December 9, 1994

The Cosmological Probability Density Function for Bianchi Class A Models in Quantum Supergravity

HUGH LUCKOCK and CHRIS OLIWA

School of Mathematics and Statistics

University of Sydney

NSW 2006

Australia

Abstract

Nicolai's theorem suggests a simple stochastic interpretation for supersymmetric Euclidean quantum theories, without requiring any inner product to be defined on the space of states. In order to apply this idea to supergravity, we first reduce to a one-dimensional theory with local supersymmetry by the imposition of homogeneity conditions. We then make the supersymmetry rigid by imposing gauge conditions, and quantise to obtain the evolution equation for a time-dependent wave function. Owing to the inclusion of a certain boundary term in the classical action, and a careful treatment of the initial conditions, the evolution equation has the form of a Fokker-Planck equation. Of particular interest is the static solution, as this satisfies all the standard quantum constraints. This is naturally interpreted as a cosmological probability density function, and is found to coincide with the square of the magnitude of the conventional wave function for the wormhole state.

PACS numbers: 98.80.H , 04.65

Introduction

Supersymmetric theories enjoy a number of appealing properties, many of which are particularly attractive in the context of quantum cosmology. For example, supersymmetry leads to the replacement of second-order constraints by first-order constraints, which are both easier to solve and more restrictive.

A feature of supersymmetry which has not yet been fully exploited in quantum cosmology arises from a theorem proved by Nicolai in 1980 [1]. In essence, this theorem states that any Euclidean theory with rigid supersymmetry can be converted into a free bosonic theory by integrating out the fermions and transforming the bosonic variables. The new variables then have the stochastic properties of white noise, and the transformation which relates them to the original bosonic variables (the "Nicolai map") is typically a stochastic differential equation of the Langevin type. This leads to a natural probabilistic intepretation of the quantum theory, without requiring the introduction of any inner product on the space of states

Recent work has shown how these ideas can be applied to quantum cosmological models with $N = 2$ supersymmetry [2]. However these models are somewhat artificial, as they are constructed by modifying purely bosonic minisuperspace models. Consequently, this kind of supersymmetry does not necessarily reflect any underlying physical symmetry of the full theory.

In this paper we overcome this limitation by applying similar ideas to supergravity. Although the full theory is not amenable to our approach (due to the breaking of supersymmetry by the imposition of initial or final conditions on the fields), we are able to make progress by focussing on homogeneous field configurations in Bianchi class A spacetimes. This reduction leads to a one-dimensional model with reparametrisation invariance and local $N = 4$ supersymmetry.

Before invoking Nicolai's theorem, it is necessary to convert the local supersymmetry to a rigid supersymmetry, and to fix the reparametrisation invariance. Both these ends are easily achieved by imposing various gauge conditions on the classical theory. This procedure also results in one of the most novel and useful features of our approach; a quantum theory which evolves with respect to clock time.

Though unorthodox, it should be noted that there is nothing unreasonable about a cosmological wave function which evolves with respect to the proper time measured by a family of observers distributed through the Universe. Of course, this evolution depends completely upon the selection of these observers; or equivalently, upon the choice of gauge.

Since the theory now has rigid supersymmetry, it must admit a Nicolai map. This will be a type of Langevin equation governing the dynamics of the bosonic variables. In principle, its explicit form could be obtained by integrating out the fermionic variables in the path integral. However, we are primarily interested in the probabilistic interpretation of the theory rather than the Nicolai maps *per se* . For this reason, we bypass the path integral formulation and use instead canonical quantisation. This leads directly to a type of Fokker-Planck equation governing the Euclidean evolution of the wave function.

We wish to emphasise that a plausible probabilistic interpretation of the quantum theory is obtained without introducing an inner product or appealing to the concept of a Hilbert space. In this sense our approach is fundamentally different from conventional quantum mechanics, even though we ultimately obtain the same expression for the static probability density function as can be obtained by a more conventional approach.

Classical Supergravity

The basic fields in supergravity are the tetrad $e^{AA'}{}_{\mu}$ and the spin 3/2 Rarita-Schwinger fields $\psi^{A}{}_{\mu}$ and $\widetilde{\psi}^{A'}{}_{\mu}$. (We use the two component notation described in [3], in which spinor indices A, A' take the values 0 and 1, and are raised or lowered with the antisymmetric quantities ϵ^{AB}, ϵ_{AB}.) The action on an unbounded region of spacetime $\mathcal{M}$ has the form

$$I = \int_{\mathcal{M}} d^4x \left[\frac{1}{2\kappa^2} eR + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} \left(\tilde{\psi}^{A'}{}_{\mu} e_{AA'\nu} D_\rho \psi^A{}_\sigma - \psi^A{}_\mu e_{AA'\nu} D_\rho \tilde{\psi}^A{}_\sigma \right) \right] \qquad (1)$$

where e is the determinant of the tetrad, R is the scalar curvature, and D_μ denotes the covariant spinor derivative. If $\mathcal{M}$ has boundary $\partial\mathcal{M}$, then the action I must also include the boundary term $\kappa^{-2}\int_{\partial\mathcal{M}} K h^{\frac{1}{2}} d^3x$, where K is the trace of the second fundamental form on $\partial\mathcal{M}$ and $h^{\frac{1}{2}} d^3x$ is the induced volume element on $\partial\mathcal{M}$ [4]. Requiring the total action to be stationary with respect to independent variation of the tetrad and the connection then leads to Einstein's equations and to the "tetrad postulate" (which relates the tetrad and connection) [5].

When the spacetime has a Lorentzian $(-+++)$ signature, the reality of the action demands that the components $e^{AA'}{}_\mu$ of the tetrad are Hermitian while the left-handed spinors $\tilde{\psi}^{A'}{}_\mu$ are the Hermitian conjugates of the right-handed spinors $\psi^A{}_\mu$. However, not wanting to restrict ourselves to Lorentzian spacetimes, we will think of all these fields as being unconstrained.

