Thermal Particle Creation in Cosmological Spacetimes: A Stochastic Approach

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The stochastic method based on the influence functional formalism introduced in an earlier paper to treat particle creation in near-uniformly accelerated detectors and collapsing masses is applied here to treat thermal and near-thermal radiance in certain types of cosmological expansions. It is indicated how the appearance of thermal radiance in different cosmological spacetimes and in the two apparently distinct classes of black hole and cosmological spacetimes can be understood under a unifying conceptual and methodological framework.

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I. INTRODUCTION

Particle creation in cosmological spacetimes was first discussed by Parker [1], Sexl and Urbantke [2], Zel'dovich and Starobinsky [3] in the late sixties. The basic mechanism can be understood as parametric amplification of vacuum fluctuations by an expanding universe [4,5]. Particle creation in black hole spacetimes was first discovered by Hawking [6] (see also [7]). A similar effect in a uniformly accelerated detector was discovered by Unruh [8] and in a moving mirror by Davies and Fulling [9]. One special class of cosmological spacetimes which shows this characteristic thermal radiance is the de Sitter Universe, as shown by Gibbons and Hawking [10]. One feature common to all these systems is that they all possess event horizons, and the conventional way to understand the thermal character of particle creation is by way of the periodicity on the propagator of quantum fields defined on the Euclidean section of the spacetime [11,12].

One would not usually think of cosmological particle creation as thermal because in general such conditions (event horizon and periodicity) do not exist. However, several authors have shown that thermal radiance can arise from cosmological particle creation in spacetimes without an event horizon [13–18]. Each case has its particular reason for generating a thermal radiance, but there is not much discussion of the common ground for these cases of cosmological spacetimes. There also seems to be a gulf between our understanding of the mechanisms giving rise to thermal radiance in these two classes of spacetimes, i.e., spacetimes with and without event horizons[1].

In some of our earlier papers we have discussed thermal radiance in the class of spacetimes which possess event horizons (uniformly accelerated detectors [20–22], moving mirrors and black holes [21,23]) using the viewpoint of exponential scaling [24,25] and the method of statistical field theory [26,27]. In a recent paper [23] we show how this method can be applied to spacetimes which possess event horizons only in some asymptotic limit, such as near-uniformly or finite-time accelerated detectors, and collapsing masses. In this paper, we study thermal particle creation from cosmological spacetimes with the aim of providing a common ground for cases where thermal radiance was reported before. Using the stochastic method, we show how to derive near-thermal radiance in spacetimes without an event horizon.

The two primary examples we picked here for analyzing this issue are that of Parker and Berger [13,14] for an exponentially expanding universe, and that of slow-roll inflationary universes, in particular, the Brandenberger-Kahn model [18]. In Sec. 2 we examine several simple cosmological expansions which admit thermal particle creation and show their common ground, and their connection with thermal radiation in the (static coordinatized) de Sitter spacetime. We point out that all cases which report thermal radiation involve an exponential scale transformation [24,25]. Thermal radiance observed in one vacuum can be understood as arising from the exponential scaling of vacuum fluctuations of the other vacuum [28]. Sec. 3 applies this method to the Parker-Berger model to show how thermal radiance can be derived from exponentially-scaled vacuum fluctuations. Sec. 4 discusses particle creation from slow-roll inflationary universe and shows how near-thermal radiance can be derived in cases which depart from strictly exponential expansion. We end with a brief discussion in Sec. 5.

II. THERMAL PARTICLE CREATION IN COSMOLOGICAL SPACETIMES: EXPONENTIAL SCALING

Consider a spatially-flat (k=0) Robertson-Walker (RW) universe with metric

$$ds^2 = dt^2 - a^2 \sum_i (dx^i)^2, \tag{2.1}$$

[1] Note that both of these effects are present in particle creation in a non-eternal black hole spacetime – over and above the thermal Hawking radiation for an eternal blackhole, there is also the contribution from backscattering of waves over a time-dependent classical effective potential, see, e.g. [19].

where t is cosmic time. A conformally-coupled massive (m) scalar field Φ obeys the wave equation (e.g., [30])

$$[\Box + m^2 + R/6]\Phi(t, x) = 0, \tag{2.2}$$

where $\Box$ is the Laplace-Beltrami operator, and $R = 6[\ddot{a}/a + (\dot{a}/a)^2]$ is the curvature scalar. In a spatially-homogeneous space, the space and time parts of the wave function separate, with mode decomposition $\Phi(t, x) = \sum_k \phi_k(t) w_k(x)$. For a spatially-flat RW universe $w_k(x) = e^{ikx}$, and the conformally-related amplitude function $\chi_k(\eta) = a\phi_k(t)$ of the k^{th} mode obeys the wave equation in conformal time $\eta = \int dt/a$:

$$\chi_k(\eta)'' + [k^2 + m^2 a^2(\eta)]\chi_k(\eta) = 0. \tag{2.3}$$

Call $\Phi_k^{in,out}(t, x)$ the modes with only positive frequency components at $t_- = -\infty$ and $t_+ = +\infty$, respectively. They are related by the Bogolubov coefficients α_k, β_k as follows:

$$\Phi_k^{in}(t, x) = \alpha_k \Phi_k^{out}(t, x) + \beta_k \Phi_{-k}^{out*}(t, x). \tag{2.4}$$

(For conformal fields it is convenient to use the conformally- related wave function $X(\eta, x) = a\Phi(t, x)$. One can define the conformal vacua at $\eta_\pm$ with $\chi^{in,out}$ in terms of the positive frequency components.) The modulus of their ratio is useful for calculating the probability $P_n(\vec{k})$ of observing n particles in mode $\vec{k}$ at late times [14]:

$$P_n(\vec{k}) = |\beta_k/\alpha_k|^{2n} |\alpha_k|^{-2}. \tag{2.5}$$

From this one can find the average number of particles $\langle N_{\vec{k}} \rangle$ created in mode $\vec{k}$ (in a comoving volume) at late times to be

$$n_k \equiv \langle N_k \rangle = \sum_{n=0}^{\infty} n P_n(\vec{k}) = |\beta_k|^2. \tag{2.6}$$

