POLYNOMIAL PROBABILITY GENERATING FUNCTIONS WITH ALL ROOTS REAL

Di Warren

School of Mathematics and Statistics, F07

University of Sydney, N.S.W. 2006

Australia

Abstract

The inherent difficulty in applying the Frobenius-Harper technique (Frobenius, 1910; Harper, 1967) to any setting, is proving that the roots of the polynomial p.g.f. are all real. We give two theorems providing sufficient conditions. This work supplies the technical proofs to Warren (1997).

Key words: generating function; real roots.

AMS 1991 subject classification: 60J10; 60C05; 60F05; 39A10

1 Introduction

Any non-negative integer-valued random variable with a polynomial p.g.f. with only real roots can be written as a sum of independent (not necessarily identically distributed) Bernoulli random variables. We call this property the *Frobenius-Harper technique* after the authors of the two seminal papers in which these ideas apparently first occur: Frobenius (1910); and Harper (1967). The inherent difficulty in applying the Frobenius-Harper technique to any setting, is proving that the roots of the polynomial p.g.f. are all real. To this end, we give the following two theorems, which we subsequently prove using three lemmas.

Theorem 1.1 Suppose that $\{X_n\}$ ($1 \leq n < M = \infty$ or $1 \leq n \leq M < \infty$) is a sequence of random variables, where $P(X_n = i) = P(n, i)$, $i = d(n), d(n)+1, \ldots, f(n)$, where $d(n) \geq 0$ is an integer which satisfies one of the following 2 conditions (though possibly not the same one for different j)

(1) $d(j+1) - d(j) = 0$

(2) $d(j+1) - d(j) = 1$

for $j \geq 1$ (if $M = \infty$) or $1 \leq j \leq M-1$ (if $M < \infty$), and where $f(n) \geq d(n)+1$ is an integer which satisfies one of the following 2 conditions (though possibly not the same one for different j)

$(1)'$ $f(j+1) - f(j) = 0$

$(2)'$ $f(j+1) - f(j) = 1$

for $j \geq 1$ (if $M = \infty$) or $1 \leq j \leq M-1$ (if $M < \infty$), and $P(n, d(n)) > 0$ and $P(n, f(n)) > 0$ ($1 \leq n \leq M$) and $P(n, i) = 0$ for $i < d(n)$, $i > f(n)$, $n \leq 0$ and $n > M$ (if $M < \infty$).

Suppose further that $\{X_n\}$ gives rise to a sequence of p.g.f.s $\{G_n(t)\}$ which satisfy

$$G_{n+1}(t) = a_n(t)G_n(t) + b_n(t)G_n^{'}(t), \tag{1}$$

for $n \geq 1$ (if $M = \infty$) or $1 \leq n \leq M-1$ (if $M < \infty$), where $a_n(t)$ and $b_n(t)$ are continuous functions on $t \epsilon (-\infty, \infty)$.

If for some $m \geq 1$ (if $M = \infty$) or $1 \leq m \leq M-1$ (if $M < \infty$) the $f(m) - d(m)$ non-zero roots of $G_m(t)$ are all real and single and also $b_n(t) < 0$ for all $t < 0$ and for $n \geq m$ (if $M = \infty$) or $m \leq n \leq M-1$ (if $M < \infty$), *then* the $f(n) - d(n)$ non-zero roots of $G_n(t)$ are all real, single (or distinct) and negative for all $n \geq m$ (if $M = \infty$) or $m \leq n \leq M$ (if $M < \infty$).

Theorem 1.2 Suppose that $\{X_n\}$ ($1 \leq n < M = \infty$ or $1 \leq n \leq M < \infty$) is a sequence of random variables satisfying the condition of Theorem 1.1, except that $f(n) > d(n) + 1$. Assume further that for some $m \geq 1$ (if $M = \infty$) or $1 \leq m \leq M-1$ (if $M < \infty$) the $f(m) - d(m) > 1$ non-zero roots of $G_m(t)$ are all real, and denoted by $\{r_i^{(m)}\}$ where $-\infty < r_1^{(m)} \leq \ldots \leq r_{f(m)-d(m)}^{(m)} < 0$. Suppose also that these roots are all single, except for $r_q^{(m)}$ ($1 \leq q \leq f(m) - d(m) - c(m) + 1$) which is a multiple root of order $c(m)$. Further suppose that $b_n'(\xi)$ exists for all ξ, $r_q^{(m)} < \xi < t < 0$, for $n \geq m$ (if $M = \infty$) or $m \leq n \leq M-1$ (if $M < \infty$).

Then:

(A) If $b_n(r_q^{(m)}) = 0$ for each $n \geq m$ (if $M = \infty$) or $m \leq n \leq M-1$ (if $M < \infty$), then $r_q^{(m)}$ is a multiple root of $G_n(t)$ of order $c(n)$ for all $n \geq m$ (if $M = \infty$) or $m \leq n \leq M$ (if $M < \infty$), and $c(n+1) \geq c(n)$ for all $n \geq m$ (if $M = \infty$) or $m \leq n \leq M-1$ (if $M < \infty$).

(B1) Suppose $q = 1$. If $b_n(r_1^{(m)}) = 0$ and $b_n(t) < 0$ for $r_1^{(m)} < t < 0$ for each $n \geq m$ (if $M = \infty$) or $m \leq n \leq M-1$ (if $M < \infty$), then for $n \geq m$ (if $M = \infty$) or $m \leq n \leq M$ (if $M < \infty$), $r_1^{(m)}$ is the smallest root of $G_n(t)$ and any non-zero roots other than $r_1^{(m)}$, are real and single. Also, $c(n+1) - c(n) = 0$ or 1 for all $n \geq m$ (if $M = \infty$) or $m \leq n \leq M-1$ (if $M < \infty$).

(B2) Suppose $q > 1$. If $b_n(r_q^{(m)}) = 0$ and $b_n(t) < 0$ for $t < 0 \cap t \neq r_q^{(m)}$ for each $n \geq m$ (if $M = \infty$) or $m \leq n \leq M-1$ (if $M < \infty$), then for $n \geq m$ (if $M = \infty$) or $m \leq n \leq M$ (if $M < \infty$), any non-zero roots of $G_n(t)$ other than $r_q^{(m)}$, are real and single. Also, $c(n+1) - c(n) = 0$ or 1 for all $n \geq m$ (if $M = \infty$) or $m \leq n \leq M-1$ (if $M < \infty$).

2 Three Important Lemmas

Lemma 2.1 Suppose that for $n \geq 1$, $G_n(t)$ is a p.g.f. $\sum_{x=d(n)}^{f(n)} p_{n,x}t^x$ where $0 \leq d(n) < f(n)$, $p_{n,d(n)} > 0$ and $p_{n,f(n)} > 0$. Suppose further that the roots are all real and are denoted by $\{r_i^{(n)}\}$, where $-\infty < r_1^{(n)} \leq \ldots \leq r_{f(n)-d(n)}^{(n)} < 0$; and $r_{f(n)-d(n)+1}^{(n)} = \ldots = r_{f(n)}^{(n)} = 0$, if $d(n) \geq 1$.

(i) If $r_1^{(n)}$ is a single root, then $G_n'(r_1^{(n)})$ has the opposite sign to $G_n(-\infty)$. In addition, if $r_2^{(n)}$ is also a single root, then $G_n'(r_2^{(n)})$ has the same sign as $G_n(-\infty)$.

(ii) If $r_i^{(n)}$ is a single root, then $G_n'(r_i^{(n)})$ has the same sign as $G_n(r_{i+1}^{(n)-})$ (for $1 \leq i \leq f(n)-1$ if $d(n)=0$, for $1 \leq i \leq f(n)-d(n)$ if $d(n) \geq 1$). In addition, if $r_{i+1}^{(n)}$ is also a single root, then $G_n(r_i^{(n)+})$ has the opposite sign to $G_n'(r_{i+1}^{(n)})$ (for $1 \leq i \leq f(n)-d(n)-1$).

(iii) If $r_i^{(n)}$ is a multiple root of order $c(n)$ ($c(n)+d(n)+1 \leq f(n)$) and $r_{i+c(n)}^{(n)}$ is a single root, then $G_n(r_i^{(n)+})$ has the opposite sign to $G_n'(r_{i+c(n)}^{(n)})$ (for $1 \leq i \leq f(n)-d(n)-c(n)$).

