DEGENERATE LYAPUNOV FUNCTIONALS AND A PREY-PREDATOR MODEL WITH INFINITE DELAYS

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Abstract. The effect of delays on the stability is one of the most fundamental problems in delay differential equations (DDEs). It is commonly believed that *small delays are negligible for a stable DDEs*, however, to have an affirmative answer is still difficult in general. In this paper, the asymptotic behavior of the classical Lotka-Volterra prey-predator integrodifferential equations with infinite delays

$$\dot{x}(t) = x(t)\big[r_1 - \int_0^\infty f_1(s)x(t-s)\,ds - a\int_0^\infty f_2(s)y(t-s)\,ds\big],$$
$$\dot{y}(t) = y(t)\big[-r_2 + b\int_0^\infty f_3(s)x(t-s)\,ds\big]$$

is concerned by an approach of perturbation and degenerate Lyapunov functionals. We first show that all the positive solutions of the system is bounded above eventually. Then, by treating the system as a perturbation of the corresponding ODE system and constructing suitable degenerate Lyapunov functionals, we are able to obtain some sufficient conditions on the global attractivity of the positive equilibrium. As a corollary, we show that small delays do not change the stability of the system.

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1. Introduction

As more realistic models of real systems, delay differential equations (DDEs) have played important roles in the history of theoretical population dynamics. One of the fundamental problems in DDEs is: what is the effect of delays on the stability of DDEs? It is commonly believed that *samll delays are negligible for a stable DDEs*. However, to have an affirmative answer for this belief and to get explicit estimate of the smallness of the delays are still very difficult problems in general.

It is well known that, for the classical autonomous Lotka-Volterra systems (without delays), its positive equilibrium (if any) is globally asymptotically stable if its community matrix satisfies some diagonal domination conditions (see Hofbauer and Sigmund [23] and Takeuchi [37]). A result of Goh [8] extends this result to allow a weaker hypothesis on the community matrix; that is the community matrix is VL-stable (see Hofbauer and Sigmund [23] also). Recently, the global stability problem of Lotka-Volterra systems with various types of delays has been studied extensively (see, for example, two recent books by Gopalsamy [10] and Kuang [26]). Among the systems considered, they are basically of two types. One type of models contains delay-independent terms (see, for example Kuang [26, 28], Leung [32], Leung and Zhou [33], Shukla [35] and Wörz-Busekros [38]), which dominate other intraspecific and interspecific interaction effects (with and/or without delays); while the other type of models contains no such delay-independent dominated terms and we refer this type of systems as *pure-delay-type* ones (see, for example, Gopalsamy [9, 10], Gopalsamy and He [11], He [14, 15, 19, 16, 17], He and Gopalsamy [20, 21, 22], Kolmanovskii et.al. [25], Kuang [26, 27] and Kuang and Smith [29, 30]). In both cases, negative feedback terms (delay dependent or independent) are required and the standard Lyapunov function(al)s method and/or Razumikhin-type arguments can be used to study the global stability. For more details, we refer to [10, 15, 19, 26, 27, 28, 29, 30] for general Lotka-Volterra autonomous and nonautonomous systems, to [9, 10, 22, 26] for autonomous competition systems, to [11, 12, 14, 21, 34] for periodic and almost periodic competition systems, to [1, 19, 16, 17, 32, 33, 35, 36]) for prey-predator systems and to [20] for mutualism systems. In particular, it was shown (see He [19]) that for the general n-species autonomous Lotka-Volterra pure-delay-type systems with both finite and infinite delays, if the coefficient matrix is VL-stable (see Goh [8] and Hofbauer and Sigmund [23]) and if the delays are small, then the positive equilibrium (if any) remains globally attractive.

However, as pointed in [19], the assumption on the VL-stability of the coefficient matrix may not necessary and this is indeed the case for the following ODEs (1.1).

Consider the well-known classical autonomous Lotka-Volterra prey-pedrator system

$$\begin{aligned} \dot{N}_1 &= N_1[b_1 - a_{11}N_1 - a_{12}N_2] \\ \dot{N}_2 &= N_2[-b_2 + a_{21}N_1] \end{aligned} \tag{1.1}$$

with $b_i, a_{ij} > 0\,(i,j = 1,2)$. It is known that if the system has a positive equilibrium, say $N^* = (N_1^*, N_2^*)$, then it must be globally asymptotically stable. In order to account for time lag effects in this model, Cushing [5, 6] considered the following prey-predator integrodifferential system with infinitely distributed delays

$$\begin{aligned} \dot{N}_1(t) &= N_1(t)\left[b_1 - a_{11}\int_{-\infty}^{t} k_{11}(t-s)N_1(s)ds - a_{12}\int_{-\infty}^{t} k_{12}(t-s)N_2(s)ds\right] \\ \dot{N}_2(t) &= N_2(t)\left[-b_2 + a_{21}\int_{-\infty}^{t} k_{21}(t-s)N_1(s)ds\right], \end{aligned} \tag{1.2}$$

where $k_{ij}(t) \in C^o(R_+, R_+), k_{ij} \geq 0, \int_0^\infty k_{ij}(t)dt = 1,\ b_i > 0,\ a_{ij} > 0\,(i,j = 1,2)$. As we known (see, for example, Cushing [5, 6, 7], Gopalsamy [9, 10], Huang et al. [24], Kuang [26], Leung [31, 32] and Stepan [36]) that, in general, the delays can complicated dynamics; equilibrium can be distabilized and periodic solutions may arise. For the system (1.2), it was shown (see Cushing [6]) that for *'small delays'* the positive equilibrium N^* is locally stable while for *'large delays'* the solutions tend to have divergent oscillations. However, the problems of whether or not *'small delays'* can change the global asymptotical stability of the positive equilibrium N^* of the system (1.2) and, if not, how small the delays can be are still open. Indeed, these are highly formidable problems to work on. Note that the coefficient matrix for (1.1) is *not* VL-stable. Also, compared with the perivous two types of systems, the system (1.2) has no such negative feedback (even with delay) for the predator and this renders the traditional Lyapunov function(al)s and Lyapunov-Razumikhin methods useless. To solve the above open problem for the system (1.2), we need to develop new techniques and methods.

As a special case of (1.2), the following prey-predator system with different discrete delays

$$\begin{aligned} \dot{N}_1(t) &= N_1(t)[b_1 - a_{11}N_1(t-\tau_{11}) - a_{12}N_2(t-\tau_{12})] \\ \dot{N}_2(t) &= N_2(t)[-b_2 + a_{21}N_1(t-\tau_{21})] \end{aligned} \tag{1.3}$$

was considered in [18] by the author recently. In [18], we first introduce concepts of degenerate Lyapunov functionals and then, by constructing suitable degenerate Lyapunov functionals, we were able to obtain explicit conditions on the both local and global asymptotical stability of the positive equilibrium N^* of (1.3). As a corollary, we shown that for '*small delays*' the positive equilibium of (1.3) remains globally asymptotically stable, which gives an affirmative answer to the above problem on the system (1.3). Moreover, our conditions lead to some explicit estimate of the smallness of the delays.

