HIGHER EXTENSIONS BETWEEN MODULES FOR SL_2

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ABSTRACT. We calculate $\mathrm{Ext}^\bullet_{\mathrm{SL}_2(k)}(\Delta(\lambda),\Delta(\mu))$, $\mathrm{Ext}^\bullet_{\mathrm{SL}_2(k)}(L(\lambda),\Delta(\mu))$, $\mathrm{Ext}^\bullet_{\mathrm{SL}_2(k)}(\Delta(\lambda),L(\mu))$, and $\mathrm{Ext}^\bullet_{\mathrm{SL}_2(k)}(L(\lambda),L(\mu))$, where $\Delta(\lambda)$ is the Weyl module of highest weight λ, $L(\lambda)$ is the simple $\mathrm{SL}_2(k)$-module of highest weight λ and our field k is algebraically closed of positive characteristic. We also get analogous results for the Dipper-Donkin quantisation.

INTRODUCTION

Suppose G is a reductive, semi-simple linear algebraic group over a field k of positive characteristic p. The category of rational G-modules is not very well understood, we do not even know the characters of the simple modules in general. We can however distinguish them by their highest weight which we use as a labelling. The category is not semi-simple and has infinite global dimension. We do have important objects, the Weyl modules, $\Delta(\lambda)$, and their duals the induced modules, $\nabla(\lambda)$, with λ a dominant weight. The simple module, $L(\lambda)$, is the head of $\Delta(\lambda)$. Understanding the decomposition numbers of the Weyl modules would determine the characters of the simple modules.

If we let $G = \mathrm{GL}_n(k)$ and truncate to the category to polynomial modules of fixed degree r then we get the module category of a quasi–hereditary algebra — namely the Schur algebra. The $\nabla(\lambda)$, $\Delta(\lambda)$, with λ a partition of r, are the costandard and standard objects respectively. Homological algebra is an important tool for studying such categories. In particular, we would like to be able to determine $\mathrm{Ext}^*_G(\Delta(\lambda),\Delta(\mu))$, $\mathrm{Ext}^*_G(\Delta(\lambda),L(\mu))$, $\mathrm{Ext}^*_G(L(\lambda),\Delta(\mu))$, and $\mathrm{Ext}^*_G(L(\lambda),L(\mu))$. Not many of these groups are known explicitly. A survey of what is known about Ext groups for algebraic groups and Hecke algebras may be found in [6].

In this paper we give recursive formulas for these groups in the case $G = \mathrm{SL}_2(k)$. The results for Weyl modules are generalisations of results of Cox and Erdmann, ([10], [4] and [5]) who determined $\mathrm{Ext}^i(\Delta(\lambda),\Delta(\mu))$ for $\mathrm{SL}_2(k)$ and the quantum group q-$\mathrm{GL}_2(k)$ for $i = 1$ and 2. (The $i = 0$ case is well known, a proof may be found in [5].) They use the Lyndon-Hochschild-Serre spectral and its associated five term exact sequence sequence to relate the G_1-cohomology to the G-cohomology. An analysis of their results shows that they have proved that the E_2 terms are the same as the E_∞ terms if they lie on or below the diagonal $i + j = 2$. We use a variant of the

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Lyndon-Hochschild-Serre spectral sequence for linear algebraic groups and show that the E_2 pages for the spectral sequences for the Ext groups we are interested in are the same as the E_∞ page. We can then just add up the qth diagonal of the E_2 page to get the required Ext group. In particular, we get nice recursion formulas for some of the Ext groups and so they may be completely calculated by this method.

We can then use the results of Donkin [8] and Cline, Parshall and Scott [3] to get some Ext groups for larger G. Let S be the set of simple roots for G, and λ and μ dominant weights for G. Set $m_\alpha = \langle \lambda, \alpha \rangle$ and $n_\alpha = \langle \mu, \alpha \rangle$ for $\alpha \in S$. If $\lambda - \mu = m\beta$ for a $\beta \in S$ and $m \in \mathbb{Z}$ then

$$\mathrm{Ext}^q_G(\Delta(\lambda), L(\mu)) \cong \mathrm{Ext}^q_{\mathrm{SL}_2}(\Delta(2m_\beta), L(2n_\beta))$$

using [3, corollary 10] and

$$\mathrm{Ext}^q_G(\Delta(\lambda), \Delta(\mu)) \cong \mathrm{Ext}^q_{\mathrm{SL}_2}(\Delta(2m_\beta), \Delta(2n_\beta))$$

using [10, section 4]. For $\mathrm{GL}_n(k)$ the condition on λ and μ is equivalent to saying that they only differ in two consecutive rows.

We finish off by showing that the results for $\mathrm{SL}_2(k)$ may be easily generalised to the quantum group q-$\mathrm{GL}_2(k)$ of Dipper and Donkin [7], thus analogous results hold there as well.

1. Notation

We first review most of the notation that we will be using. The reader is referred to [12] and [17] for further information. This material is also in [14] where is it presented in the form of group schemes.

Throughout this paper k will be an algebraically closed field of characteristic p. Let G be a linear algebraic group which is connected and reductive, and let $\mathrm{F} : G \to G$ its corresponding Frobenius morphism. We let G_1 be the first Frobenius kernel. We fix a maximal torus T of G of dimension n, the rank of G. We also fix B, a Borel subgroup of G with $B \supseteq T$ and let W_p be the affine Weyl group of G.

We will write $\mathrm{mod}(G)$ for the category of finite dimensional rational G-modules. Most G-modules considered in this paper will belong to this category. Let $X(T) = X$ be the weight lattice for G. We take R to be the roots of G and S the simple roots. Let h be the maximum of the Coxeter numbers for the connected components of R.

We have a partial order on X defined by $\mu \leqslant \lambda \Leftrightarrow \lambda - \mu \in \mathbb{N}S$. We let X^+ be the set of dominant weights. Take $\lambda \in X^+$ and let k_λ be the one-dimensional module for B which has weight

λ. We define the *induced module*, $\nabla(\lambda) = \mathrm{Ind}_B^G(k_\lambda)$. This module has formal character given by Weyl's character formula and has simple socle $L(\lambda)$, the irreducible G-module of highest weight λ. Any finite dimensional, rational irreducible G-module is isomorphic to $L(\lambda)$ for a unique $\lambda \in X^+$. We let X_1 be the p-restricted weights. Then the G-modules $L(\lambda)$ with $\lambda \in X_1$ form a complete set of simples upon restriction for G_1.

Since G is split, connected and reductive we have an antiautomorphism, τ, which acts as the identity on T ([14], II, corollary 1.16). From this morphism we may define $^\circ$, a contravariant dual. It does not change a module's character, hence it fixes the irreducible modules. We define the Weyl module, to be $\Delta(\lambda) = \nabla(\lambda)^\circ$. Thus $\Delta(\lambda)$ has simple head $L(\lambda)$. We say a module is a *tilting module* if it has a filtration by both Weyl modules and induced modules. For each $\lambda \in X^+$ there is a unique indecomposable tilting module $T(\lambda)$ with highest weight λ.

We say that λ and μ are *linked* if they belong to the same W_p orbit on X (under the dot action). If two irreducible modules $L(\lambda)$ and $L(\mu)$ are in the same G block then λ and μ are linked. We will identify a block with the set of dominant weights which label the simples in that block.

If $G = \mathrm{SL}_2(k)$ then X^+ may be identified with the natural numbers. Two regular weights $\lambda = pa + i$ and $\mu = pb + j$ with $a, b \in \mathbb{N}$ and $0 \leqslant i \leqslant p-2$ and $0 \leqslant j \leqslant p-2$ will only be in the same block if $a-b$ is even and $i = j$ or if $a-b$ is odd and $j = p-2-i$. The only non-regular weights are the Steinberg weights, two such weights $pa+p-1$ and $pb+p-1$ will only be in the same block if a and b are in the same block.

If $H = G$ or G_1 or G/G_1 then the category of rational H-modules has enough injectives and so we may define $\mathrm{Ext}^*_H(-,-)$ as usual by using injective resolutions (see [1], section 2.4 and 2.5). We also define the cohomology groups $H^*(H,-) \cong \mathrm{Ext}^*_H(k,-)$, which are the right derived functors of the fixed point functor $-^H$.

We may form the Frobenius twist of a module V, by composing the given action with the Frobenius morphism. This new module is denoted V^{F} and it is trivial as G_1-module. Conversely any G-module W which is trivial as a G_1-module, is of the form V^{F} for some G-module V which is unique up to isomorphism. If W and V are G-modules then $\mathrm{Ext}^i_{G_1}(W,V)$ has a natural structure as a G-module. Moreover when W and V are finite dimensional we have,

$$\mathrm{Ext}^i_{G_1}(W, V \otimes Y^{\mathrm{F}}) \cong \mathrm{Ext}^i_{G_1}(W,V) \otimes Y^{\mathrm{F}} \tag{1}$$

as G-modules. If $H = G$ or G_1 or G/G_1 and V, W are H-modules then we have

$$\mathrm{Ext}^i_H(W,V) \cong \mathrm{Ext}^i_H(V^*,W^*) \cong \mathrm{Ext}^i_H(k, W^* \otimes V) \cong H^i(H, W^* \otimes V)$$

where * is the ordinary dual. We also note that $(V^{\mathrm{F}})^{G/G_1} \cong V^G$, and $H^i(G/G_1, V^{\mathrm{F}}) \cong H^i(G,V)$.

2. Spectral sequences

We will use a variant of the Lyndon-Hochschild-Serre spectral sequence to show that the E_2 page is the same as the E_∞ page for the Ext groups we are interested in. Our primary reference for this section is [2, chapter 3].

Usually we would construct the Lyndon-Hochschild-Serre spectral sequence by taking G and G/G_1 injective resolutions as in [14, I, section 6.6]. We wish to analyse the E_0 and E_1 page so we construct it in a different way.

First take a G (co)resolution of a finite dimensional G-module W which is injective as a G_1 resolution. This is certainly possible as the injectives for G are also injective for G_1 upon restriction. Usually we will want much smaller, finite dimensional modules in our resolution. Thus the modules will not be injective as G-modules. This will sometimes be possible as the indecomposable G_1-modules $Q(i)$ have a G structure (proved in general for $p \geqslant 2(h-1)$ [13], and is true for all p for SL_2 and SL_3). In all the examples we take for SL_2 we will construct this resolution explicitly. So we have a G_1-injective resolution with $I_i \in \mathrm{mod}(G)$

$$0 \to W \to I_0 \to I_1 \to \cdots \to I_{m-1} \to I_m \to \cdots .$$

Now take a (minimal) injective G resolution of k.

$$0 \to k \to J_0 \to J_1 \to \cdots$$

These injective modules will have infinite dimension. We then take the Frobenius twist of the modules J_m. We end up with a G/G_1 resolution of k which is injective as G/G_1-modules.

We take a finite dimensional G-module V and form a double complex in a similar way to [2, section 3.5]

$$E_0^{mn} = \mathrm{Hom}_{G/G_1}(k, \mathrm{Hom}_{G_1}(V, I_n) \otimes J_m^{\mathrm{F}})$$

where the horizontal maps are the differentials induced by the differentials in the G/G_1 resolution of k and the vertical maps are $(-1)^m$ times the differentials induced by the G_1 injective resolution of V.

Using [2, theorem 3.4.2] we now have a spectral sequence with

$$E_1^{mn} = \mathrm{Hom}_{G/G_1}(k, \mathrm{Ext}^n_{G_1}(V, W) \otimes J_m^{\mathrm{F}})$$
$$E_2^{mn} = H^m(G/G_1, \mathrm{Ext}^n_{G_1}(V, W))$$

We have

$$\begin{aligned}\mathrm{Hom}_{G/G_1}(k, \mathrm{Hom}_{G_1}(V, I_n) \otimes J_m^{\mathrm{F}}) &\cong ((V^* \otimes I_n)^{G_1} \otimes J_m^{\mathrm{F}})^{G/G_1} \\ &\cong ((V^* \otimes I_n \otimes J_m^{\mathrm{F}})^{G_1})^{G/G_1} \\ &\cong (V^* \otimes I_n \otimes J_m^{\mathrm{F}})^{G} \\ &\cong \mathrm{Hom}_G(V, I_n \otimes J_m^{\mathrm{F}})\end{aligned}$$

as J_m^{F} is trivial as a G_1-module and I_n has a G-structure.