A. Bernard-Duncan Model

In a model studied by Bernard and Duncan [16] the scale factor $a(\eta)$ evolves like

$$** \quad Case\ 1 \ ** \qquad a^2(\eta) = A + B \tanh \rho\eta \tag{2.7}$$

which tends to constant values $a_\pm^2 \equiv A \pm B$ at asymptotic times $\eta \to \pm\infty$. Here ρ measures how fast the scale factor rises, and is the relevant parameter which enters in the temperature of thermal radiance. With this form for the scale function, α_k, β_k have analytic forms in terms of products of gamma functions. One obtains

$$|\beta_k/\alpha_k|^2 = \sinh^2(\pi\omega_-/\rho)/\sinh^2(\pi\omega_+/\rho) \tag{2.8}$$

where

$$\omega_\pm = \frac{1}{2}(\omega^{out} \pm \omega_{in}), \quad \omega_{in}^{out} = \sqrt{k^2 + m^2 a_\pm^2}. \tag{2.9}$$

For cosmological models in which $a(+\infty) \gg a(-\infty)$, the argument of sinh is very large (i.e. $(\pi/\rho)\omega_\pm \gg 1$). To a good approximation this has the form

$$|\beta_k/\alpha_k|^2 = \exp(-2\pi\omega_{in}/\rho). \tag{2.10}$$

For high momentum modes, one can recognize the Planckian distribution with temperature given by

$$k_B T_\eta = \rho/(2\pi a_+) \tag{2.11}$$

as detected by an observer (here in the conformal vacuum) at late times.

B. Parker-Berger Model

This model is similar in spirit to the one proposed by Parker earlier [14], who considered a massless, minimally coupled scalar field in a Robertson-Walker universe with metric

$$ds^2 = a^6 d\tau^2 - a^2 \sum_i (dx^i)^2 \tag{2.12}$$

where τ is a time defined by $dt = a^3 d\tau$. The scale factor a is assumed to approach a constant at $\tau \to \pm\infty$, where in these asymptotic regions one can define a vacuum with respect to τ time and construct the corresponding field theory. He considered the general class of functions for the scale factor (from Epstein and Eckart [31])

$$** \quad Case\ 2 \ ** \qquad a^4(\tau) = a_1^4 + e^{\sigma\tau}[(a_2^4 - a_1^4)(e^{\sigma\tau}+1) + b](e^{\sigma\tau}+1)^{-2} \tag{2.13}$$

where a_1, a_2, b are adjustable parameters with $a_2 > a_1$, and σ is the rise function (similar to the ρ in the earlier case in terms of the conformal vacuum). The modulus of the ratio of the Bogolubov coefficients is given by [14]

$$\left|\frac{\beta_k}{\alpha_k}\right|^2 = \frac{\sin^2 \pi d + \sinh^2(\pi\omega_-/\sigma)}{\sin^2 \pi d + \sinh^2(\pi\omega_+/\sigma)} \tag{2.14}$$

where d is a real number involving the constant b and

$$\omega_\pm \equiv k(a_1^2 \pm a_2^2). \tag{2.15}$$

For cosmological models in which $a_2 \gg a_1$, the argument of the sinh is very large, as in Case 1. Then, to a good approximation (2.14) has the form

$$|\beta_k/\alpha_k|^2 = \exp(-4\pi k a_1^2/\sigma). \tag{2.16}$$

This form is independent of the adjustable parameters b, a_2. From this one can show that the amount of particle creation is given by

$$n_k = [\exp(\mu k) - 1]^{-1} \tag{2.17}$$

where $\mu = 4\pi a_1^2/\sigma$. Converting to the physical momentum at late times $p = k/a_2$, one sees that this is a Planckian spectrum characteristic of thermal radiance with temperature

$$k_B T_\tau = \sigma/(4\pi a_1^2 a_2). \tag{2.18}$$

How sensitively does the thermal character of particle creation depend on the scale factor? From physical considerations, the period when particle creation is significant is when the nonadiabaticity parameter $\bar{\Omega}$ satisfies [5]

$$\bar{\Omega} \equiv \Omega'/\Omega^2 \geq 1. \tag{2.19}$$

Here Ω is the natural frequency (given by ka^2 in this case) and $\Omega' = d\Omega/d\tau$. Using this criterion, Parker argued that significant particle creation occurs during an early period when $e^{\tau/\sigma}$ (or in the first case, $e^{\eta/\rho}$) is small, whence a^4 has effectively the form

$$** \ Case\ 3 \ ** \qquad a^4(\tau) = a_1^4 + a_0^4 e^{\tau/\sigma} \tag{2.20}$$

where a_0 is an adjustable parameter. This form of the scale factor was used by Berger [13] for the calculation of particle creation in a Kasner universe, who also reported thermal radiation. Since particle creation vanishes at early and late times (as measured by the nonadiabaticity parameter), adiabatic vacua can be defined then and one can construct WKB positive and negative frequency solutions for the calculation of the Bogolubov coefficients. Parker [14] showed explicitly that the modulus of their ratio has the same exponential form as that of the more complicated scale function (2.13), which, as we have seen, is what gives rise to the thermal character of the spectrum. Indeed he speculated that the exponential form in $|\beta/\alpha|$ should hold for a general class of scale functions which possess the properties that: 1) they smoothly approach a constant at early time, 2) their values at late times are much larger than at initial times, and most importantly, that 3) they and their derivatives are continuous functions. The exponential factor contained in the scale functions in all three cases above at early times is thus responsible for the thermal property of particle creation, with the temperature proportional to the rise factor in the exponential function (σ, or ρ in the first case). He also noted that this property is quite insensitive to the late time asymptotically static behavior of $a(\tau)$ (could be the flattening behavior of a tanh function, or the rising behavior of an exp function).

C. Common Features

The three examples related above highlight an important common feature of thermal particle creation. That is, that a period of exponential expansion in the scale factor would give rise to thermal particle creation, i.e., an observer in the in vacuum before the expansion reports zero particles, while an observer in the out vacuum after the exponential expansion will report a thermal particle spectrum with temperature proportional to the rise factor in the exponential function in that particular time (e.g., conformal time η in Bernard and Duncan's model, τ in Parker's model, and cosmic time t in the de Sitter universe example below). The exponential scale factor and the relation between these two vacua are important to understanding particle creation in cosmological spacetimes on the same footing as that in the class of spacetimes with event horizons, including that of a uniform accelerated observer, a moving mirror, black holes and the de Sitter universe.