The purpose of this paper is to study the global attractivity of the positive equilibrium of system (1.2) and it is largely motivated by the above-mentioned work, in particular, by [18], where a constructive method on the degenerate Lyapunov functionals was indicated. The difference of the system (1.2) from the system (1.3) is that the delays are not finite. This makes the problem more difficult and hence leads to a more meticulous analysis on the construction of suitable degenerate Lyapunov functionals.

This paper is organised as follows. For convenience, we first include the concepts of degenerate Lyapunov functionals introduced in [18]. Then in section 3, we show that all the positive solutions of (1.2) are bounded above eventually and a bound estimate is indicated also. Section 4 contains our main results on the global attractivity of the positive equilibrium of the system. The results are established by an approach of perturbation and degenerate Lyapunov functionals. Our results show also that, under some conditions on the kernal functions, the positive equilibrium of (1.2) is globally attractive when delays are *small* (in the sense to be stated in section 4) and this gives an affermative answer to the above open problem for the system (1.2). Some remarks are included in the last section.

2. Degenerate Lyapunov functionals

Suppose that $0 \le r \le \infty$ is given. If $x : [\sigma - r, \sigma + A) \to R^n, A > 0$, is a given function, then for each $t \in [s, \sigma + A)$, $\theta \in [-r, 0]$, we define, as usual, $x_t(\theta) = x(t + \theta)$. It is understood here that $[\sigma - r, \sigma + A) = (-\infty, \sigma + A)$ if $r = \infty$.

It is well known that Lyapunov function(al)s method is often one of the most powerful and effective tools to establish both local (when the systems involve more than two delays, see, for example, Gopalsamy [10]) and global stability of delay differential equation

$$\dot{x}(t) = f(t, x_t), \qquad x_{t_o} = \phi(s), \tag{2.1}$$

where $f : R \times C \to R^n$ is completely continuous and $f(t, 0) = 0$. Following the notations and definition in Hale and Verduyn Lunel [13], a functional $V : R \times C \to R$ is called a Lyapunov functional for (2.1) if V satisfies

(i) $u(|\phi(0)|) \le V(t, \phi) \le v(\|\phi\|)$;

(ii) $\dot{V}(t, \phi)|_{(2.1)} \le -w(|\phi(0)|)$,

where $u(s), v(s), w(s) : R_+ \to R_+$ are continuous and nondecreasing, $u(s) > 0, v(s) > 0$ for $s > 0$ and $u(0) = v(0) = w(0) = 0$. It is known (see Theorem 5.2.1, Hale and Verduyn Lunel [13]) that the existence of such Lyapunov functional implies the uniform stability of $x = 0$. If, in addition, $u(s) \to \infty$ as $s \to \infty$, then solutions of (2.1) are uniformly bounded. On the other hand, if, in addition to (i) and (ii), $w(s) > 0$ for $s > 0$, then $x = 0$ is uniformly asymptotically stable. For more references on the Lyapunov functional method and various generalizations, we refer to Burton [2, 3, 4], Hale and Verduyn Lunel [13], Yoshizawa [39], Zhang [40] and the references cited there.

The concept of degenerate Lyapunov functionals was first introduced by Gopalsamy [9] when he considered the asymptotical stability of the trivial solution of the following nonautonomous linear delay system

$$\dot{x}_i(t) + \sum_{j=1}^{2} a_{ij}(t) x_j(t - \tau_{ij}) = 0, \qquad i = 1, 2. \tag{2.2}$$

As pointed in [9], it is usually difficult to find or construct Lyapunov function(al)s with negative definite rates of change along the solutions of the systems considered. Also, the approach of a combination of an invariance principle and Razumikhin-type arguments appears quite difficult in applications to equations such as Lotka-Volterra system. To overcome these difficult, Gopalsamy [9] used a functional $V(x_1, x_2) = (V_1 + V_2)(x_1, x_2)$ with

$$V_1(x_1, x_2) = \sum_{i=1}^{2} \left[x_i(t) + \sum_{j=1}^{2} \int_{t-\tau_{ij}}^{t} a_{ij}(s + \tau_{ij}) x_j(s) ds \right]^2 \tag{2.3}$$

and some $V_2(x_1, x_2) \ge 0$. With the help of such functional and Barbalat's lemma (see Lemma 3.1 in section 3), he was able to derive the asymptotic stability of (2.2). As an application, he obtained some sufficient conditions for the global asymptotic stability of the positive equilibrium of the competition delay system

$$\dot{N}_i(t) = N_1(t)\Big[r_i - \sum_{j=1}^{2} b_{ij}N_j(t-\tau_{ij})\Big], \qquad i = 1, 2. \tag{2.4}$$

(We refer to He [19] and He and Gopalsamy [22] for the further generalization on the system (2.4)). One can see that the functional defined by (2.3) does not satisfy the condition (i) of the Lyapunov functional defined above and this type of functionals was called a "degenerate" Lyapunov functional in Gopalsamy [9].

On the other hand, as a special case of (1.3), the global stability of the delay system

$$\begin{aligned} \dot{N}_1(t) &= N_1(t)[r_1 - aN_1(t) - bN_2(t-\tau)] \\ \dot{N}_2(t) &= N_2(t)[-r_2 + cN_1(t-\sigma)] \end{aligned} \tag{2.5}$$

was considered recently (see He [19]). By assuming (2.5) has a positive equilibrium $N^* = (N_1^*, N_2^*)$ and constructing a function V satisfying (i) and

$$\dot{V}(N_1, N_2) \le -\alpha[N_1(t) - N_1^*]^2 \tag{2.6}$$

for all large t, small delays τ and σ and some positive constant α, we were able to derive the global attractivity of N^*. Noting that there is no negative term on $[N_2(t) - N_2^*]^2$ in (2.6) and hence this functional does not fit the condition (ii) of the Lyapunov functional defined above.

Motivated by the above observation, we introduced in [18] the following definition on the degenerate Lyapunov functionals.

Definition 2.1. *A functional $V : R \times C \to R$ is called a weakly degenerate Lyapunov function for (2.1) if for all large t (say $t \ge T$ for some $T > 0$)*

(a) there exists a continuous and nondecreasing function $v(s) : R_+ \to R_+$ with $v(0) = 0$ and $v(s) > 0$ for $s > 0$ such that $0 \le V(t, \phi) \le v(\|\phi\|)$;

(b) there exists a continuous and nondecreasing function $w(s) : R_+ \to R_+$ with $w(0) = 0$ and $w(s) > 0$ for $s > 0$ such that $\dot{V}(t, \phi) \le -w(|\phi(0)|)$.