As well as coordinate and Lorentz transformations, the supergravity action is invariant (up to boundary terms) under the local supersymmetry transformations

$$\delta e^{AA'}{}_\mu = -i\kappa(\epsilon^A \tilde{\psi}^{A'}{}_\mu + \tilde{\epsilon}^{A'}\psi^A{}_\mu), \quad \delta\psi^A{}_\mu = 2\kappa^{-1} D_\mu \epsilon^A, \quad \delta\tilde{\psi}^{A'}{}_\mu = 2\kappa^{-1} D_\mu \tilde{\epsilon}^{A'} \qquad (2)$$

where the spinors $\epsilon^A, \tilde{\epsilon}^{A'}$ have anticommuting Grassmann components.

It was shown in [6] that, when initial or final conditions are imposed, the validity of Nicolai's theorem depends on the existence of an unbroken subalgebra of supersymmetry generators. The first step in our program is therefore to identify a (non-trivial) subalgebra of supersymmetry generators which preserves the hypersurfaces on which we will specify the initial and final data. In fact, the subalgebra of left-handed supersymmetry generators obtained by setting $\epsilon = 0$ in (2) is adequate for this purpose, since the anticommutator of two such generators is always zero. (The subalgebra of right-handed supersymmetry generators obtained by setting $\tilde{\epsilon} = 0$ would do equally well.)

We cannot proceed immediately with our program, as the supersymmetric variation of the action gives rise to boundary terms which invalidate Nicolai's theorem [6]. We therefore seek a boundary correction which restores the invariance of the supergravity action under the chosen subalgebra.

In fact D'Eath has shown that there is no such boundary correction for the full supergravity theory, which is therefore not amenable to our approach [7]. However, it turns out that we *can* complete the program outlined above if we restrict our attention to spatially homogeneous Bianchi class A models.

In Bianchi class A models, spacetime is parametrised by a global timelike coordinate $t = x^0$ and three locally defined spacelike coordinates x^i $(i = 1, 2, 3)$; there are also three 1-forms $\boldsymbol{\omega}^p = \omega^p{}_i \boldsymbol{d}x^i$ $(p = 1, 2, 3)$ satisfying

$$\boldsymbol{d}\boldsymbol{\omega}^p = \frac{1}{2} m^{ps} \epsilon_{sqr} \boldsymbol{\omega}^q \wedge \boldsymbol{\omega}^r \tag{3}$$

where $m^{ps} = m^{(ps)}$ is some constant symmetric matrix [8]. The tetrad and the spin 3/2 field are required to be spatially constant in the sense that

$$e^{AA'}{}_i = e^{AA'}{}_p(t)\omega^p{}_i, \qquad \psi^A{}_i = \psi^A{}_p(t)\omega^p{}_i, \qquad \tilde{\psi}^{A'}{}_i = \tilde{\psi}^{A'}{}_p(t)\omega^p{}_i \tag{4}$$

with $e^{AA'}{}_p(t)$, $\psi^A{}_p(t)$, $\tilde{\psi}^{A'}{}_p(t)$ and $e^{AA'}{}_0(t)$, $\psi^A{}_0(t)$, $\tilde{\psi}^{A'}{}_0(t)$ being independent of the spatial coordinates x^i. The 3-metric on each spatial hypersurface $\Sigma(t)$ of constant t then has the form

$$h_{pq}(t)\boldsymbol{\omega}^p \otimes \boldsymbol{\omega}^q \quad \text{where} \quad h_{pq} = -e^{AA'}{}_p e_{AA'q}. \tag{5}$$

Having imposed a homogeneous ansatz on the fields, we now need a boundary correction which restores the invariance of the action under the chosen subalgebra. Indeed, the action is found to be invariant under the left-handed subalgebra if one adds the boundary correction [9]

$$I_{\partial\mathcal{M}} = \int_{\partial\mathcal{M}} \left(i\kappa^{-2} m^{pq} h_{pq} - \frac{1}{2}\epsilon^{pqr} \tilde{\psi}^{A'}{}_p \psi^A{}_q e_{AA'r} \right) \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \tag{6}$$

where the spacetime boundary $\partial\mathcal{M}$ is assumed to consist of two disjoint hypersurfaces $\Sigma(t_{initial})$ and $\Sigma(t_{final})$ on which initial and final data are to be specified. Alternatively, the action can be made invariant under the right-handed subalgebra by subtracting $I_{\partial\mathcal{M}}$. We thus obtain two new forms of the classical action

$$I_{\pm} \equiv I \pm I_{\partial\mathcal{M}} \tag{7}$$

with I_{+} invariant under the left-handed subalgebra, and I_{-} invariant under the right-handed subalgebra.

Note that the boundary corrections are not real, and so their inclusion corresponds to a non-unitary transformation in the (Lorentzian) quantum theory. However we are ultimately concerned with the Euclidean theory, where unitarity is not required [2].

Before quantisation, we consider briefly the canonical form of the classical theory. We take the dynamical variables as $e^{AA'}{}_{p}$, $\psi^{A}{}_{p}$, $\tilde{\psi}^{A'}{}_{p}$ and $p_{AA'}{}^{p}$, the latter denoting the momenta conjugate to $e^{AA'}{}_{p}$ in the original theory defined by eq. (1). There are second-class constraints relating the spinor fields to their own conjugate momenta, and so the latter need not be treated as independent fields.

