HOMOTOPY CLASSIFICATION OF PD^3–PAIRS

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ABSTRACT. In 1977 Hendriks [6] gave a complete system of homotopy invariants of PD^3–complexes. Turaev provided an alternative proof of Hendriks' classification theorem in [12]. We generalise Turaev's proof to the case of PD^3–pairs.

1. PRELIMINARIES

1.1. **Products Relating Homology and Cohomology.** Given a group G, let Λ be the integral group ring of G, that is, $\Lambda := \mathbb{Z}[G]$. Further let $\mathrm{aug} : \Lambda \to \mathbb{Z}$ denote the augmentation homomorphism determined by $\mathrm{aug}(g) := 1$ for all $g \in G$. The kernel I of the augmentation homomorphism is called the augmentation ideal.

$$I \rightarrowtail \Lambda \twoheadrightarrow \mathbb{Z}$$

A cohomology class, $\omega \in \mathrm{H}^1(G, \mathbb{Z}/2\mathbb{Z})$, gives rise to a function from G to $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ which we also denote by ω. This function, in turn, gives rise to the anti–isomorphism

$$\overline{\ } : \Lambda \longrightarrow \Lambda;\ \lambda \longmapsto \overline{\lambda}$$

determined by

$$\overline{g} := (-1)^{\omega(g)} g^{-1} \quad \text{for } g \in G.$$

We may define a right action of Λ on $\mathbb{Z}$ by

$$n.\lambda := \mathrm{aug}(\overline{\lambda})n, \quad n \in \mathbb{Z},\ \lambda \in \Lambda,$$

and denote the right Λ–modules thus obtained by $\mathbb{Z}^\omega$.

Fixing $\omega \in \mathrm{H}^1(G, \mathbb{Z}/2\mathbb{Z})$, we may associate a left Λ–module with every right Λ–module and vice versa by means of the anti–isomorphism $\overline{\ }$. Namely, given a right Λ–module A and a left Λ–module B, define a left action on the set underlying A and a right action on the set underlying B by

$$\lambda.a := a.\overline{\lambda} \quad \text{for } a \in A,\ \lambda \in \Lambda;$$
$$b.\lambda := \overline{\lambda}.b \quad \text{for } b \in B,\ \lambda \in \Lambda.$$

We denote the modules thus obtained by $\overline{A}$ and $\overline{B}$ respectively.

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Given chain complexes C and D of right and left Λ–modules respectively, the chain complex $C \otimes_\Lambda D$ of abelian groups is given by

$$(C \otimes_\Lambda D)_n := \bigoplus_{i+j=n} C_i \otimes_\Lambda D_j$$
$$\partial = \partial^{C\otimes_\Lambda D} : (C \otimes_\Lambda D)_n \longrightarrow (C \otimes_\Lambda D)_{n-1}$$

where

$$\partial^{C\otimes_\Lambda D}|_{C_i\otimes_\Lambda D_j} = \partial^C \otimes \mathrm{id} + (-1)^i \mathrm{id} \otimes \partial^D.$$

We write just $C \otimes D$ for $C \otimes_\Lambda D$ when $\Lambda = \mathbb{Z}$. Furthermore we drop the subscript Λ in the tensor product of elements or Λ–morphisms.

There is a homomorphism

$$\alpha : \mathrm{H}C \otimes_\Lambda \mathrm{H}D \longrightarrow \mathrm{H}(C \otimes_\Lambda D) \quad \text{defined by}$$
$$\alpha([x]\otimes[y]) := [x\otimes y] \quad \text{for } x \in ZC,\ y \in ZD.$$

For details see [3] or [11].

Furthermore, given chain complexes C and D of left Λ–modules, the chain complex $\mathrm{Hom}_\Lambda(C, D)$ of abelian groups is defined by

$$\mathrm{Hom}_\Lambda(C, D)_n := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_\Lambda(C_i, D_{i+n})$$
$$\partial : \mathrm{Hom}_\Lambda(C, D)_n \longrightarrow \mathrm{Hom}_\Lambda(C, D)_{n-1}$$

where

$$\partial\big((f_i)_{i\in\mathbb{Z}}\big) := (\partial^D f_i)_{i\in\mathbb{Z}} - \big((-1)^n f_i \partial^C\big)_{i\in\mathbb{Z}}.$$

Given a chain complex C of left Λ–modules, a right Λ–module A and a left Λ–module B we put

$$\mathrm{H}_n(C, A) := \mathrm{H}_n(A\otimes_\Lambda C)$$
$$\mathrm{H}^k(C, B) := \mathrm{H}_{-k}(\mathrm{Hom}_\Lambda(C, B))$$

where we treat both A and B as chain complexes concentrated at level zero.

Now let C and D be two chain complexes of left Λ–modules and let M be a right Λ–module. On the level of chains the twisted slant operation is a group homomorphism

$$/ : \mathrm{Hom}_\Lambda(C, \overline{M})\otimes\big(\mathbb{Z}^\omega\otimes_\Lambda(C\otimes D)\big) \longrightarrow M\otimes_\Lambda D,$$

where G acts on $C\otimes D$ diagonally, that is,

$$\sum_{g\in G} n_g g(c_i\otimes d_j) := \sum_{g\in G} n_g (gc_i\otimes gd_j)$$

for $\sum_{g\in G} n_g g \in \Lambda$ and $c_i\otimes d_j \in C_i\otimes D_j$. Namely, the twisted slant operation

$$/ : \big(\mathrm{Hom}_\Lambda(C, \overline{M})\big)_{-k}\otimes\big(\mathbb{Z}^\omega\otimes_\Lambda(C\otimes D)\big)_n \longrightarrow M\otimes_\Lambda D_{n-k}$$

is determined by

$$\varphi/(m\otimes \sum_{i+j=n} c_i\otimes d_j) := m\varphi(c_k)\otimes d_{n-k}$$

where we assume from now on that elements live in the appropriate abode.

Lemma 1.1. *The slant operation is a well–defined chain map.*

Proof. Take $\lambda = \sum_{g\in G} n_g g$. Then

$$\begin{aligned}
\varphi/(m\otimes\lambda.\sum_{i+j=n} c_i\otimes d_j) &= \varphi/(m\otimes \sum_{i+j=n}\sum_{g\in G} n_g(g.c_i\otimes g.d_j) = m\sum_{g\in G} n_g\varphi(g.c_k)\otimes g.d_{n-k}\\
&= m\sum_{g\in G} n_g(g.\varphi(c_k)).g\otimes d_{n-k} = m\sum_{g\in G} n_g(-1)^{\omega(g)}g^{-1}(g\varphi(c_k))\otimes d_{n-k}\\
&= m\ \mathrm{aug}(\overline{\lambda})\varphi(c_k)\otimes d_{n-k} = \varphi/(m.\lambda\otimes \sum_{i+j=n} c_i\otimes d_j),
\end{aligned}$$

which shows that the twisted slant operation is well–defined. Furthermore

$$\begin{aligned}
/\partial\big(\varphi\otimes(m\otimes \sum_{i+j=n} c_i\otimes d_j)\big) &= /\big(\partial\varphi\otimes(m\otimes \sum_{i+j=n} c_i\otimes d_j) + (-1)^{-k}\varphi\otimes(m\otimes \sum_{i+j=n} \partial c_i\otimes d_j)\\
&\quad +(-1)^{i-k}\varphi\otimes(m\otimes \sum_{i+j=n} c_i\otimes\partial d_j)\big)\\
&= /\big((-1)^{-k+1}\varphi\circ\partial\otimes(m\otimes \sum_{i+j=n} d_i\otimes c_j)\\
&\quad +(-1)^{-k}\varphi\otimes(m\otimes \sum_{i+j=n} \partial c_i\otimes d_j)\\
&\quad +(-1)^{i-k}\varphi\otimes(m\otimes \sum_{i+j=n} c_i\otimes\partial d_j)\big)\\
&= m(-1)^{-k+1}\varphi(\partial(c_{k+1}))\otimes d_{n-k-1} + m(-1)^{-k}\varphi(\partial(c_{k+1}))\otimes d_{n-k-1}\\
&\quad +m\varphi(c_k)\otimes\partial d_{n-k}\\
&= \partial\big(\varphi/(m\otimes \sum_{i+j=n} c_i\otimes d_j)\big),
\end{aligned}$$

which shows that the twisted slant operation is indeed a chain map □

Passing to homology and composing with α, we obtain the twisted slant operation

$$/ : \mathrm{H}^k(C,\overline{M})\otimes\mathrm{H}_n(C\otimes D,\mathbb{Z}^\omega)\to \mathrm{H}_{n-k}(D,M).$$

An augmented chain complex of left Λ–modules is a chain complex P of left Λ–modules with an augmentation $\varepsilon : C \twoheadrightarrow \mathbb{Z}$. An equivariant diagonal Δ on P is a chain map

$$\Delta : P \longrightarrow P\otimes P$$

of Λ–complexes, where G acts on $P\otimes P$ diagonally, such that $(\varepsilon\otimes\mathrm{id})\Delta(c) = (\mathrm{id}\otimes\varepsilon)\Delta(c) = c$. A map of $\varphi : Q \rightarrowtail P$ of augmented chain complexes of left Λ–modules with equivariant diagonal is a chain map such that both

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & P \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ Q\otimes Q & \xrightarrow{\varphi\otimes\varphi} & P\otimes P \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Q & \xrightarrow{\varphi} & P \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} \end{array}$$

commute.