(iv) If $r_i^{(n)}$ is a multiple root of order $c(n)$ ($c(n)+d(n)+2 \leq f(n)$) and $r_{i-1}^{(n)}$ and $r_{i+c(n)}^{(n)}$ are single roots, then $G_n'(r_{i-1}^{(n)})$ has the opposite sign to $G_n'(r_{i+c(n)}^{(n)})$ (if $c(n)$ is even) and $G_n'(r_{i-1}^{(n)})$ has the same sign as $G_n'(r_{i+c(n)}^{(n)})$ (if $c(n)$ is odd) (for $2 \leq i \leq f(n)-d(n)-c(n)$).

Lemma 2.2 Let $\{X_n\}, n \geq 1$ be a sequence of non-negative, integer-valued random variables, where $P(X_n = x) = p_{n,x}, x = d(n), d(n)+1, \ldots, f(n)$, where $d(n)$ and $f(n)$ are integers and $0 \leq d(n) < f(n)$, $p_{n,d(n)} > 0$ and $p_{n,f(n)} > 0$. Denote the p.g.f. of X_n by $G_n(t) = \sum_{x=d(n)}^{f(n)} p_{n,x}t^x$ and suppose that $\{G_n(t)\}$ satisfies (1) where $a_n(t)$ and $b_n(t)$ are continuous functions for $t \epsilon (-\infty, \infty)$.

Suppose that the $f(n)$ roots of $G_n(t)$ are all real and are denoted by $\{r_i^{(n)}, i = 1, \ldots, f(n)\}$, where $-\infty < r_1^{(n)} \leq \ldots \leq r_{f(n)-d(n)}^{(n)} < 0$; and $r_{f(n)-d(n)+1}^{(n)} = \ldots = r_{f(n)}^{(n)} = 0$, if $d(n) \geq 1$.

(i) If $r_i^{(n)}$ and $r_{i+1}^{(n)}$ are both single roots and $b_n(r_i^{(n)}) < 0$ and $b_n(r_{i+1}^{(n)}) < 0$ ($1 \leq i \leq f(n)-d(n)-1$), then there is at least one root of $G_{n+1}(t)$ in $(r_i^{(n)}, r_{i+1}^{(n)})$.

(ii) Suppose that $r_1^{(n)}$ is a single root and that $f(n+1) = f(n)+1$. If $b_n(r_1^{(n)}) < 0$, then there is a root of $G_{n+1}(t)$ in $(-\infty, r_1^{(n)})$. If $b_n(r_1^{(n)}) = 0$ and $b_n(r_2^{(n)}) < 0$ where $r_2^{(n)}$ is a single root, then there is at least one root of $G_{n+1}(t)$ in either $(-\infty, r_1^{(n)})$ or

$(r_1^{(n)}, r_2^{(n)})$ or a multiple root of order 2 at $r_1^{(n)}$.

(iii) Suppose that $r_1^{(n)}$ is a multiple root of order $c(n)$ $(c(n)+d(n)+1 \leq f(n))$ and that $r_1^{(n+1)} = r_1^{(n)}$ and is a multiple root of order $c(n+1)$, where $c(n+1) = c(n)$ and $f(n+1) = f(n)+1$. If $b_n(r_{c(n)+1}^{(n)}) < 0$ where $r_{c(n)+1}^{(n)}$ is a single root, then there is at least one root of $G_{n+1}(t)$ in $(r_1^{(n)}, r_{c(n)+1}^{(n)})$.

(iv) Suppose that $r_1^{(n)}$ is a multiple root of order $c(n)$ $(c(n)+d(n) = f(n))$ and that $r_1^{(n+1)} = r_1^{(n)}$ and is a multiple root of order $c(n+1)$, where $c(n+1) = c(n)$ and $d(n+1) = d(n)$ and $f(n+1) = f(n)+1$. Then, there is precisely one root of $G_{n+1}(t)$ in $(r_1^{(n)}, 0)$.

(v) Suppose that $r_i^{(n)}$ $(2 \leq i \leq f(n)-d(n)-c(n))$ is a multiple root of order $c(n)$ $(c(n)+d(n)+2 \leq f(n))$ and is also a multiple root of $G_{n+1}(t)$ of order $c(n+1)$, where $c(n+1) = c(n)$. If $b_n(r_{i-1}^{(n)}) < 0$ and $b_n(r_{i+c(n)}^{(n)}) < 0$ where $r_{i-1}^{(n)}$ and $r_{i+c(n)}^{(n)}$ are both single roots, then there is at least one root of $G_{n+1}(t)$ in either $(r_{i-1}^{(n)}, r_i^{(n)})$ or $(r_i^{(n)}, r_{i+c(n)}^{(n)})$.

(vi) Suppose that $r_{f(n)-d(n)}^{(n)}$ (the nearest non-zero root to zero) is a single root and that $d(n+1) = d(n)$. If $b_n(r_{f(n)-d(n)}^{(n)}) < 0$, then there is at least one root of $G_{n+1}(t)$ in $(r_{f(n)-d(n)}^{(n)}, 0)$.

(vii) Suppose that $r_{f(n)-d(n)-c(n)+1}^{(n)}$ (the nearest non-zero root to zero) is a multiple root of order $c(n)$ and a multiple root of order $c(n+1)$ of $G_{n+1}(t)$, where $c(n+1) = c(n)$ and $d(n+1) = d(n)$. If $b_n(r_{f(n)-d(n)-c(n)}^{(n)}) < 0$ where $r_{f(n)-d(n)-c(n)}^{(n)}$ is a single root, then there is at least one root of $G_{n+1}(t)$ in $(r_{f(n)-d(n)-c(n)}^{(n)}, r_{f(n)-d(n)-c(n)+1}^{(n)})$.

Lemma 2.3 Let $\{X_n\}, n \geq 1$ be a sequence of non-negative, integer-valued random variables, where $P(X_n = x) = p_{n,x}, x = d(n), d(n)+1, \ldots, f(n)$, where $d(n)$ and $f(n)$ are integers and $0 \leq d(n) < f(n)$, $p_{n,d(n)} > 0$ and $p_{n,f(n)} > 0$. Denote the p.g.f. of X_n by $G_n(t) = \sum_{x=d(n)}^{f(n)} p_{n,x} t^x$ and suppose that $\{G_n(t)\}$ satisfies (1) where $a_n(t)$ and $b_n(t)$ are continuous functions on $t\epsilon(-\infty, \infty)$, and Suppose also that $b_n'(\xi)$ exists for all $\xi < 0$, for each n.

Suppose that the $f(n)$ roots of $G_n(t)$ are all real and are denoted by $\{r_i^{(n)}, i = 1, \ldots, f(n)\}$, where $-\infty < r_1^{(n)} \leq \ldots \leq r_{f(n)-d(n)}^{(n)} < 0$ and $r_{f(n)-d(n)+1}^{(n)} = \cdots = r_{f(n)}^{(n)} = 0$, if $d(n) \geq 1$.

(i) If the non-zero roots of $G_n(t)$ are all single, and $b_n(0) = 0$ and $b_n(t) < 0$ for all

$t < 0$, then $d(n+1) \geq d(n)$ and $f(n+1) \geq f(n)$.

(ii) If the non-zero roots of $G_n(t)$ are all single, except for a multiple root of order $c(n)$ at $r_q^{(n)}$ $(1 \leq q \leq f(n) - d(n) - c(n) + 1)$, and $b_n(0) = b_n(r_q^{(n)}) = 0$ and $b_n(t) < 0$ for all $t < 0 \cap t \neq r_q^{(n)}$, then $r_q^{(n)}$ is a multiple root of $G_{n+1}(t)$ of order $c(n+1)$ where $c(n+1) \geq c(n)$, and $d(n+1) \geq d(n)$ and $f(n+1) \geq f(n)$.

3 Proofs of Three Lemmas

Proof of Lemma 2.1

(i) For small $\delta > 0$, $G_n(-\infty)$ has the same sign as $G_n(r_1^{(n)} - \delta)$ (as $r_1^{(n)}$ is the smallest root of $G_n(t)$). Now $G_n(t) = (t - r_1^{(n)})H_n(t)$ for some $H_n(t)$, where $H_n(r_1^{(n)}) \neq 0$ (as $r_1^{(n)}$ is a single root). So $G_n(r_1^{(n)} - \delta) = -\delta H_n(r_1^{(n)} - \delta)$ which means that $G_n(r_1^{(n)} - \delta)$ has the opposite sign to $H_n(r_1^{(n)} - \delta)$ (as $\delta > 0$). But $H_n(r_1^{(n)} - \delta)$ has the same sign as $H_n(r_1^{(n)})$ (from the continuity of $H_n(t)$ at $r_1^{(n)}$, as $H_n(r_1^{(n)}) \neq 0$). Also, $G_n'(r_1^{(n)}) = H_n(r_1^{(n)})$, so $H_n(r_1^{(n)})$ has the same sign as $G_n'(r_1^{(n)})$. Therefore, $G_n(-\infty)$ has the opposite sign to $G_n'(r_1^{(n)})$.