Now

$$\mathrm{Ext}^q_G(V, I_n \otimes J_m^{\mathrm{F}}) \cong \mathrm{Ext}^q_{G/G_1}(k, \mathrm{Hom}_{G_1}(V, I_n) \otimes J_m^{\mathrm{F}})$$

as $V^* \otimes I_n \otimes J_m^{\mathrm{F}}$ is injective as a G_1-module and the usual Lyndon-Hochschild-Serre spectral sequence collapses to a line. But now this Ext group is zero for $q \geqslant 1$, as J_m^{F} is injective as a G/G_1-module. Thus the total homology of the double complex E_0 is $\mathrm{Ext}^*_G(V, W)$.

Hence we have constructed a spectral sequence which converges to $\mathrm{Ext}^q_G(V, W)$ and whose E_2 page has the same terms as that as the usual Lyndon-Hochschild-Serre sequence. (It will also have the same differentials - but we will not need this.)

Our results for $\mathrm{SL}_2(k)$ say that the E_2 page is often the same as the E_∞ page. We now present some lemmas which give conditions for the k_2 associated to the 2nd derived couple to be zero. For what follows we refer the reader to [2, section 3.4], where the E_1 and E_2 pages as well as the D_0 and D_1 are explicitly constructed.

We have the double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \\
E_0^{02} & \xrightarrow{d_1} & E_0^{12} & \xrightarrow{d_1} & E_0^{22} & \xrightarrow{d_1} & \cdots \\
\uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \\
E_0^{01} & \xrightarrow{d_1} & E_0^{11} & \xrightarrow{d_1} & E_0^{21} & \xrightarrow{d_1} & \cdots \\
\uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \\
E_0^{00} & \xrightarrow{d_1} & E_0^{10} & \xrightarrow{d_1} & E_0^{20} & \xrightarrow{d_1} & \cdots
\end{array}$$

with $d_1^2 = 0$, $d_0^2 = 0$ and $d_1 d_0 + d_0 d_1 = 0$.

We have

$$D_0^{mn} = \bigoplus_{m+n=e+f,\ e \geqslant m} E_0^{ef}$$

$$E_1^{mn} = H(E_0^{mn}, d_0)$$
$$D_1^{mn} = H(E_0^{mn} \oplus E_0^{m+1,n-1} \oplus \cdots, d_0 + d_1)$$

We will use square brackets to denote the class of $(x_1, x_2, \ldots)$ in the homology group.

The first derived couple has long exact sequence

$$\cdots \to E_1^{m,n-1} \xrightarrow{k_1^{m,n-1}} D_1^{m+1,n-1} \xrightarrow{i_1^{m+1,n-1}} D_1^{m,n} \xrightarrow{j_1^{m,n}} E_1^{m,n} \xrightarrow{k_1^{m,n}} \cdots$$

where $k_1[x] = [(d_1 x, 0, \ldots)]$, which is induced by taking the homology of the short exact sequence

$$0 \to D_0^{m+1,n-1} \to D_0^{m,n} \to E_0^{m,n} \to 0$$

We define the higher derived couples by taking the derived couple of the previous one. We have an exact diagram of doubly graded k-modules

$$\begin{tikzcd} D_l \arrow[rr, "i_l"] & & D_l \arrow[dl, "j_l"] \\ & E_l \arrow[ul, "k_l"] & \end{tikzcd}$$

The derived couple (for $l \geqslant 1$) is defined by

$$D_{l+1}^{mn} = \operatorname{im} i_l^{m+1,n-1} \subseteq D_l^{mn}$$
$$E_{l+1}^{mn} = H(E_l^{mn}, d_l)$$
$$i_{l+1}^{mn} = i_l^{mn}\Big|_{D_{l+1}}$$
$$j_{l+1}^{mn}(i_l^{m+1,n-1}(x)) = j_l^{m+1,n-1}(x) + \operatorname{im}(d_l)$$
$$k_{l+1}^{mn}(z + \operatorname{im}(d_l)) = k_l^{mn}(z)$$
$$d_{l+1} = j_{l+1} \circ k_{l+1}$$

Note there is a slight abuse of notation here — we have two maps called d_1. We will only use d_1 to denote the horizontal differentials on the E_0 page and use $j_1 k_1$ to refer to the differential of the E_1 page. The connection between the two maps is that $j_1 k_1[x] = [d_1 x]$. So $E_2 = H(E_1, j_1 k_1)$. Note that the kernel of $j_1 k_1 : E_1^{m,n} \to E_1^{m+1,n}$ is $\{[x] \mid d_1 x = d_0 z \text{ for some } z \in E_1^{m+1,n-1}\}$. Since we have a first quadrant spectral sequence the $E_l^{m,n}$ and $D_l^{m,n}$ must eventually stabilise, the E_∞ and D_∞ are then this stable value.

Lemma 2.1. *Suppose $d_0^{m+1,n-1}$ is zero, then $k_2^{m,n}$ is zero.*

Proof. Now since $d_0^{m+1,n-1}$ are zero, we must have that the kernel of $j_1 k_1 : E_1^{m,n} \to E_1^{m+1,n-1}$ is the set

$$\{[x] \mid x \in \ker(d_1 : E_0^{m,n} \to E_0^{m+1,n})\}.$$

Thus we have

$$k_2^{m,n}([x] + \operatorname{im} j_1k_1) = k_1^{m,n}([x]) = [(d_1x, 0, 0, \ldots)] = 0$$

as $[x] \in \ker j_1k_1$ and so $d_1x = 0$. $\square$

Lemma 2.2. *Suppose $d_0^{m+2,n-1}$ is injective, then $k_2^{m,n}$ is zero.*

Proof. The kernel of $j_1k_1 : E_1^{m,n} \to E_1^{m+1,n}$ is $\{[x] \mid d_1(x) = d_0(z) \text{ for some } z \in E_1^{m+1,n-1}\}$. Thus we have

$$k_2^{m,n}([x] + \operatorname{im} j_1k_1) = k_1^{m,n}([x]) = [(d_1x, 0, 0, \ldots)]$$

Now since $[x] \in \ker j_1k_1$, $d_1x = d_0z$ for some $z \in E_1^{m+1,n-1}$. Thus

$$[(d_1x, 0, 0, \ldots)] = [(d_0z, 0, 0, \ldots)] = [(d_0z, 0, 0, \ldots) - (d_1 + d_0)(z, 0, 0, \ldots)] = [(0, -d_1z, 0, \ldots)].$$

Now since $d_0 : E_0^{m+2,n-1} \to E_0^{m+2,n}$ is an embedding, we have $d_1z = 0$ if and only if $d_0d_1z = 0$ which is zero if and only if $-d_1d_0z = 0$. But $d_1d_0z = d_1d_1x = 0$. Thus $d_1z = 0$ and $k_2^{m,n}$ is zero.

This is illustrated in the following diagram.

$$\begin{tikzcd}
 & \vdots & & \vdots & & \vdots & \\
\cdots \arrow[r] & E_0^{m,n} \arrow[u, "d_0"] \arrow[rr, "d_1"] & & E_0^{m+1,n} \arrow[u, "d_0"] \arrow[rr, "d_1"] & & E_0^{m+2,n} \arrow[u, "d_0"] \arrow[r, "d_1"] & \cdots \\
x \arrow[ur, phantom, "\in" sloped] \arrow[rr, mapsto] & & d_1x = d_0z \arrow[ur, phantom, "\in" sloped] \arrow[rr, mapsto] & & 0 \arrow[ur, phantom, "\in" sloped] & & \\
\cdots \arrow[r] & E_0^{m,n-1} \arrow[uu, "d_0"] \arrow[rr, "d_1"] & & E_0^{m+1,n-1} \arrow[uu, "d_0"] \arrow[rr, "d_1"] & & E_0^{m+2,n-1} \arrow[uu, "d_0"] \arrow[r, "d_1"] & \cdots \\
 & & z \arrow[ur, phantom, "\in" sloped] \arrow[uu, mapsto] \arrow[rr, mapsto] & & d_1z \arrow[ur, phantom, "\in" sloped] \arrow[uu, mapsto] & & \\
 & \vdots \arrow[uu, "d_0"] & & \vdots \arrow[uu] & & \vdots \arrow[uu, "d_0"] &
\end{tikzcd}$$

$\square$

Corollary 2.3. *Suppose the d_0 are all either injections or all zero. Then the E_2 page is the same as the E_∞ page.*

Proof. The conditions on the d_0 imply that all the k_2's in the second derived couple are zero. This in particular implies that all subsequent differentials d_f are zero for $f \geqslant 2$. Since the E_{i+1} is the homology of E_i with respect to d_i this means that the homology stabilises with the E_2 page. Thus the E_2 page is the same as the E_∞ page. $\square$

Remark 2.4. The first condition of this corollary, that all the d_0 are zero implies that the E_2 page is the same as the E_∞ page is used implicitly in Evens' proof of a theorem of Nakaoka, [11, theorem 5.3.1]. I am grateful to Dave Benson for this remark.

3. THE SL_2 CASE

We now restrict our attention to $G = \mathrm{SL}_2$. We list some general facts about modules for SL_2. Here X^+ may be identified with $\mathbb{N}$, and X_1 is the set $\{i \mid 0 \leqslant i \leqslant p-1, i \in \mathbb{N}\}$.

Take i, $j \in \mathbb{N}$ with $0 \leqslant i \leqslant p-2$ and $0 \leqslant j \leqslant p-2$. We define $\bar{\imath} = p-2-i$ and similarly $\bar{\jmath} = p-2-j$. Note that if $p \geqslant 3$ then $i \neq \bar{\imath}$. The simples $L(i)$ and $L(\bar{\imath})$ are in the same G_1-block and are the only simples in this block. If $p = 2$ then $L(i) = L(\bar{\imath})$ and there is only one simple in this G_1-block. We will often need to argue separately for the $p = 2$ case for this reason, as the G_1 cohomology is more complicated. Although the results for $p = 2$ can be thought of as the results for $p \geqslant 3$ but with the $i = j$ and $i = \bar{\jmath}$ cases added together.

For this section we will assume that a, b, i and $j \in \mathbb{N}$ and $0 \leqslant i \leqslant p-2$ and $0 \leqslant j \leqslant p-2$. Thus if $p = 2$ then $i = j = 0 = \bar{\imath} = \bar{\jmath}$.

We know that the tilting module $T(pa+i)$ for $\mathrm{SL}_2(k)$ is isomorphic to $T(a-1)^{\mathrm{F}} \otimes T(p+i)$ for $a \geqslant 1$. We also know that $T(i) \cong \Delta(i) \cong \nabla(i)$. We have that $T(p+i) = Q(\bar{\imath})$, the G_1-injective hull of $L(\bar{\imath})$ as a G-module. We have the short exact sequences

$$0 \to \Delta(pa+i) \to \Delta(a-1)^{\mathrm{F}} \otimes T(p+i) \to \Delta(p(a-1)+\bar{\imath}) \to 0 \tag{2}$$

and

$$0 \to \Delta(a-1)^{\mathrm{F}} \otimes L(\bar{\imath}) \to \Delta(pa+i) \to \Delta(a)^{\mathrm{F}} \otimes L(i) \to 0 \tag{3}$$

which appear in the thesis [19]. The latter non-split sequence is a G_1 socle series for $\Delta(pa+i)$. That is $\Delta(a)^{\mathrm{F}} \otimes L(i)$ is the G_1 head of $\Delta(pa+i)$ and $\Delta(a-1)^{\mathrm{F}} \otimes L(\bar{\imath})$ is the G_1 socle, if $a \leqslant 1$.

