Saddlepoint Approximations for the Difference of Order Statistics and Studentized Sample Quantiles

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SUMMARY

For a sample from a given distribution the difference of two order statistics and the studentized quantile are statistics whose distribution is needed to obtain tests and confidence intervals for quantiles and quantile differences. This paper gives saddlepoint approximations for densities and saddlepoint approximations of the Lugannani-Rice form for tail probabilities of these statistics. The relative errors of the approximations are n^{-1} uniformly in a neighbourhood of the parameters and this uniformity is global if the densities are log-concave.

Keywords: SADDLEPOINT APPROXIMATION; LAPLACE METHOD; DIFFERENCE OF ORDER STATISTICS; STUDENTIZED SAMPLE QUANTILE.

1 INTRODUCTION

Suppose that $F(x)$ is an absolutely continuous distribution function with density f, $X_1, \cdots, X_n$ are a random sample from $F(x)$, and $X_{1:n} \leq \cdots \leq X_{n:n}$ are the corresponding order statistics. The purpose of this paper is to derive saddlepoint approximations for the difference of order statistics and studentized sample quantiles by using the Laplace approximation. Similar approximations for other statistics have been considered in the literature; see for instance, Tierney, Kass and Kadane (1989), DiCiccio, Field and Fraser (1990), Daniels and Young (1991), Diciccio and Martin (1991), Jing and Robinson (1994), and Jensen (1995).

The difference of two order statistics

$$Z_{r_1,r_2} = X_{r_2:n} - X_{r_1:n},$$

is often used to construct a confidence interval for the corresponding population quantile difference, say $\xi_{p_{02}} - \xi_{p_{01}}$, where $r_i = [np_{0i}] + 1,\ i = 1, 2$ and $[x]$ stands for the largest integer not exceeding x. Such quantities may be of interest in their own right; $X_{n:n} - X_{1:n}$, the range of a distribution, $X_{\frac{3}{4}n:n} - X_{\frac{1}{4}n:n}$, the interquartile range and $X_{\frac{9}{10}n:n} - X_{\frac{1}{10}n:n}$, the interdecile of a distribution. These measures evidently give some approximate idea of "spread" of a distribution. However, for most densities f, the density of Z_{r_1,r_2} is impossible to determine analytically, and its asymptotic normality is often used; see for example David (1981). Harvill and Newton (1995) obtain the saddlepoint approximation for the density of the difference of order statistics by using the approach of Easton and Ronchetti (1986), where the major difficulty arises in suitably approximating the related cumulant generating function so that the solution of the approximating saddlepoint equation exists and is unique. By using the Laplace method in Section 2 we derive saddlepoint approximations for both the density and the tail probability of the difference of order statistics. These approximations have relative errors of order n^{-1} uniformly in a neighbourhood of the difference of the population quantiles. Further, if the density f is log-concave, the relative errors are uniformly of order n^{-1}; results of this type for means are discussed in Jensen (1991 and 1995, Chapter 6). Interestingly enough, our saddlepoint approximation for the density of Z_{r_1,r_2} is "exact" in the case of the exponential distribution and the uniform distribution. This is related to the results of Daniels (1954, 1980) and Blaesild and Jensen (1985) who obtained the only three "exact" saddlepoint approximations for the density of an i.i.d.

sum.

Another quantity of interest is of the form

$$\frac{X_{r:n}-\xi_{p_0}}{Z_{r_1,r_2}},$$

where $r_1 < r < r_2$ are integers, ξ_{p_0} is the p_0-th population quantile $(0 < p_0 < 1)$ and

$$r = \begin{cases} np_0, & \text{if } np_0 \text{ is an integer,} \\ [np_0]+1, & \text{otherwise.} \end{cases}$$

It is called the studentized sample quantile, since its structure is somewhat similar to that of Student's t in the sense that

(i) its numerator is the difference between the sample quantile $X_{r:n}$ and the population quantile ξ_{p_0} and as an estimate of a scale parameter, its denominator is the difference of order statistics; and

(ii) it is invariant under linear transformations, so that its distribution is independent of location and scale parameters.

The studentized sample quantile has been investigated by Siddiqui (1960), Bloch and Gastwirth (1968), Birnbaum and Vincze (1973), Hall and Sheather (1988), amongst others. In particular, Hall and Sheather (1988) derived the Edgeworth expansion for the studentized sample quantile with specified $r_1 = r - r_0$, $r_2 = r + r_0$ and applied it to construct confidence intervals. In Section 3 we will derive saddlepoint approximations for both the density and the tail probability of the studentized sample quantile, with relative errors uniformly of order n^{-1} in a neighbourhood of the quantile. Again,if the density f is log-concave, the relative errors are uniformly of order n^{-1}. A numerical example shows the great accuracy of the saddlepoint approximation.

Let $\phi(\cdot)$ and $\Phi(\cdot)$ represent the standard normal density and distribution functions respectively. For a multivariate function $h(x_1, \cdots, x_k)$, denote its derivatives as follows

$$\begin{aligned} \dot{h}_{x_i}(x_1,\cdots,x_k) &= \frac{\partial h(x_1,\cdots,x_k)}{\partial x_i}, \\ \ddot{h}_{x_i x_j}(x_1,\cdots,x_k) &= \frac{\partial^2 h(x_1,x_j)}{\partial x_i \partial x_j}, \quad i,j = 1,\cdots,k, \end{aligned}$$

and let $\nabla h(x_1, \cdots, x_k)$ stands for its the Hessian matrix.

2 SADDLEPOINT APPROXIMATION FOR THE DIFFERENCE OF ORDER STATISTICS

Given $0 < p_{01} < p_{02} < 1$, consider the difference of two order statistics

$$Z_{r_1,r_2} = X_{r_2:n} - X_{r_1:n},$$

where $r_i = [np_{0i}] + 1, \ i = 1, 2$. Denote the density and tail probability of Z_{r_1,r_2} by $f_{Z_{r_1,r_2}}(z)$ and $\bar{F}_{Z_{r_1,r_2}}(z)$, respectively.

The joint density of $(X_{r_1:n}, X_{r_2:n})$ is

$$f(x_1, x_2) = \frac{n! f(x_1) f(x_2)}{(r_1 - 1)!(r_2 - r_1 - 1)!(n - r_2)!} F^{r_1-1}(x_1)\{F(x_2) - F(x_1)\}^{r_2-r_1-1}\{1 - F(x_2)\}^{n-r_2},$$
$$x_1 < x_2.$$

Based on $f(x_1, x_2)$, the joint density of $(X_{r_1:n}, Z_{r_1,r_2})$ is $f(t, t + z), \ z \geq 0$, which gives the exact density of Z_{r_1,r_2} as

$$f_{Z_{r_1,r_2}}(z) = \int_{-\infty}^{\infty} f(t, t + z)dt, \qquad z \geq 0.$$

Write $p_i = r_i/n, \ i = 1, 2$, where we have suppressed the dependence on n to simplify the notation.