Now $G_n'(r_1^{(n)})$ has the opposite sign to $G_n'(r_2^{(n)})$ (as $r_1^{(n)}$ and $r_2^{(n)}$ are distinct, successive, single roots), therefore $G_n'(r_2^{(n)})$ has the same sign as $G_n(-\infty)$.

(ii) For small $\delta > 0$, $G_n(r_{i+1}^{(n)-})$ has the same sign as $G_n(r_i^{(n)} + \delta)$ (as $r_i^{(n)}$ and $r_{i+1}^{(n)}$ are successive, single, real roots). Now $G_n(t) = (t - r_i^{(n)})H_n(t)$ for some $H_n(t)$, where $H_n(r_i^{(n)}) \neq 0$ (as $r_i^{(n)}$ is a single root). So $G_n(r_i^{(n)} + \delta) = \delta H_n(r_i^{(n)} + \delta)$ which means that $G_n(r_i^{(n)} + \delta)$ has the same sign as $H_n(r_i^{(n)} + \delta)$ (as $\delta > 0$). But $H_n(r_i^{(n)} + \delta)$ has the same sign as $H_n(r_i^{(n)})$ (from the continuity of $H_n(t)$ at $r_i^{(n)}$, as $H_n(r_i^{(n)}) \neq 0$). Also, $G_n'(r_i^{(n)}) = H_n(r_i^{(n)})$, so $H_n(r_i^{(n)})$ has the same sign as $G_n'(r_i^{(n)})$. Therefore, $G_n(r_{i+1}^{(n)-})$ has the same sign as $G_n'(r_i^{(n)})$.

For small $\delta > 0$, $G_n(r_i^{(n)+})$ has the same sign as $G_n(r_{i+1}^{(n)} - \delta)$ (as $r_i^{(n)}$ and $r_{i+1}^{(n)}$ are successive, single, real roots). Now $G_n(t) = (t - r_{i+1}^{(n)})H_n(t)$ for some $H_n(t)$, where $H_n(r_{i+1}^{(n)}) \neq 0$ (as $r_{i+1}^{(n)}$ is a single root). So $G_n(r_{i+1}^{(n)} - \delta) = -\delta H_n(r_{i+1}^{(n)} - \delta)$ which means that $G_n(r_{i+1}^{(n)} - \delta)$ has the opposite sign to $H_n(r_{i+1}^{(n)} - \delta)$ (as $\delta > 0$). But $H_n(r_{i+1}^{(n)} - \delta)$

has the same sign as $H_n(r_{i+1}^{(n)})$ (from the continuity of $H_n(t)$ at $r_{i+1}^{(n)}$, as $H_n(r_{i+1}^{(n)}) \neq 0$). Also, $G'_n(r_{i+1}^{(n)}) = H_n(r_{i+1}^{(n)})$, so $H_n(r_{i+1}^{(n)})$ has the same sign as $G'_n(r_{i+1}^{(n)})$. Therefore, $G_n(r_i^{(n)+})$ has the opposite sign to $G'_n(r_{i+1}^{(n)})$.

(iii) For small $\delta > 0$, $G_n(r_i^{(n)+})$ has the same sign as $G_n(r_{i+c(n)}^{(n)} - \delta)$ (as $r_i^{(n)}$ and $r_{i+c(n)}^{(n)}$ are successive, single, real roots). Now $G_n(t) = (t - r_{i+c(n)}^{(n)})H_n(t)$ for some $H_n(t)$, where $H_n(r_{i+c(n)}^{(n)}) \neq 0$ (as $r_{i+c(n)}^{(n)}$ is a single root). So $G_n(r_{i+c(n)}^{(n)} - \delta) = -\delta H_n(r_{i+c(n)}^{(n)} - \delta)$ which means that $G_n(r_{i+c(n)}^{(n)} - \delta)$ has the opposite sign to $H_n(r_{i+c(n)}^{(n)} - \delta)$ (as $\delta > 0$). But $H_n(r_{i+c(n)}^{(n)} - \delta)$ has the same sign as $H_n(r_{i+c(n)}^{(n)})$ (from the continuity of $H_n(t)$ at $r_{i+c(n)}^{(n)}$, as $H_n(r_{i+c(n)}^{(n)}) \neq 0$). Also, $G'_n(r_{i+c(n)}^{(n)}) = H_n(r_{i+c(n)}^{(n)})$, so $H_n(r_{i+c(n)}^{(n)})$ has the same sign as $G'_n(r_{i+c(n)}^{(n)})$. Therefore, $G_n(r_i^{(n)+})$ has the opposite sign to $G'_n(r_{i+c(n)}^{(n)})$.

(iv) Since $G_n(t) = (t - r_i^{(n)})^{c(n)} H_n(t)$ for some $H_n(t)$ where $H_n(r_i^{(n)}) \neq 0$, $G_n(r_i^{(n)-})$ has the same sign as $G_n(r_i^{(n)+})$ (when $c(n)$ is even) and $G_n(r_i^{(n)-})$ has the opposite sign to $G_n(r_i^{(n)+})$ (when $c(n)$ is odd). Now from (i), we know that $G'_n(r_{i-1}^{(n)})$ has the same sign as $G_n(r_i^{(n)-})$ and from (ii), $G_n(r_i^{(n)+})$ has the opposite sign to $G'_n(r_{i+c(n)}^{(n)})$. So clearly, $G'_n(r_{i-1}^{(n)})$ has the opposite sign to $G'_n(r_{i+c(n)}^{(n)})$ (if $c(n)$ is even) and $G'_n(r_{i-1}^{(n)})$ has the same sign as $G'_n(r_{i+c(n)}^{(n)})$ (if $c(n)$ is odd). $\square$

Proof of Lemma 2.2

(i) From (1), we note that $G_{n+1}(r_i^{(n)}) = b_n(r_i^{(n)})G'_n(r_i^{(n)})$ and $G_{n+1}(r_{i+1}^{(n)}) = b_n(r_{i+1}^{(n)})G'_n(r_{i+1}^{(n)})$ (as $G_n(r_i^{(n)}) = G_n(r_{i+1}^{(n)}) = 0$). But $b_n(r_i^{(n)}) < 0$ and $b_n(r_{i+1}^{(n)}) < 0$ and $G'_n(r_i^{(n)})$ and $G'_n(r_{i+1}^{(n)})$ have opposite signs (as $r_i^{(n)}$ and $r_{i+1}^{(n)}$ are successive, single, real roots), so $G_{n+1}(r_i^{(n)})$ and $G_{n+1}(r_{i+1}^{(n)})$ will have opposite signs. Therefore, there must be at least one root (more precisely: an odd number of roots) of $G_{n+1}(t)$ within the interval $(r_i^{(n)}, r_{i+1}^{(n)})$.

(ii) $G_{n+1}(r_1^{(n)})$ is non-zero and has the opposite sign to $G'_n(r_1^{(n)})$ (by (1), as $b_n(r_1^{(n)}) < 0$ where $r_1^{(n)}$ is a single root), which has the opposite sign to $G_n(-\infty)$ (by Lemma 2.1 (i)). But $G_{n+1}(-\infty)$ has the opposite sign to $G_n(-\infty)$ (as $f(n+1) = f(n) + 1$). Therefore, $G_{n+1}(r_1^{(n)})$ has the opposite sign to $G_{n+1}(-\infty)$ and so there is at least one root of $G_{n+1}(t)$ in $(-\infty, r_1^{(n)})$.

In the second part, $G_{n+1}(r_2^{(n)})$ has the opposite sign to $G_n'(r_2^{(n)})$ (by (1), as $b_n(r_2^{(n)}) < 0$ where $r_2^{(n)}$ is a single root), which has the same sign as $G_n(-\infty)$ (by Lemma 2.1 (i), as $r_1^{(n)}$ and $r_2^{(n)}$ are single roots). Again, $G_{n+1}(-\infty)$ has the opposite sign to $G_n(-\infty)$ (as $f(n+1) = f(n)+1$). Therefore, $G_{n+1}(-\infty)$ has the same sign as $G_{n+1}(r_2^{(n)})$, and $G_{n+1}(r_1^{(n)}) = 0$ (by (1), as $b_n(r_1^{(n)}) = 0$), so there is at least one root of $G_{n+1}(t)$ in either $(-\infty, r_1^{(n)})$ or $(r_1^{(n)}, r_2^{(n)})$ or a double root at $r_1^{(n)}$.