The addition of the boundary corrections $\pm I_{\partial\mathcal{M}}$ to the original action suggests that we should also define new momenta

$$\begin{aligned} p^{\pm}{}_{AA'}{}^{p} &\equiv p_{AA'}{}^{p} \pm \frac{\partial I_{\partial\mathcal{M}}}{\partial e^{AA'}{}_{p}} && (8) \\ &= p_{AA'}{}^{p} \mp 2i\kappa^{-2}\sigma m^{pq} e_{AA'q} \mp \frac{1}{2}\sigma\epsilon^{pqr}\tilde{\psi}_{A'q}\psi_{Ar} && (9) \end{aligned}$$

where σ is the integral of the 3-form $\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3$ over the spatial hypersurface $\Sigma(t)$. Then $p^{+}{}_{AA'}{}^{p}$ are the momenta conjugate to $e^{AA'}{}_{p}$ in the version of the theory invariant under the left-handed subalgebra, while $p^{-}{}_{AA'}{}^{p}$ are the momenta in the version invariant under the right-handed subalgebra.

Because there are second-class constraints, the usual Poisson brackets are replaced by Dirac brackets and their fermionic generalisations [3, 10, 11]. These are very simple when written in terms of the new momenta $p^{\pm}{}_{AA'}{}^p$. In particular, one finds that

$$[p^+{}_{AA'}{}^p, p^+{}_{BB'}{}^q]^* = 0 \qquad [p^-{}_{AA'}{}^p, p^-{}_{BB'}{}^q]^* = 0 \tag{10}$$

$$[p^+{}_{AA'}{}^p, \psi^B{}_q]^* = 0 \qquad [p^-{}_{AA'}{}^p, \tilde{\psi}^{B'}{}_q]^* = 0. \tag{11}$$

The vanishing of these brackets is a pleasant surprise; the corresponding brackets involving the usual momenta $p_{AA'}{}^p$ do *not* vanish.

We also find that

$$\begin{aligned}
[e^{AA'}{}_p\,, \psi^B{}_q]^* &= 0 & (12)\\
[e^{AA'}{}_p\,, \tilde{\psi}^{B'}{}_q]^* &= 0 & (13)\\
[\psi^A{}_p\,, \psi^B{}_q]^* &= 0 & (14)\\
[\tilde{\psi}^{A'}{}_p, \tilde{\psi}^{B'}{}_q]^* &= 0 & (15)\\
[e^{AA'}{}_p, p^{\pm}{}_{BB'}{}^q]^* &= \epsilon^A{}_B \epsilon^{A'}{}_{B'} \delta^q_p & (16)\\
[\psi^A{}_p, \tilde{\psi}^{A'}{}_q]^* &= -\frac{1}{\sigma} D^{AA'}{}_{pq} & (17)\\
[p^+{}_{AA'}{}^p, \tilde{\psi}^{B'}{}_q]^* &= \epsilon^{pqr} D_A{}^{B'}{}_{rs} \tilde{\psi}_{A'q} & (18)\\
[p^-{}_{AA'}{}^p, \psi^B{}_q]^* &= \epsilon^{pqr} D^B{}_{A'sq} \psi_{Ar} & (19)
\end{aligned}$$

where

$$D^{AA'}{}_{pq} = -2i(\det[h_{pq}])^{-1/2} e^{AB'}{}_q e_{BB'p} n^{BA'}, \tag{20}$$

and $n^{AA'} = e^{AA'}{}_\mu n^\mu$ is the spinor version of the future-pointing unit vector normal to the spatial hypersurface $\Sigma(t)$, satisfying $n_{AA'}n^{AA'} = 1$ and $n_{AA'}e^{AA'}{}_p = 0$. Clearly, $n^{AA'}$ depends on the dynamical variables $e^{AA'}{}_p$; its Dirac brackets are found to be

$$[n^{AA'}, e^{BB'}{}_q]^* = [n^{AA'}, \psi^A{}_p]^* = [n^{AA'}, \tilde{\psi}^{B'}{}_q]^* = 0, \tag{21}$$

$$[n^{AA'}, p^{\pm}{}_{BB'}{}^q]^* = h^{qr} e^{AA'}{}_r n_{BB'}. \tag{22}$$

In addition to the second-class constraints mentioned above, the Lorentz invariance of the theory gives rise to the first-class constraints $J_{AB} = \tilde{J}_{A'B'} = 0$ where

$$\tilde{J}_{A'B'} \equiv e^A{}_{(A'p}\, p^+{}_{AB')}{}^p \qquad J_{AB} \equiv e_{(A}{}^{A'}{}_p\, p^-{}_{B)A'}{}^p. \tag{23}$$

These quantities can be interpreted as the generators of Lorentz transformations.

One can identify the charges associated with the other symmetries of the theory. In particular, the supersymmetry generators are

$$\widetilde{S}_{A'} \equiv \frac{i}{2}\kappa^2 \psi^A{}_p p^+{}_{AA'}{}^p \qquad\qquad S_A \equiv -\frac{i}{2}\kappa^2 p^-{}_{AA'}{}^p \widetilde{\psi}^{A'}{}_p. \tag{24}$$

Indeed, if f is any function of the dynamical variables then its variation under supersymmetry transformations (2) is

$$\delta f = -\frac{2}{\kappa}[\epsilon^A S_A + \widetilde{S}_{A'}\widetilde{\epsilon}^{A'}, f]^*. \tag{25}$$

Time translations are generated by the Hamiltonian, which is found to be

$$H = -e^{AA'}{}_0 \mathcal{H}_{AA'} + \psi^A{}_0 S_A + \widetilde{S}_{A'}\widetilde{\psi}^{A'}{}_0 - \omega^{AB}{}_0 J_{AB} - \widetilde{\omega}^{A'B'}{}_0 \widetilde{J}_{A'B'} \tag{26}$$

where $\omega^{AB}{}_0$ and $\widetilde{\omega}^{A'B'}{}_0$ are the x^0 components of the spin connection 1-form [3]. The expression for $\mathcal{H}_{AA'}$ is quite complicated, but it is related to the other generators by the identity

$$\mathcal{H}_{AA'} = -\frac{2i}{\kappa^2}[S_A, \widetilde{S}_{A'}]^* \;+\; \text{terms proportional to } J \text{ and } \widetilde{J}. \tag{27}$$

It can be verified that the classical equations of motion imply the evolution equation

$$\frac{df}{dt} = [f, H]^* \tag{28}$$

for any function f of the dynamical variables.