Now assume that Q is a subcomplex of the augmented chain complex P with equivariant diagonal such that the inclusion $\iota : Q \rightarrowtail P$ is a map of augmented chain complexes with equivariant diagonal where the augmentation and equivariant diagonal on Q are given by the restrictions of the augmentation and equivariant diagonal of P to Q. Put $D := P/Q$ so that $Q \rightarrowtail P \twoheadrightarrow D$ is a short exact sequence of chain complexes. Then we may define relative equivariant diagonals

$$\Delta_{\mathrm{rel}} : D \longrightarrow P\otimes D \quad \text{and} \quad {\Delta'}_{\mathrm{rel}} : D \longrightarrow D\otimes P$$

by setting

$$\Delta_{\mathrm{rel}}(d) := (\mathrm{id}\otimes\pi)(\Delta(p)) \quad \text{where } d = \pi(p) \quad \text{and}$$
$${\Delta'}_{\mathrm{rel}}(d) := (\pi\otimes\mathrm{id})(\Delta(p)) \quad \text{where } d = \pi(p).$$

Lemma 1.2. *The relative diagonals Δ_{rel} and ${\Delta'}_{\mathrm{rel}}$ are a well–defined chain maps of Λ–modules where G acts diagonally on $P\otimes D$ and $D\otimes P$ respectively.*

Proof. Given $p, p' \in P$ with $\pi(p) = \pi(p') = d$ we have $p - p' \in \ker\pi = \mathrm{im}\iota$. Thus there is a $q \in Q$ with $\iota(q) = p - p'$. Note that $\Delta(q) \in Q\otimes Q$ so that $(\mathrm{id}\otimes\pi)(\Delta(q)) = 0$. Hence $(\mathrm{id}\otimes\pi)(\Delta(p')) = (\mathrm{id}\otimes\pi)(\Delta(p) + \Delta(q)) = (\mathrm{id}\otimes\pi)(\Delta(p))$ showing that Δ_{rel} is well–defined.

Furthermore, given $d \in D$ with $d = \pi(p)$ and $\Delta(p) = \sum p_i\otimes p'_j$ we have

$$\begin{aligned} \partial\Delta_{\mathrm{rel}}(d) &= \partial(\mathrm{id}\otimes\pi)(\Delta(p)) = \partial\sum p_i\otimes\pi(p'_j) \\ &= \sum \partial p_i\otimes\pi(p'_j) + \sum(-1)^i p_i\otimes\partial\pi(p'_j) = \sum \partial p_i\otimes\pi(p'_j) + \sum(-1)^i p_i\otimes\pi\partial(p'_j) \\ &= (\mathrm{id}\otimes\pi)(\partial\Delta(p)) = (\mathrm{id}\otimes\pi)(\Delta\partial(p)) \\ &= \Delta_{\mathrm{rel}}\pi\partial(p) = \Delta_{\mathrm{rel}}\partial\pi(p) \\ &= \Delta_{\mathrm{rel}}\partial(d) \end{aligned}$$

showing that Δ_{rel} is a chain map. The proof for ${\Delta'}_{\mathrm{rel}}$ is analogous. □

The relative twisted cap product is the composition of the relative diagonal and the twisted slant operation. As both the relative diagonal and the twisted slant operation are chain maps, the relative twisted cap product is also a chain map. On the level of chains we obtain, for a right Λ–module M,

$$\cap : \mathrm{Hom}_\Lambda(P, \overline{M})_{-k}\otimes(\mathbb{Z}^\omega\otimes_\Lambda D)_n \to (M\otimes_\Lambda D)_{n-k}$$
$$\varphi\cap(m\otimes d) := \varphi/(m\otimes\Delta_{\mathrm{rel}}(d)).$$

Passing to homology and composing with α we obtain the relative twisted cap product

$$\cap : \mathrm{H}^k(P, \overline{M}) \otimes \mathrm{H}_n(D, \mathbb{Z}^\omega) \to \mathrm{H}_{n-k}(D, M).$$

Now suppose that all chain complexes are complexes of free Λ–modules. Then the short exact sequence $Q \overset{\iota}{\rightarrowtail} P \overset{\pi}{\twoheadrightarrow} D$ splits and stays split short exact when we tensor with right modules. Let δ_* and δ'_* be the connecting homomorphisms of the short exact sequences $\mathbb{Z}^\omega \otimes Q \rightarrowtail \mathbb{Z}^\omega \otimes P \twoheadrightarrow \mathbb{Z}^\omega \otimes D$ and $M \otimes Q \rightarrowtail M \otimes P \twoheadrightarrow M \otimes D$ respectively.

Lemma 1.3. *$\delta'_*(x \cap y) = \iota^* x \cap \delta_* y$, for $x \in H^k(P, \overline{M})$ and $y \in H_n(D, \mathbb{Z}^\omega)$.*

Proof. Take a cycle $m \otimes d \in \mathrm{Hom}_\Lambda(P_k, \overline{M})$ and a cocycle $\varphi \in \mathbb{Z}^\omega \otimes_\Lambda D_n$ representing x and y respectively. Furthermore take $p \in P$ and $q \in Q$ such that $m \otimes d = m \otimes \pi(p)$ and $m \otimes \partial p = m \otimes \iota(q)$ and suppose $\Delta(q) = \sum q_i \otimes q'_j$. Then $\iota^* x \cap \delta_* y$ is represented by

$$\varphi \circ \iota / (m \otimes \Delta q) = \varphi \circ \iota / (m \otimes \sum q_i \otimes q'_j) = m\varphi(\iota(q'_k)) \otimes q_{n-k}$$

and

$$\begin{aligned} (\mathrm{id} \otimes \iota)\big(\varphi \circ \iota / (m \otimes \Delta q)\big) &= m\varphi(\iota(q'_k)) \otimes \iota(q_{n-k}) = \varphi / (m \otimes (\iota \otimes \iota)\Delta q) \\ &= \varphi / (m \otimes \Delta\iota(q)) = \varphi / (m \otimes \Delta\partial p) \\ &= \varphi / (m \otimes \partial\Delta p) = \partial(\varphi / (m \otimes \Delta p)) \end{aligned}$$

as φ is a cocycle. Furthermore $x \cap y$ is represented by

$$\varphi / (m \otimes \Delta_{\mathrm{rel}} d) = \varphi / \big(m \otimes (\pi \otimes \mathrm{id})\Delta p\big),$$

so that $\delta'_*(x \cap y)$ is represented by $m \otimes a$ where

$$(\mathrm{id} \otimes \iota)(m \otimes a) = \partial\big(\varphi / (m \otimes \Delta p)\big).$$

As $(\mathrm{id} \otimes \iota)$ is a monomorphism we may conclude that $\delta'_*(x \cap y) = \iota^* x \cap \delta_* y$. □

If M is a Λ–bimodule then so is $\overline{M}$ and $\mathrm{Hom}_\Lambda(P, \overline{M})$ carries a right Λ–module structure given by

$$(\varphi.\lambda)(p) := \varphi(p).\lambda$$

for $\varphi \in \mathrm{Hom}_\Lambda(P, \overline{M})$, $p \in P$ and $\lambda \in \Lambda$. Thus $\overline{\mathrm{Hom}_\Lambda(P, \overline{M})}$ is a left Λ–module and we may view $\overline{\mathrm{Hom}_\Lambda(-, \overline{M})}$ as a functor from the category of chain complexes of left Λ–modules to itself.

Given $1 \otimes x \in Z_n(\mathbb{Z}^\omega \otimes_\Lambda D)$ with $\Delta_{\mathrm{rel}} x = \sum p_i \otimes d_j$ and $\varphi \in \overline{\mathrm{Hom}_\Lambda(P_k, \overline{M})}$ we obtain

$$\partial(\varphi \cap 1 \otimes x) = \partial\varphi \cap 1 \otimes x + (-1)^k \varphi \cap \partial(1 \otimes x) = \partial\varphi \cap 1 \otimes x.$$

And, for $\lambda \in \Lambda$,

$$\begin{aligned} (\lambda.\varphi) \cap (1 \otimes x) &= \varphi.\overline{\lambda} / 1 \otimes \Delta_{\mathrm{rel}} x = \varphi.\overline{\lambda} / 1 \otimes \sum p_i \otimes d_j \\ &= (\varphi.\overline{\lambda})(p_k) \otimes d_{n-k} = \varphi(p_k).\overline{\lambda} \otimes d_{n-k} \\ &= \lambda.\varphi(p_k) \otimes d_{n-k} = \lambda.(\varphi(p_k) \otimes d_{n-k}) \\ &= \lambda.(\varphi \cap 1 \otimes x), \end{aligned}$$

where Λ acts on $M \otimes_\Lambda D$ from the left via

$$\lambda.(m \otimes d) := (\lambda.m) \otimes d$$

for $m \in M,\ d \in D$ and $\lambda \in \Lambda$. Summarizing we obtain the following

Lemma 1.4. *For* $1 \otimes x \in Z_n(\mathbb{Z}^\omega \otimes_\Lambda D)$

$$\cap 1 \otimes x : \overline{\mathrm{Hom}_\Lambda(P, \overline{M})} \longrightarrow M \otimes_\Lambda D$$

is a chain map of chain complexes of left Λ*–modules.*

1.2. **Projective Homotopy Theory of Modules.** In this section Λ may be any ring with unit. Unless otherwise specified, $A, B, \ldots$ will denote left Λ–modules and $\varphi, \psi, \ldots$ will denote Λ–morphisms.