(iii) $G_{n+1}(r_{c(n)+1}^{(n)})$ has the opposite sign to $G_n'(r_{c(n)+1}^{(n)})$ (from (1), as $b_n(r_{c(n)+1}^{(n)}) < 0$ where $r_{c(n)+1}^{(n)}$ is a single root), which is the opposite sign to $G_n(r_1^{(n)+})$ (by Lemma 2.1 (iii)). But $G_n(r_1^{(n)+})$ has the opposite sign to $G_{n+1}(r_1^{(n)+})$ (as $f(n+1) = f(n)+1$ and $c(n+1) = c(n)$). Therefore, $G_{n+1}(r_{c(n)+1}^{(n)})$ has the opposite sign to $G_{n+1}(r_1^{(n)+})$, so there is at least one root of $G_{n+1}(t)$ in $(r_1^{(n)}, r_{c(n)+1}^{(n)})$.

(iv) Since $c(n+1) = c(n)$, $d(n+1) = d(n)$ and $f(n+1) = f(n)+1$, there must be another non-zero real root, and it must differ from $r_1^{(n)}$, hence it must lie strictly between $r_1^{(n)}$ and 0.

(v) If $c(n)$ is even, then $G_n'(r_{i-1}^{(n)})$ and $G_n'(r_{i+c(n)}^{(n)})$ will have opposite signs (from Lemma 2.1 (iv)), so $G_{n+1}(r_{i-1}^{(n)})$ and $G_{n+1}(r_{i+c(n)}^{(n)})$ will have opposite signs (from (1), as $b_n(r_{i-1}^{(n)}) < 0$ and $b_n(r_{i+c(n)}^{(n)}) < 0$ where $r_{i-1}^{(n)}$ and $r_{i+c(n)}^{(n)}$ are single roots). Now as $c(n+1) = c(n)$, $G_{n+1}(r_i^{(n)-})$ and $G_{n+1}(r_i^{(n)+})$ will have the same sign (as $c(n)$ is even), so there is at least one real root of $G_{n+1}(t)$ in either $(r_{i-1}^{(n)}, r_i^{(n)})$ or $(r_i^{(n)}, r_{i+c(n)}^{(n)})$. Similarly, is $c(n)$ is odd, then $G_n'(r_{i-1}^{(n)})$ and $G_n'(r_{i+c(n)}^{(n)})$ will have the same sign (from Lemma 2.1 (iv)), so $G_{n+1}(r_{i-1}^{(n)})$ and $G_{n+1}(r_{i+c(n)}^{(n)})$ will have the same sign (from (1), as $b_n(r_{i-1}^{(n)}) < 0$ and $b_n(r_{i+c(n)}^{(n)}) < 0$ where $r_{i-1}^{(n)}$ and $r_{i+c(n)}^{(n)}$ are single roots). Now as $c(n+1) = c(n)$, $G_{n+1}(r_i^{(n)-})$ and $G_{n+1}(r_i^{(n)+})$ will have opposite signs (as $c(n)$ is odd), so there is at least one root of $G_{n+1}(t)$ in either $(r_{i-1}^{(n)}, r_i^{(n)})$ or $(r_i^{(n)}, r_{i+c(n)}^{(n)})$.

(vi) First, suppose that $d(n) \geq 1$. $G_{n+1}(r_{f(n)-d(n)}^{(n)})$ has the opposite sign to $G_n'(r_{f(n)-d(n)}^{(n)})$

(by (1), as $b_n(r^{(n)}_{f(n)-d(n)}) < 0$ where $r^{(n)}_{f(n)-d(n)}$ is a single root), which has the same sign as $G_n(0^-)$ (by Lemma 2.1 (ii), as $d(n) \geq 1$). But $G_{n+1}(0^-)$ has the same sign as $G_n(0^-)$ (as $d(n+1) = d(n)$). Therefore $G_{n+1}(r^{(n)}_{f(n)-d(n)})$ has the opposite sign to $G_{n+1}(0^-)$ and so there is at least one root of $G_{n+1}(t)$ in $(r^{(n)}_{f(n)-d(n)}, 0)$.

Next, suppose that $d(n) = 0$, so $r^{(n)}_{f(n)-d(n)} = r^{(n)}_{f(n)}$. Now $G_{n+1}(r^{(n)}_{f(n)}) < 0$ (from (1), as $G'_n(r^{(n)}_{f(n)}) > 0$ since $r^{(n)}_{f(n)}$ is the last root and is single, and $b_n(r^{(n)}_{f(n)}) < 0$) and $G_{n+1}(0) > 0$ (as $G_{n+1}(0) = p_{n+1,0} = p_{n+1,d(n+1)} > 0$, as $d(n+1) = d(n) = 0$), so there is at least one root of $G_{n+1}(t)$ in $(r^{(n)}_{f(n)}, 0)$.

(vii) $G'_n(r^{(n)}_{f(n)-d(n)-c(n)})$ has the same sign as $G_n(r^{(n)-}_{f(n)-d(n)-c(n)+1})$ (by Lemma 2.1 (ii)). But $G_{n+1}(r^{(n)}_{f(n)-d(n)-c(n)})$ has the opposite sign to $G'_n(r^{(n)}_{f(n)-d(n)-c(n)})$ (by (1), as $b_n(r^{(n)}_{f(n)-d(n)-c(n)}) < 0$ where $r^{(n)}_{f(n)-d(n)-c(n)}$ is a single root), and $G_{n+1}(r^{(n)-}_{f(n)-d(n)-c(n)+1})$ has the same sign as $G_n(r^{(n)-}_{f(n)-d(n)-c(n)+1})$ (as $d(n+1) = d(n)$ and $c(n+1) = c(n)$), so $G_{n+1}(r^{(n)}_{f(n)-d(n)-c(n)})$ has the opposite sign to $G_{n+1}(r^{(n)}_{f(n)-d(n)-c(n)+1})$. Therefore, there is at least one root of $G_{n+1}(t)$ in $(r^{(n)}_{f(n)-d(n)-c(n)}, r^{(n)}_{f(n)-d(n)-c(n)+1})$. $\square$

Proof of Lemma 2.3

(i) If $d(n) = 0$, then clearly $d(n+1) \geq d(n)$ (as $d(n+1) \geq 0$). If $d(n) \geq 1$, then $G_n(t) = t^{d(n)} H_n(t)$ for some polynomial $H_n(t)$ where $H_n(0) \neq 0$. Therefore, $G'_n(t) = t^{d(n)-1}\left(d(n)H_n(t) + tH'_n(t)\right)$. Now from the continuity and differentibility assumptions on $b_n(t)$, the Mean Value Theorem gives $b_n(t) = b_n(0) + tb'_n(\xi_t)$, where ξ_t is strictly between t and 0. But $b_n(0) = 0$, so $b_n(t) = tb'_n(\xi_t)$.

Substituting the expressions for $G_n(t)$, $G'_n(t)$ and $b_n(t)$ in (1), we get

$$G_{n+1}(t) = t^{d(n)}\left(a_n(t)H_n(t) + b'_n(\xi_t)\left(d(n)H_n(t) + tH'_n(t)\right)\right).$$

Hence 0 is a root of $G_{n+1}(t)$ of order $d(n+1) \geq d(n)$.

Firstly, suppose that $d(n) + 1 = f(n)$ (i.e. there is only one non-zero root, a single root, at $r^{(n)}_1$). There is at least one root of $G_{n+1}(t)$ in $(-\infty, r^{(n)}_1)$ (if $f(n+1) =$

$f(n)+1$, from Lemma 2.2(ii)) and in $(r_1^{(n)}, 0)$ (if $d(n+1) = d(n)$, from Lemma 2.2(vi), as $r_{f(n)-d(n)}^{(n)} = r_1^{(n)}$). Adding the $d(n+1)$ roots at 0, we have accounted for $I_{(f(n+1)=f(n)+1)} + I_{(d(n+1)=d(n))} + d(n+1)$ of the roots of $G_{n+1}(t)$. This has a lower bound of $I_{(f(n+1)=f(n)+1)} + d(n) + 1$ (as $d(n+1) \geq d(n)$) which is $I_{(f(n+1)=f(n)+1)} + f(n)$ (as $d(n)+1 = f(n)$). Therefore, $f(n+1) \geq f(n)$.