If, instead of (b),

(b1) there exists a continuous and nondecreasing function $w(s) : R_+^m \to R_+$ with $m < n$, $w(0) = 0$ and $w(s_1, \cdots, s_m) > 0$ for $(s_1, \cdots, s_m) \ne 0$ such that $\dot{V}(t, \phi) \le -w(\phi_1(0), \cdots, \phi_m(0))$,

then such functional V is called a strongly degenerate Lyapunov functional for (2.1).

Existence of degenerate Lyapunov functionals implies that $w(|\phi(0)|) \in L_1[T,\infty)$ in the first case and $w(\phi_1(0),\cdots,\phi_m(0)) \in L_1[T,\infty)$ in the second case. Then, with the help of the functionals constructed and the equations themselves, it is possible to examinate the uniform continuity of the solutions. Once this is confirmed, one can apply Barbalat's lemmas (Lemmas 1.2.2, 1.2.3, Gopalsamy [10]) to conclude the desired asymptotic behaviour. However, as we may have realized, the difficulty is how to construct suitable degenerate Lyapunov functionals. Quite often, the Lyapunov functions of the corresponding ODEs are very useful (as we will see in section 4). In this paper, to study the stability of the positive equilibrium of (1.2), we propose an effective technique to construct the required degenerate Lyapunov functionals when the system involves infinite delays. The construction will be carried on step by step so that one can see the basic idea and method to follow.

3. Preliminaries and boundedness

In the following sections we shall consider the system (1.2). For convenience, we rescale the variables in (1.2) and consider the following system

$$\begin{cases} \dot{x}(t) = x(t)\Big[r_1 - \displaystyle\int_0^\infty f_1(s)x(t-s)\,ds - a\int_0^\infty f_2(s)y(t-s)\,ds\Big] \\ \dot{y}(t) = y(t)\Big[-r_2 + b\displaystyle\int_0^\infty f_2(s)x(t-s)\,ds\Big]. \end{cases} \tag{3.1}$$

As in Cushing [5, 6], we always assume that a, b, r_1 and r_2 are positive constants, $f_i(s) \in C^0(R_+)$ are bounded functions satisfying

$$r_1 b > r_2, \qquad f_i(\cdot) \geq 0, \quad \int_0^\infty f_i(s)\,ds = 1, \quad \int_0^\infty s f_i(s)\,ds < \infty \quad (i=1,2,3). \tag{3.2}$$

It is easy to see that, under (3.2), the system (3.1) has a unique positive equilibrium (x^*, y^*) with

$$x^* = \frac{r_2}{b}, \qquad y^* = \frac{r_1 b - r_2}{ab}.$$

As usual, we define the average time delays as

$$T_{f_i} = \int_0^\infty s f_i(s)\,ds \qquad (i=1,2,3). \tag{3.3}$$

From the biological backgrand of the system (3.1), it is natural to seek solutions of (3.1) corresponding to nonnegative initial data belonging to the Banach space

$$BC = \{\phi = (\phi_1,\phi_2) \in C((-\infty,0],\mathbb{R}^2):\ \|\phi\|_\infty = \sup_{s\leq 0}[|\phi_1(s)| + |\phi_2(s)|] < \infty\}.$$

Hence we always assume the associated initial conditions satisfy

$$\begin{aligned} &(x(s), y(t)) = \phi(s) = (\phi_1(s), \phi_2(s)) \geq 0 \qquad \text{for} \qquad s \leq 0, \\ &\phi_i(0) > 0\,(i = 1, 2) \qquad \text{and} \qquad \phi \in BS. \end{aligned} \tag{3.4}$$

For more detailed biological explanation on the system (3.1), we refer to Cushing [5].

The following lemmas will be useful for our discussion. The first lemma is known as Barbalat's lemma. The proof of the second one is easy and we omit it.

Lemma 3.1 (Gopalsamy [10], Lemma 1.2.2). *Let f be a nonnegative function defined on $[0, \infty)$ such that f is integrable on $[0, \infty)$ and uniformly continuous on $[0, \infty)$. Then $f(t) \to 0$ as $t \to \infty$.*

Lemma 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and $g : [0, \infty) \to [0, \infty)$ satisfying $\int_0^\infty g(s)\, ds = 1$. If $\limsup_{t\to\infty} f(t) = \alpha$, then $\limsup_{t\to\infty} \int_0^\infty g(s) f(t - s)\, ds \leq \alpha$.*

Our first result is devoted to the boundedness of the positive solutions of the system (3.1). We show that all the positive solutions of (3.1) are bounded above eventually. In addition, an estimate of the ultimate upper bound of the solutions is established.

Theorem 3.1. *Let $(x(t), y(t))$ be a solution of (3.1) and (3.4). Then $(x(t), y(t)) > 0$ for all $t \geq 0$ and*

$$\limsup_{t\to\infty} x(t) \leq M_1 = \frac{r_1}{\int_0^\infty f_1(s) e^{-r_1 s} ds}. \tag{3.5}$$

Furthermore, for any $0 < \delta$ (small enough), there exists a $0 < \tilde{T}$ such that

$$y(t) \leq M_2 = \frac{(r_1 + \delta) - \frac{1}{b}(r_2 - \delta) \int_0^\infty f_1(s) e^{-r_1 s}\, ds}{a\big(\int_0^\infty f_2(s) e^{-p(\epsilon_o) s}\, ds - \delta\big)} \exp[p(\epsilon_o)\tilde{T}] \tag{3.6}$$

for all large t with $p(\epsilon_o) = -r_2 + b(M_1 + \epsilon_o)$ and some $\epsilon_o > 0$.

Proof. The positivity of the solution $(x(t), y(t))$ follows from (3.1) and (3.4) immediately. It then follows from the first equation of (3.1) that

$$\dot{x}(t) \leq x(t) \left[r_1 - \int_0^\infty f_1(s) x(t - s)\, ds \right]. \tag{3.7}$$

From (3.7) and Theorem 2.1 in Gopalsamy and He [12], we have (3.5).