Definition 1.5. *The* Λ*–morphism* $\varphi : A \to B$ *is nullhomotopic, written as* $\varphi \simeq 0$*, if and only if there is a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ & \searrow \quad \nearrow & \\ & P & \end{array} \tag{1}$$

where P *is a projective* Λ*–module.*

As every projective Λ–module is a direct summand of a free Λ–module the existence of Diagram 1 is equivalent to the existence of a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ & \searrow \quad \nearrow & \\ & \Lambda^m. & \end{array}$$

If $\varepsilon : PA \twoheadrightarrow A$ is an epimorphism and PA is projective then PA is called a path space over A (in analogy to topological homotopy theory). Since the category of left Λ–modules has enough projectives, every Λ–module A has a path space. It is not difficult to show that a Λ–morphism $\varphi : A \to B$ is nullhomotopic if and only if it factors through a given path space $\varepsilon : PB \twoheadrightarrow B$ of B, that is, if and only if there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ & \searrow \quad \nearrow & \\ & PB. & \end{array}$$

Thus, if $\varphi : A \to B$ factors through one particular path space of B, it factors through any path space of B. Hence

$$\mathrm{Nhom}_\Lambda(\mathrm{A}, \mathrm{B}) := \{\varphi : \mathrm{A} \to \mathrm{B} \mid \varphi \simeq 0\ \}$$

is a subgroup of $\mathrm{Hom}_\Lambda(\mathrm{A}, \mathrm{B})$.

Definition 1.6. *Two Λ–morphisms φ and ψ are homotopic if and only if $\varphi - \psi \simeq 0$. Furthermore the group*

$$[A, B] := \mathrm{Hom}_\Lambda(\mathrm{A}, \mathrm{B})/\mathrm{Nhom}_\Lambda(\mathrm{A}, \mathrm{B})$$

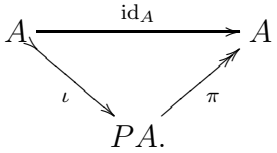
of homotopy classes of Λ–morphisms is called the homotopy group from A to B.

It is not difficult to show that homotopy respects composition of Λ–morphisms. Thus we obtain a category, called the projective homotopy category (PHOM) whose objects are left Λ–modules and whose morphisms are homtopy classes of Λ–morphisms. Furthermore $[A, B]$ is functorial in both variables and preserves direct products.

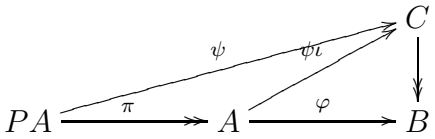
As in topological homotopy theory, we say that $\varphi : A \to B$ is a homotopy equivalence if and only if there is a Λ–morphism $\psi : B \to A$ such that $\varphi\psi \simeq \mathrm{id}_B$ and $\psi\varphi \simeq \mathrm{id}_A$. If there is a homotopy equivalence $\varphi : A \to B$ then A and B are said to be homotopy equivalent and we denote the set of homotopy equivalences from A to B by $\mathrm{Equi}(A, B)$.

Lemma 1.7. *A Λ–module A is projective if and only if $[X, A] = 0$ for every Λ–module X.*

Proof. We only need to show that $[X, A] = 0$ for every Λ–module X implies that A is projective. So assume that $[X, A] = 0$ for every Λ–module X. Then $[A, A] = 0$ which implies $\mathrm{id}_A \simeq 0$, that is, id_A factors through a path space $PA \twoheadrightarrow A$ of A. Thus there is a commutative diagram



Now let $\varphi : A \to B$ be a Λ–morphism and let $\varepsilon : C \twoheadrightarrow B$ be an epimorphism. Since PA is projective there is a Λ–morphism $\psi : PA \to C$ such that $\varepsilon\psi = \varphi\pi$. Hence $\varepsilon\psi\iota = \varphi\pi\iota = \varphi$, showing that A is projective.



□

Given a path space $\varepsilon : PB \twoheadrightarrow B$, any $\varphi : A \to B$ factors as

$$A \rightarrowtail^{\iota} A \oplus PB \overset{\varphi'}{\twoheadrightarrow} B,$$

where φ' is defined by $\varphi'(a, p) = \varphi(a) + \varepsilon(p)$ for $a \in A$ and $p \in PB$.

Theorem 1.8. *A homotopy equivalence $\varphi : A \to B$ factors as*

$$A \rightarrowtail^{\iota} A \oplus P \overset{\tilde{\varphi}}{\longrightarrow} B \oplus Q \overset{\pi}{\twoheadrightarrow} B$$

where P and Q are projective and ι and π are the natural inclusion and projection respectively.

Proof. First assume that φ is an epimorphism. Let $\psi : B \to A$ be a homotopy inverse of φ and let $\varepsilon : PA \twoheadrightarrow A$ be a path space of A. Then $\varphi\varepsilon : PA \twoheadrightarrow B$ is a path space of B and hence $\varphi\psi - \mathrm{id}_B \simeq 0$ implies that there is a Λ–morphism $\eta : B \to PA$ such that the diagram

$$\begin{array}{ccccc} PA & \xrightarrow{\varepsilon} & A & \xrightarrow{\varphi} & B \\ {\scriptstyle\eta}\uparrow & & & \nearrow_{\varphi\psi-\mathrm{id}_B} & \\ B & & & & \end{array}$$

commutes. Put $\tilde{\psi} := \psi - \varepsilon\eta$. Then $\tilde{\psi} \simeq \psi$ and

$$\varphi\tilde{\psi} = \varphi(\psi - \varepsilon\eta) = \varphi\psi - \varphi\varepsilon\eta = \varphi\psi - \varphi\psi + \mathrm{id}_B = \mathrm{id}_B.$$

Hence $\tilde{\psi}$ is a monomorphism and the short exact sequence

$$B \overset{\tilde{\psi}}{\rightarrowtail} A \overset{\pi'}{\twoheadrightarrow} \mathrm{coker}\tilde{\psi}$$

splits so that $A = \tilde{\psi}(B) \oplus Q$ where $Q = \mathrm{coker}\tilde{\psi}$. In order to show that Q is projective it is enough to show that $[X, Q] = 0$ for all X. So take any Λ–module X. Then

$$[X, B] \overset{\psi_*}{\rightarrowtail\!\!\!\!\twoheadrightarrow} [X, \tilde{\psi}(B) \oplus Q] \rightarrowtail\!\!\!\!\twoheadrightarrow [X, \tilde{\psi}(B)] \oplus [X, Q] \twoheadrightarrow [X, Q]$$

is onto. But what does this homomorphism do to the homotopy class of a Λ–morphism $\nu : X \to B$?

$$[\nu] \mapsto [\psi\nu] = [\tilde{\psi}\nu] \mapsto [\pi'\tilde{\psi}\nu] = 0.$$

Thus $[X, Q] = 0$ showing that Q is projective.

Thus φ factors as

$$A = \tilde{\psi}(B) \oplus Q \rightarrowtail\!\!\!\!\twoheadrightarrow B \oplus Q \twoheadrightarrow B.$$

Given an arbitrary homotopy equivalence $\varphi : A \to B$ we obtain

$$\begin{array}{ccccc} A & \overset{\iota}{\rightarrowtail} & A \oplus PB & \xrightarrow{\varphi'} & B \\ & & & \searrow \quad \nearrow & \\ & & & B \oplus Q & \end{array} .$$

□

1.3. Π_1–**Systems and Pairs of Eilenberg–MacLane Spaces.** Here $X, Y, \ldots$ denote CW–complexes and all maps are assumed to be cellular. Recall that a pair (X, Y) of CW–complexes is given by the inclusion $\iota : Y \to X$ of the subcomplex Y in the complex X. We will always assume that the total space X is connected. A map of pairs

$$f : (X, Y) \longrightarrow (X', Y')$$

is given by a map $f : X \to Y$ with $f(Y) \subseteq Y'$. Thus a map $f : (X, Y) \to (X', Y')$ of pairs induces a map $\widehat{f} : Y \to Y'$ with $\widehat{f}(x) = f(x)$ for every $x \in Y$. Two maps $f, g : (X, Y) \to (X', Y')$ are homotopic as maps of pairs if and only if there is a homotopy

$h : X \times I \to X'$ of f to g with $h(Y \times I) \subseteq Y'$. Such a homotopy is called a homotopy of pairs and it induces a homotopy $\widehat{h} : Y \times I \to Y'$ of $\widehat{f}$ to $\hat{g}$ with $\hat{h}(x,t) = h(x,t)$ for every $x \in Y$ and $t \in I$. Furthermore, two pairs are said to be homotopy equivalent if and only if there are maps of pairs $f : (X,Y) \to (X',Y')$ and $g : (X',Y') \to (X,Y)$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_{X'}$ as maps of pairs. The maps f and g are then called homotopy equivalences of pairs and $\hat{g}\hat{f} = \widehat{gf} \simeq \widehat{\mathrm{id}}_X = \mathrm{id}_Y$ and $\hat{f}\hat{g} = \widehat{fg} \simeq \widehat{\mathrm{id}}_{X'} = \mathrm{id}_{Y'}$ implies that $\hat{f}$ and $\hat{g}$ are also homotopy equivalences.

$$\begin{array}{ccc} Y & \xrightarrow{\widehat{f}} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

Now let $f : (X,Y) \to (X',Y')$ be a homotopy equivalence of pairs. Then Y and Y' must have the same number of connected components and f induces homotopy equivalences

$$\widehat{f}_i : Y_i \longrightarrow Y'_i$$

where $\{Y_i\}_{i\in J}$ and $\{Y'_i\}_{i\in J}$ are the connected components of Y and Y' respectively. We choose base points for X, X' and the connected components of Y and Y', as well as paths from the base points of the connected components of Y and Y' to the base point of X and X' respectively. Then we put

$$\begin{aligned} G &:= \Pi_1(X,*), \qquad & G_i &:= \Pi_1(Y_i,*) \\ G' &:= \Pi_1(X',*), \qquad & G'_i &:= \Pi_1(Y'_i,*). \end{aligned}$$

Furthermore let $\kappa_i : G_i \to G$ and $\rho_i : G'_i \to G'$ be the homomorphisms induced by the inclusion of Y in X and Y' in X' respectively. Note that different paths from the base points of the connected components to the base point of the total space will give rise to different homomorphisms but the image of G_i under κ_i is determined up to conjugacy within G. The family $\{\kappa_i : G_i \to G\}_{i\in J}$ of group homomorphisms is called the Π_1–system of the pair (X,Y) and similarly for (X',Y'). Observe that κ_i and ρ_i need not be injective. The commutative diagram

$$\begin{array}{ccc} Y_i & \xrightarrow{\widehat{f}_i} & Y'_i \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

in the category of CW–complexes gives rise to the following commutative diagram in the category of groups

$$\begin{array}{ccc} G_i & \xrightarrow{\widehat{f}_{i*}} & G'_i \\ {\scriptstyle\kappa_i}\downarrow & & \downarrow{\scriptstyle\rho_i} \\ G & \xrightarrow{f_*} & G'. \end{array}$$

Since homotopy equivalences induce isomorphisms of fundamental groups, f_* and $\widehat{f_{i*}}$ are group isomorphisms.