Secondly, suppose that $d(n)+1 < f(n)$ (i.e. there is more than one single root less than 0). Again, there is at least one root of $G_{n+1}(t)$ in $(-\infty, r_1^{(n)})$ (if $f(n+1) = f(n)+1$) and in $(r_{f(n)-d(n)}^{(n)}, 0)$ (if $d(n+1) = d(n)$), plus at least one root of $G_{n+1}(t)$ in each of the intervals $(r_1^{(k)}, r_2^{(k)}), (r_2^{(k)}, r_3^{(k)}), \ldots, (r_{f(n)-d(n)-1}^{(k)}, r_{f(n)-d(n)}^{(k)})$ (from Lemma 2.2(i), as $b_n(t) < 0$ for all $t < 0$). Adding the $d(n+1)$ roots at 0, we have accounted for $I_{(f(n+1)=f(n)+1)} + I_{(d(n+1)=d(n))} + f(n) - d(n) - 1 + d(n+1)$ [1] of the roots of $G_{n+1}(t)$. This can be rewritten as $I_{(f(n+1)=f(n)+1)} + f(n) + \big(d(n+1) - d(n) + I_{(d(n+1)=d(n))} - 1\big)$ which has a lower bound of $I_{(f(n+1)=f(n)+1)} + f(n)$ (as $d(n+1) \geq d(n)$), which is the expression above, so again $f(n+1) \geq f(n)$.

(ii) We have already shown that $d(n+1) \geq d(n)$ (as $b_n(0) = 0$). Using the same method, with 0 replaced by $r_q^{(n)}$, we can show that $r_q^{(n)}$ is a multiple root of $G_{n+1}(t)$ of order $c(n+1) \geq c(n)$ (as $b_n(r_q^{(n)}) = 0$).

I: Suppose that $q = 1$.

Firstly, suppose that $c(n) + d(n) = f(n)$ (that is, there are no single, real, non-zero roots of $G_n(t)$). There are $c(n+1)$ roots of $G_{n+1}(t)$ at $r_1^{(n)}$ and $d(n+1)$ roots at 0 and a root in $(r_1^{(n)}, 0)$ (if $c(n+1) = c(n)$, $d(n+1) = d(n)$ and $f(n+1) = f(n)+1$, from Lemma 2.2(iv)), which accounts for at least $c(n+1) + d(n+1) + I_{(c(n+1)=c(n), d(n+1)=d(n), f(n+1)=f(n)+1)}$ roots of $G_{n+1}(t)$. But $c(n+1) \geq c(n)$ and $d(n+1) \geq d(n)$, so this has a lower bound of $c(n) + d(n) + I_{(c(n+1)=c(n), d(n+1)=d(n), f(n+1)=f(n)+1)}$, which reduces to $f(n) + I_{(c(n+1)=c(n), d(n+1)=d(n), f(n+1)=f(n)+1)}$ (as $c(n) + d(n) = f(n)$). Clearly, $f(n+1) \geq f(n)$.

[1] Note that this equation actually holds for the previous case: where $f(n) - d(n) - 1 = 0$.

Secondly, suppose that $c(n)+d(n)+1=f(n)$ (i.e. there is one single, real, non-zero root $r^{(k)}_{c(k)+1}$ of $G_n(t)$, which is greater than $r^{(n)}_1$). There is at least one real root of $G_{n+1}(t)$ in $(r^{(n)}_1, r^{(n)}_{c(n)+1})$ (if $f(n+1)=f(n)+1$ and $c(n+1)=c(n)$, from Lemma 2.2(iii)) and in $(r^{(n)}_{c(n)+1}, 0)$ (if $d(n+1)=d(n)$, from Lemma 2.2(vi), as $r^{(n)}_{c(n)+1}=r^{(n)}_{f(n)-d(n)}$). Adding the $c(n+1)$ roots at $r^{(n)}_1$ and $d(n+1)$ roots at 0, we have accounted for $I_{(f(n+1)=f(n)+1,c(n+1)=c(n))}+I_{(d(n+1)=d(n))}+c(n+1)+d(n+1)$ roots of $G_{n+1}(t)$. But $c(n+1)\geq c(n)$ and $d(n+1)\geq d(n)$, so this has a lower bound of $c(n)+d(n)+1+I_{(c(n+1)=c(n),d(n+1)=d(n),f(n+1)=f(n)+1)}$, which reduces to $f(n)+I_{(f(n+1)=f(n)+1,c(n+1)=c(n))}$ (as $c(n)+d(n)+1=f(n)$). Again, $f(n+1)\geq f(n)$.

Finally, suppose that $c(n)+d(n)+1<f(n)$ (i.e. there are at least two single, real, non-zero roots of $G_n(t)$ greater than $r^{(n)}_1$). Again, there is at least one root of $G_{n+1}(t)$ in $(r^{(n)}_1, r^{(n)}_{c(n)+1})$ (if $f(n+1)=f(n)+1$ and $c(n+1)=c(n)$) and in $(r^{(n)}_{f(n)-d(n)}, 0)$ (if $d(n+1)=d(n)$), plus at least one root in each of the intervals $(r^{(n)}_{c(n)+1}, r^{(n)}_{c(n)+2}), \ldots, (r^{(n)}_{f(n)-d(n)-1}, r^{(n)}_{f(n)-d(n)})$ (from Lemma 2.2(i), as $b_k(t)<0$ for $r^{(n)}_1<t<0$). Adding the $c(n+1)$ roots at $r^{(n)}_1$ and $d(n+1)$ roots at 0, we have accounted for at least $I_{(f(n+1)=f(n)+1,c(n+1)=c(n))}+I_{(d(n+1)=d(n))}+f(n)-d(n)-c(n)-1+c(n+1)+d(n+1)$ roots of $G_{n+1}(t)$. This can be rewritten as $f(n)-1+I_{(f(n+1)=f(n)+1,c(n+1)=c(n))}+\Big(d(n+1)-d(n)+I_{(d(n+1)=d(n))}\Big)+(c(n+1)-c(n))$, which has a lower bound of $f(n)+I_{(f(n+1)=f(n)+1,c(n+1)=c(n))}$ (as $d(n+1)-d(n)=0$ or 1), as above.

(II): Suppose that $1<q<f(n)-d(n)-c(n)+1$ (i.e. there is at least one single, real, non-zero root of $G_n(t)$ less than, and greater than, $r^{(n)}_q$).

Firstly, suppose that $q=2$ and $q+c(n)=f(n)-d(n)$, which gives $c(n)+d(n)+2=f(n)$ (i.e. there is exactly one single, real, non-zero root of $G_n(t)$ less than, and another such greater than, $r^{(n)}_q$ - these are $r^{(n)}_1$ and $r^{(k)}_{c(k)+2}$ respectively). There is at least one root of $G_{n+1}(t)$ in $(-\infty, r^{(n)}_1)$ (if $f(n+1)=f(n)+1$, from Lemma 2.2(ii), as $q\neq 1$) and in $(r^{(n)}_{2+c(n)}, 0)$ (if $d(n+1)=d(n)$, from Lemma 2.2(vi), as $r^{(n)}_{f(n)-d(n)}=r^{(n)}_{2+c(n)}$) and in either $(r^{(n)}_1, r^{(n)}_2)$ or $(r^{(n)}_2, r^{(n)}_{2+c(n)})$ (if $c(n+1)=c(n)$, from Lemma 2.2(v), as $q=2$). Adding the $c(n+1)$ roots at $r^{(n)}_q$ (from (A)) and the $d(n+1)$ roots at 0 we have $I_{(f(n+1)=f(n)+1)}+I_{(d(n+1)=d(n))}+I_{(c(n+1)=c(n))}+c(n+1)+d(n+1)$ roots. This can be rewritten as $I_{(f(n+1)=f(n)+1)} + d(n) + c(n) + \big(c(n+1) - c(n) + I_{(c(n+1)=c(n))}\big) + \big(d(n+1) - d(n) + I_{(d(n+1)=d(n))}\big)$, which has a lower bound of $I_{(f(n+1)=f(n)+1)} + f(n)$ (as $c(n+1) \geq c(n)$ and $d(n+1) \geq d(n)$ and $c(n) + d(n) + 2 = f(n)$), so $f(n+1) \geq f(n)$.