We also have:

$$\Delta(pa+p-1) \cong \Delta(a)^{\mathrm{F}} \otimes \mathrm{St}$$
$$T(pa+p-1) \cong T(a)^{\mathrm{F}} \otimes \mathrm{St}$$

where $\mathrm{St} = L(p-1)$ is the Steinberg module, and natural isomorphisms

$$\mathrm{Ext}^m_G(M, N) \cong \mathrm{Ext}^m_G(M^{\mathrm{F}} \otimes \mathrm{St}, N^{\mathrm{F}} \otimes \mathrm{St})$$

for all $m \in \mathbb{N}$ and M, N G-modules.

The aim is to calculate $\mathrm{Ext}^m_G(\Delta(pb+j), \Delta(pa+i))$, $\mathrm{Ext}^m_G(L(pb+j), \Delta(pa+i))$, $\mathrm{Ext}^m_G(\Delta(pb+j), L(pa+i))$, $\mathrm{Ext}^m_G(L(pb+j), L(pa+i))$, and $\mathrm{Ext}^m_G(T(pb+j), \Delta(pa+i))$.

First note that for N and M G-modules and $q \geqslant 0$ we have

$$\mathrm{Ext}^q_{G_1}(N^{\mathrm{F}} \otimes L(i), M^{\mathrm{F}} \otimes Q(i)) \cong \begin{cases} (N^{\mathrm{F}})^* \otimes M^{\mathrm{F}} & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

and if $i \neq \bar{\imath}$ then

$$\mathrm{Ext}^q_{G_1}(N^{\mathrm{F}} \otimes L(i), M^{\mathrm{F}} \otimes Q(\bar{\imath})) \cong 0.$$

as $Q(i)$ is injective as a G_1-module. This will be extensively in this paper without further comment.

We will use the following lemmas in our calculations in the following sections. The quantum version of the next two lemmas appears in [5, proposition 2.1].

Lemma 3.1. *Let M be a G-module. Then if $p \geqslant 3$ we have*

$$\mathrm{Hom}_{G_1}(\Delta(pb+j), M^{\mathrm{F}} \otimes Q(i)) \cong \begin{cases} \nabla(b)^{\mathrm{F}} \otimes M^{\mathrm{F}} & \text{if } i = j \\ \nabla(b-1)^{\mathrm{F}} \otimes M^{\mathrm{F}} & \text{if } i = \bar{\jmath} \\ 0 & \text{otherwise.} \end{cases}$$

If $p = 2$ then

$$\mathrm{Hom}_{G_1}(\Delta(2b), M^{\mathrm{F}} \otimes Q(0)) \cong \nabla(b)^{\mathrm{F}} \otimes M^{\mathrm{F}} \oplus \nabla(b-1)^{\mathrm{F}} \otimes M^{\mathrm{F}}$$

Proof. We need only prove this for $M = k$ as the M^{F} comes out using (1). Applying the exact functor $\mathrm{Hom}_{G_1}(-, Q(i))$ to the short exact sequence (3) gives us

$$0 \to \mathrm{Hom}_{G_1}(\Delta(b)^{\mathrm{F}} \otimes L(j), Q(i)) \to \mathrm{Hom}_{G_1}(\Delta(pb+j), Q(i))$$
$$\to \mathrm{Hom}_{G_1}(\Delta(b-1) \otimes L(\bar{\jmath}), Q(i)) \to 0$$

The first Hom group is 0 if $i \neq j$ and $\nabla(b)^{\mathrm{F}}$ if $i = j$. The last Hom group is $\nabla(b-1)^{\mathrm{F}}$ if $i = \bar{\jmath}$ and 0 if $i \neq \bar{\jmath}$.

If $p = 2$ and $i = j$ then both the first and third Hom groups are non-zero. The middle Hom group has the structure of a G/G_1-module. We have

$$\mathrm{Ext}^1_{G/G_1}(\nabla(b-1)^{\mathrm{F}}, \nabla(b)^{\mathrm{F}}) \cong \mathrm{Ext}^1_G(\nabla(b-1), \nabla(b)) \cong 0.$$

So the middle Hom group is a direct sum as claimed. □

Lemma 3.2. *Let M be a G-module and $q \in \mathbb{N}$. If $p \geqslant 3$ then*

$$\mathrm{Ext}^q_G(\Delta(pb+j), M^{\mathrm{F}} \otimes Q(i)) \cong \begin{cases} \mathrm{Ext}^q_G(\Delta(b), M) & \text{if } i = j \\ \mathrm{Ext}^q_G(\Delta(b-1), M) & \text{if } i = \bar{\jmath} \\ 0 & \text{otherwise.} \end{cases}$$

If $p = 2$ then

$$\mathrm{Ext}^q_G(\Delta(2b), M^{\mathrm{F}} \otimes Q(0)) \cong \mathrm{Ext}^q_G(\Delta(b), M) \oplus \mathrm{Ext}^q_G(\Delta(b-1), M)$$

Proof.

$$\mathrm{Ext}^m_G(\Delta(pb+j), M^{\mathrm{F}} \otimes Q(i)) \cong H^m(G/G_1, \mathrm{Hom}_{G_1}(\Delta(pb+j), M^{\mathrm{F}} \otimes Q(i)))$$

as the module $\nabla(pb+j) \otimes T(p+i) \otimes M^{\mathrm{F}}$ is injective as a G_1-module so the usual Lyndon-Hochschild-Serre spectral sequence for this module collapses to a line. We then use the previous lemma. □

The following observation and proof is due to Stephen Donkin.

Lemma 3.3. *The module $\Delta(a) \otimes \nabla(b)$ has a good filtration if $b \geqslant a-1$ and it has a Weyl filtration if $b \leqslant a+1$.*

Proof. Suppose that $b \geqslant a-1$. The weights of $\Delta(a)$ are a, $a-2$, $a-4$, … , $-a+2$ and $-a$. So the weights of the B-module $\Delta(a) \otimes k_b$ are $a+b$, $a+b-2$, …, $b-a+2$ and $b-a$. Since $b-a \geqslant -1$ when we induce this module up to G we will get $\Delta(a) \otimes \nabla(b)$ and it will have a good filtration (starting at the top) by $\nabla(a+b)$,$\nabla(a+b-2)$, …, $\nabla(b-a+2)$ and $\nabla(b-a)$, where if $b-a=-1$ then the last module is taken to be the zero module. The case $b \leqslant a+1$ is proved by taking duals. □

Lemma 3.4.

$$\begin{aligned}\mathrm{Ext}^1_{G_1}(\Delta(pb+i), \Delta(pa+j)) &\cong \begin{cases} \mathrm{Ext}^1_{G_1}(\Delta(p(b-a)+i), \Delta(j)) & \text{if } b \geqslant a \\ \mathrm{Ext}^1_{G_1}(\Delta(i), \Delta(p(a-b)+j)) & \text{if } a \geqslant b \end{cases} \\ &\cong \begin{cases} \Delta(a-b-2)^{\mathrm{F}} & \text{if } a-b \geqslant 2 \text{ and } i=j \\ \nabla(b-a+1)^{\mathrm{F}} & \text{if } a-b \leqslant 1 \text{ and } i=\bar{\jmath} \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Proof. We use the techniques of [10], where the lemma is proved for $a \geqslant b$. Thus we need only prove the lemma for $b \geqslant a$. We have

$$\begin{aligned}\mathrm{Ext}^1_{G_1}(\Delta(pb+j), \Delta(pa+i)) &\cong \mathrm{Ext}^1_{G_1}(\Delta(p(b-1)+\bar{\jmath}), \Delta(p(a-1)+\bar{\imath})) \\ &\cong \mathrm{Ext}^1_{G_1}(\Delta(p(b-1)+j), \Delta(p(a-1)+i))\end{aligned}$$

for $a \geqslant 1$ and $b \geqslant 1$, using the short exact sequence (2) twice. The last isomorphism follows using the translation principle. We thus get for $b \geqslant a$ that

$$\mathrm{Ext}^1_{G_1}(\Delta(pb+j), \Delta(pa+i)) \cong \mathrm{Ext}^1_{G_1}(\Delta(p(b-a)+j), \Delta(i))$$

Now apply $\mathrm{Hom}_{G_1}(-,\Delta(i))$ to the short exact sequence (2) for $\Delta(p(b-a)+j)$. We get

$$0 \to \mathrm{Hom}_{G_1}(\Delta(p(b-a)+j),\Delta(i)) \to \mathrm{Hom}_{G_1}(\Delta(b-a)^{\mathrm{F}} \otimes Q(j),\Delta(i))$$
$$\to \mathrm{Hom}_{G_1}(\Delta(p(b-a+1)+\bar{\jmath}),\Delta(i)) \to \mathrm{Ext}^1_{G_1}(\Delta(p(b-a)+j),\Delta(i)) \to 0$$

The last zero follows as $Q(j)$ is projective. Since $\Delta(p(b-a)+j)$ has G_1-head $\Delta(b-a)^{\mathrm{F}} \otimes L(j)$ we get that the first Hom-group is $\nabla(b-a)^{\mathrm{F}}$ if $j=i$ and 0 otherwise. The second Hom-group is also $\nabla(b-a)^{\mathrm{F}}$ if $j=i$ and zero otherwise. So

$$\mathrm{Hom}_{G_1}(\Delta(p(b-a+1)+\bar{\jmath}),\Delta(i)) \cong \mathrm{Ext}^1_{G_1}(\Delta(p(b-a)+j),\Delta(i)).$$

This Hom group is $\nabla(b-a+1)^{\mathrm{F}}$ if $i=\bar{\jmath}$ and 0 otherwise. Hence the result. □

Lemma 3.5.

$$\begin{aligned}\mathrm{Ext}^1_{G_1}(\Delta(pb+i),\nabla(pa+j)) &\cong \mathrm{Ext}^1_{G_1}(\Delta(p(a+b)+i),\Delta(j)) \\ &\cong \begin{cases} \nabla(a+b+1)^{\mathrm{F}} & \textit{if } i=\bar{\jmath} \\ 0 & \textit{otherwise} \end{cases}\end{aligned}$$

Proof. A proof of this may be found in [18, proposition 1.1] and is similar to that of the previous lemma. □

Lemma 3.6. *If $p \geqslant 3$ then*

$$\mathrm{Hom}_{G_1}(\Delta(pb+\bar{\imath}),\Delta(pa+i)) \cong \Delta(a-1)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}}$$

(where $\Delta(-1)$ is interpreted as the zero module) and

$$\mathrm{Hom}_{G_1}(\Delta(pb+i),\Delta(pa+i)) \cong \begin{cases} \nabla(b-a)^{\mathrm{F}} & \textit{if } b \geqslant a \\ 0 & \textit{otherwise} \end{cases}$$

If $p=2$ then

$$\mathrm{Hom}_{G_1}(\Delta(2b),\Delta(2a)) \cong \begin{cases} \Delta(a-1)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} \oplus \nabla(b-a)^{\mathrm{F}} & \textit{if } b \geqslant a \\ \Delta(a-1)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \textit{if } b < a \end{cases}$$

(where again $\Delta(-1)$ is interpreted as the zero module)

Proof. We use the fact that the G_1-socle of $\Delta(pa+i)$ is $\Delta(a-1)^{\mathrm{F}} \otimes L(\bar{\imath})$ if $a \geqslant 1$ and the G_1-head of $\Delta(pa+i)$ is $\Delta(a)^{\mathrm{F}} \otimes L(i)$.

We first assume that $p \geqslant 3$ and so we may assume that $a \geqslant 1$ as $\Delta(i)$ has the wrong G_1 type to give a homomorphism.