At this point note that the quantities $e^{AA'}{}_0$, $\psi^A{}_0$ and $\widetilde{\psi}^{A'}{}_0$ are non-dynamical, as their time derivatives do not appear in the action and consequently the classical equations of motion tell us nothing about their evolution. If we wish, we may freely specify their values at all times, thereby eliminating some of the gauge degrees of freedom.

The Time-Dependent Quantum Theory

Many authors choose to treat the quantities $e^{AA'}{}_0$, $\psi^A{}_0$ and $\tilde{\psi}^{A'}{}_0$ as free variables, i.e. Lagrange multipliers, and then derive a set of first-class constraints by requiring that the action be stationary with respect to their variation. However, one may instead suppose that some of these quantities are fixed externally by the imposition of gauge conditions; in this case it does not make sense to require stationarity of the action and so the correponding constraints are absent from the classical theory.

To quantise the theory, we replace functions of the dynamical variables by operators and Dirac brackets by commutators or anticommutators. When the theory is quantised, a function f is then represented in the Heisenberg picture by an operator which evolves according to the equation

$$i\hbar \frac{df}{dt} = [f, H] \tag{29}$$

where $[f, H]$ denotes the commutator of two operators. However, we prefer to use the Schrödinger picture, in which the operators are constant and the wave function Ψ evolves according to the Schrödinger equation

$$\begin{aligned} i\hbar \frac{\partial}{\partial t}\Psi &= H\Psi && (30) \\ &= -e^{AA'}{}_0(t)\,\mathcal{H}_{AA'}\Psi + \psi^A{}_0(t)\,S_A\Psi - \tilde{\psi}^{A'}{}_0(t)\,\tilde{S}_{A'}\Psi. && (31) \end{aligned}$$

(The $J, \tilde{J}$ terms have been omitted from the Hamiltonian, since the wave function is required to be Lorentz invariant.) If one demands that the wave function should remain independent of $e^{AA'}{}_0$, $\psi^A{}_0$, $\tilde{\psi}^{A'}{}_0$ (i.e. independent of the choice of gauge) then differentiation with respect to these quantities leads to the standard quantum constraints $\mathcal{H}_{AA'}\Psi = S_A\Psi = \tilde{S}_{A'}\Psi = 0$.

In the present context, however, we want a time-dependent version of the quantum theory. This can be achieved by specifying the function $e^{AA'}{}_0(t)$ before quantisation; in other words, by imposing a gauge condition on the classical theory. Since

${e^{AA'}}_0$ cannot then be viewed as a Lagrange multiplier, there is no constraint on $\mathcal{H}_{AA'}$ in the classical theory or on $\mathcal{H}_{AA'}\Psi$ in the quantum theory.

However, fixing ${e^{AA'}}_0(t)$ breaks supersymmetry unless we simultaneously set

$$ {\psi^A}_0 = {\tilde{\psi}^{A'}}_0 = 0. \tag{32} $$

This has the effect of converting the local supersymmetry to a rigid supersymmetry, since ϵ and $\tilde{\epsilon}$ must now be constant so that the conditions (32) are preserved by supersymmetry transformations. Note that rigid supersymmetry is required to prove the existence of Nicolai maps.

With ${e^{AA'}}_0$, ${\psi^{A'}}_0$ and ${\tilde{\psi}^{A'}}_0$ fixed classically in the manner described above, the wave function will satisfy the evolution equation

$$ \begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} &= -{e^{AA'}}_0(t)\, \mathcal{H}_{AA'}\Psi & (33) \\ &= \frac{2}{\hbar\kappa^2} {e^{AA'}}_0(t)(S_A\tilde{S}_{A'} + \tilde{S}_{A'}S_A)\Psi. & (34) \end{aligned} $$

The last line is obtained by using the anticommutation relations which follow from (27). We thus have a straightforward quantum theory with rigid $N = 4$ supersummetry.

In order for Nicolai's theorem to be applicable, the quantum state must be invariant under a non-trivial supersymmetric subalgebra which preserves the initial and final hypersurfaces. In fact, we have a choice of possible subalgebras. For definiteness, suppose that we decide to use the I_+ form of the action; then a suitable subalgebra is the one spanned by the left-handed supersymmetry generators $\tilde{S}_{A'}$. The requirement that these correspond to unbroken symmetries leads to the additional quantum constraints

$$ \tilde{S}_{A'}\Psi = 0 \qquad (A' = 0, 1) \tag{35} $$

and so the evolution equation becomes

$$ \frac{\partial \Psi}{\partial t} = -\frac{2i}{\hbar^2\kappa^2} {e^{AA'}}_0\, \tilde{S}_{A'}S_A\Psi \tag{36} $$

There are different operator representations of the quantum theory, corresponding to the different versions of the classical action. Having chosen the I_+ form of the action, we find that $e^{AA}{}_p$ and $\psi^A{}_p$ are the canonical coordinates and so are represented in the quantum theory by multiplicative operators. Quantisation is then completed by setting

$$p^+{}_{AA'}{}^p = -i\hbar \frac{\partial}{\partial e^{AA'}{}_p}, \qquad \tilde{\psi}^{A'}{}_p = -\frac{i\hbar}{\sigma} D^{AA'}{}_{qp} \frac{\partial}{\partial \psi^A{}_q} \tag{37}$$

in accordance with the commutation and anticommutation relations which follow from the Dirac brackets (10- 19). Operator representations of $\tilde{J}_{A'B'}$ and $\tilde{S}_{A'}$ are obtained from (23,24) without any ordering ambiguities, thanks to (11,16). Moreover, (9) gives

$$p^-{}_{AA'}{}^p = -i\hbar \frac{\partial}{\partial e^{AA'}{}_p} + 4i\kappa^{-2}\sigma m^{pq} e_{AA'q} + \sigma \epsilon^{pqr}[(1-\lambda)\tilde{\psi}_{A'q}\psi_{Ar} - \lambda \psi_{Ar}\tilde{\psi}_{A'q}] \tag{38}$$

where the value of the parameter λ determines the ordering of the operators in the $\psi\tilde{\psi}$ term. Note that this is the *only* operator-ordering ambiguity which will arise.