The density $f(x_1, x_2)$ may be re-expressed as

$$f(x_1, x_2) = c_n \exp\{-n \cdot h(x_1, x_2)\} g(x_1, x_2), \qquad x_1 < x_2,$$

where

$$\begin{aligned} h(x_1, x_2) &= p_1 \ln p_1 + (p_2 - p_1) \ln(p_2 - p_1) + (1 - p_2) \ln(1 - p_2) \\ &\quad - \{p_1 \ln F(x_1) + (p_2 - p_1) \ln(F(x_2) - F(x_1)) + (1 - p_2) \ln(1 - F(x_2))\}, \\ g(x_1, x_2) &= \{F(x_1)(F(x_2) - F(x_1))\}^{-1} f(x_1) f(x_2), \end{aligned}$$

and

$$\begin{aligned} c_n &= \frac{n!}{(r_1 - 1)!(r_2 - r_1 - 1)!(n - r_2)!} \\ &\quad \cdot \exp\{n[p_1 \ln p_1 + (p_2 - p_1) \ln(p_2 - p_1) + (1 - p_2) \ln(1 - p_2)]\}, \end{aligned}$$

and again we have suppressed the dependence of $h(x_1, x_2)$ and $g(x_1, x_2)$ on n. So

$$f_{Z_{r_1,r_2}}(z) = c_n \int_{-\infty}^{\infty} \exp\{-n \cdot h(t, t + z)\} g(t, t + z)dt, \qquad z \geq 0.$$

Applying Stirling's formula to each factorial, an estimate of c_n is then given by

$$c_n = \frac{n}{2\pi} c\{1 + O(n^{-1})\},$$

where

$$c = \{\frac{p_1(p_2 - p_1)}{1 - p_2}\}^{\frac{1}{2}}.$$

As a result, $f_{Z_{r_1,r_2}}(z)$ is approximated by

$$f_{Z_{r_1,r_2}}(z) = \frac{nc}{2\pi} \int_{-\infty}^{\infty} \exp\{-n \cdot h(t, t+z)\} g(t, t+z) dt \{1 + O(n^{-1})\}, \qquad z \geq 0. \quad (2.1)$$

Next it will be further approximated by using a Laplace approximation.

To state the results, some notation is needed. Let $z^0 = x_2^0 - x_1^0$, where (x_1^0, x_2^0) is given by

$$F(x_1^0) = p_1, \quad F(x_2^0) = p_2.$$

For each z let $\tilde{t} = t(z)$ be the solution of the equation $\partial h(t, t+z)/\partial t = 0$, namely,

$$(\frac{p_1}{F(\tilde{t})} - \frac{p_2 - p_1}{F(\tilde{t}+z) - F(\tilde{t})}) f(\tilde{t}) = (\frac{1 - p_2}{1 - F(\tilde{t}+z)} - \frac{p_2 - p_1}{F(\tilde{t}+z) - F(\tilde{t})}) f(\tilde{t}+z). \quad (2.2)$$

Our saddlepoint approximations for the density and tail probability of Z_{r_1,r_2} are presented in the following theorem.

Theorem 1 Suppose that $f''(x)$ exists and is bounded, and $f(x_i^0) > 0$ $(i = 1, 2)$. Then uniformly in a neighbourhood of z^0,

$$\text{(i)} \quad f_{Z_{r_1,r_2}}(z) = \sqrt{\frac{n}{2\pi}} \exp\{-n \cdot H(z)\} G(z) \{1 + O(n^{-1})\}, \quad (2.3)$$

where

$$\begin{aligned} H(z) &= h(t(z), t(z)+z), \\ G(z) &= cg(t(z), t(z)+z) \ddot{h}_t^{-\frac{1}{2}}(t(z); z), \end{aligned}$$

and

$$\ddot{h}_t(t; z) = \frac{\partial^2 h(t, t+z)}{\partial t^2}.$$

$$\text{(ii)} \quad \bar{F}_{Z_{r_1,r_2}}(z) = 1 - \Phi(\sqrt{n} w(z)) - \frac{\phi(\sqrt{n} w(z))}{\sqrt{n}} \{\frac{1 - \psi(z)}{w(z)} + O(n^{-1})\}, \quad (2.4)$$

or alternatively,

$$\bar{F}_{Z_{r_1,r_2}}(z) = [1 - \Phi(\sqrt{n}w^*(z))]\{1 + O(n^{-1})\}, \tag{2.5}$$

where

$$\begin{aligned} w(z) &= \{2H(z)\}^{1/2} sgn(z - z^0), \\ w^*(z) &= w(z) - \frac{1}{nw(z)} \log\{\psi(z)\}, \end{aligned}$$

and

$$\psi(z) = G(z) \mid \frac{w(z)}{H'(z)} \mid .$$

To prove Theorem 1, we require some lemmas, whose proofs are given in an Appendix. Lemma 1 (i) below ensures that $H(z)$ is always nonnegative so that $w(z)$ in (2.4) and (2.5) is well-defined. To apply the Laplace method, it has to be shown that for large n, the behaviour of the integral in (2.1) is determined by behaviour near the minimum of $h(t, t + z)$, where $z > 0$ is fixed. This can be seen from Lemma 1 (iii).

Lemma 1

(i) $h(x_1, x_2)$ is nonnegative and attains its minimum 0 at (x_1^0, x_2^0).

(ii) For any constant $0 \le \lambda_0 \le \frac{1}{2}\min\{p_1, 1 - p_2\}$,

$$h(x_1, x_2) \ge \lambda_0\{[F(x_1) - p_1]^2 + [F(x_2) - p_2]^2\}, \qquad x_1 < x_2.$$

(iii) For fixed $z > 0$ and any $d_1 < z^0 < d_2$,

$$\left(\int_{-\infty}^{d_1} + \int_{d_2}^{\infty}\right) \exp\{-n \cdot h(t, t + z)\} g(t, t + z) dt \le c_1 e^{-nc_2},$$

where c_1, c_2 are positive constants independent of n.

In particular, for $z^0 = x_2^0 - x_1^0$, $t(z^0) = x_1^0$ and from Lemma 1 (i), $h(t, t + z^0)$ is minimized at x_1^0.

Lemma 2

(i) $\mid \nabla h(t(z), t(z) + z) \mid = \ddot{h}_t(t(z); z) H''(z)$.

(ii) $\nabla h(t(z), t(z) + z)$ is positive definite if and only if one of the following statements is true

(1) $\ddot{h}_t(t(z);z) > 0$ and $H''(z) > 0$;

(2) $\ddot{h}_t(t(z);z) > 0$ and $\mid \nabla h(t(z), t(z)+z) \mid > 0$;

(3) $H''(z) > 0$ and $\mid \nabla h(t(z), t(z)+z) \mid > 0$.

It is easy to see that under the conditions described in Lemma 2 (ii), for each given z, $h(t, t+z)$ as a function of t has its unique minimum at $\tilde{t} = t(z)$, and $H''(z) > 0$.

