Quantum Casimir elements and Sugawara operators

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Any element of the center $\mathrm{Z}(\mathfrak{gl}_n)$ of $\mathrm{U}(\mathfrak{gl}_n)$ is called a Casimir element.

Given an n-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_n)$, the corresponding irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_n$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \qquad \text{and}$$

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Any element $z \in \mathrm{Z}(\mathfrak{gl}_n)$ acts in $L(\lambda)$ by multiplying each vector by a scalar $\chi(z)$.

When regarded as a function of the highest weight, $\chi(z)$ is a symmetric polynomial in the variables $\ell_1, \dots, \ell_n$, where $\ell_i = \lambda_i + n - i$.

The Harish-Chandra isomorphism is the map

$$\chi : \mathrm{Z}(\mathfrak{gl}_n) \to \mathbb{C}[\ell_1, \ldots, \ell_n]^{\mathfrak{S}_n},$$

where $\mathbb{C}[\ell_1, \ldots, \ell_n]^{\mathfrak{S}_n}$ denotes the algebra of symmetric polynomials in $\ell_1, \ldots, \ell_n$.

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[Okounkov 1996, Okounkov and Olshanski 1998]:

The quantum immanants $\mathbb{S}_\mu$ form a basis of $\mathrm{Z}(\mathfrak{gl}_n)$ as μ runs over Young diagrams with at most n rows.

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The quantum immanants $\mathbb{S}_\mu$ form a basis of $\mathrm{Z}(\mathfrak{gl}_n)$ as μ runs over Young diagrams with at most n rows. Moreover,

$$\chi : \mathbb{S}_\mu \mapsto s^*_\mu,$$

the s^*_μ are the shifted (factorial) Schur polynomials.

In the case of column-diagrams μ, we recover

the Capelli determinant [1890]:

$$C(u) = \operatorname{cdet} \begin{bmatrix} u+n-1+E_{11} & E_{12} & \dots & E_{1n} \\ E_{21} & u+n-2+E_{22} & \dots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & \dots & \dots & u+E_{nn} \end{bmatrix}$$

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The coefficients $C_1, \dots, C_n$ are free generators of $\mathrm{Z}(\mathfrak{gl}_n)$.

Gelfand invariants

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Combine the generators E_{ij} into the matrix

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[Perelomov and Popov 1966]:

$$\chi(\operatorname{tr} E^m) = \sum_{k=1}^{n} \ell_k^m \frac{(\ell_1 - \ell_k + 1) \dots (\ell_n - \ell_k + 1)}{(\ell_1 - \ell_k) \dots \wedge \dots (\ell_n - \ell_k)}.$$

A short proof is based on the formula

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Under the Harish-Chandra isomorphism,

$$\chi:\frac{C(u+1)}{C(u)}\mapsto\frac{(u+\ell_1+1)\dots(u+\ell_n+1)}{(u+\ell_1)\dots(u+\ell_n)}.$$

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and

$$L^- = \begin{bmatrix} l_{11}^- & 0 & \dots & 0 \\ l_{21}^- & l_{22}^- & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}^- & l_{n2}^- & \dots & l_{nn}^- \end{bmatrix}.$$

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with subscripts of $L^{\pm}$ indicating the copies of $\mathrm{End}\,\mathbb{C}^n$ as in

$$L_1^{\pm} = \sum_{i,j} e_{ij} \otimes 1 \otimes l_{ij}^{\pm} \in \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{U}_q(\mathfrak{gl}_n),$$

$$L_2^{\pm} = \sum_{i,j} 1 \otimes e_{ij} \otimes l_{ij}^{\pm} \in \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{U}_q(\mathfrak{gl}_n).$$

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Equivalently, $\mathrm{U}_q^\circ(\mathfrak{gl}_n)$ can be regarded as the algebra generated by the entries of the matrix $L = [l_{ij}]$ subject to the reflection equation

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The quantum Gelfand invariants are defined by

$$\mathrm{tr}_q L^m = \mathrm{tr}\, D L^m,$$

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They generate the center $\mathrm{Z}_q^{\circ}(\mathfrak{gl}_n)$ of $\mathrm{U}_q^{\circ}(\mathfrak{gl}_n)$.

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The representation $L_q(\lambda)$ of $\mathrm{U}_q(\mathfrak{gl}_n)$ with $\lambda = (\lambda_1, \dots, \lambda_n)$ is generated by a nonzero vector ξ such that

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Set $\ell_i = \lambda_i - i + 1$ to have the Harish-Chandra isomorphism

$$\chi : \mathrm{Z}_q^{\circ}(\mathfrak{gl}_n) \to \mathbb{C}\,[q^{2\ell_1}, \dots, q^{2\ell_n}]^{\mathfrak{S}_n}.$$

[Joseph and Letzter 1992, Rosso 1990, Tanisaki 1990].

Theorem.

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We have

$$\chi : q^{n-1}\,\mathrm{tr}_q L^m \mapsto \sum_{k=1}^{n} q^{2\ell_k m}\,\frac{[\ell_k-\ell_1+1]_q\ldots[\ell_k-\ell_n+1]_q}{[\ell_k-\ell_1]_q\ldots\wedge\ldots[\ell_k-\ell_n]_q},$$

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The Perelomov–Popov formulas follow from the theorem by taking the limit $q \to 1$.

