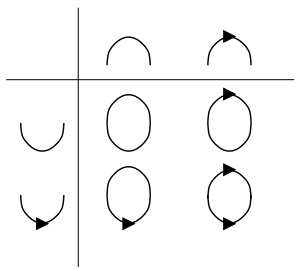

Figure 8.

A simple restatement of the inner product above is that we need only consider the concatenation of the top halves of diagrams in the diagram basis of a standard module with bottom halves in the dual. Accordingly we may compute the Gram matrix $G_2(\lambda)$ for $\Delta_2(\lambda)$ with $\lambda = \emptyset$ from the diagrams in Figure 8, which give the corresponding matrix

$$\begin{pmatrix} \delta_0 & \delta_1 \\ \delta_1 & \delta_0 \end{pmatrix}.$$

That is, $|G_2(\emptyset)| = \delta_0^2 - \delta_1^2$.

Let us consider for a moment what happens in a singular specialisation. If $\delta_1 = \delta_0$, then $\Delta_2(\emptyset)$ is not simple. Armed with this knowledge it is straightforward to construct a proper submodule. Indeed it will be evident that if we write a and b for the two basis elements depicted, then $T_2(i)(a-b) = -(a-b)$ for $i = 1, 2$, and $E_2(1)(a-b) = 0$. Thus $(a-b)$ generates a submodule of $\Delta_2(\emptyset)$ isomorphic

to $\Delta_2(1,1)$ in such a specialisation. By Theorem 1.1(i) we obtain corresponding homomorphisms

$$\Delta_n(1,1) \to \Delta_n(\emptyset)$$

for all even n.

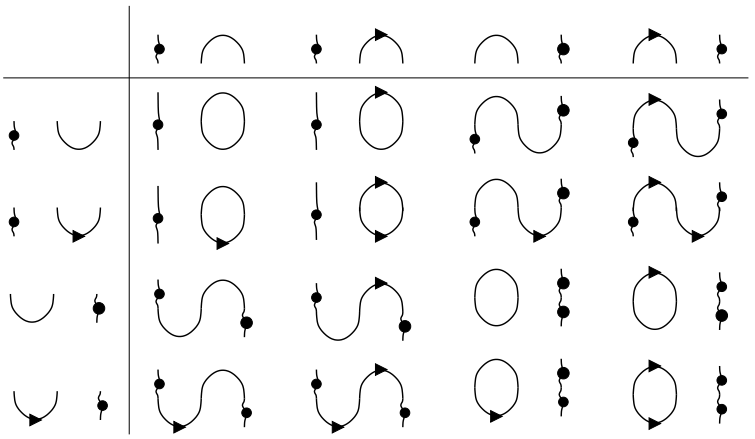

FIGURE 9.

Returning to generic parameters, for $\Delta_3(\lambda)$ with $\lambda = (1)$ or (2) we have from Figure 9 that the Gram matrix equals

$$\begin{pmatrix} \delta_0 & \delta_1 & 1 & \pm 1 \\ \delta_1 & \delta_0 & \pm 1 & 1 \\ 1 & \pm 1 & \delta_0 & \delta_1 \\ \pm 1 & 1 & \delta_1 & \delta_0 \end{pmatrix}$$

The determinant here is again easy to compute, but the details do not concern us here. Instead we return to the general case.

PROPOSITION 4.1. *Considering $\delta_0, \delta_1, \ldots, \delta_{m-1}$ as indeterminates, the determinant $|G_n(\lambda)|$ is non-zero.*

Proof. It is clear that all Gram matrix elements take the form $\xi \prod_i (\delta_i)^{\alpha_i}$ where ξ is some mth root of unity. Consider for a moment the diagonal elements of the Gram matrix, organised as indicated by our examples. In these, every upper arc meets a mirror image lower arc, and either both are undecorated, or they have 'cancelling' decorations. Thus every arc contributes positively to α_0. It follows that in each row of any Gram matrix the value of α_0 for the matrix element on the diagonal strictly exceeds any other, and hence that $|G_n(\lambda)|$ is a non-zero polynomial. □

COROLLARY 4.2. *The algebras $X^d_{n,m}$ are generically semisimple with respect to the Zariski topology for our parameter space.*

5. Discussion

Note that we have just proved generic semisimplicity of our algebras without appeal to the full strength of the machinery developed in Section 1. However, Proposition 4.1 does not provide a means for determining *which* specialisations are non-semisimple; indeed determining the zeros of $|G_n(\lambda)|$ for general λ seems a rather intractable problem. We conclude by discussing how our result can be strengthened using the machinery developed.