Proof of Theorem 1 (i) Note that

$$\begin{aligned}\ddot{h}_t(t(z^0);z^0) &= \frac{f^2(x_1^0))}{p_1} + \frac{[f(x_2^0) - f(x_1^0)]^2}{p_2 - p_1} + \frac{f(x_2^0)}{1-p_2}, \\ \mid \nabla h(x_1^0, x_2^0) \mid &= \frac{n^3}{r_1(r_2-r_1)(n-r_2)} f^2(x_1^0) f^2(x_2^0),\end{aligned}$$

which implies that $\nabla h(t(z^0), t(z^0)+z^0)$ is positive definite by Lemma 2 (ii). So there exists a neighbourhood of z^0 such that for every fixed $z > 0$ in that region, $h(t, t+z)$ as a function of t is locally convex for $\mid t - x_1^0 \mid < \delta$ with $\delta > 0$ small, and takes its unique minimum at $\tilde{t} = t(z)$. Outside that region Lemma 1(iii) says that the integrand of (2.1) is exponentially small and the integral is likewise exponentially small. Consequently, as n becomes large, the major contribution to the integral in (2.1) comes from that neighbourhood of z^0. The Laplace method is then applicable (see e.g. Jensen (1995)),

$$\begin{aligned}f_{Z_{r_1,r_2}}(z) &= \frac{n}{2\pi} c \int_{x_1^0-\delta}^{x_1^0+\delta} \exp\{-n \cdot h(t, t+z)\} g(t, t+z) dt \{1 + O(n^{-1})\} \\ &\quad + \frac{n}{2\pi} c \left(\int_{-\infty}^{x_1^0-\delta} + \int_{x_1^0+\delta}^{\infty}\right) \exp\{-n \cdot h(t, t+z)\} g(t, t+z) dt \{1 + O(n^{-1})\} \\ &= \sqrt{\frac{n}{2\pi}} \exp\{-n \cdot H(z)\} G(z) \{1 + O(n^{-1})\} + \frac{n}{2\pi} c c_1 e^{-nc_2},\end{aligned}$$

and so (2.3) is obtained.

(ii) We derive (2.4) by Temme's method; see Temme (1981) or Barndorff-Nielsen and Cox (1989, p82). Obviously,

$$\frac{dw}{dz} = \frac{H'(z)}{w(z)},$$

and from (2.3),

$$F_{Z_{r_1,r_2}}(z) = \sqrt{\frac{n}{2\pi}} \int_{w(z)}^{\infty} \psi(w) e^{-\frac{n}{2}w^2} dw \{1 + O(n^{-1})\}, \tag{2.6}$$

where

$$\psi(w) = G(z)| \frac{dw}{dz} |^{-1} = c\frac{\ddot{h}_t^{-\frac{1}{2}}(t(z); z)f(t(z)+z) \mid w(z) \mid}{\mid p_1 F(t(z)+z) - p_2 F(t(z)) \mid}.$$

Specifically, $\psi(0) = 1$. To see this, note that $w \to 0$ is equivalent to $z \to z^0$, and

$$\lim_{z \to z^0} \mid \frac{dw}{dz} \mid = \{H''(z^0)\}^{\frac{1}{2}}.$$

From Lemma 2 (i), one has

$$\lim_{w \to 0} \psi(w) = G(z^0) \lim_{z \to z^0} \mid \frac{dw}{dz} \mid^{-1} = \mid \frac{H''(z^0)\ddot{h}_t(x_1^0; z^0)}{\mid \nabla h(x_1^0, x_2^0) \mid} \mid^{-\frac{1}{2}} = 1.$$

Replacing $\psi(w)$ by $1 - \{1 - \psi(w)\}$, and then integrating by parts, from (2.6) we have

$$\bar{F}_{Z_{r_1,r_2}}(z) = 1 - \Phi(\sqrt{n}w(z)) - \frac{1}{\sqrt{2\pi}} \int_{w(z)}^{\infty} we^{-\frac{n}{2}w^2} \{\frac{1}{w} - \frac{\psi(w)}{w}\} dw\{1 + O(n^{-1})\}.$$

Finally, (2.4) is obtained by integrating by parts.

The approximation in (2.6) is equivalent to (2.5) by an approach of Jensen (1992). The proof of Theorem 1 is established.

An important special case is that $f(x)$ is log-concave. Such a density is also called a PF_2 density. Typical examples include: normal, truncated normal, exponential, double exponential (or Laplace), logistic, uniform, triangular, some of the Gamma, Beta and Weibull, and so on. For properties we refer to Ibragimov (1956), Karlin (1968) and Barlow and Proschan (1981). The following one is proved by Pratt (1981, p105).

Lemma 3 If $f(x)$ is log-concave, then

(i) both $F(x)$ and $1 - F(x)$ are log-concave;

(ii) $F(x) - F(y)$ is log-concave as a function of $(x, y) \in \{(x_1, x_2) : x_1 \geq x_2\}$;

(iii) for constants a_i and b_i $(i = 1, 2)$, $F(a_1x + b_1y) - F(a_2x + b_2y)$ is log-concave as a function of (x, y) on such a set that $(a_1x + b_1y, a_2x + b_2y) \in \{(x_1, x_2) : x_1 \geq x_2\}$.

Corollary Under the assumption of Lemma 3, $f(x_1, x_2)$ is log-concave; and for every fixed $z > 0$, $h(t, t + z)$ is a convex function of t and takes its unique minimum at $\tilde{t} = t(z)$, and $H''(z) > 0$. Furthermore, (2.3)-(2.5) hold uniformly for every z if $f(\cdot)$ is log-concave.

Example 1 Consider an exponential distribution with the hazard rate $\lambda > 0$ and

$$f(x) = \lambda \exp(-\lambda x), \qquad x \geq 0.$$

It is well-known that the normalized spacings

$$T_k = \lambda(n-k+1)(X_{k:n} - X_{k-1:n}), \qquad k = 1, \cdots, n, \tag{2.7}$$

are independent and exponentially distributed with the hazard rate 1, where $X_{0:n} \equiv 0$. Consequently,

$$Z_{r_1,r_2} = \frac{1}{\lambda} \sum_{i=r_1+1}^{r_2} \frac{T_i}{n-i+1} \stackrel{d}{=} \frac{1}{\lambda} \sum_{i=1}^{r_2-r_1} \frac{T_i}{(n-r_1)-i+1} \stackrel{d}{=} X_{r_2-r_1:n-r_1},$$

where $\stackrel{d}{=}$ signifies that the two sides have the same distribution. So the exact density of Z_{r_1,r_2} is

$$f_{Z_{r_1,r_2}}(z) = \frac{(n-r_1)!}{(r_2-r_1-1)!(n-r_2)!} [F(z)]^{r_2-r_1-1} [1-F(z)]^{n-r_2} f(z), \qquad z \geq 0. \tag{2.8}$$