To put the physics in a more general context, consider two vacua related by some transformation. Let us define the asymptotic in-vacuum as $|0\rangle_t$, the asymptotic out-vacuum as $|0\rangle_{out}$ and the vacuum of an observer undergoing exponential expansion as $|0\rangle_s$. The in and out vacua are well defined because the scale factor approaches a constant at asymptotic past and future times, thus imparting the space with a Killing vector ∂_t with respect to which one can define particle states in terms positive frequency modes. The s-vacuum is defined with respect to a different set of mode functions (like the Fulling- Rindler vacuum vis-a-vis the Minkowski vacuum for a uniformly- accelerated observer). The above examples calculate the particle creation between an in and out vacuum, but they also illustrate the important fact that the number of particles created is insensitive to the behavior of the scale function at late times – e.g., the result for the flattening tanh function which gives an asymptotically static universe is the same as that of the exponential function. Furthermore, the thermal nature of particle creation depends only on the initial stage of exponential expansion. In [28,29] these findings were used to connect the result of thermal radiance in these two classes of spacetimes. The assertion is that the more basic cause of thermality lies in the exponential scaling behavior rather than the existence of an event horizon [24,25]. (The latter necessarily implies the former, but the converse is not always true). The fine distinction between these two ways of understanding (kinematic versus geometric) thermal radiance will enable us to treat non-thermal cases in spacetimes which do not possess event horizons [23], and to explore the stochastic nature of the Hawking-Unruh effect.

At this point it is perhaps also useful for us to adopt this kinematic viewpoint to reexamine the cause of thermal radiance in a de Sitter universe.

D. de Sitter Universe

The de Sitter universe metric can be expressed in many different coordinates (see, e.g. [30]). There is the so-called closed (k=1) Robertson-Walker (RW) coordinatization which covers the whole space, $a(t) = \cosh(Ht)$, the flat (k=0) RW coordinatization which covers only half of de Sitter space with scale factor $a(t) = e^{Ht}$, and the static coordinate shown below, to name just the commonly encountered ones. The vacuum states defined with respect to different coordinatization and normal mode decomposition have been studied by many authors [32]. In the static coordinate the metric is given by

$$ds^2 = [1 - (H\tilde{r})^2]d^2 - \frac{\mathtt{d\tilde{r}}^2}{[1-(\mathtt{H\tilde{r}})^2]}. \tag{2.21}$$

Note that an event horizon exists at $\tilde{r} = H^{-1}$ for observers at $\tilde{r} = 0$, following the trajectory generated by the Killing vector ∂. This is similar in form to a Schwarzschild metric for a massive (M) object, restricted to 2 dimensions:

$$ds^2 = (1 - \frac{2M}{r})dt^2 - \frac{dr^2}{(1-\frac{2M}{r})}, \tag{2.22}$$

for which Hawking [6] first reported the famous black hole thermal radiance effect.

Calculation of particle creation in the static de Sitter universe was carried out by Gibbons and Hawking [10] (GH) using the periodicity condition in the field propagator. Lapedes [33] gave derivations based on the use of Bogolubov transformations, and Brandenberger and Kahn [18] treated the asymptotically de Sitter case. Let us analyze the relation between particle creation calculated in the Gibbons-Hawking vacuum (defined with respect to the 'static' de Sitter time) $|0\rangle$ and that in the Robertson-Walker (k=0) vacuum (defined with respect to cosmic time t) $|0\rangle_t$. We will see that the $|0\rangle$ vacuum bears the same relation to $|0\rangle_t$ vacuum as that between the exponential vacuum $|0\rangle_s$ defined earlier and the asymptotic in-vacuum $|0\rangle_t$ in the cosmological cases above.

E. Exponential scaling: a kinematic viewpoint

Starting from special relativity, assuming that two coordinate systems $S = (t, r)$ and $\tilde{S} = (, \mathtt{\tilde{r}})$ (these are not the black hole coordinates) coincide at the origin, so the initial vacuum of $\tilde{S}$ is the same as the RW vacuum, but the final vacuum in $\tilde{S}$ is the GH vacuum. The two systems are related by the following conditions:

$$(i) \quad \tilde{r} = a(t)r \tag{2.23}$$

$$(ii) \quad a(t) = e^{Ht} \tag{2.24}$$

$$(iii) \quad H\tilde{r} = Har = \dot{a}r = \beta, \tag{2.25}$$

$$(iv) \quad \frac{a()}{a(t)} = \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2.26}$$

The meaning of these conditions is explained in [28], which uses this example to illustrate the existence of exponential scaling in all cases which report thermal radiance. The two systems are related by a scale transformation *(i)*, in this case, an exponential scaling *(ii)* such that an observer in $\tilde{S}$ is seen as receding from S with a velocity of β, with H the red-shift or Hubble parameter *(iii)*, and a Lorentz factor γ *(iv)*. Condition *(iv)* is called relativistic exponential transformation [28], which plays a central role in the understanding of the Hawking effect in terms of scaling concepts [24,25,34]. With these correspondences, it is not difficult to see the analogy with the cosmological particle creation cases studied before. The initial (asymptotic in-) vacuum $|0\rangle_t$ defined with respect to t time here (or η, τ time in the earlier examples) and the vacuum $|0\rangle$ defined in the exponentially receding system $\tilde{S}$ bear the same relation. It is no surprise that the modulus of the ratio between the Bogolubov coefficients have the same form characteristic of a thermal spectrum (2.10), but with ρ, σ replaced by H. [One can find explicit calculation of thermal particle creation in Eq. (33) of [33], using Bogolubov transformations replacing R by our H^{-1}.] The Hawking temperature for the de Sitter universe is given by

$$k_B T_{dS} = \frac{H}{2\pi}. \tag{2.27}$$

Once the relation between the de Sitter universe (which belongs to the class of spacetimes which show the Hawking-Unruh effect), and that of some general cosmological spacetimes (with specific scale functions such as in (2.7, 2.13, 2.20) is established, it is easy to generalize to the black hole and accelerated detector cases.

III. THERMAL RADIANCE IN THE PARKER-BERGER MODEL

We now use the influence functional formalism (see e.g., [21]) to investigate particle creation in the Parker-Berger model [13,14]. The line element is given by

$$ds^2 = a^6 d\tau^2 - a^2 \sum_i (dx^i)^2 \tag{3.1}$$

where

$$a^4(\tau) = 1 + e^{\rho\tau}. \tag{3.2}$$

We consider the action of a massless, minimally coupled real scalar field ϕ, which forms an environment acting upon a detector coupled to this field at some point in space. The field can be decomposed into a collection of oscillators of time-dependent frequency. Using the influence functional formalism, we can determine the effect of such an environment on the detector, which is also modelled by an oscillator.