Definition 1.9. *A homomorphism $(\varphi, \{\varphi_i\}_{i\in J}) : \{\kappa_i : G_i \to G\} \to \{\rho_i : G_i' \to G'\}$ of Π_1–systems consists of a homomorphism $\varphi : G \to G'$ and a family of homomorphisms $\{\varphi_i : G_i \to G_i'\}$ such that the the diagram*

$$\begin{array}{ccc} G_i & \xrightarrow{\varphi_i} & G_i' \\ {\scriptstyle\kappa_i}\downarrow & & \downarrow{\scriptstyle\rho_i} \\ G & \xrightarrow{\varphi} & G' \end{array}$$

commutes. Furthermore, $\{\kappa_i : G_i \to G\}$ and $\{\rho_i : G_i' \to G'\}$ are said to be isomorphic if and only if φ_i is an isomorphism for every $i \in J$ and $\varphi : G \to G$ is an isomorphism.

With this definition we may say that a homotopy equivalence f of pairs induces an isomorphism of Π_1–systems. Thus it is a necessary condition for two pairs of CW–complexes to be homotopy equivalent that their Π_1–systems be isomorphic.

Now let $\{\kappa_i : G_i \to G\}_{i\in J}$ be a family of group homomorphisms and let (X, Y) be a pair of CW–complexes with Π_1–system $\{\kappa_i : G_i \to G\}_{i\in J}$. Put $\Lambda := \mathbb{Z}[G]$ and let I be the augmentation ideal, that is, the kernel of the augmentation homomorphism $\mathrm{aug} : \Lambda \to \mathbb{Z}$ given by $\mathrm{aug}(g) = 1$. Then define

$$\Lambda_{\mathrm{rel}} := \bigoplus_{i\in J} \mathbb{Z}[G] \otimes_{\kappa_i(G_i)} \mathbb{Z} = \bigoplus_{i\in J} \mathbb{Z}[G/\kappa_i(G_i)],$$

that is, Λ_{rel} is the free abelian group with the cosets $xG_i, x \in G, i \in J$, as generators together with the G–action induced by left multiplication. We define the relative augmentation homomorphism $\mathrm{aug}_{\mathrm{rel}} : \Lambda_{\mathrm{rel}} \to \mathbb{Z}$ by $\mathrm{aug}_{\mathrm{rel}}(xG_i) = 1$ for every $x \in X, i \in J$, and denote the kernel of $\mathrm{aug}_{\mathrm{rel}}$ by I_{rel} and call it the relative augmentation ideal.

$$I_{\mathrm{rel}} \rightarrowtail \Lambda_{\mathrm{rel}} \xrightarrow{\mathrm{aug}_{\mathrm{rel}}} \mathbb{Z}$$

Let $p : \tilde{X} \to X$ be the universal covering of X and let P denote the cellular chain complex of $\tilde{X}$ viewed as a complex of Λ–modules. We denote the subcomplex of P generated by the cells lying above Y by Q and put $D := P/Q$, so that $Q \rightarrowtail P \twoheadrightarrow D$ is a short exact sequence of left Λ–modules. We call D the relative cellular complex and $Q \rightarrowtail P \twoheadrightarrow D$ the short exact sequence of cellular chain complexes of the pair (X, Y). Note that we may choose P and Q such that $P_0 = Q_0 = \bigoplus_{i\in J} \Lambda$, that is, X has exactly one zero cell for each connected component of Y. Then $D_0 = P_0/Q_0 = 0$.

Lemma 1.10.

$$H_0(Q) \cong \Lambda_{\mathrm{rel}}$$

and the inclusion $Y \to X$ induces the relative augmentation $\mathrm{aug}_{\mathrm{rel}} : \Lambda_{\mathrm{rel}} \to \mathbb{Z}$.

Proof. Adapt the proof of the fact that the zeroth homology group of a connected space is isomorphic to $\mathbb{Z}$. □

The above Lemma together with $\mathrm{H}_1(P) = 0$ and the long exact homology sequence of $Q \rightarrowtail P \twoheadrightarrow D$ implies

Corollary 1.11. *The diagram*

$$\begin{array}{ccccc} I_{\mathrm{rel}} & \rightarrowtail & \Lambda_{\mathrm{rel}} & \xrightarrow{\mathrm{aug}_{\mathrm{rel}}} & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}_1(D) & \xrightarrow{\partial_*} & \mathrm{H}_0(Q) & \xrightarrow{\iota_*} & \mathrm{H}_0(P) \end{array}$$

commutes and all vertical maps are isomorphisms.

Given a group G we may construct a CW–complex which has G as fundamental group and no other non–trivial homotopy groups. Such a CW–complex is called an Eilenberg–Mac Lane space of type G and denoted by $K(G;1)$. Given a family $\{\kappa_i : G_i \to G\}_{i\in J}$ of group homomorphisms we may ask whether there is a pair (X,Y) which has Π_1–system $\{\kappa_i : G_i \to G\}_{i\in J}$. The answer is yes, namely, for $i \in J$ take $K(G_i;1)$ spaces Y_i and a $K(G;1)$ space X. Then the family $\kappa_i : G_i \to G$ of homomorphisms determines a map $f : \coprod_{i\in J} Y_i \to X$. Let K be the mapping cylinder of f and identify $\coprod_{i\in J} Y_i$ with its image under the inclusion in K. Then (K,Y) is a pair with Π_1–system $\{\kappa_i : G_i \to G\}_{i\in J}$.

As we do not require the homomorphisms κ_i to be injective we will adopt the following non–standard definition for the purpose of this paper.

Definition 1.12. *Let $\{\kappa_i : G_i \to G\}_{i\in J}$ be a family of group homomorphisms. Then (X,Y) is an Eilenberg–Mac Lane pair of type $(G, \{G_i\}_{i\in J}; 1)$ if and only if X is an Eilenberg–Mac Lane space of type $K(G;1)$ and the connected components $\{Y_i\}_{i\in J}$ of Y are Eilenberg–Mac Lane spaces of type $K(G_i;1)$ and $\{\kappa_i : G_i \to G\}_{i\in J}$ is the Π_1–system of (X,Y).*

In the standard definition of Eilenberg–Mac Lane pairs given by Bieri–Eckmann in [1] the homomorphisms κ_i are required to be injective.

An Eilenberg–Mac Lane pair of type $(G, \{G_i\}_{i\in J}; 1)$ is determined up to homotopy of pairs and we write $K(G, \{G_i\}_{i\in J}; 1)$ for any such pair. With this definition we proved the following lemma.

Lemma 1.13. *Let $\{\kappa_i : G_i \to G\}_{i\in J}$ be a family of group homomorphisms. Then there is an Eilenberg–Mac Lane pair (X,Y) of type $(G, \{G_i\}_{i\in J}; 1)$.*

1.4. **A Certain Functor and a Certain Homomorphism.** Given a chain complex

$$\cdots \longrightarrow C_{r+1} \xrightarrow{\partial_r} C_r \longrightarrow \cdots$$

of left Λ–modules put

$$\mathrm{G}_r(C) := \mathrm{coker}\partial_r = C_r/\mathrm{im}\partial_r.$$

If $f : C \to D$ is a chain map of chain complexes of left Λ–modules then $f_r(\mathrm{im}\partial_r^C) \subseteq \mathrm{im}\partial_r^D$. Hence there is an induced Λ–morphism of cokernels $\mathrm{G}_r(f) : \mathrm{G}_r(C) \to \mathrm{G}_r(D)$ such that the

diagram

$$\begin{array}{ccccc} \mathrm{im}\partial_r^C & \rightarrowtail & C_r & \xrightarrow{f_r} & \mathrm{G}_r(C) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{G}_r(f)} \\ \mathrm{im}\partial_r^D & \rightarrowtail & D_r & \twoheadrightarrow & \mathrm{G}_r(D) \end{array}$$

commutes. It is not difficult to check that $G = G_*$ is a functor from the category of chain complexes of left Λ–modules to itself.