For a given z, solving (2.2) gives

$$t(z) = -\frac{1}{\lambda} \log(1 - \frac{r_1}{n}) \equiv x_1^0.$$

Also

$$\ddot{h}_t(t(z), t(z)+z) = \frac{1-p_1}{p_1} \lambda^2$$

is independent of z.

$$H(z) = -\{(p_2 - p_1) \log F(z) + (1-p_2) \log[1-F(z)]\} - c,$$

where

$$c = (p_2 - p_1) \log \frac{1-p_1}{p_2-p_1} + (1-p_2) \log \frac{1-p_1}{1-p_2},$$

and

$$G(z) = \{\frac{(1-p_1)(p_2-p_1)}{1-p_2}\}^{\frac{1}{2}} \frac{f(z)}{F(z)}.$$

Hence, from (2.3),

$$\hat{f}_{Z_{r_1,r_2}}(z) = \frac{1}{\sqrt{2\pi}} \frac{(n-r_1)^{n-r_1+\frac{1}{2}}}{(r_2-r_1)^{r_2-r_1-\frac{1}{2}} (n-r_2)^{n-r_2+\frac{1}{2}}} [F(z)]^{r_2-r_1-1} [1-F(z)]^{n-r_2} f(z),$$

$$z \geq 0,$$

which is the same as (2.8) except for a scale constant, obtained on replacing the factorials by Stirling's approximation.

The exact tail probabilities are compared in Table 1 with the saddlepoint approximation provided by the formula (2.5). Saddlepoint approximations are seen to have remarkably small relative errors.

[Insert Table 1 about here.]

Example 2 For a uniform distribution

$$f(x) = 1, \qquad 0 \leq x \leq 1,$$

it is easy to verify that $Z_{r_1,r_2} \overset{d}{=} X_{r_2-r_1:n}$, and so

$$f_{Z_{r_1,r_2}}(z) = \frac{n!}{(r_2-r_1-1)!(n-r_2+r_1)!} z^{r_2-r_1-1}(1-z)^{n-r_2+r_1}, \qquad 0 \leq z \leq 1. \quad (2.9)$$

From (2.3), the saddlepoint approximation for the density of Z_{r_1,r_2} is

$$\hat{f}_{Z_{r_1,r_2}}(z) = \frac{1}{\sqrt{2\pi}} \frac{n^{n+\frac{1}{2}}}{(r_2-r_1)^{r_2-r_1-\frac{1}{2}}(n-r_2+r_1)^{n-r_2+r_1+\frac{1}{2}}} \cdot z^{r_2-r_1-1}(1-z)^{n-r_2+r_1},$$
$$0 \leq z \leq 1,$$

which is the same as (2.9) except for a scale constant, obtained on replacing the factorials by Stirling's approximation.

Table 2 compares the exact tail probability $\bar{F}_{Z_{r_1,r_2}}(z)$ with the saddlepoint approximation (2.5), which has remarkably small relative error.

[Insert Table 2 about here.]

3 SADDLEPOINT APPROXIMATION FOR THE STUDENTIZED SAMPLE QUANTILE

For a given $p_0 \in (0,1)$, denote the p_0-th population quantile by ξ_{p_0}; namely, $F(\xi_{p_0}) = p_0$. We are interested in the distribution of the studentized sample quantile

$$Y = \sqrt{n} \frac{X_{r:n} - \xi_{p_0}}{\sigma_p S_{n,r_0}},$$

where r and r_0 are integers, $r = [np_0] + 1$, $r_0 = [c_1 n^{1-\delta_0}] + 1$ for some $c_1 > 0$ and $0 < \delta_0 < 1$ so that $\frac{r_0}{n} \to 0$ $(n \to \infty)$, $\sigma_p^2 = p(1-p)$ and

$$S_{n,r_0} = \frac{n}{2r_0}(X_{r+r_0:n} - X_{r-r_0:n}).$$

For simplicity of notation, denote $p = r/n$ and $p_0^* = r_0/n$. Y is alternatively expressed as

$$Y = s(X_{r-r_0:n}, X_{r:n}, X_{r+r_0:n}),$$

where

$$s(x_1, x_2, x_3) = \sigma_{n,r_0}^{-1} \frac{x_2 - \xi_p}{x_3 - x_1}, \qquad x_1 < x_2 < x_3,$$

and

$$\sigma_{n,r_0} = \frac{\sqrt{n}\sigma_p}{2r_0}.$$

The joint density of $(X_{r-r_0:n}, X_{r:n}, X_{r+r_0:n})$ is, for $x_1 < x_2 < x_3$,

$$\begin{aligned} f(x_1, x_2, x_3) = & \frac{n!}{(r - r_0 - 1)!(r_0 - 1)!(r_0 - 1)!(n - r - r_0)!} f(x_1) f(x_2) f(x_3) \\ & F^{r-r_0-1}(x_1)\{F(x_2) - F(x_1)\}^{r_0-1}\{F(x_3) - F(x_2)\}^{r_0-1}\{1 - F(x_3)\}^{n-r-r_0}, \end{aligned}$$

or alternatively,

$$f(x_1, x_2, x_3) = c_n \exp\{-n \cdot h(x_1, x_2, x_3)\} g(x_1, x_2, x_3),$$

where

$$\begin{aligned} c_n = & \frac{n!}{(r - r_0 - 1)!(r_0 - 1)!(r_0 - 1)!(n - r - r_0)!} \\ & \cdot \exp\{-[(r - r_0)\ln(p - p_0^*) + 2r_0 \ln p_0^* + (n - r - r_0)\ln(1 - p - p_0^*)]\}, \end{aligned}$$

$$g(x_1, x_2, x_3) = [F(x_1)(F(x_2) - F(x_1))(F(x_3) - F(x_2))]^{-1} f(x_1) f(x_2) f(x_3),$$

and

$$\begin{aligned} h(x_1, x_2, x_3) \quad = \quad & (p - p_0^*)\ln\frac{p - p_0^*}{F(x_1)} + p_0^* \ln\frac{p_0^*}{F(x_2) - F(x_1)} + p_0^* \ln\frac{p_0^*}{F(x_3) - F(x_2)} \\ & + (1 - p - p_0^*)\ln\frac{1 - p - p_0^*}{1 - F(x_3)}. \end{aligned}$$