To do this we calculate the noise ν and dissipation μ produced by the field. These are given by:

$$\zeta \equiv \nu + i\mu = \int_0^\infty dk \; I(k, s, s') X(s) X^*(s') \tag{3.3}$$

where I is the spectral density describing the system/environment interaction, and X is a sum of Bogolubov coefficients satisfying the classical equation of motion for the field oscillators. First we decompose the field into its modes; the Lagrangian density is

$$\begin{aligned}\mathcal{L}(x) &= \frac{\sqrt{-g}}{2}\phi^{,\mu}\phi_{,\mu} \\ &= \frac{1}{2}\left[\phi_{,\tau}^2 - a^4 \sum_i (\phi_{,i})^2\right].\end{aligned} \tag{3.4}$$

In terms of normal modes the Lagrangian becomes

$$L(\tau) = \sum_{k,\sigma=\pm} \frac{1}{2}\left[(q^\sigma_{\mathbf{k},\tau})^2 - a^4 k^2 (q^\sigma_{\mathbf{k}})^2\right]. \tag{3.5}$$

We see then that the bath can be described as a set of oscillators with mass and frequency

$$m = 1 \quad , \quad \omega^2 = a^4 k^2. \tag{3.6}$$

Now as was already mentioned, X satisfies the classical equation of motion for an oscillator with the given parameters, and being a sum of Bogolubov coefficients, its initial values are predetermined. So we need to solve

$$X''(\tau) + k^2(1 + e^{\rho\tau})X = 0, \tag{3.7}$$

with initial conditions:

$$X(\tau_0) = 1 \quad , \quad X'(\tau_0) = -ik. \tag{3.8}$$

With a change of variables $z = \ln(2k/\rho) + \rho\tau/2$ we find the solutions in terms of the Bessel functions:

$$X(\tau) = c_1 J_{\frac{2ik}{\rho}}\left(\frac{2k}{\rho}e^{\rho\tau/2}\right) + c_2 J_{-\frac{2ik}{\rho}}\left(\frac{2k}{\rho}e^{\rho\tau/2}\right) \tag{3.9}$$

To fix the constants c_1, c_2 consider that the initial time is $\tau_0 \to -\infty$; however the complex index Bessel functions oscillate infinitely often as their arguments approach zero, and so for now we leave τ_0 unspecified. In that case we can calculate[2] c_1 and c_2; the final expression for X becomes, with

$$f(\tau) \equiv \frac{2k}{\rho} e^{\rho\tau/2},$$

and Bessel indices labelled by $\nu \equiv 2ik/\rho$:

$$X(\tau) = \frac{\pi k}{\rho} \mathrm{csch} \frac{2\pi k}{\rho} \left\{ ie^{\rho\tau_0/2} \begin{vmatrix} J_\nu(f(\tau)) & J_{-\nu}(f(\tau)) \\ J'_\nu(f(\tau_0)) & J'_{-\nu}(f(\tau_0)) \end{vmatrix} - \begin{vmatrix} J_\nu(f(\tau)) & J_{-\nu}(f(\tau)) \\ J_\nu(f(\tau_0)) & J_{-\nu}(f(\tau_0)) \end{vmatrix} \right\}. \tag{3.10}$$

In the limiting case of $\tau \to \infty$ ($\tau_0 \to -\infty$) we can use first order and asymptotic expressions for J and J' to write (which defines the phases α and β)

$$\begin{aligned} J_\nu(f(\tau_0)) &\simeq \sqrt{\frac{\sinh 2\pi k/\rho}{2\pi k/\rho}} e^{i\alpha(k)} \\ J'_\nu(f(\tau_0)) &\simeq e^{-\rho\tau_0/2} \sqrt{\frac{\sinh 2\pi k/\rho}{2\pi k/\rho}} e^{i\beta(k)} \\ J_\nu(f(\tau)) &\simeq \sqrt{\frac{2}{\pi z}} \cos\left(f(\tau) - \frac{\nu\pi}{2} - \frac{\pi}{4}\right). \end{aligned} \tag{3.11}$$

In evaluating $X(\tau)X^*(\tau')$ we obtain various products of the Bessel functions with their derivatives (note: $J^*_\nu = J_{-\nu}$); in particular we need

$$\begin{aligned} \beta - \alpha &= \arg\Gamma\left(1 + \frac{2ik}{\rho}\right) - \arg\Gamma\left(\frac{2ik}{\rho}\right) \\ &= \arg\frac{2ik}{\rho} = \frac{\pi}{2} \text{ provided } k \neq 0. \end{aligned} \tag{3.12}$$

Also, when calculating the Bessel products, there arise sines and cosines with argument $f(\tau) + f(\tau') \equiv 2k/\rho\left(e^{\rho\tau/2} + e^{\rho\tau'/2}\right)$; when $\tau \to \infty$ and we ultimately integrate over k, these terms won't contribute to the integral and so can be discarded. Changing to sum and difference variables defined by

$$\Sigma \equiv (\tau + \tau')/2 \quad , \quad \Delta \equiv \tau - \tau' \tag{3.13}$$

we finally obtain

$$\zeta = e^{-\rho\Sigma/2} \int_0^\infty dk\ I(k,\tau,\tau') \left[\cos\frac{2k}{\rho}\left(e^{\rho\tau/2} - e^{\rho\tau'/2}\right) \coth\frac{2\pi k}{\rho} - i \sin\frac{2k}{\rho}\left(e^{\rho\tau/2} - e^{\rho\tau'/2}\right)\right]. \tag{3.14}$$

We can now equate ζ with the standard form:

$$\zeta = \int_0^\infty dk\ I_{\mathit{eff}}(k,\Sigma)\ \left[C(k,\Sigma)\cos k\Delta - i \sin k\Delta\right]. \tag{3.15}$$

This is the form of ζ which, with the function C replaced by $\coth\frac{k}{2T}$, would describe a thermal bath of static oscillators each in a coherent state. We will show that the unknown function C does indeed have the form

[2] To calculate these coefficients the wronskian of $J_{2ik/\rho}$ and $J_{-2ik/\rho}$ is needed; note that there is a misprint in Gradshteyn and Ryzhik §8.474: the relevant quantity should be $-\frac{2}{\pi z}\sin\nu\pi$.

of a coth, and can then deduce the temperature of the radiation seen by the detector. Here $I_{eff}(k,\Sigma)$ is the effective spectral density, also to be determined. We can always write ζ in this way since ν is even in Δ while μ is odd. By equating the real and imaginary parts of the two forms of ζ and Fourier inverting, we obtain

$$\begin{aligned} I_{eff}C &= \frac{1}{\pi}\int_{-\infty}^{\infty} d\Delta\ \cos k\Delta\ \nu(\Sigma,\Delta), \\ I_{eff} &= -\frac{1}{\pi}\int_{-\infty}^{\infty} d\Delta\ \sin k\Delta\ \mu(\Sigma,\Delta). \end{aligned} \tag{3.16}$$

These expressions will be used throughout this paper to calculate C for the various cases of induced radiance that we consider. In order to use these we need to calculate the dissipation and noise, μ and ν.