We may compose the functors G and $\overline{\mathrm{Hom}_\Lambda(_, \overline{\Lambda})}$ to obtain the functor F (see [12] p.265) given by

$$F_r(C) = G_{-r}(\overline{\mathrm{Hom}_\Lambda(C, \overline{\Lambda})}) = \overline{\mathrm{Hom}_\Lambda(C, \overline{\Lambda})}/\mathrm{im}\partial_{r-1}^*.$$

Now let $\{\kappa_i : G_i \to G\}_{i\in J}$ be a family of group homomorphisms and let (X, Y) be an Eilenberg–MacLane pair of type $(G, \{G_i\}_{i\in J}; 1)$. Furthermore let $Q \rightarrowtail P \twoheadrightarrow D$ be the short exact sequence of cellular complexes of the pair (X, Y) as in Section 1.3. Then the diagram

$$\begin{array}{ccccc} I_{\mathrm{rel}} & \rightarrowtail & \Lambda_{\mathrm{rel}} & \xrightarrow{\mathrm{aug}_{\mathrm{rel}}} & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}_1(D) & \xrightarrow{\partial_*} & \mathrm{H}_0(Q) & \xrightarrow{\iota_*} & \mathrm{H}_0(P) \end{array}$$

commutes. By Lemma 1.4, every cycle $1\otimes x \in Z_{l+1}(\mathbb{Z}^\omega \otimes_\Lambda D)$ yields a chain map

$$\cap 1\otimes x : \overline{\mathrm{Hom}_\Lambda(P, \overline{\Lambda})} \to \Lambda \otimes_\Lambda D \cong D$$

of degree $l + 1$ and hence a chain map

$$G(\cap 1\otimes x) : F(P) \longrightarrow G(D).$$

In particular, we obtain a Λ–morphism

$$G_l(\cap 1\otimes x) : F^l(P) = \overline{\mathrm{Hom}_\Lambda(P_l, \overline{\Lambda})}/\mathrm{im}\partial_{l-1}^* \longrightarrow D_1/\mathrm{im}\partial_1.$$

For a boundary $1\otimes x = \partial(1\otimes y) = 1\otimes\partial y$ with $1\otimes y \in \mathbb{Z}^\omega \otimes_\Lambda D_{l+1}$ and $[\varphi] \in F^l(P)$, we obtain

$$\begin{aligned} G_l(\cap 1\otimes x)([\varphi]) &= [\varphi \cap 1\otimes x] = [\varphi \cap \partial(1\otimes y)] \\ &= (-1)^k[\partial_l^*\varphi \cap 1\otimes y] + (-1)^{k+1}[\partial(\varphi \cap 1\otimes y)] \\ &= (-1)^k[\partial_l^*\varphi \cap 1\otimes y]. \end{aligned}$$

Thus $G_l(\cap 1\otimes x)$ factorizes as

$$F^l(P) \xrightarrow{\tilde{\partial}_l^*} \overline{\mathrm{Hom}_\Lambda(P_{l+1}, \overline{\Lambda})} \xrightarrow{\cap 1\otimes y} D_1 \twoheadrightarrow D_1/\mathrm{im}\partial_1$$

where $\tilde{\partial}_l^*$ is the Λ–morphism induced by ∂_l^*. Hence $G_l(\cap 1\otimes x)$ factors through a free (and therefore projective) Λ–module and is thus nullhomotopic. Now recall that we may chooose

P and Q such that $D_0 = 0$ so that $D_1/\mathrm{im}\partial_1 = \mathrm{H}_1(D) \cong I_{\mathrm{rel}}$. Thus there is a well–defined group homomorphism

$$\nu_r : \mathrm{H}_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda D) \longrightarrow [F^r(C), I_{\mathrm{rel}}]$$

where $[A, B]$ denotes the group of homotopy classes of Λ–morphisms from A to B (see Section 1.2).

Lemma 1.14. *Let $f, g : C \to D$ be chain homotopic maps of chain complexes over Λ. If D_n is projective, then $G_n(f) \simeq G_n(g)$ as Λ–morphisms.*

Proof. Let χ be a chain homotopy from f to g. Observe that, for all $n \in \mathbb{Z}$, the boundary operators ∂^C_{n-1} and ∂^D_{n-1} factor as

$$\begin{array}{ccc} C_n & \xrightarrow{\partial^C_{n-1}} & C_{n-1} \\ {\scriptstyle\sigma^C_{n-1}}\downarrow & \nearrow_{\rho^C_{n-1}} & \\ G_n(C) & & \end{array} \quad \text{and} \quad \begin{array}{ccc} D_n & \xrightarrow{\partial^D_{n-1}} & D_{n-1} \\ {\scriptstyle\sigma^D_{n-1}}\downarrow & \nearrow_{\rho^D_{n-1}} & \\ G_n(D) & & \end{array}$$

respectively. Then

$$\begin{aligned} \sigma^D_{n-1}(f_n - g_n) &= \sigma^D_{n-1}(\chi_{n-1}\partial^C_{n-1} + \partial^D_n\chi_n) = \sigma^D_{n-1}\chi_{n-1}\rho^C_{n-1}\sigma^C_{n-1} + \sigma^D_{n-1}\partial^D_n\chi_n \\ &= \sigma^D_{n-1}\chi_{n-1}\rho^C_{n-1}\sigma^C_{n-1} \end{aligned}$$

Thus the diagram

$$\begin{array}{ccccccc} C_{n+1} & & \xrightarrow{f_{n+1}-g_{n+1}} & & & & D_{n+1} \\ {\scriptstyle\partial^C_n}\downarrow & & & & & & {\scriptstyle\partial^D_n}\downarrow \\ C_n & & \xrightarrow{f_n-g_n} & & & & D_n \\ {\scriptstyle\sigma^C_{n-1}}\downarrow & & & & & & {\scriptstyle\sigma^D_{n-1}}\downarrow \\ G_n(C) & \xrightarrow{\rho^C_{n-1}} & C_{n-1} & \xrightarrow{\chi_{n-1}} & D_n & \xrightarrow{\sigma^D_{n-1}} & G_n(D) \end{array}$$

commutes. As the induced map of cokernels is uniquely determined, this implies

$$G_n(f) - G_n(g) = G_n(f - g) = \sigma^D_{n-1}\chi_{n-1}\rho^C_{n-1} \simeq 0$$

as D_n is projective. □

Corollary 1.15. *Let $f : C \to D$ be a chain homotopy equivalence of chain complexes over Λ. If C_n and D_n are projective, then $G_n(f)$ is a homotopy equivalence of Λ–modules.*

2. Homotopy Classification of Poincaré Duality Pairs

2.1. **Definition of Poincaré Duality Pairs.** Let X be a CW–complex and let $p : \tilde{X} \to X$ be the universal covering of X. Fix a base point in X and put $G := \Pi_1(X; *)$ and $\Lambda := \mathbb{Z}[G]$.

Denote the cellular chain complex of $\tilde{X}$, viewed as a complex of left Λ–modules, by P. Then, for a right Λ–module A and a left Λ–module B,

$$\begin{aligned} \mathrm{H}_r(X;A) &:= \mathrm{H}_r(P;A) = \mathrm{H}_r(A\otimes_\Lambda P), \\ \mathrm{H}^r(X;B) &:= \mathrm{H}^r(P;B) = \mathrm{H}_{-r}(\mathrm{Hom}_\Lambda(P,B)). \end{aligned}$$

Definition 2.1. *A connected* PD^n*–complex is a triple* $(X,\omega,[X])$ *where* X *is a connected CW–complex dominated by a finite complex ,* $\omega \in H^1(X;\mathbb{Z}/2\mathbb{Z})$ *and* $[X] \in H_n(X;\mathbb{Z}^\omega)$ *such that*

$$\cap[X] : H^r(X;B) \to H_{n-r}(X,\bar{B}); \quad \alpha \mapsto \alpha \cap [X]$$

is an isomorphism of abelian groups for every $r \in \mathbb{Z}$ *and every left* Λ*–module* B*. We often write* X *for the triple* $(X,\omega,[X])$ *and call* ω *the orientation character and* $[X]$ *the fundamental class of* X*.*

A PD^n*–complex* $(X,\omega,[X])$ *is a finite disjoint union of connected* PD^n*–complexes* $(X_i,\omega_i,[X]_i)$*,* $i \in J$*, where* $\Lambda = \bigoplus_{i\in J}\mathbb{Z}[X_i]$*,* $\omega = (\omega_i)_{i\in J}$ *and* $[X] = ([X_i])_{i\in J}$*.*

With this definition every n–dimensional manifold determines a PD^n–complex. There are examples of PD^n–complexes which are not homotopy equivalent to n–dimensional manifolds (see [12]).

Definition 2.2. *The* PD^3*–complexes* $(X_1,\omega_1,[X_1])$ *and* $(X_2,\omega_2,[X_2])$ *are orientedly homotopy equivalent if and only if there is a homotopy equivalence* $f : X_1 \to X_2$ *with* $f^*(\omega_2) = \omega_1$ *and* $f_*([X_1]) = [X_2]$*.*

In fact, it follows from the Wu formulae that $f^*(\omega_2) = \omega_1$ if $f : X_1 \to X_2$ is a homotopy equivalence of PD^n–complexes.

If $(M,\partial M)$ is a manifold with boundary then ∂M is a manifold with fundamental class and orientation character induced by the fundamental class and orientation character of $(M,\partial M)$.