The joint density of $(X_{r-r_0:n}, X_{r+r_0:n}, Y)$ is derived from $f(x_1, x_2, x_3)$ by making a transformation

$$\begin{cases} t_1 = x_1, \\ t_2 = x_3, \\ y = s(x_1, x_2, x_3), \end{cases}$$

whose Jacobian is

$$\mid J \mid = \mid \frac{\partial s(x_1, x_2, x_3)}{\partial x_2} \mid^{-1} = \sigma_{n,r_0}(x_3 - x_1) = \sigma_{n,r_0}(t_2 - t_1),$$

under which $f(x_1, x_2, x_3)$ transforms to

$$c_n \exp\{-n \cdot \tilde{h}(t_1, t_2, y)\}\tilde{g}(t_1, t_2, y) \mid J \mid,$$

where

$$\tilde{h}(t_1, t_2, y) = h(t_1, \xi_p + (t_2 - t_1)\sigma_{n,r_0} y, t_2),$$

and

$$\tilde{g}(t_1, t_2, y) = g(t_1, \xi_p + (t_2 - t_1)\sigma_{n,r_0} y, t_2).$$

Hence, the exact density of $Y = s(X_{r-r_0:n}, X_{r:n}, X_{r+r_0:n})$ is

$$f_Y(y) = c_n \int\int_{D(t_1,t_2;y)} \exp\{-n \cdot \tilde{h}(t_1, t_2, y)\}\tilde{g}(t_1, t_2, y) \mid J \mid dt_1 dt_2,$$

where $D(t_1, t_2; y)$ is a convex set defined by

$$D(t_1, t_2; y) = \{(t_1, t_2) : t_1 \leq \xi_p + (t_2 - t_1)\sigma_{n,r_0} y \leq t_2\}.$$

Applying Stirling's approximation to each factorial of c_n gives

$$c_n = (\frac{n}{2\pi})^{\frac{3}{2}} c^* \{1 + O(r_0^{-1})\},$$

where

$$c^* = (\frac{p - p_0^*}{1 - p - p_0^*})^{\frac{1}{2}} p_0^*.$$

Hence,

$$f_Y(y) = (\frac{n}{2\pi})^{\frac{3}{2}} c^* \int\int_{D(t_1,t_2;y)} \exp\{-n \cdot \tilde{h}(t_1, t_2, y)\}\tilde{g}(t_1, t_2, y) \mid J \mid dt_1 dt_2 \{1 + O(r_0^{-1})\}, \tag{3.1}$$

which will be approximated by using a Laplace approximation.

In the sequel, for each y, let $(\tilde{t}_1, \tilde{t}_2) = (t_1(y), t_2(y))$ be the solution of the equations

$$\begin{cases} \dot{h}_{x_1}(\tilde{t}_1, \xi_p + (\tilde{t}_2 - \tilde{t}_1)\sigma_{n,r_0} y, \tilde{t}_2) - \sigma_{n,r_0} y \dot{h}_{x_2}(\tilde{t}_1, \xi_p + (\tilde{t}_2 - \tilde{t}_1)\sigma_{n,r_0} y, \tilde{t}_2) = 0, \\ \dot{h}_{x_3}(\tilde{t}_1, \xi_p + (\tilde{t}_2 - \tilde{t}_1)\sigma_{n,r_0} y, \tilde{t}_2) + \sigma_{n,r_0} y \dot{h}_{x_2}(\tilde{t}_1, \xi_p + (\tilde{t}_2 - \tilde{t}_1)\sigma_{n,r_0} y, \tilde{t}_2) = 0. \end{cases} \tag{3.2}$$

Let $y^0 = s(x_1^0, x_2^0, x_3^0)$, where

$$F(x_1^0) = p - p_0^*, \quad F(x_2^0) = p, \quad F(x_3^0) = p + p_0^*.$$

Clearly, $x_i^0 \to \xi_p$ $(i = 1,2,3)$ as $n \to \infty$. It is easy to verify that for a fixed y, the Hessian matrix of $\tilde{h}(t_1, t_2, y)$ is

$$\nabla\tilde{h}(t_1, t_2; y) = \begin{pmatrix} 1 & 0 \\ -\sigma_{n,r_0} y & \sigma_{n,r_0} y \\ 0 & 1 \end{pmatrix}^T \nabla h(x_1, x_2, x_3) \begin{pmatrix} 1 & 0 \\ -\sigma_{n,r_0} y & \sigma_{n,r_0} y \\ 0 & 1 \end{pmatrix},$$

where $\quad x_1 = t_1,\ x_2 = \xi_p + (t_2 - t_1)\sigma_{n,r_0} y,\ x_3 = t_2.$

Our saddlepoint approximations are demonstrated in the following theorem.

Theorem 2 Suppose that $f''(x)$ exists and is bounded, and $f(\xi_p) > 0$. Then uniformly in a neighbourhood of ξ_p,

$$\text{(i)}\ \ f_Y(y) = \sqrt{\frac{n}{2\pi}} \exp\{-n \cdot H(y)\} G(y) \{1 + O(r_0^{-1})\}, \tag{3.3}$$

where

$$\begin{aligned} H(y) &= \tilde{h}(t_1(y), t_2(y), y), \\ G(y) &= c^* \tilde{g}(t_1(y), t_2(y), y) \mid J \mid \cdot \mid \nabla\tilde{h}(t_1(y), t_2(y); y) \mid^{-\frac{1}{2}}, \end{aligned}$$

and

$$\mid J \mid = \sigma_{n,r_0}(t_2(y) - t_1(y)).$$

$$\text{(ii)}\ \ \bar{F}_Y(y) = 1 - \Phi(\sqrt{n} w(y)) - \frac{\phi(\sqrt{n} w(y))}{\sqrt{n}} \{\frac{1 - \psi(y)}{w(y)} + O(r_0^{-1})\}, \tag{3.4}$$

or alternatively,

$$\bar{F}_Y(y) = [1 - \Phi(\sqrt{n} w^*(y))]\{1 + O(r_0^{-1})\}, \tag{3.5}$$

where

$$\begin{aligned} w(y) &= \{2H(y)\}^{1/2} sgn(y - y^0), \\ w^*(y) &= w(y) - \frac{1}{n w(y)} \log\{\psi(y)\}, \end{aligned}$$

and

$$\psi(y) = G(y) \mid \frac{w(y)}{H'(y)} \mid .$$

Corollary (3.3)-(3.5) hold uniformly for every y if $f(\cdot)$ is log-concave.

The following Lemmas 4 and 5 are used to prove Theorem 2. The proofs are similar to those of Lemma 1 and 2 and are thus omitted.

Lemma 4

(i) $h(x_1, x_2, x_3)$ is nonnegative and attains its minimum 0 at (x_1^0, x_2^0, x_3^0).

(ii) For any constant $0 \leq \lambda_0 \leq \frac{p-p_0^*}{2}$,

$$h(x_1, x_2, x_3) \geq \lambda_0\{[F(x_1) - (p - p_0^*)]^2 + [F(x_2) - p]^2 + [F(x_3) - (p + p_0^*)]^2\},$$

$$x_1 < x_2 < x_3.$$

Let

$$\nabla h[y] = \nabla h(x_1, x_2, x_3),$$

at $x_1 = t_1(y),\ x_2 = \xi_p + (t_2(y) - t_1(y))\sigma_{n,r_0} y,\ x_3 = t_2(y)$.