The supersymmetry generators are now represented by the operators

$$\tilde{S}_{A'} = \frac{\hbar\kappa^2}{2} \frac{\partial}{\partial e^{AA'}{}_p} \psi^A{}_p, \tag{39}$$

$$S_A = -\frac{\hbar\kappa^2}{2} \tilde{\psi}^{A'}{}_p \left(\frac{\partial}{\partial e^{AA'}{}_p} - \frac{4\sigma}{\hbar\kappa^2} m^{pq} e_{AA'q} + 2\lambda e_{AA'q} h^{qp} \right) \tag{40}$$

where h^{pq} is the inverse of h_{pq}. Thus, expression (36) gives

$$\frac{\partial \Psi}{\partial t} = i\frac{\kappa^2}{2} e^{AA'}{}_0 \frac{\partial}{\partial e^{BA'}{}_p} \left[\psi^B{}_p \tilde{\psi}^{B'}{}_q \left(\frac{\partial}{\partial e^{AB'}{}_q} - \frac{4\sigma}{\hbar\kappa^2} m^{qr} e_{AB'r} + 2\lambda h^{qr} e_{AB'r} \right) \Psi \right] \tag{41}$$

where the wave function Ψ depends on the canonical coordinates $e^{AA'}{}_p$ and $\psi^A{}_p$.

The Probability Density Function of the Universe

A general solution of (41) is the sum of a number of distinct components, each of different order in the Grassmann variables $\psi^A{}_p$. It turns out that certain components are singled out by the choice of particular boundary conditions.

The Nicolai map will be realised as a stochastic differential equation relating $\dot{e}^{AA'}{}_p$ to a white noise process. Consequently, some kind of boundary conditions on $e^{AA'}{}_p$ are needed to ensure that this map is invertible. In particular, if we are interested in evolution from a particular 3-geometry, then we must specify $e^{AA'}{}_p$ at some initial time. With these boundary conditions, it will turn out that only the $O(\psi^6)$ component of the wave function is non-zero.

The argument for a probabilistic interpretation depends on the existence of a Nicolai map, which in this case is assured only if the quantum state – and hence any initial conditions – are invariant under the subalgebra of left-handed supersymmetry generators $\widetilde{S}_{A'}$. In particular, to ensure that the specified initial values of $e^{AA'}{}_p$ are invariant under this subalgebra, we must simultaneously impose Dirichlet initial conditions on the right-handed spinors $\psi^A{}_p$ (since $[\widetilde{S}_{B'}, e^{AA'}{}_p]^* = \frac{i}{2}\kappa^2\psi^A{}_p\epsilon_{B'}{}^{A'}$). Together these initial conditions on $e^{AA'}{}_p$ and $\psi^A{}_p$ are $\widetilde{S}_{A'}$-invariant, and will give rise to a quantum state satisfying the constraint (35).

However, if all the $\psi^A{}_p$ vanish initially, then their equations of motion imply that they will also vanish at later times. Consequently, the quantum states arising from the initial conditions described above will satisfy the six conditions

$$\psi^A{}_p\Psi(e,\psi;t) = 0 \qquad (p = 1,2,3; \quad A = 0,1). \tag{42}$$

It follows that any $\widetilde{S}_{A'}$-invariant states which evolve from specified initial values of $e^{AA'}{}_p$ must have the form

$$\Psi(e^{AA'}{}_p, \psi^A{}_p; t) = (\psi^A{}_p\delta^{pq}\psi_{Aq})^3\, F(e^{AA'}{}_p; t) \tag{43}$$

where the function $F(e^{AA'}{}_p; t)$ is independent of the fermionic variables. Our argument indicates that these states should admit Nicolai maps, and therefore probabilistic interpretations.

Making use of the anticommutation relation $\psi^B{}_p\tilde{\psi}^{B'}{}_q = -\frac{i\hbar}{\sigma}D^{BB'}{}_{pq} - \tilde{\psi}^{B'}{}_q\psi^B{}_p$ (which follows from (17)) and using (42) we obtain the bosonic evolution equation

$$\frac{\partial F}{\partial t} = \frac{\kappa^2}{2\sigma}e^{AA'}{}_0\frac{\partial}{\partial e^{BA'}{}_p}\left[D^{BB'}{}_{pq}\left(\hbar\frac{\partial F}{\partial e^{AB'}{}_q} + \frac{\partial \Phi}{\partial e^{AB'}{}_q}F\right)\right] \tag{44}$$

where we have defined

$$\Phi = \frac{2\sigma}{\kappa^2}m^{pq}h_{pq} - \lambda\hbar\log\det[h_{pq}]. \tag{45}$$

It is clear from (44) that the integral of $F(e^{AA'}{}_p;t)$ will be conserved in time. It can also be shown that (44) is a real equation when restricted to Euclidean tetrads. (We are interested in Euclidean spacetimes because only Euclidean theories admit Nicolai maps with stochastic interpretations).

Given these facts, it is tempting to interpret (44) as a Fokker-Planck equation describing the evolution of the probability density function $F(e^{AA'}{}_p;t)$ for the location of a Brownian particle moving randomly in a 12-dimensional minisuperspace with coordinates $e^{AA'}{}_p$ and a potential function $\Phi(e^{AA'}{}_p)$. We will tentatively adopt this intepretation for now, postponing a more careful investigation until the next section.