For a given small $\epsilon_0 > 0$, it is found from (3.5) that there exists a $T_1 > 0$ such that

$$x(t) \leq M_1 + \epsilon_0 \qquad \text{for} \qquad t \geq T_1. \tag{3.8}$$

This, together with the boundedness of the initial function $\phi_1(s)$, implies that

$$x(t) \le m \qquad \text{for} \qquad t \in \mathbb{R} \qquad \text{and some} \qquad 0 < m < \infty. \tag{3.9}$$

From (3.7), $\dot{x}(t) \le r_1 x(t)$ for all $t \ge 0$, which leads to

$$x(t-s) \ge e^{-r_1 s} x(t), \qquad t \ge s \ge 0. \tag{3.10}$$

Also, using Lemma 3.2, we know from (3.5), (3.9) and (3.1) that there exists a $T_2 \ge T_1$ such that

$$\dot{y}(t) \le y(t)[-r_2 + b(M_1 + \epsilon_0)] = p(\epsilon_o) y(t) \qquad \text{for} \qquad t \ge T_2 \tag{3.11}$$

with $p(\epsilon_o) = -r_2 + b(M_1 + \epsilon_o) \ge p(0) = -r_2 + bM_1 > 0$. By (3.11),

$$y(t-s) \ge y(t) e^{-p(\epsilon_o)s} \qquad \text{for} \qquad t \ge s \ge T_2. \tag{3.12}$$

It then follows from (3.1), (3.10) and (3.12) that for $t > T_2$

$$\begin{aligned} \dot{x}(t) \;&\le x(t)\Big[r_1 - \int_0^t f_1(s) x(t-s)\, ds - a \int_0^{t-T_2} f_2(s) y(t-s)\, ds\Big] \\ &\le x(t)\Big[r_1 - \Big(\int_0^t f_1(s) e^{-r_1 s}\, ds\Big) x(t) - a\Big(\int_0^{t-T_2} f_2(s) e^{-p(\epsilon_o)s}\, ds\Big) y(t)\Big]. \end{aligned} \tag{3.13}$$

Noting

$$\lim_{t\to\infty} \int_t^\infty f_1(s) e^{-r_1 s} ds = \lim_{t\to\infty} \int_{t-T_2}^\infty f_2(s) e^{-p(\epsilon_o)}\, ds = 0,$$

for $0 < \delta < \min\{r_2, \int_0^\infty f_1(s) e^{-r_1 s}\, ds, \int_0^\infty f_2(s) e^{-p(\epsilon_o)s}\, ds\}$, there exists a $T_3 \ge T_2$ such that for $t \ge T_3$

$$\begin{aligned} \dot{x}(t) \;&\le x(t)\Big[r_1 - \Big(\int_0^\infty f_1(s) e^{-r_1 s}\, ds - \frac{\delta}{M_1 + \epsilon_o}\Big) x(t) \\ &\qquad - a\Big(\int_0^\infty f_2(s) e^{-p(\epsilon_o)s}\, ds - \delta\Big) y(t)\Big] \\ &\le x(t)\Big[(r_1 + \delta) - \Big(\int_0^\infty f_1(s) e^{-r_1 s}\, ds\Big) x(t) \\ &\qquad - a\Big(\int_0^\infty f_2(s) e^{-p(\epsilon_o)s}\, ds - \delta\Big) y(t)\Big]. \end{aligned} \tag{3.14}$$

Denote $y_o = \frac{1}{b}(r_2 - \delta)$ and

$$M = \frac{(r_1 + \delta) - \frac{1}{b}(r_2 - \delta) \int_0^\infty f_1(s) e^{-r_1 s}\, ds}{a\big(\int_0^\infty f_2(s) e^{-p(\epsilon_o)s}\, ds - \delta\big)}. \tag{3.15}$$

Obviousely, $y_0 > 0$, $M > 0$. For this M, the solution $y(t)$ is either oscillatory about M or not eventually.

Case (i). $y(t)$ does not oscillate about M eventually; that is, there exists a $T_4 > T_3$ such that either $y(t) \leq M$ or $y(t) > M$ for $t \geq T_4$. We claim in this case that $y(t)$ is bounded above by M eventually. Otherwise, $y(t) > M$ for $t \geq T_4$. It follows from (3.14) that for $t \geq T_4$

$$\begin{aligned} \dot{x}(t) \quad &\leq x(t)\Big[(r_1+\delta) - a\Big(\int_0^\infty f_2(s)e^{-p(\epsilon_o)s}\,ds - \delta\Big)M \\ &\qquad -\Big(\int_0^\infty f_1(s)e^{-r_1 s}\,ds\Big)x(t)\Big]. \end{aligned} \tag{3.16}$$

It turns out from (3.16) that for $t > T_4$

$$\limsup_{t\to\infty} x(t) \leq \frac{(r_1+\delta) - a\big(\int_0^\infty f_2(s)e^{-p(\epsilon_o)s}\,ds - \delta\big)M}{\int_0^\infty f_1(s)e^{-r_1 s}\,ds} = y_o.$$

For $\delta/(2b)$, using Lemma 3.2, we know that there exists a $T_5 \geq T_4$ such that

$$\int_0^\infty f_3(s)x(t-s)\,ds \leq y_o + \frac{\delta}{2b} \qquad \text{for} \qquad t \geq T_5.$$

This, together with (3.1), implies

$$\dot{y}(t) \leq [-r_2 + b(y_o + \frac{\delta}{2b})]y(t) = -\frac{\delta}{2}y(t) \tag{3.17}$$

for $t \geq T_5$. This leads to $y(t) \to 0$ as $t \to \infty$ which contradicts $y(t) > M$ for all large t. This shows that, if $y(t)$ does not oscillate about M eventually, $y(t)$ must be bounded above by M eventually.

Case (ii). $y(t)$ oscillates about M eventually. From $\int_t^\infty f_3(s)\,ds \to 0$ as $t \to \infty$, we know that for $\frac{\delta}{4bm}$, there exists $\tilde{T}$ such that $\int_{\tilde{T}}^\infty f_3(s)\,ds < \frac{\delta}{4bm}$. Denote $M_2 = Me^{p(\epsilon_o)\tilde{T}}$. We claim that, in this case, $y(t)$ is bounded above by M_2 eventually. Otherwise, for $t \geq T_3$, there must exist two sequences $\{s_n\}$ and $\{t_n\}$ satisfying $T_3 < s_n < t_n < s_{n+1}$ such that

$$y(s_n) = M, \qquad y(t_n) = M_2, \qquad \dot{y}(t_n) \geq 0 \qquad \text{and} \qquad y(t) \geq M \qquad \text{for} \qquad t \in [s_n, t_n].$$

It is found from (3.12) that

$$t_n - s_n \geq \frac{1}{p(\epsilon_o)} ln \frac{y(t_n)}{y(s_n)} = \tilde{T}.$$

Note that for $t \in [s_n, t_n]$, $y(t) \geq M$. Hence, as shown in case (i), there exists an n_o satisfying $s_n > \tilde{T}$ for $n \geq n_o$ such that $x(t) \leq y_o + \frac{\delta}{2b}$ for $t \in [s_n, t_n]$ and $n \geq n_o$. This, together with (3.1), implies

$$\begin{aligned} \dot{y}(t_n) &= y(t_n)\Big[-r_2 + b\int_0^{\tilde{T}} f_3(s)x(t_n - s)\,ds + b\int_{\tilde{T}}^{\infty} f_3(s)x(t_n - s)\,ds\Big] \\ &\leq y(t_n)[-r_2 + b(y_o + \frac{\delta}{2b}) + bm\frac{\delta}{4bm}] = -\frac{\delta}{4}y(t_n) < 0 \end{aligned}$$

This contradicts $\dot{y}(t_n) \geq 0$. Noting that $M < M_2$. Thus $y(t)$ is bounded above eventually by M_2. This completes the proof.