Now let $(X,\partial X)$ be a pair of CW complexes. As before, let $p : \tilde{X} \to X$ be the universal covering of X, fix a base point in X and put $G := \Pi_1(X;*)$ and $\Lambda := \mathbb{Z}[G]$. Let $Q \rightarrowtail P \twoheadrightarrow D$ be the short exact sequence of cellular chain complexes of $(X,\partial X)$. Then, for a right Λ–module A and a left Λ–module B,

$$\begin{aligned} \mathrm{H}_r(X,\partial X;A) &:= \mathrm{H}_r(D;A) = \mathrm{H}_r(A\otimes_\Lambda D), \\ \mathrm{H}^r(X,\partial X;B) &:= \mathrm{H}^r(D;B) = \mathrm{H}_{-r}(\mathrm{Hom}_\Lambda(D,B)). \end{aligned}$$

Note that P, Q and D are complexes of free Λ–modules. Hence the split short exact sequence $Q \rightarrowtail P \twoheadrightarrow D$ stays split short exact when tensored with $\mathbb{Z}^\omega$. Denoting the connecting homomorphism of $\mathbb{Z}^\omega\otimes_\Lambda Q \rightarrowtail \mathbb{Z}^\omega\otimes_\Lambda P \twoheadrightarrow \mathbb{Z}^\omega\otimes_\Lambda D$ by δ_* we define

Definition 2.3. *The sextuple* $(X,\omega_X,[X,\partial X],\partial X,\omega_{\partial X},[\partial X])$ *is a connected* PD^n*–pair if and only if*

(i) X *is a connected CW–complex dominated by a finite complex,*

(ii) *∂X is a subcomplex of X and $(\partial X, \omega_{\partial X}, [\partial X])$ is a* PD^{n-1}*–complex,*

(iii) *ω induces ω_i on the connected components ∂X_i of ∂X,*

(iv) *$[X, \partial X] \in H_n(X, \partial X; \mathbb{Z}^\omega)$ with $\delta_*[X, \partial X] = [\partial X] \in H_{n-1}(\partial X; \mathbb{Z}^\omega)$ such that*

$$\cap[X, \partial X] : H^r(X; \bar{B}) \to H_{n-r}(X, \partial X; B), \quad \alpha \mapsto \alpha \cap [X, \partial X]$$

is an isomorphism for every left Λ–module B and every $r \in \mathbb{Z}$.

We often write $(X, \partial X)$ for the PD^n*–pair and call ω the orientation character and $[X, \partial X]$ the fundamental class of $(X, \partial X)$.*

As for PD^n–complexes we may define a general PD^n–pair as a finite disjoint union of connected PD^n–pairs.

With this definition every manifold with boundary determines a PD^n–pair.

Given a Poincaré pair $(X, \partial X)$, the cap product with a representative of the fundamental class is in fact a chain homotopy equivalence of degree n (see [8]).

Lemma 2.4. *Let $(X, \partial X)$ be a* PD^n*–pair and let $e \in D$ be a chain such that $[1 \otimes e] = [X, \partial X]$. Then*

$$\cap 1 \otimes e : \overline{\mathrm{Hom}_\Lambda(P, \overline{\Lambda})} \longrightarrow \Lambda \otimes_\Lambda D \cong D$$

is a chain homotopy equivalence of degree n.

Definition 2.5. *The* PD^3*–pairs $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$ are orientedly homotopy equivalent if and only if there is a homotopy equivalence $f : (X_1, \partial X_1) \to (X_2, \partial X_2)$ of pairs with $f^*(\omega_{X_2}) = \omega_{X_1}$ and $f_*([X_1, \partial X_1]) = [X_2, \partial X_2]$.*

2.2. **Statement of the Classification Theorem.** First we review the homotopy classification theorem in the absolute case, that is, when the boundary is empty.

In 1977 Hendriks [6] gave a complete system of homotopy invariants of PD^3–complexes.

Theorem 2.6. *The* PD^3*–complexes $(X_1, \omega_1, [X_1])$ and $(X_2, \omega_2, [X_2])$ are orientedly homotopy equivalent if and only if there is an isomorphism*

$$\varphi : \Pi_1(X_1, *) \longrightarrow \Pi_1(X_2, *)$$

of fundamental groups such that

$$\varphi^*(\omega_2) = \omega_1 \quad \text{and} \quad \varphi_*(c_{1*}([X_1])) = c_{2*}([X_2])$$

where

$$c_i : X_i \longrightarrow K(\Pi_1(X_i, *); 1)$$

are classifying maps for $i = 1, 2$.

Turaev provided an alternative proof in [12] with the following ingredients.

1. Given PD^3–complexes $(X_1, \omega_1, [X_1])$ and $(X_2, \omega_2, [X_2])$ let $f : X_1 \to X_2$ be a map inducing an isomorphism of fundamental groups with $f^*(\omega_2) = \omega_1$ and $f_*([X_1]) = [X_2]$. Then Poincaré Duality and Whitehead's Theorem imply that f is a homotopy

equivalence.

2. C.T.C. Wall showed in 1967 [8] that up to homotopy every PD^3–complex X may be obtained by glueing a three–dimensional cell e^3 along its boundary to a homologically two–dimensional cell–complex X', that is,

$$X \simeq X' \cup_g e^3 \quad \text{where} \quad g : \partial e^3 \to X'.$$

By [8] we may in fact assume that X' is geometrically of dimension less than or equal to three. Before we continue we recall what it means for a cell–complex to be of homological dimension n.

Definition 2.7. *A cell–complex X is said to be homologically n–dimensional if and only if the cellular chain complex of the universal covering $\tilde{X}$ of X viewed as complex of (left) $\mathbb{Z}[\Pi_1(X)]$–modules is chain homotopy equivalent to an n–dimensional chain complex.*

By a comment in [10] p.137, every cell–complex X which satisfies Wall's condition D_2, that is, $\mathrm{H}_i(\tilde{X}) = 0$ for $i > 2$ and $\mathrm{H}^3(X; B) = 0$ for all left Λ–modules B, is homologically two–dimensional.

3. Then Turaev uses obstruction theory and homological algebra for the construction of a map $f : X_1 \to X_2$ satisfying the hypotheses of 1.

The proof of the Classification Theorem for PD^3– pairs provided in Sections 2.3 and 2.4 is an adaption of Turaev's proof for PD^3–complexes to the relative situation, that is, to PD^3–pairs.

What do we know about a PD^3–pair $(X, \partial X)$ with $\partial X \neq \emptyset$? By definition the boundary ∂X is a PD^2–complex. Work by Eckmann, Müller and Linnell (see [5] and [4]) shows that every connected PD^2–complex is homotopy equivalent to a closed surface. Applying the mapping cylinder construction we may assume that the components $\{\partial X_i\}_{i \in J}$ of ∂X are closed surfaces. It is not difficult to show that X satisfies D_2 and is thus homologically two–dimensional.

The following Lemma by Hendriks [6] replaces the first ingredient in the relative case.

Lemma 2.8. *Suppose $(X, \partial X)$ and $(Y, \partial Y)$ are PD^3–pairs and $f : (X, \partial X) \to (Y, \partial Y)$ is a map of pairs such that*

(i) *f induces a homotopy equivalence $\hat{f} : \partial X \to \partial Y$;*

(ii) *$f_* : \Pi_1(X, *) \to \Pi_1(Y, *)$ is an isomorphism respecting the orientation character;*

(iii) *f has degree one, that is, $f_*([X, \partial X]) = [Y, \partial Y]$.*

Then f is a homotopy equivalence of PD^3–pairs.

By the mapping cylinder construction, we may also assume that the components of ∂X are collared in X, the collar being a three–dimensional manifold with boundary. Then we

may split off a three–cell from the collar of one of the boundary components and hence obtain the following lemma corresponding to 2.

Lemma 2.9. *$X = X' \cup_g e^3$ where $g : S^2 \to X'$ and X' is homologically two–dimensional.*

Proof. As before it is enough to show that X' satisfies D_2, that is, $\mathrm{H}_i(\tilde{X}') = 0$ for $i > 2$ and $\mathrm{H}^3(X'; B) = 0$ for all left Λ–modules B. Note that $\Pi_1(X') = \Pi_1(X)$ so that $\mathrm{H}_i(\tilde{X}') = \mathrm{H}_i(X'; \Lambda)$. By excision in $\tilde{X}$, we obtain $\mathrm{H}_i(\tilde{X}, \tilde{X}') = 0$ for $i \neq 3$ and $\mathrm{H}_3(\tilde{X}, \tilde{X}') = \Lambda$. Thus the homology sequence of (X, X') yields $\mathrm{H}_i(\tilde{X}') = \mathrm{H}_i(\tilde{X}, \tilde{X}') = 0$ for $i > 2$. Now let B be any left Λ–module. Then $\mathrm{H}^4(X, X'; B) = \mathrm{H}^3(X; B) = 0$ and hence the cohomology sequence of the pair (X, X') implies that $\mathrm{H}^3(X'; B) = 0$. □

As in the absolute case, Lemma 2.9 together with obstruction theory and homological algebra allows us to construct a map of pairs satisfying the hypotheses of Lemma 2.8 and we obtain the classification theorem which will be proved in Sections 2.3 and 2.4.

Theorem 2.10. *Let $(X, \partial X)$ and $(Y, \partial Y)$ be* PD^3*–pairs such that the components of ∂X and ∂Y are aspherical. Then $(X, \partial X)$ and $(Y, \partial Y)$ are orientedly homotopy equivalent if and only if there is an isomorphism $(\varphi, \{\varphi_i\}_{i \in J})$ of their Π_1–systems such that*

$$\varphi^*(\omega_Y) = \omega_X \quad \text{and} \quad \varphi_*(c_{X*}([X, \partial X])) = c_{Y*}([Y, \partial Y])$$

where c_X and c_Y are classifying maps.

2.3. **Proof of the Classification Theorem: Part I.** In this section we study the structure of the cellular chain complexes arising from a PD^3–pair $(X, \partial X)$. We use the following fact from homological algebra.

Lemma 2.11. *Let C be a chain complex of projective (left) Λ–modules such that $H^k(C, B) = 0$ for $k \geq n$ and for every (left) Λ–module B. Then C is the direct sum of the following two chain complexes of projective Λ–modules*

$$A' : \qquad \cdots \longrightarrow A_{n+1} \longrightarrow A_n \twoheadrightarrow \mathrm{im}\partial_{n-1}$$

$$A'' : \qquad 0 \longrightarrow \mathrm{coker}\partial_n \rightarrowtail A_{n-2} \longrightarrow \cdots$$

and A' is chain homotopic to the trivial chain complex. Moreover, if C is a free complex, then $\mathrm{im}\partial_{n-1}$ and $\mathrm{coker}\partial_n$ are stably free.