We first evaluate μ as given by (3.14); substituting it into (3.16) will then give us the effective spectral density $I_{eff}(k,\Sigma)$. Define

$$\sigma \ \equiv\ \frac{2}{\rho}\left(e^{\rho s/2} - e^{\rho s'/2}\right) \ =\ \frac{4}{\rho}e^{\rho\Sigma/2}\sinh\frac{\rho\Delta}{4}. \tag{3.17}$$

To proceed, we need to specify a form for the spectral density. This has been calculated in [21], and in 3+1 dimensions it is

$$I(k,s,s') = \frac{c^2 k}{4\pi^2}, \tag{3.18}$$

where c is the coupling strength of the detector to the field. Then from (3.14) we have

$$\begin{aligned} \mu &= -\frac{c^2}{4\pi^2}e^{-\rho\Sigma/2}\int_0^{\infty} dk\ k\sin\sigma k \\ &= \frac{c^2}{4\pi^2}e^{-\rho\Sigma/2}\ \pi\delta'(\sigma) \end{aligned} \tag{3.19}$$

where the last result follows from the discussion in [35]. Substituting this form for μ into (3.16) gives the following result:

$$I_{eff}(k,\Sigma) = \frac{c^2 k}{4\pi^2}e^{-3\rho\Sigma/2}. \tag{3.20}$$

Evaluating the noise kernel ν is a more complicated affair. From (3.14) we write

$$\begin{aligned} \nu &= \frac{c^2}{4\pi^2}e^{-\rho\Sigma/2}\int_0^{\infty} dk\ k\cos\sigma k\coth\frac{2\pi k}{\rho} \\ &= \frac{c^2}{4\pi^2}e^{-\rho\Sigma/2}\left[\frac{d}{d\sigma}P(1/\sigma) + \frac{1}{\sigma^2} - \frac{\rho^2}{16}\mathrm{csch}^2\frac{\rho\sigma}{4}\right] \end{aligned} \tag{3.21}$$

where again the last integral has been calculated in [35]. Upon substituting this into (3.16) we obtain

$$\begin{aligned} C(k,\Sigma) = {} & \frac{e^{\rho\Sigma}}{\pi k}\int_{-\infty}^{\infty} d\Delta\ \cos k\Delta\frac{d}{d\sigma}P(1/\sigma) \\ & + \frac{\rho^2}{16\pi k}\int_{-\infty}^{\infty} d\Delta\ \cos k\Delta\left[\mathrm{csch}^2\,\frac{\rho\Delta}{4} - e^{\rho\Sigma}\mathrm{csch}^2\left(e^{\rho\Sigma/2}\sinh\frac{\rho\Delta}{4}\right)\right]. \end{aligned} \tag{3.22}$$

The first integral can be done by parts to get

$$\int_{-\infty}^{\infty} d\Delta \; \cos k\Delta \frac{d}{d\sigma} P(1/\sigma) = \int_{-\infty}^{\infty} d\sigma \; \frac{d\Delta}{d\sigma} \cos k\Delta \frac{d}{d\sigma} P(1/\sigma)$$
$$= -\mathrm{PV} \int_{-\infty}^{\infty} \frac{d\Delta}{\sigma} \frac{d}{d\Delta} \left[\frac{d\Delta}{d\sigma} \cos k\Delta \right]$$
$$= \frac{4\pi k}{\rho} \coth \frac{2\pi k}{\rho}. \tag{3.23}$$

The second integral in (3.22) does not appear to be expressible in terms of known functions. Suppose we call it $B(k, \rho, \Sigma)$, and write

$$B(k, \rho, \Sigma) = 2 \int_0^{\infty} d\Delta \; \cos k\Delta \left[\mathrm{csch}^2 \frac{\rho\Delta}{4} - e^{\rho\Sigma} \mathrm{csch}^2 \left(e^{\rho\Sigma/2} \sinh \frac{\rho\Delta}{4} \right) \right]. \tag{3.24}$$

Then we have

$$C(k, \rho, \Sigma) = e^{\rho\Sigma} \left[\frac{4}{\rho} \coth \frac{2\pi k}{\rho} + \frac{\rho^2}{16\pi k} e^{-\rho\Sigma} B(k, \rho, \Sigma) \right]. \tag{3.25}$$

We need to examine the second term in the last brackets. The function B tends to zero for large k (by the Riemann-Lebesgue lemma), and attains a maximum at $k = 0$ (since the cos term stops oscillating there). However, numerical work shows that this maximum value increases roughly with $e^{\rho\Sigma}$ which means that on first glance the second term in the brackets does not necessarily vanish at late times ($\Sigma \to \infty$). So we need to examine the value of B at $k = 0$ more closely, to see precisely how it changes with Σ. To this end we can consider $B(0, \rho, \Sigma)$ as a function of $e^{\rho\Sigma}$ and analyze its concavity; i.e. with $x \equiv e^{\rho\Sigma}$ we need $\partial^2 B/\partial x^2$. Differentiating twice under the integral sign gives an integrand which is everywhere negative, and so we conclude that $\partial^2 B/\partial x^2 < 0$, which means that B as a function of x is everywhere concave down. But B increases with x, and thus $B/x \equiv e^{-\rho\Sigma} B \to 0$ as $\Sigma \to \infty$. In that case the second term in the brackets gives no contribution in the large time limit.

Finally, from (3.15) we can write ζ in a form which reveals the thermal nature of the detected radiation:

$$\zeta = \int_0^{\infty} dk \; I_{\mathit{eff}}(k, \Sigma) \; \left[\frac{4e^{\rho\Sigma}}{\rho} \coth \frac{2\pi k}{\rho} \cos k\Delta - i \sin k\Delta \right]. \tag{3.26}$$

The temperature of the radiation is then

$$k_B T = \frac{\rho}{4\pi}. \tag{3.27}$$

Inspection of (3.14) suggests that an alternative time variable can be chosen:

$$t = \frac{2}{\rho} e^{\rho\tau/2}. \tag{3.28}$$

The metric becomes

$$ds^2 = \frac{4a^6}{\rho^2 t^2} dt^2 - a^2 \sum_i (dx^i)^2 \tag{3.29}$$

with

$$a^4 = 1 + \frac{\rho^2 t^2}{4}. \tag{3.30}$$

Again following the previous formalism, we arrive at a description of the environment field in terms of oscillators, now with time dependent mass and frequency:

$$m = \frac{\rho t}{2} \quad , \quad \omega^2 = \frac{4a^4k^2}{\rho^2 t^2}. \tag{3.31}$$

Solutions for X in this case are the same as before, and the calculations carry through in much the same way. With now Σ and Δ defined as mean and differences of t and t' we again arrive at thermal forms for the noise and dissipation:

$$\zeta = e^{-\rho\Sigma/2} \int_0^\infty dk\ I(k,s,s') \left[\cos k\Delta \coth \frac{2\pi k}{\rho} - i \sin k\Delta\right] \tag{3.32}$$

and the detected temperature is the same as (3.27).