By Theorem 1.1(ii), we have the much simpler condition

Corollary 5.1. *The algebra $X^d_{n,m}$ is semisimple over k if and only if $(\delta_0, \dots, \delta_{m-1})$ is such that*

$$\prod_{n' \leqslant n} \prod_{\lambda \in \Lambda^{n'-2}_{n'}} |G_{n'}(\lambda)| \neq 0.$$

Remark 5.2. For $X^\infty_{n,m}$ the Gram matrices in Corollary 5.1 are precisely those calculated in [RX, Proposition 8.1]. The answer given there is a complicated but explicit polynomial in the defining parameters. Thus, using the polynomial in [RX, Proposition 8.1], we can determine precisely which specialisations of $X^\infty_{n,m}$ are semisimple. Very similar explicit results may be obtained for the algebras $X^d_{n,m}$; for $d = 0$ these were calculated in [Mar91], and for $d = 1$ in [MS94].

The theory developed in Section 1 also provides a means for studying non-semisimple specialisations, as it provides a means for determining a large number of homomorphisms. In the interests of brevity we do not pursue the structure of the non-semisimple cases of the contour algebras further here. Note, however, that much (in some cases essentially all) of the structure of the other algebras mentioned in Section 1 has been derived in the literature using methods which are entirely based on (ad hoc formulations of) (A1-6). Similar efficacy may be anticipated here.

The second author and Ryom-Hansen recently made play with an interesting tensor space representation of the blob algebra, which they show in [MRH] to be a full tilting module in quasi-hereditary specialisations of that algebra. It is worth noting that the bulk of the machinery they use in their proof follows from our (A1-6).

In particular, suppose that we have a tower of algebras A_n satisfying (A1-6), together with a contravariant duality ^{o} on each A_n. For each n let T_n be an A_n-module such that

(A7) *(i) T_0 and T_1 are tilting modules.*
(ii) For each $n \geqslant 2$ we have $F_n(T_n) \cong T_{n-2}$ and $T_n^o \cong T_n$.
(iii) The natural map $G_{n-2}F_n(T_n) \to T_n$ is injective.

Then by the results in [MRH, Proposition 5] we have that T_n is a tilting module for each n.

Diagram algebras typically have a contravariant duality given by inverting the individual diagrams. Thus the examples discussed in Section 1 (together with the contour algebras) do satisfy the conditions before (A7). In many examples modules satisfying (A7) arise by constructing analogues of 'tensor space' representations for the corresponding families of algebras. We do not have a candidate for a full tilting module here, but if one were forthcoming then a similar analysis should be possible.

Note that the contour algebras can be further generalised by allowing diagrams to have more than one line from a given node and/or dropping the non-crossing condition. An obvious example would be a decorated version of the partition algebra. The notion of depth is no longer meaningful, and the proof of quasi-heredity is slightly more complicated, but otherwise our machinery continues to apply. The most significant complication is the replacement of the cyclic group in our analysis by other, more complicated, group algebras.

We conclude with some remarks on our choice of axiom scheme. In (A1), the choice of $N = 2$ in the definition of $\Phi : A_{n-N} \longrightarrow A_n$ could be varied. However, for larger values of N the analysis of the interplay between induction/restriction and globalisation/localisation becomes more complicated, and the case $N = 2$ seems to cover all diagram algebra examples introduced to date. The reason for having intermediate layers is to ensure that Δ-restriction is multiplicity free — a useful feature in practical calculations (see [VO96]).

Note that the heredity chain for any quasi-hereditary algebra gives rise to a tower satisfying (A1) and (A2). It is the extra structure imposed by the remaining axioms that we wish to emphasise

here. In particular the metric structure induced on our set of weights by the local behaviour (A5) justifies the use of the term weights, by analogy with [Jan87].

Quasi-heredity is quite a strong property for an algebra to possess, and there have been several alternatives proposed for the study of wider classes of algebras. Important examples are cellular algebras [GL96] (but see also [KX98]), tabular algebras [Gre02], and various types of stratified algebras [CPS96, Dla00]. It would be interesting to consider how axiom (A2) might be weakened in these (or other) settings.

Acknowledgements

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