Remark 1. One can see from the proof of Theorem 3.1 that, in general, $\tilde{T}$ is depended on the solution. When the kernel function f_3 has a distribution on a finite interval, say $f_3(t)$ staisfies $f_3(s) \geq 0$ for $s \in [0, \gamma]$ and $f_3(s) = 0$ for $s > \gamma$, then, for any positive solutions of the system (3.1), one can always choose $\tilde{T} = \gamma$. Then, letting $\epsilon_0 \to 0$, $\delta \to 0$, we have

$$\limsup_{t\to\infty} y(t) \leq \tilde{M}_2 = \frac{r_1 b - r_2\int_0^{\infty} f_1(s)e^{-r_1 s}\,ds}{ab\int_0^{\infty} f_2(s)\exp[(r_2 - bM_1)s]\,ds}e^{(-r_2+bM_1)\gamma}.$$

Therefore, all the positive solutions of (3.1) are uniformly ultimately bounded above by $(M_1, \tilde{M}_2)$. However, how to find an ultimate uniform upper bound on the solutions $y(t)$ for the general kernal function $f_3(s)$ is still an interesting problem.

4. Global attractivity

In this section, we consider the global asymptotical behavior of the positive solutions of (3.1). Even though the system (3.1) is a much simplified model and its local stability and bifurcation problem has been studied extensively (see, for example, Cushing [5]), as we will see in this section, the global asymptotic behavior of the system is still a problem of formidable complexity. However, our tractable analysis does give some of the ideas on the construction of degenerate Lyapunov functionals. We will show that, under the assumption (3.2) and some conditions, the positive equilibrium of the system (3.1) attracts the positive solutions of the system.

The construction of degenerate Lyapunov functionals is motivated by the following simple observation. Let $(x(t), y(t))$ be the positive solution of (3.1) and (x^*, y^*) be the unique positive equilibrium of (3.1). Denote

$$u(t) = ln\frac{x(t)}{x^*}, \qquad v(t) = ln\frac{y(t)}{y^*}. \tag{4.1}$$

Then $w(t)=(u(t),v(t))$ satisfies

$$\begin{cases} \dot u(t) = -x^*\displaystyle\int_0^\infty f_1(s)[e^{u(t-s)}-1]\,ds - ay^*\int_0^\infty f_2(s)[e^{v(t-s)}-1]\,ds \\ \dot v(t) = bx^*\displaystyle\int_0^\infty f_3(s)[e^{u(t-s)}-1]\,ds. \end{cases} \tag{4.2}$$

Then the corresponding ODE system of (4.2) (without delays) has the form

$$\begin{cases} \dot u(t) = -x^*[e^{u(t)}-1] - ay^*[e^{v(t)}-1] \\ \dot v(t) = bx^*[e^{u(t)}-1]. \end{cases} \tag{4.3}$$

Let $P_1(u) = \int_0^u [e^s-1]\,ds$ and $P_2(v) = \int_0^v [e^s-1]\,ds$. Then $P_i(s) > 0$ for $s \neq 0$ and $P_i(0)=0\,(i=1,2)$. Along the solutions of (4.3), we have

$$\dot P_1(u) = -x^*[e^{u(t)}-1]^2 + ay^*[e^{u(t)}-1][e^{v(t)}-1], \qquad \dot P_2(v) = bx^*[e^{u(t)}-1][e^{v(t)}-1].$$

Then for $P(u,v) = ay^*P_1(u) + bx^*P_2(v)$

$$\dot P(u(t),v(t)) = -ax^*y^*[e^{u(t)}-1]^2. \tag{4.4}$$

This, together with La Salle's invariance principle, leads to $(u(t),v(t)) \to (0,0)$ as $t\to\infty$, that is (x^*,y^*) attracts all the positive solutions of (4.3). Note that, in (4.4), there is no negative term for $[e^{v(t)}-1]^2$, which makes hard to use stardand Lyapunov functionals method to study the global attractivity of the corresponding delay systems. Basically, in the following discussion, we will treat (4.2) as a perturbation (in terms of delays) of the ODE system (4.3) and develop a tractable technique on the construction of degenerate Lyapunov functionals which have the properties of $\dot P_1$, $\dot P_2$ and P in (4.4).

Due to the complexity of the following analysis, we first consider a special case of the system (3.1)

$$\begin{cases} \dot x(t) = x(t)\Big[r_1 - x(t) - a\displaystyle\int_0^\infty f_2(s)y(t-s)\,ds\Big] \\ \dot y(t) = y(t)\Big[-r_2 + b\displaystyle\int_0^\infty f_3(s)x(t-s)\,ds\Big]. \end{cases} \tag{4.5}$$

We will see that, comparing with the system (3.1), the construction of the required degenerate Lyapunov functionals for the system (4.5) is simpler, although some meticulous estimates will be needed, while in Theorem 4.2, we will pay more attention to the construction of degenerate Lyapunov functionals for the system (3.1), instead of the detailed estaimtes.

Theorem 4.1. *Assume that $t^2 f_2(t)$, $t^2 f_3(t) \in L_1[0,\infty)$. Let $(x(t), y(t))$ be a solution of (4.5) and (3.4). If*

$$abM_2[T_{f_2} + e^{\beta} T_{f_3}] < 1 \tag{4.6}$$

with $\beta = bM_1 T_{f_3}$ and M_1, M_2 defined by Theorem 3.1, then

$$\lim_{t\to\infty}(x(t), y(t)) = (x^*, y^*).$$

Proof. Let $(x(t), y(t))$ be the positive solution of (4.5) and (3.4). For a sufficiently small $\epsilon > 0$ (to be selected), we have from Theorem 3.1 that there exists a $T > 0$ such that

$$x(t) \le M_1 + \epsilon, \qquad y(t) \le M_2 + \epsilon \qquad \text{for} \qquad t \ge T \tag{4.7}$$

with M_1 and M_2 defined by Theorem 3.1. Let $w(t) = (u(t), v(t))$ be defined by (4.1). Then it satisfies

$$\begin{cases} \dot{u}(t) = -x^*[e^{u(t)} - 1] - ay^* \displaystyle\int_0^\infty f_2(s)[e^{v(t-s)} - 1]\, ds \\ \dot{v}(t) = bx^* \displaystyle\int_0^\infty f_3(s)[e^{u(t-s)} - 1]\, ds. \end{cases} \tag{4.8}$$