Apply $\mathrm{Hom}_{G_1}(-,\Delta(pa+i))$ to the short exact sequence (3) for $\Delta(pb+\bar{\imath})$.

$$0 \to \mathrm{Hom}_{G_1}(\Delta(b)^{\mathrm{F}} \otimes L(\bar{\imath}), \Delta(pa+i)) \to \mathrm{Hom}_{G_1}(\Delta(pb+\bar{\imath}), \Delta(pa+i)) \\ \to \mathrm{Hom}_{G_1}(\Delta(b-1) \otimes L(i), \Delta(pa+i))$$

The first Hom group is $\nabla(b)^{\mathrm{F}} \otimes \Delta(a-1)^{\mathrm{F}}$. The third Hom group is 0. Hence the first result.

To prove the second we consider $\mathrm{Hom}_{G_1}(\Delta(pb+i),-)$ applied to the short exact sequence (2) for $\Delta(a)^{\mathrm{F}} \otimes Q(i)$.

$$0 \to \mathrm{Hom}_{G_1}(\Delta(pb+i), \Delta(p(a+1)+\bar{\imath})) \to \mathrm{Hom}_{G_1}(\Delta(pb+i), \Delta(a)^{\mathrm{F}} \otimes Q(i)) \\ \to \mathrm{Hom}_{G_1}(\Delta(pb+i), \Delta(pa+i)) \to \mathrm{Ext}^1_{G_1}(\Delta(pb+i), \Delta(p(a+1)+\bar{\imath})) \to 0$$

The last zero follows as $Q(i)$ is injective. We use the first result and lemma 3.1 to deduce that the first two Hom groups are isomorphic to $\nabla(b)^{\mathrm{F}} \otimes \Delta(a)^{\mathrm{F}}$. Thus

$$\mathrm{Hom}_{G_1}(\Delta(pb+i), \Delta(pa+i)) \cong \mathrm{Ext}^1_{G_1}(\Delta(pb+i), \Delta(p(a+1)+\bar{\imath}))$$

This latter Ext group is $\nabla(b-a)^{\mathrm{F}}$ using lemma 3.4.

If $p=2$ then we prove the lemma by induction on a. It is clear for $a=0$ as we know that the G_1-head of $\Delta(2b)$ is $\Delta(b)^{\mathrm{F}}$.

We consider $\mathrm{Hom}_{G_1}(\Delta(2b),-)$ applied to the short exact sequence (2) for $\Delta(a)^{\mathrm{F}} \otimes Q(0)$.

$$0 \to \mathrm{Hom}_{G_1}(\Delta(2b), \Delta(2(a+1))) \to \mathrm{Hom}_{G_1}(\Delta(2b), \Delta(a)^{\mathrm{F}} \otimes Q(0)) \\ \to \mathrm{Hom}_{G_1}(\Delta(2b), \Delta(2a)) \to \mathrm{Ext}^1_{G_1}(\Delta(2b), \Delta(2(a+1))) \to 0$$

The last zero follows as $Q(0)$ is injective. By induction and lemmas 3.1 and 3.4 we have that the first Hom group has the required character. If $b \geqslant a$ then we have

$$0 \to \mathrm{Hom}_{G_1}(\Delta(2b), \Delta(2(a+1))) \to \nabla(b)^{\mathrm{F}} \otimes \Delta(a)^{\mathrm{F}} \oplus \nabla(b-1)^{\mathrm{F}} \otimes \Delta(a)^{\mathrm{F}} \\ \xrightarrow{\phi} \nabla(b)^{\mathrm{F}} \otimes \Delta(a-1)^{\mathrm{F}} \oplus \nabla(b-a)^{\mathrm{F}} \to \nabla(b-a)^{\mathrm{F}} \to 0$$

Now if $b \geqslant a$ then $\nabla(b)^{\mathrm{F}} \otimes \Delta(a-1)^{\mathrm{F}}$ has a filtration by modules (in order starting at the top) $\nabla(a+b-1)^{\mathrm{F}}$, $\nabla(a+b-3)^{\mathrm{F}}$, …, $\nabla(b-a+1)^{\mathrm{F}}$ and $\nabla(b-1)^{\mathrm{F}} \otimes \Delta(a)^{\mathrm{F}}$ has a filtration by modules (in order starting at the top) $\nabla(a+b-1)^{\mathrm{F}}$, $\nabla(a+b-3)^{\mathrm{F}}$, …, $\nabla(b-a-1)^{\mathrm{F}}$, using lemma 3.3. Also $\nabla(b)^{\mathrm{F}} \otimes \Delta(a)^{\mathrm{F}}$ has a filtration by modules (in order starting at the top) $\nabla(a+b)^{\mathrm{F}}$, $\nabla(a+b-2)^{\mathrm{F}}$, …, $\nabla(b-a)^{\mathrm{F}}$ by lemma 3.3. Note that this doesn't intersect with any factor in $\nabla(b)^{\mathrm{F}} \otimes \Delta(a-1)^{\mathrm{F}}$. Consequently the kernel of the map ϕ is $\nabla(b)^{\mathrm{F}} \otimes \Delta(a)^{\mathrm{F}} \oplus \nabla(b-a-1)^{\mathrm{F}}$, which is the required Hom group.

We may do a similar argument if $b < a$ only now $\nabla(b)^{\mathrm{F}} \otimes \Delta(a)^{\mathrm{F}}$ has a filtration by twisted Weyl modules. □

Lemma 3.7.

$$\mathrm{Hom}_{G_1}(\Delta(pb+i), \nabla(pa+j)) \cong \begin{cases} \nabla(b)^{\mathrm{F}} \otimes \nabla(a)^{\mathrm{F}} & \textit{if } i = j \\ 0 & \textit{otherwise} \end{cases}$$

Proof. A proof of this may be found in [18, proposition 1.2] and is similar to that of the previous lemma. □

We finish this section by noting some vanishing results for the G-cohomology.

Proposition 3.8. *Let M, $N \in \mathrm{mod}(G)$ and set $pm_1 + m_0$ to be the highest weight of M and $pn_1 + n_0$ to be the highest weight of N with m_0, $n_0 \in X_1$ and m, $n \in X^+$, then*

$$\mathrm{Ext}^i_G(M, N) = 0 \quad \textit{if } i > m_1 + n_1$$

$$\mathrm{Ext}^i_G(M, \nabla(pn_1 + n_0)) = 0 \quad \textit{if } i > m_1$$

$$\mathrm{Ext}^i_G(\Delta(pm_1 + m_0), N) = 0 \quad \textit{if } i > n_1$$

Proof. This follows using the results of [15], as m_1 is an upper bound for the Weyl filtration dimension of M and n_1 is an upper bound for the good filtration dimension of N. □

4. Applying to modules for $\mathrm{SL}_2(k)$ I

We now apply the theory of section 2 and look more closely at the spectral sequences for $\mathrm{Ext}^q_G(V, \Delta(pa+i))$.

We first explicitly construct the G_1 injective resolution of $\Delta(pa+i)$. This is easy using the sequence (2), and its dual, as the middle term is projective as a G_1-module. We have

$$0 \to \Delta(pa+i) \xrightarrow{\delta_{-1}} I_0 \xrightarrow{\delta_0} I_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{m-1}} I_m \xrightarrow{\delta_m} I_{m+1} \xrightarrow{\delta_{m+1}} \cdots$$

with

$$I_m \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes Q(i) & \text{if } m \text{ odd and } m \leqslant a-1 \\ \Delta(a-m-1)^{\mathrm{F}} \otimes Q(\bar{\imath}) & \text{if } m \text{ even and } m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes Q(\bar{\imath}) & \text{if } m \text{ odd and } m \geqslant a \\ \nabla(m-a)^{\mathrm{F}} \otimes Q(i) & \text{if } m \text{ even and } m \geqslant a \end{cases}$$

We define $M_m = \ker \delta_m$. We have

$$M_m \cong \begin{cases} \Delta(p(a-m)+\bar{\imath}) & \text{if } m \text{ odd and } m \leqslant a-1 \\ \Delta(p(a-m)+i) & \text{if } m \text{ even and } m \leqslant a-1 \\ \nabla(p(m-a)+\bar{\imath}) & \text{if } m \text{ odd and } m \geqslant a \\ \nabla(p(m-a)+i) & \text{if } m \text{ even and } m \geqslant a \end{cases}$$

We have for N a G-module and $p \geqslant 3$ that

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(i), I_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \text{ odd and } m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \text{ even and } m \geqslant a \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(\bar{\imath}), I_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \text{ even and } m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \text{ odd and } m \geqslant a \\ 0 & \text{otherwise.} \end{cases}$$

If $p = 2$ then

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}}, I_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \geqslant a. \end{cases}$$

If $p \geqslant 3$ then we also have

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(i), M_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \text{ odd and } m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \text{ even and } m \geqslant a \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(\bar{\imath}), M_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \text{ even and } m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \text{ odd and } m \geqslant a \\ 0 & \text{otherwise.} \end{cases}$$

If $p = 2$ then

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}}, M_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes (N^{\mathrm{F}})^* & \text{if } m \geqslant a \end{cases}$$

using lemmas 3.6 and 3.7.

We have exact sequences

$$\begin{aligned} 0 \to \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), M_m) &\xrightarrow{\delta^*_{m-1}} \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), I_m) \\ &\xrightarrow{\delta^*_m} \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), M_{m+1}) \to \mathrm{Ext}^1_{G_1}(N^{\mathrm{F}} \otimes L(j), M_m) \to 0. \end{aligned}$$

Now since we have

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), I_m) \cong \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), M_m)$$

for all m, j and p, we must have

$$\begin{aligned}\mathrm{Ext}^m_{G_1}(N^{\mathrm{F}} \otimes L(j), \Delta(pa+i)) &\cong \mathrm{Ext}^1_{G_1}(N^{\mathrm{F}} \otimes L(j), M_{m-1})\\ &\cong \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), M_m)\\ &\cong \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), I_m).\end{aligned}$$

Thus all the induced maps δ^*_m in the following complex are zero.

$$\begin{aligned}&\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), I_0) \xrightarrow{\delta^*_0} \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), I_1) \xrightarrow{\delta^*_1} \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), I_2)\\ &\quad\xrightarrow{\delta^*_2} \cdots \xrightarrow{\delta^*_{m-2}} \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), I_{m-1}) \xrightarrow{\delta^*_{m-1}} \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), I_m)\\ &\qquad\qquad\xrightarrow{\delta^*_m} \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), I_{m+1}) \xrightarrow{\delta^*_{m+1}}\end{aligned}$$

Since all the differentials in the Hom_{G_1} complex are zero, the induced differentials d_0 on the E_0 page are also zero. Thus all the k_2's associated to the 2nd derived couple are zero using lemma 2.1. Hence the E_2 page is the same as the E_∞ page. using corollary 2.3.

Now the E_2 page for the spectral sequence associated to $\mathrm{Ext}^q_G(N^{\mathrm{F}} \otimes L(i), \Delta(pa+i))$ and $p \geqslant 3$ is

$$\begin{aligned}E_2^{m,n} &= H^m(G/G_1, \mathrm{Ext}^n_{G_1}(N^{\mathrm{F}} \otimes L(i), \Delta(pa+i)))\\ &\cong \begin{cases}\mathrm{Ext}^m_G(N, \Delta(a-n-1)) & \text{if } n \text{ odd and } n \leqslant a-1\\ \mathrm{Ext}^m_G(N, \nabla(n-a)) & \text{if } n \text{ even and } n \geqslant a\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

and for $\mathrm{Ext}^q_G(\nabla(pa+i), N^{\mathrm{F}} \otimes L(\bar{\imath}))$ and $p \geqslant 3$ we have

$$\begin{aligned}E_2^{m,n} &= H^m(G/G_1, \mathrm{Ext}^n_{G_1}(\nabla(pa+i), N^{\mathrm{F}} \otimes L(\bar{\imath})))\\ &\cong \begin{cases}\mathrm{Ext}^m_G(N, \Delta(a-n-1)) & \text{if } n \text{ even and } n \leqslant a-1\\ \mathrm{Ext}^m_G(N, \nabla(n-a)) & \text{if } n \text{ odd and } n \geqslant a\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

If $p = 2$ then the E_2 page for the spectral sequence associated to $\mathrm{Ext}^q_G(N^{\mathrm{F}}, \Delta(2a))$ is

$$\begin{aligned}E_2^{m,n} &= H^m(G/G_1, \mathrm{Ext}^n_{G_1}(N^{\mathrm{F}}, \Delta(2a)))\\ &\cong \begin{cases}\mathrm{Ext}^m_G(N, \Delta(a-n-1)) & \text{if } n \leqslant a-1\\ \mathrm{Ext}^m_G(N, \nabla(n-a)) & \text{if } n \geqslant a.\end{cases}\end{aligned}$$

We thus have the following theorem.