Secondly, suppose that $q = 2$ and $q + c(n) < f(n) - d(n)$, which gives $c(n) + d(n) + 2 < f(n)$ (i.e. there is exactly one single, real, non-zero root of $G_n(t)$ less than, and more than one root, each single, greater than, $r_q^{(n)}$). Again, there is at least one root of $G_{n+1}(t)$ in $(-\infty, r_1^{(n)})$ (if $f(n+1) = f(n) + 1$) and in $(r_{f(n)-d(n)}^{(n)}, 0)$ (if $d(n+1) = d(n)$) and in either $(r_1^{(n)}, r_2^{(n)})$ or $(r_2^{(n)}, r_{2+c(n)}^{(n)})$ (if $c(n+1) = c(n)$), plus roots in the intervals $(r_{2+c(n)}^{(n)}, r_{2+c(n)+1}^{(n)}), \ldots, (r_{f(n)-d(n)-1}^{(n)}, r_{f(n)-d(n)}^{(n)})$ (from Lemma 2.2(i), as $b_k(t) < 0$ for $t < 0 \cap t \neq r_q^{(m)}$). Adding the $c(n+1)$ roots at $r_q^{(n)}$ and the $d(n+1)$ roots at 0, we have $I_{(f(n+1)=f(n)+1)} + I_{(d(n+1)=d(n))} + I_{(c(n+1)=c(n))} + f(n) - d(n) - c(n) - 2 + c(n+1) + d(n+1)$ roots. This can be rewritten as $I_{(f(n+1)=f(n)+1)} + f(n) - 2 + \big(d(n+1) - d(n) + I_{(d(n+1)=d(n))}\big) + \big(c(n+1) - c(n) + I_{(c(n+1)=c(n))}\big)$, which has a lower bound of $I_{(f(n+1)=f(n)+1)} + f(n)$ (as $c(n+1) \geq c(n)$ and $d(n+1) \geq d(n)$), as above.

Thirdly, suppose that $q > 2$ and $q + c(n) = f(n) - d(n)$, (i.e. there are $q - 1$ real, single non-zero roots of $G_n(t)$ less than, and precisely one $r_{q+c(n)}^{(n)}$ greater than, $r_q^{(n)}$). Again, there is at least one root of $G_{n+1}(t)$ in $(-\infty, r_1^{(n)})$ (if $f(n+1) = f(n) + 1$) and in $(r_{f(n)-d(n)}^{(n)}, 0)$ (if $d(n+1) = d(n)$) and in either $(r_{q-1}^{(n)}, r_q^{(n)})$ or $(r_q^{(n)}, r_{q+c(n)}^{(n)})$ (if $c(n+1) = c(n)$) plus roots in the intervals $(r_1^{(n)}, r_2^{(n)}), \ldots, (r_{q-2}^{(n)}, r_{q-1}^{(n)})$. Adding the $c(n+1)$ roots at $r_q^{(n)}$ and the $d(n+1)$ roots at 0, we have $I_{(f(n+1)=f(n)+1)} + I_{(d(n+1)=d(n))} + I_{(c(n+1)=c(n))} + q - 2 + c(n+1) + d(n+1)$ roots. This can be rewritten as $I_{(f(n+1)=f(n)+1)} + q - 2 + c(n) + d(n) + \big(c(n+1) - c(n) + I_{(c(n+1)=c(n))}\big) + \big(d(n+1) - d(n) + I_{(d(n+1)=d(n))}\big)$, which has a lower bound of $I_{(f(n+1)=f(n)+1)} + f(n)$ (as $c(n+1) \geq c(n)$ and $d(n+1) \geq d(n)$ and $q + c(n) = f(n) - d(n)$), as above.

Finally, suppose that $q > 2$ and $q + c(n) < f(n) - d(n)$, (i.e. there are $q - 1$ single, real, non-zero roots of $G_n(t)$ less than, and more than one greater than, $r_q^{(n)}$). Again, there are roots in $(-\infty, r_1^{(n)})$ (if $f(n+1) = f(n) + 1$) and in $(r_{f(n)-d(n)}^{(n)}, 0)$ (if $d(n+1) = d(n)$) and in either $(r_{q-1}^{(n)}, r_q^{(n)})$ or $(r_q^{(n)}, r_{q+c(n)}^{(n)})$ (if $c(n+1) = c(n)$), plus at least one root in each of the intervals $(r_1^{(n)}, r_2^{(n)}), \ldots, (r_{q-2}^{(n)}, r_{q-1}^{(n)})$ and $(r_{q+c(n)}^{(n)}, r_{q+c(n)+1}^{(n)}), \ldots,$ $(r_{f(n)-d(n)-1}^{(n)}, r_{f(n)-d(n)}^{(n)})$. Adding the $c(n+1)$ roots at $r_q^{(n)}$ (from (A)) and the $d(n+1)$ roots at 0, we have $I_{(f(n+1)=f(n)+1)} + I_{(d(n+1)=d(n))} + I_{(c(n+1)=c(n))} + f(n) - d(n) - c(n) - 2 + c(n+1) + d(n+1)$ roots. This can be rewritten as $I_{(f(n+1)=f(n)+1)} + f(n) - 2 + \left(d(n+1) - d(n) + I_{(d(n+1)=d(n))}\right) + \left(c(n+1) - c(n) + I_{(c(n+1)=c(n))}\right)$, which has a lower bound of $I_{(f(n+1)=f(n)+1)} + f(n)$, (as $c(n+1) \geq c(n)$ and $d(n+1) \geq d(n)$), as above.

(III): Suppose that $q = f(n) - d(n) - c(n) + 1$ (i.e. there are no non-zero roots of $G_n(t)$ greater than $r_q^{(n)}$).

Firstly, suppose that $q = 2$, that is $c(n) + d(n) + 1 = f(n)$ (since $q = 2$, $r_1^{(n)}$ is the only non-zero, single root of $G_n(t)$ less than $r_q^{(n)} = r_2^{(n)}$). There is at least one root of $G_{n+1}(t)$ in $(-\infty, r_1^{(n)})$ (if $f(n+1) = f(n) + 1$, from Lemma 2.2(ii)) and in $(r_1^{(n)}, r_2^{(n)})$ (if $d(n+1) = d(n)$ and $c(n+1) = c(n)$, from Lemma 2.2(vii), as $r_{f(n)-d(n)-c(n)}^{(n)} = r_1^{(n)}$). Adding the $c(n+1)$ roots at $r_{f(n)-d(n)-c(n)+1}^{(n)}$ and the $d(n+1)$ roots at 0, we have $I_{(f(n+1)=f(n)+1)} + I_{(d(n+1)=d(n),c(n+1)=c(n))} + c(n+1) + d(n+1)$ roots. This can be rewritten as $I_{(f(n+1)=f(n)+1)} + c(n) + d(n) + \left(c(n+1) - c(n) + d(n+1) - d(n) + I_{(d(n+1)=d(n),c(n+1)=c(n))}\right)$, which has a lower bound of $I_{(f(n+1)=f(n)+1)} + f(n)$ (as $c(n+1) \geq c(n)$ and $d(n+1) \geq d(n)$ and $c(n) + d(n) + 1 = f(n)$), so $f(n+1) \geq f(n)$.

Secondly, suppose that $q > 2$, that is $c(n) + d(n) + 1 < f(n)$ (since $q > 2$, there are at least two non-zero, single roots of $G_n(t)$ less than $r_q^{(n)}$). Again, there is at least one root of $G_{n+1}(t)$ in the intervals in $(-\infty, r_1^{(n)})$ (if $f(n+1) = f(n) + 1$) and in $(r_{f(n)-d(n)-c(n)}, r_{f(n)-d(n)-c(n)+1})$ (if $d(n+1) = d(n)$ and $c(n+1) = c(n)$), plus at least one root in each of the intervals $(r_1^{(n)}, r_2^{(n)}), \ldots, (r_{f(n)-d(n)-c(n)-1}^{(n)}, r_{f(n)-d(n)-c(n)}^{(n)})$ (from Lemma 2.2(i)). Adding the $c(n+1)$ roots at $r_{f(n)-d(n)-c(n)+1}^{(n)}$ and the $d(n+1)$ roots at 0, we have $I_{(f(n+1)=f(n)+1)} + I_{(d(n+1)=d(n),c(n+1)=c(n))} + f(n) - d(n) - c(n) - 1 + c(n+1) + d(n+1)$ roots. This can be rewritten as $I_{(f(n+1)=f(n)+1)} + f(n) - 1 + \left((c(n+1) - c(n)) + (d(n+1) - d(n)) + I_{(d(n+1)=d(n),c(n+1)=c(n))}\right)$, as above. □

4 Proof of Theorem 1.1

We know that for some $m \geq 1$, the $f(m) - d(m)$ non-zero roots of $G_m(t)$ are all real, distinct (and negative). Now assume for $k \geq m$, that the non-zero roots of $G_k(t)$ are all real and distinct and are denoted in ascending order by $\{r_i^{(k)}\}, i = 1, 2, \ldots, f(k) - d(k)$. We must prove that the $f(k+1) - d(k+1)$ non-zero roots of $G_k(t)$ are all real, distinct (and negative).