As a perturbation of ODE system (4.3), we rewrite (4.8) in the form

$$\begin{cases} \dot{u}(t) = -x^*[e^{u(t)} - 1] - ay^*[e^{v(t)} - 1] + ay^* \displaystyle\int_0^\infty f_2(s)[e^{v(t)} - e^{v(t-s)}]\, ds \\ \dfrac{d}{dt}\left[v(t) + bx^* \displaystyle\int_0^\infty f_3(s) \int_{t-s}^t [e^{u(p)} - 1]\, dp\, ds\right] = bx^*[e^{u(t)} - 1]. \end{cases} \tag{4.9}$$

Define

$$V_1(u) = \int_0^u [e^s - 1] ds. \tag{4.10}$$

Then, along the solution $w(t) = (u(t), v(t))$,

$$\dot{V}_1(u(t)) = [e^{u(t)} - 1]\dot{u}(t) = -x^*[e^{u(t)} - 1]^2 - ay^*[e^{u(t)} - 1][e^{v(t)} - 1] + J_1(w_t), \tag{4.11}$$

where

$$J_1(w_t) = ay^*[e^{u(t)} - 1] \int_0^\infty f_2(s)[e^{v(t)} - e^{v(t-s)}]\, ds. \tag{4.12}$$

Considering the defferentiability of the solution, we can split J_1 so that

$$J_1(w_t) = J_2(w_t) + J_3(w_t) \tag{4.13}$$

with

$$J_2(w_t) = ay^*[e^{u(t)} - 1] \int_0^t f_2(s) \int_{t-s}^t e^{v(p)} \dot{v}(p) dp ds \tag{4.14}$$

and

$$J_3(w_t) = ay^*[e^{u(t)} - 1] \int_t^\infty f_2(s)[e^{v(t)} - e^{v(t-s)}]\, ds. \tag{4.15}$$

From (4.15) and (4.1), we have

$$J_3(w_t) \le \frac{1}{x^*}|x(t) - x^*|[y(t) + |\phi_2|_\infty] \int_t^\infty f_2(s) ds, \tag{4.16}$$

where we denote $|\phi|_\infty = \sup_{s\le 0}|\phi(s)|$ for $\phi : (-\infty, 0] \to \mathbb{R}$. For J_2, using (4.8) and inequality $\alpha\beta \le \frac{1}{2}[\alpha^2 + \beta^2]$, we have

$$\begin{aligned} J_2(u_t, v_t) &= abx^*y^*[e^{u(t)} - 1] \int_0^t f_2(s) \int_{t-s}^t e^{v(p)} \int_0^\infty f_3(q)[e^{u(p-q)} - 1]\, dq\, dp\, ds \\ &\le \frac{1}{2} abx^*y^* \Big\{ \Big(\int_0^t f_2(s) \int_{t-s}^t e^{v(p)}\, dp\, ds \Big) [e^{u(t)} - 1]^2 \\ &\qquad + \int_0^\infty f_2(s) \int_{t-s}^t e^{v(p)} \int_0^\infty f_3(q)[e^{u(p-q)} - 1]^2\, dq\, dp\, ds \Big\}. \end{aligned}$$

We now define

$$\begin{aligned} V_2(w_t) = \tfrac{1}{2} abx^*y^* \Big\{ & \int_0^\infty f_2(s) \int_{t-s}^t \int_\ell^t e^{v(p)} \int_0^\infty f_3(q)[e^{u(p-q)} - 1]^2\, dq\, dp\, d\ell\, ds \\ & + T_{f_2} \int_0^\infty f_3(s) \Big(\int_{t-s}^t e^{v(p+s)} [e^{u(p)} - 1]^2\, dp \Big)\, ds \Big\}. \end{aligned} \tag{4.17}$$

By the boundedness of the solution and the assumption $t^2 f_2(t) \in L_1[0, \infty)$, one can see that $V_2 \ge 0$ is well defined. With this V_2, one can verify that

$$\begin{aligned} (J_2 + \dot{V}_2)(w_t) \le \tfrac{1}{2} abx^*y^* \Big[& \Big(\int_0^t f_2(s) \int_{t-s}^t e^{v(p)}\, dp\, ds \Big) \\ & + T_{f_2} \Big(\int_0^\infty f_3(s) e^{v(t+s)}\, ds \Big) \Big] [e^{u(t)} - 1]^2. \end{aligned} \tag{4.18}$$

It then follows from (4.11)-(4.18) that

$$\begin{aligned} \tfrac{d}{dt}[V_1 + V_2](w_t) \le\ & -x^*[e^{u(t)} - 1]^2 - ay^*[e^{u(t)} - 1][e^{v(t)} - 1] \\ & + B(t)[e^{u(t)} - 1]^2 + J_3(w_t), \end{aligned} \tag{4.19}$$

where

$$B(t)=\frac{1}{2}abx^*y^*\Big[\Big(\int_0^t f_2(s)\int_{t-s}^t e^{v(p)}\,dp\,ds\Big)+T_{f_2}\Big(\int_0^\infty f_3(s)e^{v(t+s)}\,ds\Big)\Big]. \tag{4.20}$$

Noting that for $t>T$

$$\begin{aligned}\int_0^t f_2(s)\int_{t-s}^t & e^{v(p)}\,dp\,ds\\ &=\frac{1}{y^*}\Big[\int_0^{t-T} f_2(s)\int_{t-s}^t y(p)\,dp\,ds+\int_{t-T}^t f_2(s)\int_{t-s}^t y(p)\,dp\,ds\Big]\\ &\le\frac{1}{y^*}\Big[(M_2+\epsilon)\int_0^\infty sf_2(s)\,ds+\big(\sup_{s\in[0,t]}y(s)\big)\int_{t-T}^t sf_2(s)\,ds\Big]\end{aligned} \tag{4.21}$$

and

$$\int_0^\infty f_3(s)e^{v(t+s)}\,ds\le\frac{1}{y^*}(M_2+\epsilon). \tag{4.22}$$

Then, using the boundedness of $y(t)$ and the assumption that $sf_2(s)\in L_1[0,\infty)$, we know that there exists $T_1\ge T$ such that $\sup_{s\in[0,t]}y(s)\int_{t-T}^t sf_2(s)\,ds<2\epsilon$ for $t>T_1$ and hence

$$B(t)\le abx^*[T_{f_2}(M_2+\epsilon)+\epsilon]\qquad\text{for}\qquad t>T_1. \tag{4.23}$$