Theorem 4.1. *Let a, $q \in \mathbb{N}$, $0 \leqslant i \leqslant p-2$, and $N \in \operatorname{mod}(G)$. If $p \geqslant 3$ we have*

$$\begin{aligned}\operatorname{Ext}^q_G(&N^{\mathrm{F}} \otimes L(i),\ \Delta(pa+i)) \\ &\cong \bigoplus_{\substack{n \text{ odd} \\ 0 \leqslant n \leqslant \min\{q, a-1\}}} \operatorname{Ext}^{q-n}_G(N, \Delta(a-n-1)) \oplus \bigoplus_{\substack{n \text{ even} \\ a \leqslant n \leqslant q}} \operatorname{Ext}^{q-n}_G(N, \nabla(n-a))\end{aligned}$$

and

$$\begin{aligned}\operatorname{Ext}^q_G(&N^{\mathrm{F}} \otimes L(\bar{\imath}),\ \Delta(pa+i)) \\ &\cong \bigoplus_{\substack{n \text{ even} \\ 0 \leqslant n \leqslant \min\{q, a-1\}}} \operatorname{Ext}^{q-n}_G(N, \Delta(a-n-1)) \oplus \bigoplus_{\substack{n \text{ odd} \\ a \leqslant n \leqslant q}} \operatorname{Ext}^{q-n}_G(N, \nabla(n-a))\end{aligned}$$

If $p = 2$ then

$$\operatorname{Ext}^q_G(N^{\mathrm{F}}, \Delta(2a))) \cong \bigoplus_{n=0}^{n=\min\{q,a-1\}} \operatorname{Ext}^{q-n}_G(N, \Delta(a-n-1)) \oplus \bigoplus_{n=a}^{n=q} \operatorname{Ext}^{q-n}_G(N, \nabla(n-a))$$

Corollary 4.2. *Let $a \geqslant 1$, $0 \leqslant i \leqslant p-2$, $q \in \mathbb{N}$ and $N \in \operatorname{mod}(G)$. If $p \geqslant 3$ we have*

$$\operatorname{Ext}^q_G(N^{\mathrm{F}} \otimes L(i), \Delta(pa+i)) \cong \operatorname{Ext}^{q-1}_G(N^{\mathrm{F}} \otimes L(i), \Delta(p(a-1)+\bar{\imath}))$$

$$\operatorname{Ext}^q_G(N^{\mathrm{F}} \otimes L(i), \Delta(pa+\bar{\imath})) \cong \operatorname{Ext}^{q-1}_G(N^{\mathrm{F}} \otimes L(i), \Delta(p(a-1)+i)) \oplus \operatorname{Ext}^q_G(N, \Delta(a-1))$$

where Ext^{-1} is interpreted as the zero module.

If $p = 2$ then

$$\operatorname{Ext}^q_G(N^{\mathrm{F}}, \Delta(2a))) \cong \operatorname{Ext}^{q-1}_G(N^{\mathrm{F}}, \Delta(2a-2)) \oplus \operatorname{Ext}^q_G(N, \Delta(a-1))$$

Hence if $N = L(b)$ then this would completely determine $\operatorname{Ext}^q_G(L(pb+j), \Delta(pa+i))$ by induction, once we knew what $\operatorname{Ext}^q_G(L(b), \nabla(n-a))$ was. Similarly for $N = \nabla(b)$ or $\Delta(b)$. (This is as $\operatorname{Ext}^q_G(N, \Delta(a))$ will be zero if q is larger than the highest weight of N divided by p plus $\lfloor \frac{a}{p} \rfloor$ using lemma 3.8.)

If $N = k$ then the $\operatorname{Ext}^m_G(k, \nabla(n-a))$ vanish unless $n = a$ and $m = 0$, so this is enough to work out $\operatorname{Ext}^q_G(\Delta(j), \Delta(pa+i))$. This particular Ext group is also a special case of the results in section 5.

5. Applying to modules for $\mathrm{SL}_2(k)$ part II

We now do a similar procedure — we still use the G_1-injective resolution for $\Delta(pa+i)$ constructed in the previous section but now we take $V = \Delta(pb+j)$.

We first assume that $p \geqslant 3$. We have

$$\mathrm{Hom}_{G_1}(\Delta(pb+i), I_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } m \text{ odd and } m \leqslant a-1 \\ \Delta(a-m-1)^{\mathrm{F}} \otimes \nabla(b-1)^{\mathrm{F}} & \text{if } m \text{ even and } m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } m \text{ even and } m \geqslant a \\ \nabla(m-a)^{\mathrm{F}} \otimes \nabla(b-1)^{\mathrm{F}} & \text{if } m \text{ odd and } m \geqslant a \end{cases}$$

and

$$\mathrm{Hom}_{G_1}(\Delta(pb+\bar{\imath}), I_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes \nabla(b-1)^{\mathrm{F}} & \text{if } m \text{ odd and } m \leqslant a-1 \\ \Delta(a-m-1)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } m \text{ even and } m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes \nabla(b-1)^{\mathrm{F}} & \text{if } m \text{ even and } m \geqslant a \\ \nabla(m-a)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } m \text{ odd and } m \geqslant a \end{cases}$$

So we have using lemmas 3.6 and 3.7 that

$$\mathrm{Hom}_{G_1}(\Delta(pb+i), M_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } m \text{ odd and } m \leqslant a-1 \\ \nabla(m-a+b)^{\mathrm{F}} & \text{if } m \text{ even} \\ & \qquad \text{and } a-b \leqslant m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } m \text{ even and } m \geqslant a \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Hom}_{G_1}(\Delta(pb+\bar{\imath}), M_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } m \text{ even and } m \leqslant a-1 \\ \nabla(m-a+b)^{\mathrm{F}} & \text{if } m \text{ odd} \\ & \qquad \text{and } a-b \leqslant m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } m \text{ odd and } m \geqslant a \\ 0 & \text{otherwise} \end{cases}$$

So

$$\mathrm{Hom}_{G_1}(\Delta(pb+i), I_m) \cong \mathrm{Hom}_{G_1}(\Delta(pb+i), M_m)$$

if m odd and $m \leqslant a-1$ or if m even and $m \geqslant a$ and

$$\mathrm{Hom}_{G_1}(\Delta(pb+\bar{\imath}), I_m) \cong \mathrm{Hom}_{G_1}(\Delta(pb+\bar{\imath}), M_m)$$

if m even and $m \leqslant a-1$ or if m odd and $m \geqslant a$

Thus the induced differential δ_m^* on the Hom_{G_1} complex is zero in the above cases.

We have exact sequences

$$0 \to \mathrm{Hom}_{G_1}(\Delta(pb+j), M_m) \xrightarrow{\delta_m^*} \mathrm{Hom}_{G_1}(\Delta(pb+j), I_m) \xrightarrow{\delta_{m+1}^*} \mathrm{Hom}_{G_1}(\Delta(pb+j), M_{m+1}) \to \mathrm{Ext}^1_{G_1}(\Delta(pb+j), M_m) \to 0$$

We thus get for $m \geqslant 1$ using lemmas 3.4 and 3.5

$$\mathrm{Ext}^m_{G_1}(\Delta(pb+i), \Delta(pa+i)) \cong \mathrm{Ext}^1_{G_1}(\Delta(pb+i), M_{m-1}) \cong \begin{cases} \Delta(a-m-b-1)^{\mathrm{F}} & \text{if } m \text{ odd and } m \leqslant a-b-1 \\ \nabla(m-a+b)^{\mathrm{F}} & \text{if } m \text{ even and } m \geqslant a-b \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Ext}^m_{G_1}(\Delta(pb+\bar{\imath}), \Delta(pa+i)) \cong \mathrm{Ext}^1_{G_1}(\Delta(pb+\bar{\imath}), M_{m-1}) \cong \begin{cases} \Delta(a-m-b-1)^{\mathrm{F}} & \text{if } m \text{ even and } m \leqslant a-b-1 \\ \nabla(m-a+b)^{\mathrm{F}} & \text{if } m \text{ odd and } m \geqslant a-b \\ 0 & \text{otherwise.} \end{cases}$$

We now consider the E_1 page corresponding to $\mathrm{Ext}^q_G(\Delta(pb+i), \Delta(pa+j))$. As in the previous section, we will show that the k_2's are all zero.

We consider the case for $\mathrm{Ext}^q_G(\Delta(pb+i), \Delta(pa+i))$, the other case is similar. We have that the differential $d_0 : E_0^{m,n} \to E_0^{m,n+1}$ is zero for n odd and $n \leqslant a-1$ or for n even and $n \geqslant a$. We also know that $E_1^{m,n+1} = \mathrm{Hom}_{G/G_1}(k, \mathrm{Ext}^{n+1}_{G_1}(\Delta(pb+i), \Delta(pa+i)) \otimes J_m^{\mathrm{F}})$ is zero for n odd and $n \leqslant a-b-2$ or for n even and $n \geqslant a-b-1$. But since $E_1^{m,n+1}$ is the homology at $E_0^{m,n+1}$ with respect to d_0 we must have that the kernel of $d_0 : E_0^{m,n+1} \to E_0^{m,n+2}$ is the image of $d_0 : E_0^{m,n} \to E_0^{m,n+1}$ which is zero for n odd and $n \leqslant a-b-2 < a-1$. Thus the map $d_0 : E_0^{m,n+1} \to E_0^{m,n+2}$ is an embedding, for n odd and $n \leqslant a-b-2$. Note this is independent of m.

Thus for n odd and $n \leqslant a-b-2$ we have that the map $d_0 : E_0^{m,n} \to E_0^{m,n+1}$ is zero and that the map $d_0 : E_0^{m+2,n-1} \to E_0^{m+2,n}$ is an embedding. Now by lemmas 2.1 and 2.2 the maps $k_2 : E_2^{m-1,n+1} \to D_2^{m,n+1}$ and $k_2 : E_2^{m,n} \to D_2^{m+1,n}$ are zero.

Now $k_2^{m,n}$ is of course zero if $E_2^{m,n}$ is. We have

$$\begin{aligned}
E_2^{m,n} &= H^m(G/G_1, \mathrm{Ext}^n_{G_1}(\Delta(pb+i), \Delta(pa+i)) \\
&= \begin{cases} \mathrm{Ext}^m_G(k, \Delta(a-n-b-1)) & \text{if } n \text{ odd and } n \leqslant a-b-1 \\ \mathrm{Ext}^m_G(k, \nabla(n-a-b)) & \text{if } n \text{ even and } n \geqslant a-b \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathrm{Ext}^m_G(k, \nabla(a-n-b-1)) & \text{if } n \text{ odd and } n \leqslant a-b-1 \\ k & \text{if } n \text{ even, } n = a-b \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Thus the only k_2 left to show is zero is $k_2^{0,a-b} : E_2^{0,a-b} \to D_2^{1,a-b}$, but now $d_0 : E_0^{1,a-b-1} \to E_0^{1,a-b}$ is zero and so lemma 2.1 shows that $k_2^{0,a-b}$ is zero.