Lemma 5 $H''(y),\ t_1'(y)$ and $t_2'(y)$ exist, and

$$\mid \nabla h[y] \mid = \mid \nabla \tilde{h}(t_1(y), t_2(y); y) \mid \cdot \frac{H''(y)}{\mid J \mid}. \tag{3.6}$$

$\nabla h[y]$ is positive definite if and only if $H''(y) > 0$ and $\nabla \tilde{h}(t_1(y), t_2(y); y)$ is positive definite.

Lemma 6

(i) Assume that $f(x)$ is continuously differentiable near ξ_p and $f(\xi_p) > 0$. Then there exists a neighbourhood of ξ_p such that for every fixed $y > 0$ in that region, $\tilde{h}(t_1, t_2, y)$ as a function of (t_1, t_2) is locally convex with $\mid t_1 - x_1^0 \mid$ and $\mid t_2 - x_3^0 \mid$ small and attains its unique minimum at $(t_1(y), t_2(y))$, where $\tilde{t}_1 = t_1(y), \tilde{t}_2 = t_2(y)$ are the solution of the equations (3.2). In addition, $H''(y) > 0$ for y in that region and $H'(y) \neq 0$ unless $y = y^0$.

(ii) If $f(x)$ is continuously differentiable and log-concave, then for a fixed y, $\tilde{h}(t_1, t_2, y)$ is convex on $D(t_1, t_2, y)$ and attains its unique minimum at $(t_1(y), t_2(y))$. Furthermore, $H''(y) > 0$ and $H'(y) = 0$ if and only if $y = y^0$.

Proof of Theorem 2 (i) Obviously, $(t_1(y^0), t_2(y^0)) = (x_1^0, x_3^0)$, at which $\tilde{h}(t_1, t_2, y^0)$ takes its minimum. Lemma 4 and Lemma 6 (i) imply that there exists a neighbourhood of ξ_{p_0} such that as n becomes large, the major contribution to the integral in (3.1) comes from such a region. By using the Laplace approximation (see e.g. Barndorff-Nielsen and Cox (1989), Ch.6) , we approximate $f_Y(y)$ by (3.3).

(ii) From the definition of $w(y)$, one has

$$w^2(y) = H(y), \quad w(y)\frac{dw}{dy} = H'(y),$$

and so

$$\frac{dw}{dy} = \frac{1}{w(y)} \mid J \mid \dot{h}_{x_2}(t_1(y), \xi_p + (t_2(y) - t_1(y))\sigma_{n,r_0} y, t_2(y)).$$

Therefore, from (3.3),

$$\bar{F}_Y(y) = \sqrt{\frac{n}{2\pi}} \int_{w(y)}^{\infty} \psi(w) e^{-\frac{n}{2}w^2} dw \{1 + O(n^{-1})\}, \tag{3.7}$$

where

$$\begin{aligned} \psi(y) &= G(y) \cdot \mid \frac{dw}{dy} \mid^{-1} \\ &= \frac{c^*}{p_0^*} \frac{f(t_1(y)) f(t_2(y)) w(y) \mid \nabla \tilde{h}(t_1(y), t_2(y); y) \mid^{-\frac{1}{2}}}{F(t_1(y)) \mid 2F(\xi_p + (t_2(y) - t_1(y))\sigma_{n,r_0} y) - F(t_1(y)) - F(t_2(y)) \mid}. \end{aligned}$$

Observe that $w \to 0$ is equivalent to $y \to y^0$, and

$$\lim_{y \to y^0} \mid \frac{dw}{dy} \mid^2 = H''(y^0).$$

We obtain, using (3.6),

$$\psi(0) = \lim_{w \to 0} \psi(w) = G(y^0) \lim_{y \to y^0} \mid \frac{dw}{dy} \mid^{-1} = \mid J_0 \mid \{ \frac{\mid \nabla \tilde{h}(x_1^0, x_3^0; y^0) \mid H''(y^0) \mid J_0 \mid}{\mid \nabla h(x_1^0, x_2^0, x_3^0) \mid} \}^{-\frac{1}{2}} = 1.$$

Proceeding as in the proof of Theorem 1, one obtains (3.4) and (3.5) from (3.7).

Example 3 For an exponential distribution with the hazard rate $\lambda > 0$ and

$$f(x) = \lambda \exp(-\lambda x), \qquad x \geq 0,$$

the evaluation of the distribution of the studentized sample quantile Y reduces to that of a weighted sum of independent exponentials; more precisely,

$$P(Y \geq y) = P(\sum_{i=1}^{r+r_0} \gamma_j(y) T_j \geq \lambda \xi_p), \tag{3.8}$$

where $T_1, \cdots, T_n$ are defined in (2.7), and

$$\gamma_j(y) = \begin{cases} \frac{1}{n-j+1}, & j = 1, \cdots, r - r_0, \\ \frac{1 - y\sigma_{n,r_0}}{n-j+1}, & j = r - r_0 + 1, \cdots, r, \\ -\frac{y\sigma_{n,r_0}}{n-j+1}, & j = r + 1, \cdots, r + r_0. \end{cases}$$

The probability of the right side of (3.8) can be calculated exactly by using a formula of Knight and Satchell (1996); see also Johnson and Kotz (1970, Ch.18). Except for the extreme case where y satisfies

$$(j-j') = y\sigma_{n,r_0}(n-j'+1), \;\; (n-j'+1) = -y\sigma_{n,r_0}(j-j'), \;\; \text{or} \;\; n-j+1 = -y\sigma_{n,r_0}(n-j'+1),$$

for some positive integers $j > j'$, the γ's are distinct, and thus

$$P(Y \geq y) = \begin{cases} \sum_{j=1}^{r+r_0} \left[\prod_{k\neq j}^{r+r0} (1 - \frac{\theta_j(y)}{\theta_k(y)})\right]^{-1} (1-p)^{\lambda\theta_j(y)}, & y < 0, \\ \sum_{j=1}^{r} \left[\prod_{k\neq j}^{r+r0} (1 - \frac{\theta_j(y)}{\theta_k(y)})\right]^{-1} (1-p)^{\lambda\theta_j(y)}, & 0 \leq y < \sigma_{n,r_0}^{-1}, \\ \sum_{j=1}^{r-r_0} \left[\prod_{k\neq j}^{r+r0} (1 - \frac{\theta_j(y)}{\theta_k(y)})\right]^{-1} (1-p)^{\lambda\theta_j(y)}, & y \geq \sigma_{n,r_0}^{-1}, \end{cases}$$

where $\theta_i(y) = \gamma_i^{-1}(y), i = 1, \ldots, r + r_0$.

Table 3 compares the exact tail probability $\bar{F}_Y(y)$ with the saddlepoint approximation (3.5) and the Edgeworth expansion given by Hall and Sheather (1988). As usual, the Edgeworth expansion works only in the centre of the distribution. The saddlepoint approximation has remarkably small relative error.