Therefore, for $t>T_1$,

$$\begin{aligned}\frac{d}{dt}[V_1+V_2](w_t)\le&-x^*[e^{u(t)}-1]^2-ay^*[e^{u(t)}-1][e^{v(t)}-1]\\&+abx^*[T_{f_2}(M_2+\epsilon)+\epsilon][e^{u(t)}-1]^2+J_3(w_t),\end{aligned} \tag{4.24}$$

On the other hand, let

$$z(t)=v(t)+bx^*\int_0^\infty f_3(s)\int_{t-s}^t[e^{u(p)}-1]\,dp\,ds$$

and

$$W_1(z)=\int_0^z[e^s-1]\,ds.$$

Obviousely, W_1 is well defined and $W_1(z)\ge 0$. It then follows from the second equation of (4.9) that

$$\begin{aligned}\dot W_1(w_t)&=bx^*[e^{z(t)}-1][e^{u(t)}-1]\\&=bx^*[e^{u(t)}-1][e^{v(t)}-1]+bx^*[e^{u(t)}-1][e^{z(t)}-e^{v(t)}].\end{aligned} \tag{4.25}$$

Using the mean value theorem, there exists $\beta(t)$ lies between 0 and $bx^*\int_0^\infty f_3(s)\int_{t-s}^t[e^{u(p)}-1]\,dp\,ds$ such that

$$e^{z(t)}-e^{v(t)}=e^{v(t)}e^{\beta(t)}bx^*\int_0^\infty f_3(s)\int_{t-s}^t[e^{u(p)}-1]\,dp\,ds.$$

Note that

$$\begin{aligned}\beta(t)\ &\le bx^*\int_0^\infty f_3(s)\int_{t-s}^t|e^{u(p)}-1|\,dp\,ds\\ &\le b\max\{x^*T_{f_3},\int_0^\infty f_3(s)\int_{t-s}^t x(p)\,dp\,ds\}.\end{aligned}\tag{4.26}$$

Also, from the boundedness of the solution and $f_3\in L_1[0,\infty)$, there exists a $T_2\ge 2T_1$ such that for $t>T_2$,

$$\begin{aligned}&\int_0^\infty f_3(s)\int_{t-s}^t x(p)\,dp\,ds\\ &\quad=\int_0^{T_2} f_3(s)\int_{t-s}^t x(p)\,dp\,ds+\int_{T_2}^\infty f_3(s)\int_{t-s}^t x(p)\,dp\,ds\\ &\quad\le (M_1+\epsilon)T_{f_3}+\big(sup_{s\le t}x(s)\big)\int_{T_2}^\infty sf_3(s)\,ds\le (M_1+\epsilon)T_{f_3}+\epsilon.\end{aligned}\tag{4.27}$$

Hence for $t>T_2$

$$\beta(t)\le b[(M_1+\epsilon)T_{f_3}+\epsilon]=\beta+b(1+T_{f_3})\epsilon$$

with $\beta=bM_1T_{f_3}$. Thus for $t>T_2$

$$\begin{aligned}\dot W_1(w_t)\ &=bx^*[e^{u(t)}-1][e^{v(t)}-1]\\ &\quad+(bx^*)^2[e^{u(t)}-1]e^{v(t)}e^{\beta(t)}\int_0^\infty f_3(s)\int_{t-s}^t[e^{u(p)}-1]\,dp\,ds\\ &\le bx^*[e^{u(t)}-1][e^{v(t)}-1]\\ &\quad+\frac12(bx^*)^2e^{v(t)}e^{\beta(t)}\int_0^\infty f_3(s)\int_{t-s}^t\Big([e^{u(t)}-1]^2+[e^{u(p)}-1]^2\Big)\,dp\,ds\\ &\le bx^*[e^{u(t)}-1][e^{v(t)}-1]+\frac12(bx^*)^2\frac{M_2+\epsilon}{y^*}e^{\beta+b(1+T_{f_3})\epsilon}\\ &\qquad\times\Big(T_{f_3}[e^{u(t)}-1]^2+\int_0^\infty f_3(s)\int_{t-s}^t[e^{u(p)}-1]^2\,dp\,ds\Big).\end{aligned}$$

Define

$$W(w_t)=(W_1+W_2)(w_t)\tag{4.28}$$

with

$$W_2(w_t) = \frac{1}{2}(bx^*)^2 \frac{M_2+\epsilon}{y^*} e^{\beta+b(1+T_{f_3})\epsilon} \int_0^\infty f_3(s) \int_{t-s}^t \int_p^t [e^{u(p)}-1]^2 \, dp \, dq \, ds$$

One can check that, under the assumption $t^2 f_3(t) \in L_1[0,\infty)$, W_2 is well defined and also

$$\dot{W}(w_t) \quad \leq bx^*[e^{u(t)}-1][e^{v(t)}-1] + (bx^*)^2 \frac{M_2+\epsilon}{y^*} e^{\beta+b(1+T_{f_3})\epsilon} T_{f_3} [e^{u(t)}-1]^2 \tag{4.29}$$

for $t > T_2$.

Finally, we define

$$H(w_t) = bx^*(V_1+V_2)(w_t) + ay^* W(w_t). \tag{4.30}$$

It follows from (4.30), (4.24) and (4.29) that for $t > T_2$

$$\dot{H}(w_t) \leq -b[1-\alpha(\epsilon)][x(t)-x^*]^2 + bx^* J_3(t), \tag{4.31}$$

where

$$\alpha(\epsilon) = ab[(M_2+\epsilon)(T_{f_2} + e^{\beta+b(1+T_{f_3})\epsilon} T_{f_3}) + \epsilon].$$

From the assumption (4.6), $\alpha(0) < 1$ and hence there exists a sufficiently small $\epsilon > 0$ such that $\alpha(\epsilon) < 1$. We now integrate (4.31) from T_2 to $t > T_2$,

$$H(w_t) + b[1-\alpha(\epsilon)] \int_{T_2}^t [x(s)-x^*]^2 \, ds \leq H(w_{T_2}) + bx^* \int_{T_2}^t J_3(s) \, ds. \tag{4.32}$$

Note that $\int_{T_2}^t \int_s^\infty f_2(p) \, dp \, ds \leq \int_0^\infty s f_2(s) ds = T_{f_2}$, it follows from (4.16) and the boundedness of the solution and the initial function that $J_3(w_t) \in L[T_1,\infty)$. This, together with (4.32) and the nonnegativity of H, implies that $[x(t)-x^*]^2 \in L_1[T_2,\infty)$. It follows from the boundedness of the solution and (4.5) that both $\dot{x}(t)$ and $\dot{y}(t)$ are bounded and hence $x(t)$ and $y(t)$ are uniformly continuous. Since $[x(t)-x^*]^2 \in L_1[T_2,\infty)$, one can use Lemma 3.1 to conclude that $x(t) \to x^*$ as $t \to \infty$. One can also show that $[y(t)-y^*]^2 \in L_1[T_2,\infty)$, so once again we have $y(t) \to y^*$ as $t \to \infty$. The proof of $[y(t)-y^*]^2 \in L_1[T_2,\infty)$ can be found in the appendix. This completes the proof.