Hence all the k_2's in the second derived couple are zero. This in particular implies that all subsequent differentials d_f are zero for $f \geqslant 2$. Thus the E_2 page is the same as the E_∞ page.

We may argue is exactly the same way to show that the k_2's for the second derived couple for $\mathrm{Ext}^q_G(\Delta(pb+\bar{\imath}), \Delta(pa+i))$ are also zero and that the E_2 page is the same as the E_∞ page.

We have

$$\begin{aligned}
E_2^{m,n} &= H^m(G/G_1, \mathrm{Ext}^n_{G_1}(\Delta(pb+\bar{\imath}), \Delta(pa+i)) \\
&= \begin{cases} \mathrm{Ext}^m_G(k, \Delta(a-n-b-1)) & \text{if } n \text{ even and } 0 < n \leqslant a-b-1 \\ \mathrm{Ext}^m_G(\Delta(b), \Delta(a-1)) & \text{if } n = 0 \leqslant a-b-1 \\ \mathrm{Ext}^m_G(k, \nabla(n-a-b)) & \text{if } n \text{ odd and } n \geqslant a-b \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathrm{Ext}^m_G(k, \Delta(a-n-b-1)) & \text{if } n \text{ even and } 0 < n \leqslant a-b-1 \\ \mathrm{Ext}^m_G(\Delta(b), \Delta(a-1)) & \text{if } n = 0 \leqslant a-b-1 \\ k & \text{if } n \text{ even, } n = a-b \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

We may thus deduce theorem 5.1 for $p \geqslant 3$.

We now assume that $p = 2$. We have

$$\mathrm{Hom}_{G_1}(\Delta(2b), I_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes \big(\nabla(b-1)^{\mathrm{F}} \oplus \nabla(b)^{\mathrm{F}}\big) & \text{if } m \leqslant a-1 \\ \nabla(m-a)^{\mathrm{F}} \otimes \big(\nabla(b-1)^{\mathrm{F}} \oplus \nabla(b)^{\mathrm{F}}\big) & \text{if } m \geqslant a. \end{cases}$$

We also have using lemmas 3.6 and 3.7 that

$$\operatorname{Hom}_{G_1}(\Delta(2b), M_m) \cong \begin{cases} \Delta(a-m-1)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } m < a-b \\ \Delta(a-m-1)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } a-b \leqslant m \leqslant a-1 \\ \quad \oplus \nabla(m-a+b)^{\mathrm{F}} & \\ \nabla(m-a)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}} & \text{if } m \geqslant a. \end{cases}$$

We have exact sequences

$$0 \to \operatorname{Hom}_{G_1}(\Delta(2b), M_m) \xrightarrow{\delta^*_{m-1}} \operatorname{Hom}_{G_1}(\Delta(2b), I_m)$$
$$\xrightarrow{\delta^*_m} \operatorname{Hom}_{G_1}(\Delta(2b), M_{m+1}) \to \operatorname{Ext}^1_{G_1}(\Delta(2b), M_m) \to 0.$$

We assume that $a > b$ and that $m < a-b$ and we write $\operatorname{Hom}_{G_1}(\Delta(2b), I_m)$ as $A_m \oplus B_m$ where $A_m \cong \Delta(a-m-1)^{\mathrm{F}} \otimes \nabla(b)^{\mathrm{F}}$ and $B_m \cong \Delta(a-m-1)^{\mathrm{F}} \otimes \nabla(b-1)^{\mathrm{F}}$. Note that $\operatorname{Hom}_{G_1}(\Delta(2b), M_m) \cong A_m$.

We claim that the map δ^*_m is the natural projection onto B_m followed by the natural embedding into $\operatorname{Hom}_{G_1}(\Delta(2b), M_{m+1})$.

This is clear by considering the G/G_1-module structure of A_m and B_m. Since $b < a-m$ the module A_m has filtration by twisted Weyl modules, namely (starting at the bottom) $\Delta(a+b-m-1)^{\mathrm{F}}$, $\Delta(a+b-m-3)^{\mathrm{F}}$, $\Delta(a+b-m-5)^{\mathrm{F}}$, …, $\Delta(a-b-m-1)^{\mathrm{F}}$. The module B_m also has a filtration by twisted Weyl modules, namely (starting at the bottom) $\Delta(a+b-m-2)^{\mathrm{F}}$, $\Delta(a+b-m-4)^{\mathrm{F}}$, $\Delta(a+b-m-6)^{\mathrm{F}}$, …, $\Delta(a-b-m)^{\mathrm{F}}$. Thus the parities of the highest weights of the twisted Weyl modules are distinct. In particular they are in different blocks of G/G_1. Hence the map $\delta^*_{m-1} : \operatorname{Hom}_{G_1}(\Delta(2b), M_m) \to \operatorname{Hom}_{G_1}(\Delta(2b), I_m)$ is the natural injection into A_m and has image A_m (as it is a G/G_1-module homomorphism). So A_m is the kernel of δ^*_m and the map δ^*_m is as described.

We now claim that the induced map $d_0^{n,m}$ on the E_0 page is zero when $a-b$ is even, m is odd and $m < a-b$, or if $a-b$ is odd, m is even and $m < a-b$.

Now $E_0^{n,m} = \operatorname{Hom}_{G/G_1}(k, A_m \otimes J_n^{\mathrm{F}} \oplus B_m \otimes J_n^{\mathrm{F}})$. The map $d_0^{n,m}$ takes a homomorphism $\phi \in E_0^{n,m}$ to the homomorphism $(\delta^*_m \otimes \mathrm{id}) \circ \phi$ where id is the identity map on J_n^{F}. So the image of $d_0^{n,m}$ is contained in $\operatorname{Hom}_{G/G_1}(B_m \otimes J_n^{\mathrm{F}})$ if $m < a-b$.

We have

$$\operatorname{Hom}_{G/G_1}(k, B_m \otimes J_n^{\mathrm{F}}) \cong \operatorname{Hom}_{G/G_1}(B^*_m, J_n^{\mathrm{F}}) \cong 0$$

if $a-b-m$ is odd and $m < a-b$ as then we know that the modules appearing in the filtration of B^*_m have the wrong highest weight to give a G/G_1-homomorphism to J_n^{F}. (We may assume that the components of J_n^{F} are all in the block of k.) Thus the map $d_0^{n,m}$ is zero if $a-b$ is even and $m<a-b$ is odd or if $a-b$ is odd, m is even and $m<a-b$,

We get for $m \geqslant 1$ using lemmas 3.4 and 3.5

$$\begin{aligned}\operatorname{Ext}^m_{G_1}(\Delta(2b),\Delta(2a)) &\cong \operatorname{Ext}^1_{G_1}(\Delta(2b), M_{m-1})\\ &\cong \begin{cases} \Delta(a-m-b-1)^{\mathrm{F}} & \text{if } m \leqslant a-b-1\\ \nabla(m-a+b)^{\mathrm{F}} & \text{if } m \geqslant a-b.\end{cases}\end{aligned}$$

Note for $m+1<a-b$ that

$$\begin{aligned}E_1^{n,m+1} &= \operatorname{Hom}_{G/G_1}(k, \Delta(a-m-b-2)^{\mathrm{F}} \otimes J_n^{\mathrm{F}})\\ &\cong \operatorname{Hom}_{G/G_1}(\nabla(a-m-b-2)^{\mathrm{F}}, J_n^{\mathrm{F}})\end{aligned}$$

and that this is zero if $a-m-b-2$ is odd. So it is zero if $a-b$ is even and m is odd or if $a-b$ is odd and m is even.

Now the same argument as for $p \geqslant 3$ allows us to deduce that the map $d_0^{n,m+1}$ is an embedding, if $a-m-b-2$ is odd. Thus the argument then proceeds as in the $p \geqslant 3$ case and the E_2 page is the same as the E_∞ page.

Suppose $a>b$ then we have

$$\begin{aligned}E_2^{m,n} &= H^m(G/G_1, \operatorname{Ext}^n_{G_1}(\Delta(2b),\Delta(2a)))\\ &= \begin{cases} \operatorname{Ext}^m_G(\Delta(b),\Delta(a-1)) & \text{if } n=0\\ \operatorname{Ext}^m_G(k,\Delta(a-n-b-1)) & \text{if } 1 \leqslant n \leqslant a-b-1\\ \operatorname{Ext}^m_G(k,\nabla(n-a+b)) & \text{if } n \geqslant a-b \geqslant 1\end{cases}\\ &= \begin{cases} \operatorname{Ext}^m_G(\Delta(b),\Delta(a-1)) & \text{if } n=0\\ \operatorname{Ext}^m_G(k,\Delta(a-n-b-1)) & \text{if } a-b-n-1 \text{ is even and } 1 \leqslant n \leqslant a-b-1\\ k & \text{if } n=a-b \text{ and } m=0\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

since the weight $a-n-b-1$ is only in the same block as the weight 0 if it is even.

We thus deduce the following theorem.

Theorem 5.1. . *Suppose $a-b$ is even then*

$$\begin{aligned}
&\operatorname{Ext}^q_G(\Delta(pb+i),\Delta(pa+i))\\
&\qquad\cong\begin{cases}
\bigoplus_{n=0}^{n=(a-b-2)/2}\operatorname{Ext}^{q-2n-1}_G(k,\Delta(a-2n-b-2)) & \text{if } q\leqslant a-b-1\\
k & \text{if } q=a-b\\
0 & \text{otherwise.}
\end{cases}
\end{aligned}$$

Suppose $a-b$ is odd then

$$\begin{aligned}
&\operatorname{Ext}^q_G(\Delta(pb+\bar{\imath}),\Delta(pa+i))\\
&\qquad\cong\begin{cases}
\operatorname{Ext}^q_G(\Delta(b),\Delta(a-1)) & \text{if } q\leqslant a-b-1\\
\qquad\oplus\bigoplus_{n=0}^{n=(a-b-1)/2}\operatorname{Ext}^{q-2n-2}_G(k,\Delta(a-2n-b-3)) & \\
k & \text{if } q=a-b\\
0 & \text{otherwise.}
\end{cases}
\end{aligned}$$

(Remark: this is true for $q=0,1,2$ by the results of [5, 10])

Corollary 5.2. *For $a\geqslant b$ and $a-b$ odd we have*

$$\begin{aligned}
&\operatorname{Ext}^m_G(\Delta(pb+\bar{\imath}),\ \Delta(pa+i))\\
&\qquad\cong\operatorname{Ext}^{m-1}_G(\Delta(pb+\bar{\imath}),\Delta(p(a-1)+\bar{\imath}))\oplus\operatorname{Ext}^m_G(\Delta(b),\Delta(a-1))
\end{aligned}$$

For $a\geqslant b$ and $a-b$ even we have

$$\begin{aligned}
\operatorname{Ext}^m_G(\Delta(pb+i),\Delta(pa+i))&\cong\operatorname{Ext}^m_G(\Delta(j),\Delta(p(a-b)+j))\\
&\cong\operatorname{Ext}^{m-1}_G(\Delta(j),\Delta(p(a-b-1)+\bar{\jmath}))
\end{aligned}$$

where the last equality follows if $m\geqslant 1$.

Thus $\operatorname{Ext}^q_G(\Delta(pa+i),\Delta(pb+j))$ may now be determined by induction as $\operatorname{Ext}^q_G(\Delta(b),\Delta(a-1))$ will be zero if $q>a-b-1$ using [16].