Now let $(X, \partial X)$ be a connected PD^3–pair such that the components of ∂X are aspherical and let $Q \rightarrowtail P \twoheadrightarrow D$ be the short exact sequence of cellular chain complexes of $(X, \partial X)$. By Lemma 2.9, $X = X' \cup_g e^3$ where $g : S^2 \to X'$ and X' is homologically two–dimensional. Without loss of generality we may assume that $\partial X \cap e^3 = \emptyset$. Let $p : \tilde{X} \to X$ be the universal covering of X and denote the subcomplex of P generated by the cells lying in $\tilde{X}' := p^{-1}(X')$ by P'. As X' is homologically two–dimensional, the chain complex P'

satisfies the hypotheses of Lemma 2.11 for $n = 3$. Noting that $P'_i = P_i$ for $i \neq 3$ we conclude that P' decomposes into the direct sum of the two subcomplexes

$$\cdots \longrightarrow P_4 \longrightarrow P'_3 \twoheadrightarrow \mathrm{im}\partial'_2$$

$$0 \longrightarrow S \longrightarrow P_1 \longrightarrow P_0,$$

where $S = \mathrm{coker}\partial'_2$. As we may also assume that X and X' are geometrically three–dimensional, we may in fact assume that $P_i = 0$ for $i \geq 4$ and that P'_3 is isomorphic to $\mathrm{im}\partial'_2$. Hence P decomposes as

$$0 \longrightarrow P'_3 \rightarrowtail \mathrm{im}\partial'_2$$

$$0 \longrightarrow S \longrightarrow P_1 \longrightarrow P_0,$$

Let $\tilde{e}^3$ be an oriented three–cell of the space $\tilde{X}$ lying above e^3 and let $e \in P_3$ be the element represented by the cell $\tilde{e}^3$. By definition, $\mathrm{im}\partial'_2$ is generated by chains which are represented by the gluing maps of three–cells of $\tilde{X}'$. Therefore, replacing g by a homotopic map if necessary, we may assume that $\partial_2(e) \in S$. Thus the chain complex P is the direct sum of the two chain complexes

$$0 \longrightarrow P'_3 \rightarrowtail \mathrm{im}\partial'_2$$

$$0 \longrightarrow \Lambda[e] \longrightarrow S \longrightarrow P_1 \longrightarrow P_0$$

where $\Lambda[e]$ denotes the free Λ-module with generator e. We know that we may assume that the connected components of ∂X are closed surfaces which implies $Q_i = 0$ for $i > 2$.

Lemma 2.12. *Let $(X, \partial X)$ be a PD^3–pair and let $p : \tilde{X} \to X$ be the universal covering. If all components of ∂X are closed aspherical surfaces, then the components of $p^{-1}(\partial X)$ are open surfaces and hence the boundary operator $Q_2 \to Q_1$ is injective.*

Lemma 2.12 implies $\mathrm{im}\partial'_2 \cap Q_2 = 0$ and hence the relative chain complex of the pair $(X, \partial X)$ decomposes into the direct sum of the two subcomplexes

$$\hat{D}: \qquad 0 \longrightarrow P'_3 \longrightarrow \mathrm{im}\partial'_2$$

$$\check{D}: \qquad 0 \longrightarrow \Lambda[e] \longrightarrow \tilde{S} \longrightarrow D_1 \longrightarrow D_0$$

where $\tilde{S} = S/Q_2$. Lemma 2.11 applied to the relative chain complex of the pair $(X', \partial X)$ also tells us that $\hat{D} \simeq 0$.

Now let $\{e^2_m\}_{m\in M}$ be a collection of two–cells of $\tilde{X}$ such that above every two–cell in $X\setminus\partial X$ there lies exactly one cell from this collection. Then the collection $\{e_m\}_{m\in M}\subset D_2$ of chains represented by these cells comprise a basis of the Λ–module D_2. Suppose

$$\partial_2(e)=\sum a_m e_m,$$

where $\partial_2: D_3\to D_2$ denotes the boundary operator of the relative complex and $a_m\in\Lambda$ for $m\in M$.

Next consider $\mathbb{Z}^\omega\otimes_\Lambda\check{D}$, that is,

$$0\longrightarrow\mathbb{Z}^\omega\otimes_\Lambda\Lambda[e]\xrightarrow{1\otimes\partial_2}\mathbb{Z}^\omega\otimes_\Lambda\tilde{S}\longrightarrow\mathbb{Z}^\omega\otimes_\Lambda D_1\longrightarrow\mathbb{Z}^\omega\otimes_\Lambda D_0$$

and note that

$$\begin{aligned}\mathbb{Z} &= \mathrm{H}_3(X,\partial X;\mathbb{Z}^\omega)=\mathrm{H}_3(\mathbb{Z}^\omega\otimes_\Lambda D)=\mathrm{H}_3(\mathbb{Z}^\omega\otimes_\Lambda(\hat{D}\oplus\check{D}))\\ &= \mathrm{H}_3(\mathbb{Z}^\omega\otimes_\Lambda\hat{D})\oplus\mathrm{H}_3(\mathbb{Z}^\omega\otimes_\Lambda\check{D})=\mathrm{H}_3(\mathbb{Z}^\omega\otimes_\Lambda\check{D}).\end{aligned}$$

Thus $\mathrm{H}_3(\mathbb{Z}^\omega\otimes_\Lambda\check{D})\neq 0$ which implies that $1\otimes\partial_2$ maps a multiple of the generator of $\mathbb{Z}^\omega\otimes_\Lambda\Lambda[e]\cong\mathbb{Z}$ to zero, that is, there is an $n\in\mathbb{N}$ such that

$$\begin{aligned}0 &= 1\otimes\partial_2(n(1\otimes e))=n(1\otimes\partial_2(e))=n(1\otimes\sum a_m e_m)\\ &= n\sum 1.a_m\otimes e_m=n\sum\mathrm{aug}(\overline{a_m})\otimes e_m.\end{aligned}$$

Since $\mathbb{Z}^\omega\otimes_\Lambda D_2=\mathbb{Z}^\omega\otimes_\Lambda\bigoplus_{m\in M}\Lambda[e_m]\cong\bigoplus_{m\in M}\mathbb{Z}^\omega\otimes_\Lambda\Lambda[e_m]=\bigoplus_{m\in M}\mathbb{Z}$ is free as abelian group, this implies $\mathrm{aug}(\overline{a_m})=0$ and hence $\overline{a_m}\in I$ for every $m\in M$. Therefore $1\otimes\partial_2(1\otimes e)=0$ and $1\otimes e\in\mathbb{Z}^\omega\otimes_\Lambda D_3$ is a cycle representing a generator of the group $\mathrm{H}_3(X,\partial X;\mathbb{Z}^\omega)$. Without loss of generality we may assume that e is oriented such that $1\otimes e$ represents the homology class $[X,\partial X]$.

Next we observe that

$$\begin{aligned}\mathrm{H}^3(\overline{\mathrm{Hom}_\Lambda(\check{D},\overline{\Lambda})}) &= \mathrm{H}^3(\overline{\mathrm{Hom}_\Lambda(D,\overline{\Lambda})})=\mathrm{H}^3(X,\partial X;\Lambda)\\ &\cong \mathrm{H}_0(X;\Lambda)=\mathrm{H}_0(\tilde{X},\mathbb{Z})=\mathbb{Z}.\end{aligned}$$

Hence the sequence

$$0\longleftarrow\mathbb{Z}\xleftarrow{\mathrm{aug}}\overline{\mathrm{Hom}_\Lambda(\Lambda[e],\overline{\Lambda})}\xleftarrow{\check{\partial}_2^*}\overline{\mathrm{Hom}_\Lambda(S,\overline{\Lambda})}$$

is exact, in particular, $I=\ker\mathrm{aug}=\mathrm{im}\check{\partial}_2^*$. The Λ–module $\overline{\mathrm{Hom}_\Lambda(D_2,\overline{\Lambda})}$ is generated by the collection $\{\tilde{e}^m\}_{m\in M}$ of Λ–morphisms where

$$\tilde{e}^m: D_2\to\Lambda,\quad e_k\mapsto\begin{cases}1 & \text{if } k=m;\\ 0 & \text{if } k\neq m.\end{cases}$$

Given $\varphi\in\overline{\mathrm{Hom}_\Lambda(S,\overline{\Lambda})}$ we obtain $\varphi=\varphi\pi\iota$ where ι and π are the natural inclusion of and projection onto the direct summand S of D_2 respectively. Then there are $\varphi_m\in\Lambda$, $m\in M$,

such that $\varphi\pi = \sum \varphi_m \tilde{e}^m$ and hence $\varphi = \sum \varphi_m \tilde{e}^m \circ \iota$ which shows that $\overline{\mathrm{Hom}_\Lambda(S, \overline{\Lambda})}$ is generated by the collection $\{\tilde{e}^m \circ \iota\}_{m\in M}$ of Λ–morphisms. But

$$\begin{aligned}(\check{\partial}_2^*(\tilde{e}^m \circ \iota))(e) &= \tilde{e}^m \circ \iota(\check{\partial}_2(e)) = \tilde{e}^m(\partial_2(e)) \\ &= \tilde{e}^m(\sum a_j e_j) = a_m = \tilde{e}(e).a_m = (\overline{a_m}\tilde{e})(e).\end{aligned}$$

where $\tilde{e} : \Lambda[e] \to \Lambda, e \mapsto 1$ generates $\overline{\mathrm{Hom}_\Lambda(\Lambda[e], \overline{\Lambda})}$. Hence

$$\check{\partial}_2^*(\tilde{e}^m \circ \iota) = \overline{a_m}.\tilde{e}$$

which implies that, as a left Λ–module, $I = \mathrm{im}\check{\partial}_2^*$ and is generated by $\{\overline{a_m}\}_{m\in M}$. Summarizing we obtain a stronger version of Lemma 2.9

Lemma 2.13. *Let $(X, \partial X)$ be a PD^3–pair. Then $X = X' \cup_g e^3$ where $g : S^2 \to X'$ and X' is homologically two–dimensional. If $e \in D_3$ is the chain corresponding to the three cell e^3 then $1\otimes e$ is a relative cycle which represents the homology class $[X, \partial X]$. Further if*

$$\partial_2(e) = \sum a_m e_m, \quad a_m \in \Lambda$$

where $\partial_2 : D_3 \to D_2$ denotes the boundary operator of the relative complex and $\{e_m\}_{m\in M} \subset D_2$ comprises a basis of the Λ–module D_2, then $I = \mathrm{im}\check{\partial}_2^*$ and is generated by $\{\overline{a_m}\}_{m\in M}$.