IV. INFLATIONARY UNIVERSE

A. Eternal versus Slow-roll Inflation

In this section we consider particle creation of a massless conformally coupled quantum scalar field at zero temperature in a spatially flat FRW universe undergoing a near-exponential (inflationary) expansion. The example of de Sitter space which corresponds to the exact exponential case has been treated in [21]. Here we first solve for a general scale factor $a(t)$ using a slightly different language from [21]. We then specialise to a spacetime (the Brandenberger-Kahn metric [18]) which has initial de Sitter behavior but with scale factor tending toward a constant at late (cosmic) times. We can also define a parameter h which measures the departure from an exact exponential expansion.

As before we first derive the noise and dissipation kernels by calculating X, the solution to the equation of motion of the field modes. The spatially-flat FRW metric is

$$ds^2 = dt^2 - a^2(t) \sum_i \left(dx^i\right)^2. \tag{4.1}$$

The Lagrangian density of the scalar field is

$$\mathcal{L}(x) = \frac{a^3}{2}\left[\dot{\Phi}^2 - \frac{1}{a^2}\sum_i \Phi_{,i}^2 - \left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)\Phi^2\right] \tag{4.2}$$

which leads to a Lagrangian in terms of the normal modes $q_{\mathbf{k}}$

$$L(t) = \sum_{k,\sigma=\pm} \frac{a^3}{2}\left[\left(\dot{q}_{\mathbf{k}}^\sigma\right)^2 + 2\frac{\dot{a}}{a}\dot{q}_{\mathbf{k}}^\sigma q_{\mathbf{k}}^\sigma - \left(\frac{k^2}{a^2} - \frac{\dot{a}^2}{a^2}\right)\left(q_{\mathbf{k}}^\sigma\right)^2\right]. \tag{4.3}$$

(We have added a surface term $1/a^3\ d/dt\ (\dot{a}a^2q^2)$ to the Lagrangian. See [21] for the rationale.) The classical equation of motion, and hence that of X, is

$$\ddot{X} + 3\frac{\dot{a}}{a}\dot{X} + \left(\frac{k^2}{a^2} + \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)X = 0 \tag{4.4}$$

with initial conditions

$$X(t_i) = 1 \quad , \quad X'(t_i) = -ik - a'(t_i). \tag{4.5}$$

We find that

$$X(t) = \frac{1}{a}e^{-ik\eta} \tag{4.6}$$

where η is the usual conformal time.

Now we use (3.3) to construct the influence kernel ζ, which contains the noise and dissipation kernels. From [21] the spectral density for the field is

$$I(k,t,t') = \frac{\varepsilon^2 k}{4\pi^2}. \tag{4.7}$$

For the rest of this section we introduce for brevity:

$$\begin{aligned} x(\Sigma,\Delta) &\equiv a(t) + a(t'), \\ y(\Sigma,\Delta) &\equiv \eta(t) - \eta(t'). \end{aligned} \tag{4.8}$$

Then we find from (3.3)

$$\begin{aligned} \zeta \equiv \nu + i\mu &= \frac{1}{a(t)a(t')}\frac{\varepsilon^2}{4\pi^2}\int_0^\infty k\, e^{-iky}\, dk \\ &= \frac{1}{a(t)a(t')}\frac{\varepsilon^2}{4\pi^2}\left[\frac{d}{dy}P(1/y) + i\pi\delta'(y)\right]. \end{aligned} \tag{4.9}$$

As before we calculate the spectrum and temperature by Fourier transforming ζ using (3.16). (Note that Σ, Δ are defined in terms of cosmic time t). We change the variable of integration from Δ to y, using

$$\frac{d\Delta}{dy} = \frac{2a(t)a(t')}{x} \tag{4.10}$$

to write

$$\begin{aligned} I_{\mathit{eff}} &= -\frac{\varepsilon^2}{2\pi^2}\int_{-\infty}^\infty \delta'(y)\frac{\sin k\Delta}{x}\, dy \\ &= \frac{\varepsilon^2 k}{4\pi^2} \end{aligned} \tag{4.11}$$

independently of the value of $a(t)$. The temperature now follows from (3.16): again we change to a y-integration by parts, remembering that $\Delta = 0 \Longleftrightarrow y = 0$. We obtain

$$I_{\mathit{eff}}\, C = \frac{\varepsilon^2}{4\pi^3}\int_{-\infty}^\infty \frac{\cos k\Delta}{a(t)a(t')}\frac{d\Delta}{dy}\frac{d}{dy}P(1/y)\, dy \tag{4.12}$$

which leads to

$$C = -\frac{4}{\pi k}\int_0^\infty \left[\frac{d}{d\Delta}\frac{\cos k\Delta}{x}\right]\frac{d\Delta}{y}. \tag{4.13}$$

This equation is the central result of this section, in that it allows us to compute the spectrum corresponding to an arbitrary scale factor. For example, in the de Sitter case with $a = e^{Ht}$ we have

$$x = 2e^{H\Sigma}\cosh\frac{H\Delta}{2} \quad ; \quad y = \frac{2e^{-H\Sigma}}{H}\sinh\frac{H\Delta}{2} \tag{4.14}$$

which when substituted into (4.13) gives

$$C = \coth\frac{\pi k}{H}. \tag{4.15}$$

So for this case we can infer the temperature seen to be

$$k_B T = \frac{H}{2\pi} \tag{4.16}$$

as was calculated with a slightly different approach in [21].

As an aside, we note that from the above analysis for a general scale factor, the noise kernel is

$$\nu = -\frac{\varepsilon^2}{\pi^3} \int_0^\infty dk \cos k\Delta \int_0^\infty \frac{du}{y} \frac{d}{du} \left[\frac{\cos ku}{a(t) + a(t')} \right] \tag{4.17}$$

with dissipation

$$\mu = \frac{\varepsilon^2 \delta'(\Delta)}{4\pi}. \tag{4.18}$$

An often-used alternative to our principal part prescription is the introduction of a cutoff in the $\int_0^\infty$ expressions; unfortunately following this procedure doesn't lead to tractable integrals even for the relatively simple de Sitter case. Note that in equation (4.13) for the temperature in the general case, we are essentially dealing with products of a and η, and it's therefore not surprising that for de Sitter expansion, where $a \propto 1/\eta$, that (4.13) can be evaluated analytically. For other forms of a, even very simple ones, (4.13) becomes very complicated.