It is also now possible to completely determine $\operatorname{Ext}^m_G(T(pb+j),\Delta(pa+i))$. Since $T(pb+j)\cong T(b-1)^{\mathrm{F}}\otimes T(p+j)$ we have

$$\begin{aligned}
&\operatorname{Ext}^m_G(T(pb+j),\Delta(pa+i))\cong\operatorname{Ext}^m_G(T(b-1)^{\mathrm{F}}\otimes T(p+j),\Delta(pa+i))\\
&\qquad\cong\operatorname{Ext}^m_G(T(b-1),\Delta(a-1)\otimes\delta_{i,j}\ k)\oplus\operatorname{Ext}^m_G(T(b-1),\Delta(a)\otimes\delta_{i,\bar{\jmath}}\ k)
\end{aligned}$$

and we keep on going until we get a tilting module that is p-restricted. We then use corollary 5.2.

6. APPLYING TO MODULES FOR $\mathrm{SL}_2(k)$ PART III

We now construct the G_1-injective resolution for $M^{\mathrm{F}} \otimes L(i)$. We can just take the G_1-injective resolution for $\Delta(i)$ and tensor it with M^{F}. We thus get

$$0 \to M^{\mathrm{F}} \otimes L(i) \xrightarrow{\delta_{-1}} I_0 \xrightarrow{\delta_0} I_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{m-1}} I_m \xrightarrow{\delta_m} I_{m+1} \xrightarrow{\delta_{m+1}} \cdots$$

with

$$I_m \cong \begin{cases} M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} \otimes Q(\bar{\imath}) & \text{if } m \text{ odd} \\ M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} \otimes Q(i) & \text{if } m \text{ even} \end{cases}$$

We define $M_m = \operatorname{im} \delta_{m-1}$, and $M_0 = M^{\mathrm{F}} \otimes L(i)$. We have

$$M_m \cong \begin{cases} M^{\mathrm{F}} \otimes \nabla(pm + \bar{\imath}) & \text{if } m \text{ odd} \\ M^{\mathrm{F}} \otimes \nabla(pm + i) & \text{if } m \text{ even} \end{cases}$$

If $p \geqslant 3$ then we have for $N \in \mathrm{mod}(G)$ that

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(i), I_m) \cong \begin{cases} (N^{\mathrm{F}})^* \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} & \text{if } m \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(\bar{\imath}), I_m) \cong \begin{cases} (N^{\mathrm{F}})^* \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} & \text{if } m \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

We also have

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(i), M_m) \cong \begin{cases} (N^{\mathrm{F}})^* \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} & \text{if } m \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(\bar{\imath}), M_m) \cong \begin{cases} (N^{\mathrm{F}})^* \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} & \text{if } m \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

If $p = 2$ then we have for $N \in \mathrm{mod}(G)$ that

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}}, I_m) \cong (N^{\mathrm{F}})^* \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}}.$$

We also have

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}}, M_m) \cong (N^{\mathrm{F}})^* \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}}.$$

Thus

$$\mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), I_m) \cong \mathrm{Hom}_{G_1}(N^{\mathrm{F}} \otimes L(j), M_m)$$

for all m, j and p and thus all the induced differentials d_0 on the E_0 page are zero.

We also get that

$$\begin{aligned}\operatorname{Ext}^m_{G_1}(N^{\mathrm{F}}\otimes L(j), M^{\mathrm{F}}\otimes L(i)) &\cong \operatorname{Hom}_{G_1}(N^{\mathrm{F}}\otimes L(j), M_m)\\ &\cong \begin{cases} (N^{\mathrm{F}})^*\otimes M^{\mathrm{F}}\otimes\nabla(m)^{\mathrm{F}} & \text{if } m \text{ even and } i=j\\ (N^{\mathrm{F}})^*\otimes M^{\mathrm{F}}\otimes\nabla(m)^{\mathrm{F}} & \text{if } m \text{ odd and } i=\bar{\jmath}\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus as in section 4 we get that the E_2-page for $\operatorname{Ext}^q_G(M^{\mathrm{F}}\otimes L(i), N^{\mathrm{F}}\otimes L(j))$ is the same as the E_∞ page.

We thus deduce the following theorem.

Theorem 6.1. *Let N and M be in $\operatorname{mod}(G)$ and $0\leqslant i\leqslant p-2$. Then if $p\geqslant 3$*

$$\operatorname{Ext}^q_G(N^{\mathrm{F}}\otimes L(i), M^{\mathrm{F}}\otimes L(i)))\cong\bigoplus_{\substack{n\text{ even}\\ 0\leqslant n\leqslant q}}\operatorname{Ext}^{q-n}_G(N,\nabla(n)\otimes M)$$

and

$$\operatorname{Ext}^q_G(N^{\mathrm{F}}\otimes L(i), M^{\mathrm{F}}\otimes L(\bar{\imath})))\cong\bigoplus_{\substack{n\text{ odd}\\ 0\leqslant n\leqslant q}}\operatorname{Ext}^{q-n}_G(N,\nabla(n)\otimes M)$$

If $p=2$ then

$$\operatorname{Ext}^q_G(N^{\mathrm{F}}, M^{\mathrm{F}})\cong\bigoplus_{n=0}^{q}\operatorname{Ext}^{q-n}_G(N,\nabla(n)\otimes M).$$

So this would completely determine $\operatorname{Ext}^q_G(L(pa+i), L(pb+j)$ if we knew what $\operatorname{Ext}^q_G(L(a),\nabla(n)\otimes L(b))$ was. This in principle we could calculate using the following:

$$\operatorname{Ext}^q_G(L(a),\nabla(n)\otimes L(b))\cong\operatorname{Ext}^q_G(L(a)\otimes L(b),\nabla(n))$$

since $L(b)^*\cong L(b)$. The structure of the tensor product $L(a)\otimes L(b)$ has been determined in [9]. We get a direct sum of modules of the form $M^{\mathrm{F}}\otimes T(l)$ where l is either p-restricted, in which case we could use theorem 7.1 to calculate the Ext group or $p-1\leqslant l\leqslant 2p-2$ in which case we would use the dual version of lemma 3.2 to calculate the Ext group.

We can get a complete formula if $N=\Delta(a)$ and $M=\nabla(b)$. Here $\operatorname{Ext}^m_G(\Delta(a),\nabla(n)\otimes\nabla(b))$ is zero if $m\geqslant 1$, and $\operatorname{Hom}_G(\Delta(a),\nabla(n)\otimes\nabla(b))$ is either 0 or k and it is k if and only if $\nabla(a)$ is a section of $\nabla(n)\otimes\nabla(b)$. This happens if a is of the same parity as $n+b$ and $n+b\geqslant a\geqslant\max\{n-b,b-n\}$

Thus we get

$$\operatorname{Ext}^q_G(\Delta(a)^{\mathrm{F}} \otimes L(i), \nabla(b)^{\mathrm{F}} \otimes L(j))) \cong \begin{cases} k & \text{if } q \text{ even, } a+b \text{ even, } i = j \\ & \qquad \text{and } q+b \geqslant a \geqslant \max\{q-b, b-q\} \\ k & \text{if } q \text{ odd, } a+b \text{ odd, } i = \bar{\jmath} \\ & \qquad \text{and } q+b \geqslant a \geqslant \max\{q-b, b-q\} \\ 0 & \text{otherwise} \end{cases}$$

We consider the special case where $pa+i$ and $pb+j$ are in the Jantzen region and so both a and b are at most $p-1$. We have $L(pa+i) \cong \Delta(a)^{\mathrm{F}} \otimes L(i)$ and $L(pb+j) \cong \nabla(b)^{\mathrm{F}} \otimes L(j)$ this then gives the Ext between simples and hence the coefficients of the dual Khazdhan-Lusztig polynomials.

7. APPLYING TO MODULES FOR $\mathrm{SL}_2(k)$ PART IV

We now want to use the G_1 injective resolution of $M^{\mathrm{F}} \otimes L(i)$ constructed in the previous section to determine $\operatorname{Ext}^q_G(\Delta(pb+j), M^{\mathrm{F}} \otimes L(i))$. If $p=2$ we will need to assume that M is indecomposable so that all its weights are of the same parity.

We first assume that $p \geqslant 3$. We have

$$\operatorname{Hom}_{G_1}(\Delta(pb+i), I_m) \cong \begin{cases} \nabla(b)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} & \text{if } m \text{ even} \\ \nabla(b-1)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} & \text{if } m \text{ odd} \end{cases}$$

and

$$\operatorname{Hom}_{G_1}(\Delta(pb+\bar{\imath}), I_m) \cong \begin{cases} \nabla(b-1)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} & \text{if } m \text{ even} \\ \nabla(b)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} & \text{if } m \text{ odd} \end{cases}$$

We also have

$$\operatorname{Hom}_{G_1}(\Delta(pb+i), M_m) \cong \begin{cases} \nabla(b)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} & \text{if } m \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\operatorname{Hom}_{G_1}(\Delta(pb+\bar{\imath}), M_m) \cong \begin{cases} \nabla(b)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} & \text{if } m \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

So we are in a similar situation to section 5.

We have

$$\operatorname{Hom}_{G_1}(\Delta(pb+i), I_m) \cong \operatorname{Hom}_{G_1}(\Delta(pb+i), M_m)$$

if m even and

$$\operatorname{Hom}_{G_1}(\Delta(pb+\bar{\imath}), I_m) \cong \operatorname{Hom}_{G_1}(\Delta(pb+\bar{\imath}), M_m)$$

if m odd. Thus the induced differential δ^*_m on the Hom_{G_1} complex is zero in the above cases.

We have for $m \geqslant 1$

$$\begin{aligned}\operatorname{Ext}^m_{G_1}(\Delta(pb+i), M^{\mathrm{F}} \otimes L(i)) &\cong \operatorname{Ext}^1_{G_1}(\Delta(pb+i), M_{m-1}) \\ &\cong \begin{cases} M^{\mathrm{F}} \otimes \nabla(m+b)^{\mathrm{F}} & \text{if } m \text{ even} \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

$$\begin{aligned}\operatorname{Ext}^m_{G_1}(\Delta(pb+\bar{\imath}), M^{\mathrm{F}} \otimes L(i)) &\cong \operatorname{Ext}^1_{G_1}(\Delta(pb+\bar{\imath}), M_{m-1}) \\ &\cong \begin{cases} M^{\mathrm{F}} \otimes \nabla(m+b)^{\mathrm{F}} & \text{if } m \text{ odd} \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

We now consider the exact couple corresponding to $\operatorname{Ext}^q_G(\Delta(pb+j), M^{\mathrm{F}} \otimes L(i))$. The above shows that the differentials $d_0 : E_0^{m,n} \to E_0^{m,n+1}$ are zero if n is even and $i = j$ or if n is odd and $i = \bar{\jmath}$.

Since $\operatorname{Ext}^{n+1}_{G_1}(\Delta(pb+j), M^{\mathrm{F}} \otimes L(i))$ is zero if n is even and $i = j$ or if n is odd and $i = \bar{\jmath}$, we have $E_2^{m,n+1}$ is zero. The same argument as in section 5 now shows that all the k_2's in the second derived couple are zero. Thus the E_2 page is the same as the E_∞ page.

We thus have proved theorem 7.1 for $p \geqslant 3$.

We now assume that $p = 2$. We have

$$\operatorname{Hom}_{G_1}(\Delta(2b), I_m) \cong \nabla(b)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} \oplus \nabla(b-1)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}}$$

We also have

$$\operatorname{Hom}_{G_1}(\Delta(2b), M_m) \cong \nabla(b)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}}$$

So again we are in a similar situation to section 5.

We have that $\operatorname{Hom}_{G_1}(\Delta(2b), M_m)$ is isomorphic to a direct summand of $\operatorname{Hom}_{G_1}(\Delta(2b), I_m)$. If $M = k$ then we know by a similar argument to that of section 5 that the induced homomorphism δ_m^* is the projection onto the other summand of $\operatorname{Hom}_{G_1}(\Delta(2b), I_m)$. Tensoring with M^{F} does not change this. (The isomorphism $\operatorname{Hom}_{G_1}(\Delta(2b), X \otimes M^{\mathrm{F}}) \cong \operatorname{Hom}_{G_1}(\Delta(2b), X) \otimes M^{\mathrm{F}}$ is canonical.)