[*Insert Table 3 about here.*]

APPENDIX : PROOFS OF LEMMAS

Proof of Lemma 1

Part (i) is a special case of Part (ii) with $\lambda_0 = 0$. To prove (ii), write

$$\begin{aligned} h_u(u_1, u_2) &= p_1 \ln p_1 + (p_2 - p_1)\ln(p_2 - p_1) + (1 - p_2)\ln(1 - p_2) \\ &\quad - \{p_1 \ln u_1 + (p_2 - p_1)\ln(u_2 - u_1) + (1 - p_2)\ln(1 - u_2))\}, \end{aligned}$$

where $0 < u_1 < u_2 < 1$. Then

$$h(x_1, x_2) = h_u(F(x_1), F(x_2)).$$

So it suffices to consider the uniform distribution with $F(x) = x,\ 0 < x < 1$, and show

$$H_u(u_1, u_2) \geq 0, \qquad 0 < u_1 < u_2 < 1,$$

with equality if and only if $u_i = p_i, \quad i = 1, 2$, where

$$H_u(u_1, u_2) = h_u(u_1, u_2) - \lambda_0\{(u_1 - p_1)^2 + (u_2 - p_2)^2\}, \qquad 0 < u_1 < u_2 < 1.$$

The Hessian matrix of $H_u(u_1, u_2)$ is

$$\nabla H_u(u_1, u_2) = \begin{pmatrix} \frac{p_2-p_1}{(u_2-u_1)^2} + \frac{p_1}{u_1^2} - 2\lambda_0 & \frac{p_2-p_1}{(u_2-u_1)^2} \\ \frac{p_2-p_1}{(u_2-u_1)^2} & \frac{p_2-p_1}{(u_2-u_1)^2} + \frac{1-p_2}{(1-u_2)^2} - 2\lambda_0 \end{pmatrix},$$

which is positive definite since for $0 < u_1 < u_2 < 1$,

$$\begin{aligned} \frac{p_2 - p_1}{(u_2 - u_1)^2} + \frac{p_1}{u_1^2} - 2\lambda_0 > \frac{p_2 - p_1}{(u_2 - u_1)^2} + p_1 - 2\lambda_0 \geq \frac{p_2 - p_1}{(u_2 - u_1)^2}, \\ \frac{p_2 - p_1}{(u_2 - u_1)^2} + \frac{1 - p_2}{(1 - u_2)^2} - 2\lambda_0 > \frac{p_2 - p_1}{(u_2 - u_1)^2} + (1 - p_2) - 2\lambda_0 \geq \frac{p_2 - p_1}{(u_2 - u_1)^2}, \end{aligned}$$

and $\mid \nabla H_u(u_1, u_2) \mid > 0$. Thus $H_u(u_1, u_2)$ is convex and attains its unique minimum 0 at (p_1, p_2), where $[\partial H_u(u_1, u_2)/\partial u_i]_{u_i = p_i} = 0, \quad i = 1, 2$.

The proof of (iii) is obtained by using (ii). In fact, for $t < d_1$ one has $p_1 - F(t) \geq p_1 - F(d_1)$, and

$$h(t, t+z) \geq \lambda_0\{[F(t) - p_1]^2 + [F(t+z) - p_2]^2\} \geq \lambda_0\{[p_1 - F(d_1)]^2\},$$

where $\lambda_0 > 0$. So

$$\begin{aligned} & \int_{-\infty}^{d_1} \exp\{-n \cdot h(t, t+z)\} g(t, t+z) dt \\ = & \int_{-\infty}^{d_1} \exp\{-(n-2) \cdot h(t, t+z)\} \cdot \exp\{-2 \cdot h(t, t+z)\} g(t, t+z) dt \\ \leq & \exp\{-(n-2)\lambda_0 [p_1 - F(d_1)]^2\} \cdot \int_{-\infty}^{d_1} \exp\{-2h(t, t+z)\} g(t, t+z) dt, \end{aligned}$$

and similarly,

$$\begin{aligned} & \int_{d_2}^{\infty} \exp\{-n \cdot h(t, t+z)\} g(t, t+z) dt \\ \leq & \exp\{-(n-2)\lambda_0 [F(d_2) - p_2]^2\} \cdot \int_{d_2}^{\infty} \exp\{-2h(t, t+z)\} g(t, t+z) dt. \end{aligned}$$

Thus, on writing $c_2 = \lambda_0 \min\{[p_1 - F(d_1)]^2, [F(d_2) - p_2]^2\}$, we have

$$(\int_{-\infty}^{d_1} + \int_{d_2}^{\infty}) \exp\{-n \cdot h(t, t+z)\} g(t, t+z) dt \le c_1 e^{-nc_2},$$

where

$$c_1 = e^{2c_2} \int_{-\infty}^{\infty} \exp\{-2h(t, t+z)\} g(t, t+z) dt < \infty.$$

The proof of Lemma 1 is complete.

Proof of Lemma 2

We will prove the following identity, from which Lemma 2 follows immediately.

$$\begin{pmatrix} 1 & t'(z) \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^T \nabla h(t(z), t(z)+z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & t'(z) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \ddot{h}_t(t(z); z) & 0 \\ 0 & H''(z) \end{pmatrix}. \tag{A.1}$$

In fact, one has

$$\begin{aligned} H'(z) &= \{\dot{h}_{x_1}(t(z), t(z)+z) + \dot{h}_{x_2}(t(z), t(z)+z)\} t'(z) + \dot{h}_{x_2}(t(z), t(z)+z) \\ &= \dot{h}_{x_2}(t(z), t(z)+z) \\ &= \frac{f(t(z))\ [p_1 F(t(z)+z) - p_2 F(t(z))]}{F(t(z))\ [F(t(z)+z) - F(t(z))]}, \end{aligned}$$

where the second equality follows from (2.2), and

$$\begin{aligned} \ddot{h}_t(t(z); z) &= \ddot{h}_{x_1x_1}(t(z), t(z)+z) + 2\ddot{h}_{x_1x_2}(t(z), t(z)+z) + \ddot{h}_{x_2x_2}(t(z), t(z)+z) \\ &= p_1 [\frac{f(t(z))}{F(t(z))}]^2 + (p_2 - p_1)[\frac{f(t(z)+z) - f(t(z))}{F(t(z)+z) - F(t(z))}]^2 + (1-p_2)[\frac{f(t(z)+z)}{1 - F(t(z)+z)}]^2 \\ &\quad -(\frac{p_1}{F(t(z))} - \frac{p_2 - p_1}{F(t(z)+z) - F(t(z))}) f'(t(z)) \\ &\quad +(\frac{1-p_2}{1 - F(t(z)+z)} - \frac{p_2 - p_1}{F(t(z)+z) - F(t(z))}) f'(t(z)+z) \\ &= p_1 [\frac{f(t(z))}{F(t(z))}]^2 + (p_2 - p_1)[\frac{f(t(z)+z) - f(t(z))}{F(t(z)+z) - F(t(z))}]^2 \\ &\quad +(1-p_2)[\frac{f(t(z)+z)}{1 - F(t(z)+z)}]^2 - H'(z)[\frac{f'(t(z))}{f(t(z))} - \frac{f'(t(z)+z)}{f(t(z)+z)}], \end{aligned}$$

where the last equality follows from (2.2).