Of particular interest is the static solution

$$F_0(e^{AA'}{}_p) = A\exp-\frac{1}{\hbar}\Phi(e^{AA'}{}_p) \tag{46}$$

as this state satisfies all the conventional constraints of homogeneous supergravity. (It is easily checked that the wave function $\Psi_0 = (\psi^A{}_p\delta^{pq}\psi_{Aq})^3F_0$ is annihilated by all four supersymmetry generators (39, 40) as well as the Lorentz generators.)

Furthermore, $F_0(e^{AA'}{}_p)$ is a plausible candidate for the the probability density function of a Universe in the wormhole state. Indeed, precisely the same p.d.f. can be obtained by squaring the modulus of the wormhole wave function of Asano *et al* [13], in accordance with the interpretation advocated by Bene and Graham [14].

A natural question is whether the time-dependent solutions of (44) have any physical significance. According to the traditional view, the answer is no: One is interested only in the static solution $F_0(e^{AA'}{}_p)$ described above, since this is in fact the only normalisable state satisifying the standard constraints of supergravity. From this perspective, it might appear that our approach merely supports the argument of Bene and Graham [14] that the cosmological probability density function should be the square the modulus of the conventional wave function.

However it should be noted that our approach, unlike that of Bene and Graham, provides a plausible probabilistic interpretation of the quantum theory *without any reference at all to the concept of an inner product.* In this sense, our approach is radically different from the usual Hilbert space formulation of quantum mechanics. It thus enjoys a clear advantage in the arena of quantum cosmology, where the choice of a satisfactory inner product has long been viewed as one of the most vexing and fundamental problems.

Time Dependent Solutions

A more radical approach would be to suppose that the wave function of the Universe really does evolve with respect to Euclidean time. The Euclidean nature of time would not necessarily be apparent to the inhabitants of the Universe, any more than the time-independence of the cosmological wave function is apparent to the inhabitants in the standard Wheeler-DeWitt approach. It might therefore be argued that such an interpretation is quite compatible with our observations and experience.

To better understand the dynamics implied by the evolution equation (44), it is convenient to eliminate the nine remaining gauge degrees of freedom from the 12-dimensional minisuperspace parametrised by the variables $e^{AA'}{}_p$. (There are six degrees of freedom associated with Lorentz transformations, and three associated with the spatial diffeomorphisms corresponding to redefinitions of the 1-forms $\boldsymbol{\omega}^p$.)

To eliminate the Lorentz degrees of freedom, the first step is to expand the x^0 component of the tetrad as

$$e^{AA'}{}_0 = -iN(t)n^{AA'} + N^p(t)e^{AA'}{}_p \tag{47}$$

where $N(t)$ is the Euclidean lapse function and the $N^p(t)$ are the components of the shift vector. The spacetime metric is then

$$g = (N^2 + N^p N_p)\boldsymbol{dt} \otimes \boldsymbol{dt} + N_p(\boldsymbol{dt} \otimes \boldsymbol{\omega}^p + \boldsymbol{\omega}^p \otimes \boldsymbol{dt}) + h_{pq}\boldsymbol{\omega}^p \otimes \boldsymbol{\omega}^q \tag{48}$$

where $N_p = h_{pq}N^q$.

Since the wave function is invariant under spatial rotations, its non-vanishing component F can be written as a function of the 3-metric h_{pq}. The evolution equation (44) can then be shown to imply that

$$\begin{aligned} \frac{\partial F}{\partial t} &= 2\frac{\kappa^2}{\sigma} N(t) G^{-\frac{1}{2}} \frac{\partial}{\partial h_{pr}} \left[G^{\frac{1}{2}} G_{(pr)(qs)} \left(\hbar \frac{\partial F}{\partial h_{qs}} + \frac{\partial \Phi}{\partial h_{qs}} F \right) \right] \\ & \quad -4N^p(t) \frac{\partial}{\partial h_{qt}} \left(\epsilon_{pqr} h_{st} m^{rs} F \right) \end{aligned} \tag{49}$$

where

$$G_{(pr)(qs)} = \frac{1}{2}(\det[h_{pq}])^{-1/2} \left(h_{pq}h_{rs} + h_{ps}h_{qr} - h_{pr}h_{qs} \right) \tag{50}$$

is the (contravariant) Wheeler-DeWitt metric, and $G = \frac{1}{2}(\det[h_{pq}])^{-1}$ is the absolute value of the determinant of its inverse [15].

Note that the new evolution equation (49) is manifestly real, and ensures that the integral of $G^{\frac{1}{2}}F$ is conserved in time. This supports the tentative interpretation of F as a probability density function on the 6-dimensional minisuperspace with coordinates h_{pq}.

Using (49) to evolve the wave function forward in time, we will clearly obtain an expression for F which depends on the gauge; i.e. on the choice of the N^p. On the other hand, if we do not wish to specify a gauge we should leave these quantities free as Lagrange multipliers. In this case, the invariance of the wave function with respect to changes in the N^p leads to the additonal constraints

$$0 = \epsilon_{trq} h_{sp} m^{qs} \frac{\partial F}{\partial h_{pr}}. \tag{51}$$

When these constraints are imposed, the evolution equation takes the form

$$\frac{\partial F}{\partial t} = \frac{2\kappa^2}{\sigma} N(t)\, G^{-\frac{1}{2}} \frac{\partial}{\partial h_{pr}} \left[G^{\frac{1}{2}}\, G_{(pr)(qs)} \left(\hbar \frac{\partial F}{\partial h_{qs}} + \frac{\partial \Phi}{\partial h_{qs}} F \right) \right]. \tag{52}$$

In the Bianchi IX model, the matrix m^{pq} is invertible and so (51) gives three independent constraints. These ensure that F is independent of the three remaining gauge degrees of freedom contained in h_{pq} associated with the possibility of redefining the invariant 1-forms $\boldsymbol{\omega}^p$ consistently with (3). Consequently, there are just three physical degrees of freedom, corresponding to the eigenvalues of the matrix $Z_p{}^q \equiv h_{pq} m^{qr}$. These eigenvalues are real and positive, since $Z_p{}^q$ is symmetric and positive-definite in the Bianchi IX case. In fact these eigenvalues are just the scale factors associated with the three principle axes of the metric h_{pq}.