So the induced map $d_0^{n,m}$ on the E_0 page takes the homomorphism

$$\phi \in \operatorname{Hom}_{G/G_1}(k, \operatorname{Hom}_{G_1}(\Delta(2b), I_m) \otimes J_n^{\mathrm{F}})$$

to the map $(\delta_m^* \otimes \mathrm{id}) \circ \phi$ where id is the identity map on J_n^{F}. The map $d_0^{n,m}$ has image contained in $\operatorname{Hom}_{G/G_1}(k, \nabla(b-1)^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes J_n^{\mathrm{F}})$. In particular the map $d_0^{n,m}$ is zero if $\operatorname{Hom}_{G/G_1}(k, \nabla(b-1)^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes J_n^{\mathrm{F}})$ is also zero. Now

$$\operatorname{Hom}_{G/G_1}(k, \nabla(b-1)^{\mathrm{F}} \otimes \nabla(m)^{\mathrm{F}} \otimes M^{\mathrm{F}} \otimes J_n^{\mathrm{F}}) \cong \operatorname{Hom}_G(\Delta(b-1) \otimes \Delta(m), M \otimes J_n)$$

So this can only be non-zero if $b-1+m$ is of the same parity as the weights in $M\otimes J_n$ which is the same parity of the weights in M as J_n is in the block of k. Thus the map $d_0^{n,m}$ is zero if $b+m$ is of the same parity as the weights in M. Let $\epsilon(M)$ be 0 if the weights of M are even and 1 if the weights of M are odd.

We have for $m\geqslant 1$

$$\begin{aligned}\mathrm{Ext}^m_{G_1}(\Delta(2b),M^{\mathrm{F}}) &\cong \mathrm{Ext}^1_{G_1}(\Delta(2b),M_{m-1})\\ &\cong M^{\mathrm{F}}\otimes\nabla(m+b)^{\mathrm{F}}.\end{aligned}$$

We now consider the E_1 page corresponding to $\mathrm{Ext}^q_G(\Delta(2b),M^{\mathrm{F}})$. The above shows that the differentials $d_0: E_0^{n,m}\to E_0^{n,m+1}$ are zero if $\epsilon(M)+b+m$ is even. Now

$$E_1^{n,m+1}=\mathrm{Hom}_{G/G_1}(k,M^{\mathrm{F}}\otimes\nabla(m+b+1)^{\mathrm{F}}\otimes J_n^{\mathrm{F}})$$

is zero if $\epsilon(M)+m+b+1$ is odd.

The same argument as in section 5 now shows that all the k_2's in the second derived couple are zero. Thus the E_2 page is the same as the E_∞ page. The assumption that M is indecomposable can now be removed as the Ext groups split into direct sums of the Ext's of the indecomposable components. We have thus proved the following theorem.

Theorem 7.1. *Let M be in $\mathrm{mod}(G)$, $b\in\mathbb{N}$ and $0\leqslant i\leqslant p-2$. If $p\geqslant 3$ then*

$$\mathrm{Ext}^q_G(\Delta(pb+i),M^{\mathrm{F}}\otimes L(i))\cong\bigoplus_{\substack{n\text{ even}\\ 0\leqslant n\leqslant q}}\mathrm{Ext}^{q-n}_G(\Delta(n+b),M)$$

and

$$\mathrm{Ext}^q_G(\Delta(pb+i),M^{\mathrm{F}}\otimes L(\bar{\imath}))\cong\bigoplus_{\substack{n\text{ odd}\\ 0\leqslant n\leqslant q}}\mathrm{Ext}^{q-n}_G(\Delta(n+b),M)$$

If $p=2$ then

$$\mathrm{Ext}^q_G(\Delta(2b),M^{\mathrm{F}})\cong\bigoplus_{n=0}^{n=q}\mathrm{Ext}^{q-n}_G(\Delta(n+b),M).$$

Corollary 7.2. *Let $b\in\mathbb{N}$, $0\leqslant i\leqslant p-2$, $q\in\mathbb{N}$ $M\in\mathrm{mod}(G)$. If $p\geqslant 3$ then we have*

$$\mathrm{Ext}^q_G(\Delta(pb+i),M^{\mathrm{F}}\otimes L(i))\cong\mathrm{Ext}^{q-1}_G(\Delta(p(b+1)+\bar{\imath}),M^{\mathrm{F}}\otimes L(i))\oplus\mathrm{Ext}^q_G(\Delta(b),M)$$

$$\mathrm{Ext}^q_G(\Delta(pb+i),M^{\mathrm{F}}\otimes L(\bar{\imath}))\cong\mathrm{Ext}^{q-1}_G(\Delta(p(b+1)+\bar{\imath}),M^{\mathrm{F}}\otimes L(\bar{\imath}))$$

where Ext^{-1} is interpreted as the zero module.

If $p=2$ then

$$\mathrm{Ext}^q_G(\Delta(2b),M^{\mathrm{F}})\cong\mathrm{Ext}^{q-1}_G(\Delta(2b+2),M^{\mathrm{F}})\oplus\mathrm{Ext}^q_G(\Delta(b),M).$$

Thus if $M=\nabla(a)$, $\Delta(a)$ or $L(a)$ we may now completely determine $\operatorname{Ext}^q_G(\Delta(pb+j), M^{\mathrm{F}}\otimes L(i))$ and $\operatorname{Ext}^q_G(M^{\mathrm{F}}\otimes L(i), \Delta(pb+j))$ using the results of the previous sections. (This is as $\operatorname{Ext}^q_G(\Delta(a), M)$ will be zero if q is larger than the highest weight of M divided by p using lemma 3.8.)

8. The quantum case

All of the results may now be readily generalised to the quantum case, where we use the quantum group of [7]. We need to remember that the Frobenius morphism now goes from the classical group $\mathrm{GL}_2(k)$ to the quantum group q-$\mathrm{GL}_2(k)$ and so when we 'untwist' a twisted module we get a module for GL_2 and not q-$\mathrm{GL}_2(k)$. The G_1 cohomology looks slightly different as we need to keep track of the degree of the modules. Most of the results of section 3 have already been generalised to the quantum case and appear in [4] and [5]. The other results may be proved in a similar fashion. Thus the resolutions we constructed in sections 4 and 6 may be generalised to the quantum case and we get essentially the same spectral sequences appearing. Thus the quantum version of sections 4 to 7 all hold. Below are stated the quantum versions of the main corollaries of these sections for the convenience of the reader. We will distinguish the modules for classical $\mathrm{GL}_2(k)$ and the quantum group q-$\mathrm{GL}_2(k)$ by putting a bar on the modules for the classical groups. We define $\bar{\imath}=l-i-2$ where q is a primitive lth root of unity.

Theorem 8.1. *Let $\overline{N}\in\operatorname{mod}(\mathrm{GL}_2(k))$ be indecomposable and let b be the highest weight of $\overline{N}$. For $a\geqslant b$ with $a-b$ odd, $0\leqslant i\leqslant l-2$ and $m\in\mathbb{N}$ we have*

$$\begin{aligned}&\operatorname{Ext}^m_{q-\mathrm{GL}_2(k)}(\overline{N}^{\mathrm{F}}\otimes L(\bar{\imath}+d,d), \Delta(la+i,0))\\ &\qquad\cong \operatorname{Ext}^{m-1}_{q-\mathrm{GL}_2(k)}(\overline{N}^{\mathrm{F}}\otimes L(\bar{\imath}+d,d), \Delta(la-1,i+1))\oplus \operatorname{Ext}^m_{\mathrm{GL}_2(k)}(\overline{N}\otimes L(f,f), \overline{\Delta}(a-1,0))\end{aligned}$$

where Ext^{-1} is interpreted as the zero module, $f=\frac{a-b-1}{2}$ and $d=lf+i+1$.

For $a\geqslant b$ with $a-b$ even, $0\leqslant i\leqslant l-2$ and $m\in\mathbb{N}$ we have

$$\operatorname{Ext}^m_{q-\mathrm{GL}_2(k)}(\overline{N}^{\mathrm{F}}\otimes L(i+d,d), \Delta(la+i,0))\cong \operatorname{Ext}^{m-1}_{q-\mathrm{GL}_2(k)}(\overline{N}^{\mathrm{F}}\otimes L(i+d,d), \Delta(la-1,i+1))$$

where $d=l(\frac{a-b}{2})$.

Theorem 8.2. *For $a\geqslant b$ with $a-b$ odd, $0\leqslant i\leqslant l-2$ and $m\in\mathbb{N}$ we have*

$$\begin{aligned}&\operatorname{Ext}^m_{q-\mathrm{GL}_2(k)}(\Delta(lb+\bar{\imath}+d,d), \Delta(la+i,0))\\ &\qquad\cong \operatorname{Ext}^{m-1}_{q-\mathrm{GL}_2(k)}(\Delta(lb+\bar{\imath}+d,d), \Delta(la-1,i+1))\oplus \operatorname{Ext}^m_{\mathrm{GL}_2(k)}(\overline{\Delta}(b+f,f), \overline{\Delta}(a-1,0))\end{aligned}$$

where Ext^{-1} is interpreted as the zero module, $f=\frac{a-b-1}{2}$ and $d=lf+i+1$.

For $a\geqslant b$ with $a-b$ even, $0\leqslant i\leqslant l-2$ and $m\in\mathbb{N}$ we have

$$\operatorname{Ext}^m_{q-\mathrm{GL}_2(k)}(\Delta(lb+i+d,d), \Delta(la+i,0))\cong \operatorname{Ext}^m_{q-\mathrm{GL}_2(k)}(\Delta(i+d,d), \Delta(l(a-b)+i,0))$$

where $d = l(\frac{a-b}{2})$. *If* $m \geqslant 1$ *then also*

$$\mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\Delta(i+d,d),\Delta(l(a-b)+i,0)) \cong \mathrm{Ext}^{m-1}_{q-\mathrm{GL}_2(k)}(\Delta(i+d,d),\Delta(l(a-b)-1,i+1)).$$

Theorem 8.3. *Let* $\overline{M} \in \mathrm{mod}(\mathrm{GL}_2(k))$ *be indecomposable and let* a *be the highest weight of* $\overline{M}$. *For* $a \geqslant b$ *with* $a-b$ *even,* $0 \leqslant i \leqslant l-2$ *and* $m \in \mathbb{N}$ *we have*

$$\begin{aligned}&\mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\Delta(lb+i+d,d),\overline{M}^{\mathrm{F}} \otimes L(i,0))\\ &\quad\cong \mathrm{Ext}^{m-1}_{q-\mathrm{GL}_2(k)}(\Delta(lb+l-1+d,d-l+i+1),\overline{M}^{\mathrm{F}} \otimes L(i,0)) \oplus \mathrm{Ext}^m_{\mathrm{GL}_2(k)}(\overline{\Delta}(b+f,f),\overline{M})\end{aligned}$$

where Ext^{-1} *is interpreted as the zero module,* $f = \frac{a-b}{2}$ *and* $d = lf+i+1$.

For $a \geqslant b$ *with* $a-b$ *odd,* $0 \leqslant i \leqslant l-2$ *and* $m \in \mathbb{N}$ *we have*

$$\begin{aligned}&\mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\Delta(lb+i+d,d),\overline{M}^{\mathrm{F}} \otimes L(\bar{\imath}+d,d))\\ &\qquad\qquad\cong \mathrm{Ext}^{m-1}_{q-\mathrm{GL}_2(k)}(\Delta(lb+l-1+d,d-l+i+1),\overline{M}^{\mathrm{F}} \otimes L(\bar{\imath}+d,d))\end{aligned}$$

where $d = l(\frac{a-b-1}{2})$.

Block considerations mean that the above cases are the only possible non-zero Ext groups of that form.

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