B. Near-exponential expansion

We now consider the case of a near de Sitter universe with a scale factor composed of the usual de Sitter one together with a factor that decays exponentially. We show that the spectrum seen is near-thermal tending toward thermal at late times.

We start by considering the Hubble parameter to have a constant value (characterising de Sitter space) plus an exponentially decaying term:

$$H(t) = H_0 \left(1 + \alpha e^{-\beta H_0 t}\right), \tag{4.19}$$

and from this our aim is to calculate C using (4.13). The scale factor results from integrating H, and is

$$a(t) = \exp\left(H_0 t - \frac{\alpha}{\beta} e^{-\beta H_0 t}\right). \tag{4.20}$$

We define the parameter h which measures the departure from exact exponential expansion to be

$$h(t) \equiv \frac{\dot{H}(t)}{H(t)^2} \to -\alpha\beta e^{-\beta H_0 t} \tag{4.21}$$

as $\beta t \to \infty$, and as we might expect it is exponentially decaying at late times.

To proceed, we indicate the de Sitter quantities by a subscript zero as well as writing

$$\widetilde{\Sigma} \equiv H_0 \Sigma \quad , \quad \widetilde{\Delta} \equiv H_0 \Delta. \tag{4.22}$$

Then immediately we have from (4.14):

$$x_0 = 2e^{\widetilde{\Sigma}} \cosh \frac{\widetilde{\Delta}}{2} \quad ; \quad y_0 = \frac{2e^{-\widetilde{\Sigma}}}{H} \sinh \frac{\widetilde{\Delta}}{2}. \tag{4.23}$$

We wish to perturb these by using the new scale factor. Suppose we define perturbations f_1, f_2 by writing

$$x = x_0 \Big(1 + f_1(\widetilde{\Sigma}, \widetilde{\Delta})\Big) \quad , \quad y = y_0 \Big(1 + f_2(\widetilde{\Sigma}, \widetilde{\Delta})\Big). \tag{4.24}$$

We first have

$$x = a(t) + a(t') = e^{H_0 t - \frac{\alpha}{\beta} e^{-\beta H_0 t}} + e^{H_0 t' - \frac{\alpha}{\beta} e^{-\beta H_0 t'}}, \tag{4.25}$$

which in the late time limit can be approximated by

$$x \simeq x_0 - \frac{2\alpha}{\beta}\, e^{(1-\beta)\widetilde{\Sigma}} \, \cosh \frac{(1-\beta)\widetilde{\Delta}}{2}, \tag{4.26}$$

which yields f_1:

$$f_1 = \frac{-\alpha}{\beta}\, e^{-\beta \widetilde{\Sigma}} \, \frac{\cosh \frac{(1-\beta)\widetilde{\Delta}}{2}}{\cosh \frac{\widetilde{\Delta}}{2}}. \tag{4.27}$$

Next we write

$$\begin{aligned} y = \eta(t) - \eta(t') &= \int_{t'}^{t} \frac{dt}{a(t)} \\ &= \int_{t'}^{t} \exp\left(-H_0 t + \frac{\alpha}{\beta} e^{-\beta H_0 t}\right), \end{aligned} \tag{4.28}$$

and by making the same late time approximation as for x we get

$$\begin{aligned} y &\simeq \int_{t'}^{t} e^{-H_0 t}\left(1 + \frac{\alpha}{\beta} e^{-\beta H_0 t}\right) \\ &= y_0 + \frac{2\alpha\, e^{-(1+\beta)\widetilde{\Sigma}}}{\beta(1+\beta)H_0} \sinh \frac{(1+\beta)\widetilde{\Delta}}{2}. \end{aligned} \tag{4.29}$$

This leads to

$$f_2 = \frac{\alpha\, e^{-\beta\widetilde{\Sigma}}}{\beta(1+\beta)} \frac{\sinh \frac{(1+\beta)\widetilde{\Delta}}{2}}{\sinh \frac{\widetilde{\Delta}}{2}}. \tag{4.30}$$

Note that at late times f_1, f_2 tend to zero. In that case to calculate C we write (4.13) in the form

$$C \simeq -\frac{4}{\pi k} \int_0^\infty \frac{d}{d\Delta}\left[\frac{\cos k\Delta}{x_0}(1 - f_1)\right] \frac{1 - f_2}{y_0}\, d\Delta, \tag{4.31}$$

and so write, to first order in f_1, f_2:

$$C \simeq \underbrace{\frac{-4}{\pi k} \int_0^\infty \frac{d\Delta}{y_0} \frac{d}{d\Delta} \frac{\cos k\Delta}{x_0}}_{=\coth \pi k/H_0} \;+\; \underbrace{\frac{4}{\pi k} \int_0^\infty \frac{d\Delta}{y_0}\left[f_2 \frac{d}{d\Delta}\frac{\cos k\Delta}{x_0} + \frac{d}{d\Delta}\frac{f_1 \cos k\Delta}{x_0}\right]}_{\equiv \Delta C, \text{ the perturbation}}. \tag{4.32}$$

Evaluating ΔC is lengthy but straightforward so we merely write the answer in integral form:

$$\begin{aligned} \Delta C = \frac{H_0^2 \alpha\, e^{-\beta\widetilde{\Sigma}}}{2\pi k \beta} \int_0^\infty d\Delta \Bigg[&\frac{-2k}{H_0(1+\beta)} \frac{\sinh \frac{(1+\beta)\widetilde{\Delta}}{2} \, \sin k\Delta}{\sinh^2 \frac{\widetilde{\Delta}}{2} \cosh \frac{\widetilde{\Delta}}{2}} - \frac{1}{1+\beta} \frac{\sinh \frac{(1+\beta)\widetilde{\Delta}}{2} \, \cos k\Delta}{\cosh^2 \frac{\widetilde{\Delta}}{2} \sinh \frac{\widetilde{\Delta}}{2}} \\ &+ \frac{2k}{H_0} \frac{\cosh \frac{(1-\beta)\widetilde{\Delta}}{2} \sin k\Delta}{\cosh^2 \frac{\widetilde{\Delta}}{2} \sinh \frac{\widetilde{\Delta}}{2}} - \frac{(1-\beta)\sinh \frac{(1-\beta)\widetilde{\Delta}}{2} \cos k\Delta}{\cosh^2 \frac{\widetilde{\Delta}}{2} \sinh \frac{\widetilde{\Delta}}{2}} + \frac{2\cosh \frac{(1-\beta)\widetilde{\Delta}}{2} \cos k\Delta}{\cosh^3 \frac{\widetilde{\Delta}}{2}} \Bigg]. \end{aligned} \tag{4.33}$$

The important point is that the factor $e^{-\beta\widetilde{\Sigma}}$ ensures that this perturbation to the thermal spectrum dies off exponentially at late times.