Suppose that the eigenvalues of $Z_p{}^q$ are arranged in decreasing order and denoted $e^{2\beta_1}, e^{2\beta_2}, e^{2\beta_3}$ (so that $\beta_1 \geq \beta_2 \geq \beta_3$). Then the physical degrees of freedom of the 3-geometry are most conveniently parametrised by the variables

$$\alpha = \frac{1}{3}(\beta_1 + \beta_2 + \beta_3), \qquad \beta_+ = \frac{1}{6}(\beta_1 + \beta_2) - \frac{1}{3}\beta_3, \qquad \beta_- = \frac{1}{2\sqrt{3}}(\beta_1 - \beta_2). \tag{53}$$

Together, $(\alpha, \beta_+, \beta_-)$ form a coordinate system on the minisuperspace of homogeneous 3-geometries; α is associated with the overall scale factor, while β_+ and β_- measure the anistropy of a given 3-geometry. Note that our decision to label the eigenvalues in descending order means that the coordinates $(\alpha, \beta_+, \beta_-)$ satisfy the inequalities

$$\beta_+ \geq \frac{1}{\sqrt{3}} \beta_- \geq 0 \tag{54}$$

Note also that the gauge invariance implied by the constraints (51) implies that the probability density F can be expressed as a function of the three minisuperspace coordinates $(\alpha, \beta_+, \beta_-)$ and the time parameter t.

By integrating out the gauge degrees of freedom in the space of 3-metrics, one finds that the volume element on the minisuperspace of 3-geometries parametrised by $(\alpha, \beta_+, \beta_-)$ is $dV = \Omega\, \boldsymbol{d}\alpha \wedge \boldsymbol{d}\beta_+ \wedge \boldsymbol{d}\beta_-$ where

$$\begin{aligned} \Omega(\alpha, \beta_+, \beta_-) &= e^{(\beta_1+\beta_2+\beta_3)}(e^{2\beta_1} - e^{2\beta_2})(e^{2\beta_2} - e^{2\beta_3})(e^{2\beta_1} - e^{2\beta_3}) \\ &= 8e^{9\alpha} \sinh(2\sqrt{3}\beta_-) \sinh(3\beta_+ - \sqrt{3}\beta_-) \sinh(3\beta_+ + \sqrt{3}\beta_-) \quad (55) \end{aligned}$$

The constraints (51) ensure that F is invariant under gauge transformations and therefore depends only on the physical degrees of freedom $(\alpha, \beta_+, \beta_-)$ and on t. However, instead of dealing with the combination FdV it is convenient to define a new probability density function

$$P(\alpha, \beta_+, \beta_-; t) \equiv \Omega(\alpha, \beta_+, \beta_-)\, F(\alpha, \beta_+, \beta_-; t) \qquad (56)$$

so that $FdV = P\boldsymbol{d}\alpha \wedge \boldsymbol{d}\beta_+ \wedge \boldsymbol{d}\beta_-$. The finiteness of F then implies that P satisfies the Dirichlet boundary conditions

$$P(\alpha, \beta_+, \beta_-; t) = 0 \quad \text{if} \quad \beta_+ = \frac{1}{\sqrt{3}}\beta_- \quad \text{or} \quad \beta_- = 0. \qquad (57)$$

From (49), it follows that $P(\alpha, \beta_+, \beta_-; t)$ obeys the evolution equation

$$\begin{aligned} \frac{\partial P}{\partial t} = \frac{\kappa^2}{12\sigma} N(t) \Bigg[&\frac{\partial}{\partial \beta_+}\left(\hbar \frac{\partial P}{\partial \beta_+} + \frac{\partial U}{\partial \beta_+} P\right) + \frac{\partial}{\partial \beta_-}\left(\hbar \frac{\partial P}{\partial \beta_-} + \frac{\partial U}{\partial \beta_-} P\right) \\ &- \frac{\partial}{\partial \alpha}\left(\hbar \frac{\partial P}{\partial \alpha} + \frac{\partial U}{\partial \alpha} P\right)\Bigg] \qquad (58) \end{aligned}$$

where we have introduced the modified potential $U = \Phi - \hbar \log \Omega$, which in terms of the coordinates α, β_+, β_- has the form

$$\begin{aligned} U = &\frac{2\sigma}{\kappa^2} e^{2\alpha}\left[2e^{2\beta_+}\cosh(\sqrt{12}\beta_-) + e^{-4\beta_-}\right] - \hbar(9+\lambda)\alpha - \hbar \log\sinh(\sqrt{12}\beta_-) \\ &- \hbar \log\sinh(3\beta_+ - \sqrt{3}\beta_-) - \hbar \log\sinh(3\beta_+ + \sqrt{3}\beta_-). \qquad (59) \end{aligned}$$

The evolution equation (58) is manifestly real, and ensures that the integral of P is constant. It has the form of a conventional three-dimensional Fokker-Planck equation, but with one unusual feature; the $-$ sign preceeding the $(\partial/\partial\alpha)^2$ term indicates that the diffusion matrix is not positive-definite. Equations of this kind are known sometimes as "pseudo-Fokker-Planck" equations, and do not always admit normalisable non-singular solutions when subject to boundary conditions of the kind discussed above [17]. (However, it should be noted that (44) has at least one well-behaved solution; namely, the static and supersymmetric solution $P_0 = \Omega F_0 = \exp(-U/\hbar)$.)