From Lemma 2.3 (i), the number of roots of $G_{k+1}(t)$ is at least $I_{(f(k+1)=f(k)+1)} + I_{(d(k+1)=d(k))} + f(k) - d(k) - 1 + d(k+1)$. But $f(k+1) - f(k) = 0$ or 1 and $d(k+1) - d(k) = 0$ or 1, so clearly we have accounted for the total number of roots of $G_{k+1}(t)$.

Therefore, using the working in Lemma 2.3 (i), the $f(k+1) - d(k+1)$ non-zero roots of $G_{k+1}(t)$ are found within the intervals $(r_1^{(k)}, r_2^{(k)}), (r_2^{(k)}, r_3^{(k)}) \ldots, (r_{f(k)-d(k)-1}^{(k)}, r_{f(k)-d(k)}^{(k)})$ (if $d(k) + 1 < f(k)$) and $(-\infty, r_1)$ (if $f(k+1) = f(k) + 1$) and $(r_{f(k)-d(k)}^{(k)}, 0)$ (if $d(k+1) = d(k)$). These intervals are based on the real, single and negative roots of $G_k(t)$ and so the non-zero roots of $G_{k+1}(t)$ roots are all real, single and negative. □

5 Proof of Theorem 1.2

We shall now use proof by induction to establish in sequence the 3 parts of Theorem 1.2.

Proof by Induction for (A) We know that that for some $m \geq 1$, $r_q^{(m)}$ is a multiple real root of $G_m(t)$ of order $c(m)$. Now assume that for some $k \geq m$, $r_q^{(m)}$ is a multiple root of $G_k(t)$ of order $c(k) \geq c(m)$. By Lemma 2.3 (ii), $r_q^{(m)}$ is a multiple root of $G_{k+1}(t)$ of order $c(k+1) \geq c(k)$. □

Proof by Induction for (B1) We know that for some $m \geq 1$ the $f(m) - d(m)$ non-zero roots of $G_m(t)$ are all real, and single except the smallest root $r_1^{(m)}$ which is a multiple root of order $c(m)$. Assume for some $k \geq m$, that the $f(k) - d(k)$ non-zero roots of $G_k(t)$ are all real, and single except the smallest root $r_1^{(m)}$ (which, from (A), is a multiple root of order $c(k)$).

Denote the $f(k)$ roots of $G_k(t)$, in ascending order, by $r_1^{(m)}, r_2^{(k)} \ldots, r_{f(k)}^{(k)}$, where

$r_1^{(m)} = r_2^{(k)} = \ldots = r_{c(k)}^{(k)}$ are the roots at $r_1^{(m)}$

$r_{c(k)+1}^{(k)}, \ldots, r_{f(k)-d(k)}^{(k)}$ are the single roots, if $c(k)+d(k)+1 \leq f(k)$, and

$r_{f(k)-d(k)+1}^{(k)} = r_{f(k)-d(k)+2}^{(k)} = \ldots = r_{f(k)}^{(k)} = 0$ are the roots at 0, if $d(k) \geq 1$.

Firstly, suppose that $c(k) + d(k) = f(k)$ (that is, there are no single, non-zero real roots of $G_k(t)$). From Lemma 2.3 (ii) I, the number of roots of $G_{k+1}(t)$ is at least $c(k+1) + d(k+1) + I_{(c(k+1)=c(k), d(k+1)=d(k), f(k+1)=f(k)+1)}$. This can be rewritten as $c(k)+d(k)+I_{(c(k+1)=c(k), d(k+1)=d(k), f(k+1)=f(k)+1)}+(c(k+1)-c(k))+(d(k+1)-d(k))$, which reduces to $f(k)+I_{(c(k+1)=c(k), d(k+1)=d(k), f(k+1)=f(k)+1)}+(c(k+1)-c(k))+(d(k+1)-d(k))$ (as $c(k)+d(k)=f(k)$).

Altogether, $G_{k+1}(t)$ has a total of $f(k+1)$ roots (where $f(k+1)-f(k)=0$ or 1) and $c(k+1) \geq c(k)$ (from (A)) and $d(k+1)-d(k) = 0$ or 1 from (1) and (2), so we have accounted for all the roots of $G_{k+1}(t)$, and $c(k+1)$ must satisfy $c(k+1)-c(k)=0$ or 1 (only 1 if $d(k+1)=d(k)$ and $f(k+1)=f(k)+1$). Clearly, $r_1^{(m)}$ is the smallest root.

Secondly, suppose that $c(k)+d(k)+1 = f(k)$ (i.e. there is one single, real, non-zero root $r_{c(k)+1}^{(k)}$ of $G_k(t)$, which is greater than $r_1^{(m)}$). Again from Lemma 2.3 (ii) I, the number of roots of $G_{k+1}(t)$ is at least $I_{(f(k+1)=f(k)+1, c(k+1)=c(k))}+I_{(d(k+1)=d(k))}+c(k+1)+d(k+1)$, which can be rewritten as $I_{(f(k+1)=f(k)+1, c(k+1)=c(k))}+c(k)+d(k)+(c(k+1)-c(k))+\left(d(k+1)-d(k)+I_{(d(k+1)=d(k))}\right)$. This reduces to $f(k)+I_{(f(k+1)=f(k)+1, c(k+1)=c(k))}+(c(k+1)-c(k))$ (as $d(k+1)-d(k)=0$ or 1, and $c(k)+d(k)+1=f(k)$).

Now $c(k+1) \geq c(k)$ (from (A)) and $f(k+1)-f(k)=0$ or 1, so we have accounted for all the roots of $G_{k+1}(t)$ and $c(k+1)-c(k)=0$ or 1 (always 0 when $f(k+1)=f(k)$). Clearly, $r_1^{(m)}$ is the smallest root and the non-zero roots, other than $r_1^{(m)}$, are real and single.

Finally, suppose that $c(k)+d(k)+1<f(k)$ (i.e. there are at least two single, real, non-zero roots of $G_k(t)$ greater than $r_1^{(m)}$). Again, from Lemma 2.3 (ii) I, the number of roots of $G_{k+1}(t)$ is at least $I_{(f(k+1)=f(k)+1,c(k+1)=c(k))}+I_{(d(k+1)=d(k))}+f(k)-d(k)-c(k)-1+c(k+1)+d(k+1)$. This can be rewritten as $f(k)-1+I_{(f(k+1)=f(k)+1,c(k+1)=c(k))}+\Big(d(k+1)-d(k)+I_{(d(k+1)=d(k))}\Big)+(c(k+1)-c(k))$, which reduces to $f(k)+I_{(f(k+1)=f(k)+1),c(k+1)=c(k))}+(c(k+1)-c(k))$ (as $d(k+1)-d(k)=0$ or 1), as above. $\square$

Proof by Induction for (B2) We know that for some $m\geq 1$ the $f(m)-d(m)$ non-zero roots of $G_m(t)$ are all real, and single except for $r_q^{(m)}$, which is a multiple root of order $c(m)$. Assume for some $k\geq m$, that the $f(k)-d(k)$ non-zero roots of $G_k(t)$ are all real, and single except for $r_q^{(m)}$ (which, from (A), is a multiple root of order $c(k)$).

(i) Suppose that $1<q<f(m)-d(m)-c(m)+1$ (i.e. there is at least one single, real, non-zero root of $G_k(t)$ less than, and greater than $r_q^{(m)}$).

Denote the $f(k)$ roots of $G_k(t)$, in ascending order, by $r_1^{(k)},r_2^{(k)},\ldots,r_{f(k)}^{(k)}$, where

$r_1^{(k)},r_2^{(k)},\ldots,r_{q-1}^{(k)}$ and $r_{q+c(k)}^{(k)},r_{q+c(k)+1}^{(k)},\ldots,r_{f(k)-d(k)}^{(k)}$ are the single roots,
$r_q^{(k)}=r_{q+1}^{(k)}=\ldots=r_{q+c(k)-1}^{(k)}$ are the roots at $r_q^{(m)}$ and
$r_{f(k)-d(k)+1}^{(k)}=r_{f(k)-d(k)+2}^{(k)}=\ldots=r_{f(k)}^{(k)}=0$ are the roots at 0, if $d(k)\geq 1$.

In the sequel, we separate the cases $q=2$ and $q>2$ for greater clarity (as in Lemma 2.3 I and II), though the proofs are very similar.