2.4. **Proof of the Classification Theorem: Part II.** The obstruction theory we use in this section can be found in [7].

Let $(X, \partial X)$ and $(Y, \partial Y)$ be connected PD^3–pairs such that the components of ∂X and ∂Y are aspherical. Let $\{\kappa_i : G_i \to G\}$ and $\{\rho_i : H_i \to H\}$ be the Π_1–systems of $(X, \partial X)$ and $(Y, \partial Y)$ respectively and let $(\varphi, \{\varphi_i\}_{i\in J}) : \{\kappa_i : G_i \to G\} \to \{\rho_i : H_i \to H\}$ be an isomorphism of Π_1–systems such that

$$\varphi^*(\omega_X) = \omega_Y \quad \text{and} \quad \varphi_*\left(c_{X*}([X, \partial X])\right) = c_{Y*}([Y, \partial Y])$$

where

$$\begin{aligned} c_X &: (X, \partial X) \longrightarrow K(G, \{G_i\}_{i\in J}; 1) \quad \text{and} \\ c_Y &: (Y, \partial Y) \longrightarrow K(H, \{H_i\}_{i\in J}; 1)\end{aligned}$$

are classifying maps. By the construction of an Eilenberg–Mac Lane pair of a given Π_1–system provided earlier, we may assume that $Y \subset K(H, \{H_i\}_{i\in J}; 1)$. We denote the Eilenberg–Mac Lane pair $K(H, \{H_i\}_{i\in J}; 1)$ by $(K, \partial Y)$ and let $q : \tilde{K} \to K$ be the universal covering. By Lemma 2.13, $X = X' \cup e^3$ and $Y = Y' \cup e'^3$ where X' and Y' are homologically two–dimensional and e, e' are chains representing the three cells e^3 and e'^3 such that $1\otimes e$ and $1\otimes e'$ represent $[X, \partial X]$ and $[Y, \partial Y]$ respectively. We put $\Lambda_X := \mathbb{Z}[X], \Lambda_Y := \mathbb{Z}[H]$ and denote the short exact sequence of cellular chain complexes of (left) modules of the pairs $(X, \partial X)$ and $(K, \partial Y)$ by $Q \rightarrowtail R \twoheadrightarrow C$ and $S \rightarrowtail P \twoheadrightarrow D$ respectively.

We want to construct a map

$$f : (X, \partial X) \longrightarrow (Y, \partial Y)$$

such that

(i) f induces a homotopy equivalence $\hat{f} : \partial X \to \partial Y$;

(ii) $f_* : \Pi_1(X, *) \to \Pi_1(Y, *)$ is an isomorphism respecting the orientation character;

(iii) f has degree one, that is, $f_*([X, \partial X]) = [Y, \partial Y]$.

Then Lemma 2.8 implies that f is a homotopy equivalence of PD^3–pairs.

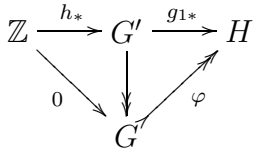
Recall that we may assume that the connected components of ∂X and ∂Y are closed surfaces. Thus there is a homeomorphism

$$\tilde{g} : \partial X \longrightarrow \partial Y$$

which induces the isomorphisms $\varphi_i : G_i \to H_i$ on the fundamental groups of the connected components of ∂X and ∂Y respectively. Now we obtain the one–skeleton $(X, \partial X)^{[1]}$ of X relative to ∂X by attaching one–cells to ∂X. Each of these one–cells corresponds to a generator of G. Thus the isomorphism φ gives rise to a map

$$g_1 : (X, \partial X)^{[1]} \longrightarrow Y'$$

such that $g_1|_{\partial X} = \tilde{g}$ and such that the diagram



commutes for every attaching map $h : S^1 \to (X, \partial X)^{[1]}$, where G' denotes the fundamental group of $(X, \partial X)^{[1]}$. Since φ is an isomorphism, g_1 extends to the two–skeleton and since $\mathrm{H}^n(X', \partial X; B) = 0$ for all $n > 2$ and all left Λ–modules B, this extension extends to all of X'. Thus we obtain a map of pairs

$$g' : (X', \partial X) \longrightarrow (Y', \partial Y)$$

with $g'|_{\partial X} = \tilde{g}$. Since $\Pi_2(K, *) = 0$ the map g' gives rise to a cellular map of pairs

$$g : (X, \partial X) \longrightarrow (K, \partial Y)$$

with $g|_{\partial X} = \tilde{g}$. Let

$$g_* : C \longrightarrow D$$

be the chain homomorphism induced by g. Then $\varphi_*(c_{X*}([X, \partial X])) = c_{Y*}([Y, \partial Y])$ implies

$$1 \otimes g_*(e) - 1 \otimes e' = 0 \quad \text{in } \mathrm{H}_3(K, \partial Y; \mathbb{Z}^{\omega_Y})$$

where we identify e' with its inclusion in K. Thus

$$g_*(e) - e' \in \mathrm{im}(\partial_3 : C_4 \to C_3) + \overline{I_Y} C_3$$

where I_Y is the kernel of the augmentation $\mathrm{aug}_Y : \Lambda_Y \to \mathbb{Z}$ and $\overline{\ } : \Lambda_Y \to \Lambda_Y$ is the canonical anti–involution induced by ω_Y. Therefore

$$\partial_2(g_*(e) - e') \in \overline{I_Y} \partial_2(C_3). \tag{2}$$

The group isomorphism $\varphi : G \to H$ induces a ring isomorphism which we denote by the same symbol, that is, $\varphi : \Lambda_X \to \Lambda_Y$. Then $\varphi^*(\omega_Y) = \omega_X$ implies that $\varphi(\overline{\lambda}) = \overline{\varphi(\lambda)}$ for every $\lambda \in \Lambda_X$ and hence $\overline{I_Y} = \overline{(\varphi(I_X))} = \varphi(\overline{I_X})$. By Lemma 2.13, $\overline{I_X}$ is generated by $\{a_m\}_{m\in M}$ where $\partial_2(e) = \sum_{m\in M} a_m e_m$ with $a_m \in \Lambda_X$. Hence $\overline{I_Y}$ is generated by $\{\varphi(a_m)\}_{m\in M}$ and (2) implies

$$\partial_2(g_*(e)) - \partial_2(e') = g_*(\partial_2(e)) - \partial_2(e') = \sum_{m\in M} \varphi(a_m) d_m$$

where $d_m = \partial_2 c_m$ for some $c_m \in D_3$, that is, d_m is represented by the gluing map $f_m : S^2 \to q^{-1}(Y'\backslash\partial Y)$ of a three–cell in $q^{-1}(Y'\backslash\partial Y)$. Thus

$$\partial_2(e') = \sum_m \varphi(a_m)(g_*(e_m) - d_m).$$

Using the fact that $D^2 \# S^2 \simeq D^2$, we may modify $g'|_{X^{[2]}}$ on the interior of two–cells to obtain a map of pairs

$$k : (X', \partial X) \longrightarrow (Y', \partial Y)$$

with

$$k_*(\partial_2(e)) = k_*(\sum_{m\in M} a_m e_m) = \sum_{m\in M} \varphi(a_m)(g_*(e_m) - d_m) = \partial_2(e'). \tag{3}$$

As all components of ∂Y are aspherical, the components of $q^{-1}(\partial Y)$ are open surfaces by Lemma 2.12 and thus the long exact homology sequence of the pair $(Y', \partial Y)$ yields an injective homomorphism $\mathrm{H}_2(q^{-1}(Y'); \Lambda_Y) \rightarrowtail \mathrm{H}_2(q^{-1}(Y'), q^{-1}(\partial Y); \Lambda_Y)$. Further, $\delta_*[Y, \partial Y] = [\partial Y]$ implies that $[\partial_2 e']$ is contained in the image of this homomorphism. Thus $\Pi_2(Y') \cong \mathrm{H}_2(q^{-1}(Y'); \Lambda_Y)$ together with (3) implies that the composition of the attaching map of the three cell e^3 (represented by e) and the map k is homotopy eqivalent to the attaching map of the three cell ${e'}^3$ (represented by e').

Thus k extends to a map of pairs

$$f : (X, \partial X) \longrightarrow (Y, \partial Y)$$

with the following properties

(i) f induces the isomorphism $\{\varphi, \varphi_i\}_{i\in J}$ on the Π_1–systems;

(ii) $f^*(\omega_Y) = \omega_X$;

(iii) $f_*([X, \partial X]) = [Y, \partial Y]$;

(iv) $f|_{\partial X} = \tilde{g}$ which is a homeomorphism.

Hence f satisfies the hypotheses of Lemma 2.8 and the PD^3–pairs $(X, \partial X)$ and $(Y, \partial Y)$ are homotopy equivalent. This concludes the proof of the Classification Theorem 2.10.

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