From a geometric point of view, the problem arises because the minisuperspace metric has a Lorentzian signature $(-++)$. This is a direct consequence of the fact that the Euclidean Einstein-Hilbert action is not positive definite [18]; analogous difficulties arise in all approaches to the quantisation of gravity. In the path-integral prescription, these problems are often dealt with by analytic continuation, with the contour of integration for the conformal part of the metric being rotated in the complex plane [18].

In the present case, the non-positive-definiteness of the diffusion matrix means that normalisable solutions to (44) will generally develop singularities after a finite time. This precludes a naive intepretation of $P(\alpha, \beta_+, \beta_-; t)$ as a time-dependent probability density function. Recall, however, that the primary use of probabilities is for calculating expectation values; a very natural approach to the problem is therefore to analytically continue the definition of the expectation value. (Note that analytic continuation has been applied successfully in quite different contexts to make sense of probabilities which are formally defined as integrals of non-normalisable density functions. For example, see Appendix C of ref. [19].)

To see how this approach might work, suppose that $P(\alpha, \beta_+, \beta_-; t)$ is some unnormalised probability density function for the random variables $\alpha(t), \beta_+(t), \beta_-(t)$ at a given time t. Then the expectation value at this time of a quantity Q which depends on α, β_+, β_- is given by the ratio

$$E[Q] = \frac{\int d\alpha \int d\beta_+ \int d\beta_- \, Q(\alpha, \beta_+, \beta_-) \, P(\alpha, \beta_+, \beta_-; t)}{\int d\alpha \int d\beta_+ \int d\beta_- \, P(\alpha, \beta_+, \beta_-; t)} \tag{60}$$

where the contours of integration are along the real axes.

However, if $P(\alpha, \beta_+, \beta_-; t)$ is a solution to (44) evolving from typical initial data then this simple definition of the expectation value may break down. Indeed, if P is initially supported inside a small region of $(\alpha, \beta_+, \beta_-)$-space, then it will quickly develop singularities at points along the real α axis. It will then become impossible to evaluate the integrals appearing in (60), and so this expression will fail to give a well-defined expectation value. A natural way to avoid this problem is by shifting the contour of the α-integration in (60) away from the real axis so that the singularities are avoided. Hopefully a contour $\mathcal{C}$ can be found which gives rise to expectation values with all the standard properties.

Certainly, any such analytically continued expectation value $E[Q]$ will depend linearly on Q, in accordance with the standard rules of probability theory. Moreover, provided that the contour $\mathcal{C}$ is symmetric under reflection in the real axis, $E[Q]$ will be real for any function $Q(\alpha, \beta_+, \beta_-)$ which takes real values along the real α-axis. The only remaining requirement for bona-fide expectation values is that, if the function $Q(\alpha, \beta_+, \beta_-)$ is positive for all real α, β_+, β_-, then $E[Q]$ should also be positive. At present, it is not clear whether a contour can be found to satisfy this third criterion.

If an appropriate contour $\mathcal{C}$ exists, then this prescription leads to a satisfactory probabilistic intepretation even for solutions of (44) which are non-normalisable or develop singularities on the real α axis. A similar type of analytic continuation was used in [2], where the contour $\mathcal{C}$ was rotated to the imaginary α axis.

On the other hand, it is possible that a satisfactory contour may not exist. This would mean that, whatever analytic continuation scheme was employed, negative results might sometimes be obtained when evaluating expectation values of positive quantities. This would contradict the usual rules of probability theory, and thus call into question the feasibility of a probabilistic intepretation for the time-dependent states.

At present, it is not clear whether a satisfactory analytic continuation procedure can be found, or if there is any other way in which (58) can be converted into the evolution equation for a bona-fide probability density function. Consequently, the physical significance of the time-dependent solutions remains uncertain.

Discussion

We have seen how, by imposing certain gauge and coordinate conditions on the classical theory, the quantisation of homogeneous (Euclidean) supergravity leads to an evolution equation for the wave function which closely resembles a conventional Fokker-Planck equation.

However, because the Einstein-Hilbert action is unbounded below, the diffusion matrix is not positive definite and consequently the class of non-singular normalisable solutions is drastically reduced. In fact it is quite possible that there are no well-behaved time-dependent solutions at all, which raises interesting questions about the nature of the supposed evolution.

Nonetheless, the evolution equation (58) is known to admit a well-behaved static solution, which can be consistently intepreted as a cosmological probability density function. This solution has the form $P_0 = \exp(-U/\hbar)$, and is known to satisfy all the standard quantum constraints for the theory. It is encouraging that this probability density function coincides with that obtained for the wormhole state by more conventional approaches to the quantum theory.

An important question is whether the Hartle-Hawking state can, like the wormhole state, be interpreted in the manner described above. At present the answer to this question is unclear. It is known that an alternative definition of homogeneity must be used if one wants to find such a state in a minisuperspace theory [16]; however, the imposition of this alternative ansatz on the fields may prevent them from obeying their classical equations of motion. Thus, classical trajectories in the minisuperspace model might not correspond to classical trajectories in the full theory.

Clearly, the absence of such a correspondence would raise grave doubts about the relevance of minisuperspace as a tool for modelling the dynamics of the full theory.

In general, one would hope eventually to go beyond the strictures of minisuperspace. In this context, it should be emphasised that the approach outlined above should be applicable to a very wide range of supersymmetric models; the only requirement is that the action can be made precisely invariant under a boundary-preserving subalgebra of supersymmetry transformations.

It is unfortunate that, without the imposition of homogeneity conditions, pure $N = 1$ supergravity fails this test. However this failure cannot be viewed as a serious limitation on the applicability of our approach, since supergravity also fails the more elementary test of consistency in the finite fermion sector [12]. Indeed, it seems likely that a fully consistent supersymmetric theory *would* be amenable to the approach described above, without the need for reduction to minisuperspace. We hope to investigate this question in future work.

Acknowledgements

H.L. is grateful to Stephen Poletti and Chris Cosgrove for helpful and illuminating discussions.

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