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The Hecke algebra $\mathcal{H}_m$ is generated by elements $T_1, \dots, T_{m-1}$ subject to the relations

$$(T_i - q)(T_i + q^{-1}) = 0,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

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The Jucys–Murphy elements are defined by

$$y_k = T_{k-1} \dots T_2 T_1^2 T_2 \dots T_{k-1}, \qquad k = 1, \dots, m.$$

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[Cherednik 1987], [Dipper and James 1987].

The Hecke algebra $\mathcal{H}_m$ is semisimple,

$$\mathcal{H}_m \cong \bigoplus_{\mu \vdash m} \mathrm{Mat}_{f_\mu}(\mathbb{C}),$$

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The diagonal matrix units $e_{\mathcal{U}} = e_{\mathcal{U}\mathcal{U}} \in \mathrm{Mat}_{f_\mu}(\mathbb{C})$ with $\mathrm{sh}(\mathcal{U}) = \mu$ are primitive idempotents of $\mathcal{H}_m$.

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They can be expressed explicitly in terms of the generators T_i or the Jucys–Murphy elements y_k.

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$$\check{R} = PR = \sum_{i,j} q^{\delta_{ij}}\, e_{ji} \otimes e_{ij} + (q - q^{-1}) \sum_{i<j} e_{jj} \otimes e_{ii}.$$

Definition. Given any standard tableau $\mathcal{U}$ of shape μ, the associated q-immanant polynomial is

$$\mathbb{S}_\mu(z) = \mathrm{tr}_{q\,(1,\dots,m)} \, (L_1^+ + zq^{-2c_1(\mathcal{U})} L_1^-) \dots (L_m^+ + zq^{-2c_m(\mathcal{U})} L_m^-) \\ \times (L_m^-)^{-1} \dots (L_1^-)^{-1} \, \mathcal{E}_\mathcal{U},$$

where $\mathcal{E}_\mathcal{U}$ is the image of $e_\mathcal{U}$, while $c_k(\mathcal{U}) = j - i$ is the content of the box $\alpha = (i, j)$ occupied by k.

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The expression under the trace belongs to

$$\underbrace{\mathrm{End}\,\mathbb{C}^n \otimes \dots \otimes \mathrm{End}\,\mathbb{C}^n}_{m} \otimes \mathrm{U}_q(\mathfrak{gl}_n).$$

Equivalent definition.

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The q-immanant polynomial is given by

$$\mathbb{S}_\mu(z) = \mathrm{tr}_{q\,(1,\dots,m)}\,(L_{\overline{1}} + z q^{-2\,c_1(\mathcal{U})}) \dots (L_{\overline{m}} + z q^{-2\,c_m(\mathcal{U})})\,\mathcal{E}_{\mathcal{U}},$$

where $L = L^{+}(L^{-})^{-1}$.

Given a matrix X, we set $X_{\overline{1}} = X_1$ and

$$X_{\overline{k}} = \check{R}_{k-1\,k} \dots \check{R}_{1\,2} X_1 \check{R}_{1\,2}^{-1} \dots \check{R}_{k-1\,k}^{-1}, \qquad k \geqslant 2.$$

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Note that in the specialization $q = 1$ we have $\check{R} = P$

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- For any fixed $z \in \mathbb{C}$, the elements $\mathbb{S}_\mu(z)$ form a basis of the center of $\mathrm{U}_q^\circ(\mathfrak{gl}_n)$.

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The limit value of $\mathbb{S}_\mu$ as $q \to 1$ coincides with the quantum immanant $\mathbb{S}_\mu$ for $\mathfrak{gl}_n$ [Okounkov 1996].

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$$R(u/v)L_1^{\pm}(u)L_2^{\pm}(v) = L_2^{\pm}(v)L_1^{\pm}(u)R(u/v),$$

$$R(uq^{-c}/v)L_1^{+}(u)L_2^{-}(v) = L_2^{-}(v)L_1^{+}(u)R(uq^{c}/v).$$

We consider the matrices $L^{\pm}(u) = \left[l_{ij}^{\pm}(u) \right]$ with

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$$R(x) = \frac{f(x)}{q - q^{-1}x}\left(R + x R_{21}\big|_{q \mapsto q^{-1}}\right),$$

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[Reshetikhin and Semenov-Tian-Shansky 1990],

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Its completion $\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ is defined as the inverse limit

$$\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}} = \varprojlim \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}/J_p, \qquad p > 0,$$

where J_p is the left ideal of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ generated by all elements $l_{ij}^-[r]$ with $r \geqslant p$.

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A general construction based on the universal R-matrix: [Ding and Etingof 1994].

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summed over semistandard tableaux $\mathcal{T}$ of shape μ with entries in $\{1, 2, \dots, n\}$, where

$$x_i(z) = q^{2-2i}\, \frac{l^{+}_{ii}(z)\, l^{-}_{11}(zq^{-n+2}) \dots l^{-}_{i-1\, i-1}(zq^{-n+2i-2})}{l^{-}_{11}(zq^{-n}) \dots l^{-}_{ii}(zq^{-n+2i-2})}.$$

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- The theorem yields the eigenvalues of quantum Sugawara operators on the q-deformed Wakimoto modules over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ at the critical level.
  [Awata, Odake and Shiraishi 1994].