Obviously, both $H''(z)$ and $t'(z)$ exist, and

$$\begin{aligned} H''(z) &= \frac{d}{dz}H'(z) = \frac{d}{dz}\dot{h}_{x_2}(t(z), t(z)+z) \\ &= \ddot{h}_{x_2x_2}(t(z), t(z)+z) + [\ddot{h}_{x_1x_2}(t(z), t(z)+z) + \ddot{h}_{x_2x_2}(t(z), t(z)+z)]t'(z). \end{aligned}$$

Observe that (2.2) is equivalent to

$$\dot{h}_{x_1}(t(z), t(z)+z) + \dot{h}_{x_2}(t(z), t(z)+z) = 0.$$

Differentiating both sides of the above equality with respect to z produces

$$\begin{aligned} & \{\ddot{h}_{x_1x_1}(t(z), t(z)+z) + 2\ddot{h}_{x_1x_2}(t(z), t(z)+z) + \ddot{h}_{x_2x_2}(t(z), t(z)+z)\}t'(z) \\ + \ & \ddot{h}_{x_1x_2}(t(z), t(z)+z) + \ddot{h}_{x_2x_2}(t(z), t(z)+z) = 0, \end{aligned}$$

that is,

$$\ddot{h}_t(t(z); z)t'(z) + \ddot{h}_{x_1x_2}(t(z), t(z)+z) + \ddot{h}_{x_2x_2}(t(z), t(z)+z) = 0.$$

(A.1) is then derived by the matrix multiplication.

Proof of Lemma 6

(i) It is easy to verify that

$$\nabla h(x_1^0, x_2^0, x_3^0) = \frac{1}{p_0^*}\begin{pmatrix} \frac{p}{p-p_0^*}f^2(x_1^0) & -f(x_1^0)f(x_2^0) & 0 \\ -f(x_1^0)f(x_2^0) & 2f^2(x_2^0) & -f(x_2^0)f(x_3^0) \\ 0 & -f(x_2^0)f(x_3^0) & \frac{1-p}{1-p-p_0^*}f^2(x_3^0) \end{pmatrix},$$

and

$$\mid \nabla h(x_1^0, x_2^0, x_3^0) \mid = \frac{1}{p_0^{*2}(p-p_0^*)(1-p-p_0^*)} f^2(x_1^0)f^2(x_2^0)f^2(x_3^0).$$

Since $f(\xi_p) > 0$, there exists a neighbourhood of ξ_p, say $\mid x - \xi_p \mid < \delta$ with small $\delta > 0$, such that for each x in this region $f(x) > 0$. Thus one may assume that $f(x_i^0) > 0$ $(i = 1, 2, 3)$, which implies that $\nabla h(x_1^0, x_2^0, x_3^0)$ is positive definite, and furthermore, $\nabla h(x_1, x_2, x_3)$ is also positive definite for $\mid x_i - \xi_p \mid < \delta$ $(i = 1, 2, 3)$. Lemma 5 in turns implies that for each y in a certain neighbourhood of ξ_p, $\nabla \tilde{h}(t_1, t_2; y)$ is positive definite with $\mid t_1 - x_1^0 \mid$ and $\mid t_2 - x_3^0 \mid$ small, and $H''(y) > 0$.

(ii) From Lemma 3, $F(t_1)$, $F(\xi_p + (t_2 - t_1)\sigma_{n,r_0}y) - F(t_1)$, $F(t_2) - F(\xi_p + (t_2 - t_1)\sigma_{n,r_0}y)$ and $1 - F(t_2)$ are convex on $D(y)$ for each fixed y. So is $\tilde{h}(t_1, t_2, y)$.

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TABLE 1

Comparison of the exact value of $\bar{F}_{Z_{r_1,r_2}}(z)$ *with*

the saddlepoint approximation (2.5).

$(\lambda = 1,\ n = 12,\ r_1 = 4,\ r_2 = 8)$

z	exact tail prob.	saddlepoint approx	relative error (%)
0.5	.6092	.6079	.2153
1.0	.1280	.1270	.7490
1.5	.0168	.0165	1.463
2.0	1.778×10^{-3}	1.736×10^{-3}	2.349
2.5	1.688×10^{-4}	1.630×10^{-4}	3.411
3.0	1.509×10^{-5}	1.439×10^{-5}	4.644
3.5	1.303×10^{-6}	1.224×10^{-6}	6.039
4.0	1.102×10^{-7}	1.019×10^{-7}	7.577
4.5	9.214×10^{-9}	8.363×10^{-9}	9.238

TABLE 2

Comparison of the exact value of $\bar{F}_{Z_{r_1,r_2}}(z)$ *with*

the saddlepoint approximation (2.5).

$(n = 20,\ r_1 = 7,\ r_2 = 16)$

z	exact tail prob.	saddlepoint approx	relative error (%)
.46	.3793	.3790	.0772
.50	.2517	.2515	.1025
.55	.1308	.1306	.1405
.60	.0565	.0564	.1869
.65	.0196	.0195	.2441
.70	5.138×10^{-3}	5.122×10^{-3}	.3160
.75	9.354×10^{-4}	9.316×10^{-4}	.4089
.80	1.017×10^{-4}	1.012×10^{-4}	.5344
.85	4.983×10^{-6}	4.947×10^{-6}	.7156
.90	5.815×10^{-8}	5.756×10^{-8}	1.009
.95	2.108×10^{-11}	2.074×10^{-11}	1.613

TABLE 3

Comparison of the exact value of $\bar{F}_Y(y)$ *with*

the saddlepoint approximation (3.5).

$(\lambda = 1,\ n = 25,\ r = 13, r_0 = 5,\ p = 0.5)$

y	exact tail prob.	saddlepoint app.	r.e. (%)	Edgeworth	r.e. (%)
0.2	.4165	.4094	1.706	.3964	4.836
0.5	.2989	.2990	.0396	.2793	6.535
2.2	.0141	.0143	1.224	.0183	29.73
2.5	7.253×10^{-3}	7.355×10^{-3}	1.411	.0112	54.59
3.2	1.579×10^{-3}	1.607×10^{-3}	1.762	2.846×10^{-3}	80.23
3.5	8.493×10^{-4}	8.651×10^{-4}	1.863	1.362×10^{-3}	60.41
4.2	2.210×10^{-4}	2.254×10^{-4}	1.999	1.636×10^{-4}	26.00
4.5	1.296×10^{-4}	1.323×10^{-4}	2.038	5.540×10^{-5}	57.26
5.2	4.091×10^{-5}	4.178×10^{-5}	2.114	2.968×10^{-6}	92.74
5.5	2.586×10^{-5}	2.642×10^{-5}	2.138	7.150×10^{-7}	97.24
6.2	9.540×10^{-6}	9.752×10^{-6}	2.221	1.750×10^{-8}	99.82