Firstly, suppose that $q=2$ and $q+c(k)=f(k)-d(k)$, which gives $c(k)+d(k)+2=f(k)$ (i.e. there is exactly one single, real, non-zero root of $G_k(t)$ less than, and another such greater than, $r_q^{(m)}$- these are $r_1^{(k)}$ and $r_{2+c(k)}^{(k)}$ respectively). From Lemma 2.3 (ii) II, the number of roots of $G_{k+1}(t)$ is at least $I_{(f(k+1)=f(k)+1)}+I_{(d(k+1)=d(k))}+$

$I_{(c(k+1)=c(k))} + c(k+1) + d(k+1)$, which can be rewritten as $I_{(f(k+1)=f(k)+1)} + f(k) - 2 + \left(c(k+1) - c(k) + I_{(c(k+1)=c(k))}\right) + \left(d(k+1) - d(k) + I_{(d(k+1)=d(k))}\right)$. This reduces to $I_{(f(k+1)=f(k)+1)} + f(k) - 1 + \left(c(k+1) - c(k) + I_{(c(k+1)=c(k))}\right)$ (as $d(k+1) - d(k) = 0$ or 1 and $c(k) + d(k) + 2 = f(k)$).

Now $c(k+1) \geq c(k)$, from (A)) and $f(k+1) - f(k) = 0$ or 1, so we have accounted for the total $f(k+1)$ roots of $G_{k+1}(t)$, and $c(k+1) - c(k) = 0$ or 1. Clearly, the non-zero roots, other than $r_2^{(m)}$, are real and single.

Secondly, suppose that $q = 2$ and $q + c(k) < f(k) - d(k)$, which gives $c(k) + d(k) + 2 < f(k)$ (i.e. there is exactly one single, real, non-zero root of $G_k(t)$ $r_1^{(k)}$ less than, and more than one root, each single, greater than, $r_q^{(m)}$). Again, from Lemma 2.3 (ii) II, the number of roots of $G_{k+1}(t)$ is at least $I_{(f(k+1)=f(k)+1)} + I_{(d(k+1)=d(k))} + I_{(c(k+1)=c(k))} + f(k) - d(k) - c(k) - 2 + c(k+1) + d(k+1)$. This can be rewritten as $I_{(f(k+1)=f(k)+1)} + f(k) - 2 + \left(d(k+1) - d(k) + I_{(d(k+1)=d(k))}\right) + \left(c(k+1) - c(k) + I_{(c(k+1)=c(k))}\right)$, which reduces to $I_{(f(k+1)=f(k)+1)} + f(k) - 1 + \left(c(k+1) - c(k) + I_{(c(k+1)=c(k))}\right)$ (as $d(k+1) - d(k) = 0$ or 1), as above.

Thirdly, suppose that $q > 2$ and $q + c(k) = f(k) - d(k)$, (i.e. there are $q - 1$ single, real, non-zero roots of $G_k(t)$ less than, and precisely one greater than, $r_q^{(m)}$). Again from Lemma 2.3 (ii) II, the number of roots of $G_{k+1}(t)$ is at least $I_{(f(k+1)=f(k)+1)} + I_{(d(k+1)=d(k))} + I_{(c(k+1)=c(k))} + q - 2 + c(k+1) + d(k+1)$. This can be rewritten as $I_{(f(k+1)=f(k)+1)} + q - 2 + c(k) + d(k) + \left(c(k+1) - c(k) + I_{(c(k+1)=c(k))}\right) + \left(d(k+1) - d(k) + I_{(d(k+1)=d(k))}\right)$, which reduces to $I_{(f(k+1)=f(k)+1)} + f(k) - 1 + \left(c(k+1) - c(k) + I_{(c(k+1)=c(k))}\right)$ (as $d(k+1) - d(k) = 0$ or 1 and $q + c(k) = f(k) - d(k)$), which is the same result as above.

Finally, suppose that $q > 2$ and $q + c(k) < f(k) - d(k)$, (i.e. there are $q - 1$ single, real, non-zero roots of $G_k(t)$ less than, and more than one greater than, $r_q^{(m)}$). From Lemma 2.3 (ii) II, the number of roots of $G_{k+1}(t)$ is at least $I_{(f(k+1)=f(k)+1)} + I_{(d(k+1)=d(k))} + I_{(c(k+1)=c(k))} + f(k) - d(k) - c(k) - 2 + c(k+1) + d(k+1)$. This can be rewritten as $I_{(f(k+1)=f(k)+1)} + f(k) - 2 + \left(d(k+1) - d(k) + I_{(d(k+1)=d(k))}\right) +$

$\Big(c(k+1) - c(k) + I_{(c(k+1)=c(k))}\Big)$, which again reduces to $I_{(f(k+1)=f(k)+1)} + f(k) - 1 + \Big(c(k+1) - c(k) + I_{(c(k+1)=c(k))}\Big)$, as above.

(ii) Now suppose that $q = f(m) - d(m) - c(m) + 1$ (i.e. there are no non-zero roots of $G_k(t)$ greater than $r_q^{(m)}$).

Denote the $f(k)$ roots of $G_k(t)$, in ascending order, by $r_1^{(k)}, r_2^{(k)}, \ldots, r_{f(k)}^{(k)}$, where

$r_1^{(k)}, r_2^{(k)}, \ldots, r_{f(k)-d(k)-c(k)}^{(k)}$ are the single roots ($< r_q^{(m)}$),
$r_{f(k)-d(k)-c(k)+1}^{(k)} = \ldots = r_{f(k)-d(k)}^{(k)}$ are the roots at $r_q^{(m)}$ and
$r_{f(k)-d(k)+1}^{(k)} = r_{f(k)-d(k)+2}^{(k)} = \ldots = r_{f(k)}^{(k)} = 0$ are the roots at 0, if $d(k) \geq 1$.

Firstly, suppose that $q = 2$, that is $c(k)+d(k)+1 = f(k)$ (since $q = 2$, $r_1^{(k)}$ is the only non-zero, single root of $G_k(t)$ less than $r_q^{(m)} = r_2^{(k)}$). From Lemma 2.3 (ii) III, the number of roots of $G_{k+1}(t)$ is at least $I_{(f(k+1)=f(k)+1)} + I_{(d(k+1)=d(k),c(k+1)=c(k))} + c(k+1) + d(k+1)$. This can be rewritten as $I_{(f(k+1)=f(k)+1)} + c(k) + d(k) + (c(k+1) - c(k) + d(k+1) - d(k) + I_{(d(k+1)=d(k),c(k+1)=c(k))}\Big)$, which reduces to $I_{(f(k+1)=f(k)+1)} + f(k) - 1 + (c(k+1) - c(k) + d(k+1) - d(k) + I_{(d(k+1)=d(k),c(k+1)=c(k))}\Big)$ (as $c(k) + d(k) + 1 = f(k)$).

But $c(k+1) \geq c(k)$ (from (A)) and $d(k+1) - d(k) = 0$ or 1 and $f(k+1) - f(k) = 0$ or 1, so we have accounted for the total $f(k+1)$ roots of $G_{k+1}(t)$, and $c(k+1) - c(k) = 0$ or 1 (only 1 if $d(k+1) = d(k)$ and $f(k+1) = f(k) + 1$).

Secondly, suppose that $q > 2$, that is $c(k) + d(k) + 1 < f(k)$ (since $q > 2$, there are at least two non-zero, single roots of $G_k(t)$ less than $r_q^{(m)}$). Again from Lemma 2.3 (ii) III,the number of roots of $G_{k+1}(t)$ is at least $I_{(f(k+1)=f(k)+1)} + I_{(d(k+1)=d(k),c(k+1)=c(k))} + f(k) - d(k) - c(k) - 1 + c(k+1) + d(k+1)$. This can be rewritten as $I_{(f(k+1)=f(k)+1)} + f(k) - 1 + ((c(k+1) - c(k)) + (d(k+1) - d(k)) + \ I_{(d(k+1)=d(k),c(k+1)=c(k))}\Big)$, as above.

$\square$

Bibliography

Frobenius, G. (1910) Über die BERNOULLI'schen Zahlen und die EULERschen Polynome. *Sitzungsberichte der Preußischen Akademie der Wissenschaften*, pp829-830.

Harper, L.H. (1967) Stirling behaviour is asymptotically normal. *Ann. Math. Statist.* **38**, 410-414.

Warren, Di (1997) The Frobenius-Harper technique in a general recurrence model. (Positively reviewed for the *Applied Probability* journals.)