C. Brandenberger-Kahn model

We are now in a position to derive the function $C(k, \Sigma)$ for the Brandenberger-Kahn model. In this case,

$$a(t) = e^{\frac{2H_0}{\alpha}\left(1-e^{-\alpha t/2}\right)} \tag{4.34}$$

with H_0, α constants. As t tends toward zero and infinity, $a(t)$ tends toward e^{Ht} and $e^{2H/\alpha}$ respectively. The Hubble expansion function is

$$H(t) \equiv \frac{\dot{a}}{a} = H_0 e^{-\alpha t/2} \tag{4.35}$$

and the parameter which measures the departure from exact exponential expansion $h(t)$ is

$$h \equiv \frac{\dot{H}(t)}{H(t)^2} = -\frac{\alpha}{2H_0} e^{\alpha t/2} = -\frac{\alpha}{2H_0} + O(\alpha^2 t^2). \tag{4.36}$$

We assume that $|\alpha t| \ll 1$. Eqn (4.13) is much too difficult to evaluate analytically here, but we can get some insight by calculating it as a first order correction in h to the de Sitter case.

At this point, we also mention an alternative perturbation of de Sitter space, given by the scale factor

$$a(t) = e^{\int_0^t H(t)dt} \tag{4.37}$$

which describes a solution of the vacuum Einstein equations with a time-dependent cosmological constant $\Lambda(t) = 3H^2(t)$. One may expand $H(t)$ in a power series about $t = 0$. Defining h as in (4.36), this form of perturbation turns out to be identical to the Brandenberger-Kahn model to first order in h. We have, to first order,

$$H(t) = H_0 + H_0^2 h t, \tag{4.38}$$

the correspondence between h and α being given by (4.36). We will therefore calculate the detector response for the Brandenberger-Kahn model only, keeping in mind its correspondence with the model mentioned above.

Again define f_1, f_2 (now with h included) such that

$$x = x_0 + h f_1(\Delta) \quad , \quad y = y_0 + h f_2(\Delta). \tag{4.39}$$

The corrections are then written as

$$\begin{aligned} f_1(\Delta) &= e^{\widetilde{\Sigma}} \left[\left(\widetilde{\Sigma}^2 + \frac{\widetilde{\Delta}^2}{4} \right) \cosh \frac{\widetilde{\Delta}}{2} + \widetilde{\Sigma}\widetilde{\Delta} \sinh \frac{\widetilde{\Delta}}{2} \right], \\ f_2(\Delta) &= -\frac{e^{-\widetilde{\Sigma}}}{H_0} \left[\left(\widetilde{\Sigma}^2 + \frac{\widetilde{\Delta}^2}{4} + 2\widetilde{\Sigma} + 2 \right) \sinh \frac{\widetilde{\Delta}}{2} - \left(\widetilde{\Sigma} + 1 \right) \widetilde{\Delta} \cosh \frac{\widetilde{\Delta}}{2} \right]. \end{aligned} \tag{4.40}$$

After some computation we obtain the spectrum to be

$$C(k, \Sigma) = (1 + h\Gamma_1) \coth \frac{\pi k}{H_0}, \tag{4.41}$$

a form which shows its approximately thermal nature, with

$$\Gamma_1 = -\frac{\widetilde{\Sigma}\frac{\pi k}{H_0}}{\sinh \frac{\pi k}{H_0}} + \frac{\widetilde{\Sigma}}{\cosh \frac{\pi k}{H_0}} + (\widetilde{\Sigma} + 1) \left[1 - \left(\tanh \frac{\pi k}{H_0} \right) \frac{2}{\pi} \int_0^\infty \frac{u \sin \frac{2ku}{H_0}}{\sinh^2 u} du \right]. \tag{4.42}$$

As a function of k/H_0 the integral looks much like $\tan^{-1}$, tending to $\pi/2$ as $k/H_0 \to \infty$.

In the low frequency limit the departure from a thermal spectrum is, to $O(k^2)$:

$$h\Gamma_1 \simeq h\left[\widetilde{\Sigma} + 1 - (\widetilde{\Sigma} + 2/3)\left(\frac{\pi k}{H_0}\right)^2\right] \sim h\widetilde{\Sigma}. \tag{4.43}$$

Note that we stipulated that $|h\widetilde{\Sigma}| \sim |\alpha t| \ll 1$, so that $h\Gamma_1$ remains small as time passes. In the high frequency limit the departure is given by

$$h\Gamma_1 \to -2h\widetilde{\Sigma}\, e^{-\pi k/H_0}\left(\frac{\pi k}{H_0} - 1\right) \tag{4.44}$$

which again remains small, and is especially close to zero for high frequencies.

V. DISCUSSION

This paper continues the theme of our previous papers on the stochastic approach to particle creation [21–23,29] with focus on two main points: 1) *A unified approach* to treat thermal particle creation from both spacetimes with and without event horizons based on the interpretation proposed by one of us [24,25] that the thermal radiance can be viewed as quantum noises of the field amplified by an exponential scale transformation in these systems (in specific vacuum states) [28]. In contradistinction to viewing these as global, geometric effects, this viewpoint emphasizes the kinematic effect of scaling on the vacuum. 2)*An approximation scheme* to show that near-thermal radiation is emitted from systems in spacetimes undergoing near-exponential expansion. We wish to demonstrate the relative ease in constructing perturbation theory using the stochastic approach.

The emphasis of the statistical field theory is on how quantum and thermal fluctuations of the matter fields are affected by different kinematic or dynamic conditions. For particle creation in spacetimes with event horizons, such as for an accelerated observer and black holes, this method derives the Hawking and Unruh effect [20,21] from the viewpoint of amplification of quantum noise and exploits the fluctuation- dissipation relation which measures the balance between fluctuations in the detector and dissipation in the field [22]. For spacetimes without event horizons, such as that in near-uniformly accelerated detectors or collapsing masses [23], and wide classes of cosmological models, some studied here, one can describe them with a single parameter measuring the deviation from uniformity (acceleration) or stationarity (expansion) which enters in the near-thermal spectrum of particle creation in all these systems. The fact that we can understand all thermal radiation generating processes in these two apparently distinct classes of (cosmological and black hole) spacetimes [28,29] and be able to calculate near-thermal radiance in this and earlier papers testifies to the conceptual unity and methodological capability of this approach.

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