Remark 2. As we know that T_{f_i} $(i = 1,2,3)$ measure the sizes of delays. It is easy to see that (4.6) holds when T_{f_2} and T_{f_3} are small enough. This implies the following corollary on the global attractivity of the positive equilibrium when the delays are small enough. It is in this sense that the small delays do not change the global attractivity of the positive equilibrium of the system (4.5).

Corollary 4.1. *Assume that $t^2 f_2(t),\ t^2 f_3(t) \in L_1[0,\infty)$. Then the positive equilibrium of (4.5) is globally attractive provided T_{f_2} and T_{f_3} are sufficiently small.*

Now we consider the system (3.1). The following result can be treated as a generalisation of the previous Theorem 4.1.

Theorem 4.2. *Let $T^*_{f_1} = \int_0^\infty s^2 f_1(s)\,ds$, $T^*_{f_2} = \frac{1}{r_2}\int_0^\infty f_2(s)[e^{r_2 s}-1]\,ds$ and $T^*_{f_3} = \int_0^\infty s^2 f_3(s)\,ds$. Assume that $T^*_{f_i} < \infty\,(i=1,2,3)$. Let $(x(t),y(t))$ be a solution of (3.1) and (3.4). If*

$$\begin{aligned}(M_1 + aM_2)T_{f_1} + abM_2\big[T_{f_2} \quad &+ (1 + x^* T_{f_1})e^{\beta} T_{f_3} \\ &+ \frac{1}{2}M_1(T_{f_2} + T^*_{f_2})T_{f_1} + \frac{3}{4}x^* T^*_{f_1}\big] < 1 \end{aligned} \tag{4.33}$$

with $\beta = bM_1T_{f_3}$ and M_1, M_2 defined by Theorem 3.1, then

$$\lim_{t\to\infty}(x(t),y(t)) = (x^*,y^*).$$

Proof. Let $(x(t),y(t))$ be the positive solution of (3.1) and (3.4) and $w(t) = (u(t),v(t))$ be defined by (4.1). Then $w(t)$ satisfies (4.2). Also, for a sufficiently small $\epsilon > 0$ (to be selected), (4.7) holds.

As a perturbation of ODE system (4.3), we rewrite (4.2) in the form

$$\begin{cases} \dot u(t) = -x^*[e^{u(t)}-1] - ay^*[e^{v(t)}-1] \\ \qquad\qquad + x^*\displaystyle\int_0^\infty f_1(s)[e^{u(t)} - e^{u(t-s)}]\,ds + ay^*\int_0^\infty f_2(s)[e^{v(t)} - e^{v(t-s)}]\,ds \\ \dfrac{d}{dt}\left[v(t) + bx^*\displaystyle\int_0^\infty f_3(s)\int_{t-s}^t [e^{u(p)}-1]\,dp\,ds\right] = bx^*[e^{u(t)}-1]. \end{cases} \tag{4.34}$$

Let V_1 be defined by (4.10). Then, following from the argument in (4.11)- (4.16), we have the following claim.

Claim 1. Along the solution $w(t)$,

$$\dot V_1(u(t)) = -x^*[e^{u(t)}-1]^2 - ay^*[e^{u(t)}-1][e^{v(t)}-1] + (I_1+I_2+I_3)(w_t), \tag{4.35}$$

where

$$\begin{aligned} I_1(u_t,v_t) =& x^*[e^{u(t)}-1]\int_t^\infty f_1(s)[e^{u(t)} - e^{u(t-s)}]\,ds \\ &+ ay^*[e^{u(t)}-1]\int_t^\infty f_2(s)[e^{v(t)} - e^{v(t-s)}]\,ds, \end{aligned} \tag{4.36}$$

$$I_2(u_t, v_t) = ay^*[e^{u(t)} - 1] \int_0^t f_2(s) \int_{t-s}^t e^{v(p)} \dot{v}(p) dp ds \tag{4.37}$$

and

$$I_3(u_t, v_t) = x^*[e^{u(t)} - 1] \int_0^t f_1(s) \int_{t-s}^t e^{u(p)} \dot{u}(p) dp ds. \tag{4.38}$$

Also,

$$\begin{aligned} I_1(w_t) \leq & \frac{1}{x^*}|x(t) - x^*|[x(t) + |\phi_1|_\infty] \int_t^\infty f_1(s) ds \\ & + \frac{1}{x^*}|x(t) - x^*|[y(t) + |\phi_2|_\infty] \int_t^\infty f_2(s) ds. \end{aligned} \tag{4.39}$$

From (4.14), (4.17) and (4.18), we have shown the following claim 2.

Claim 2. Let V_2 be defined by (4.17). Then along the solution

$$(I_2 + \dot{V}_2)(w_t) \leq B(t)[e^{u(t)} - 1]^2$$

with $B(t)$ defined by (4.20).

Now we turn to I_3. Using the first equation in (4.2), we have

$$\begin{aligned} I_3(w_t) \quad &= x^*[e^{u(t)} - 1] \int_0^t f_1(s) \int_{t-s}^t e^{u(p)} \Big(- x^* \int_0^\infty f_1(q)[e^{u(p-q)} - 1] \, dq \\ & - ay^* \int_0^\infty f_2(q)[e^{v(p-q)} - 1] \, dq \Big) \, dp \, ds. \end{aligned} \tag{4.40}$$

Claim 3. Let

$$I_3(w_t) = (I_4 + I_5)(w_t)$$

with

$$I_4(w_t) = -(x^*)^2[e^{u(t)} - 1] \int_0^t f_1(s) \int_{t-s}^t e^{u(p)} \int_0^\infty f_1(q)[e^{u(p-q)} - 1] \, dq \, dp \, ds \tag{4.41}$$

and

$$I_5(w_t) = -ax^*y^*[e^{u(t)} - 1] \int_0^t f_1(s) \int_{t-s}^t e^{u(p)} \int_0^\infty f_2(q)[e^{v(p-q)} - 1] \, dq \, dp \, ds. \tag{4.42}$$

Then there exists a nonnegative functional $V_3(w_t)$ such that

$$\begin{aligned} (I_4 + \dot{V}_3)(w_t) \leq \tfrac{1}{2}(x^*)^2 \Big[& \quad \Big(\int_0^t f_1(s) \int_{t-s}^t e^{u(p)} \, dp \, ds \Big) \\ & + T_{f_1} \Big(\int_0^\infty f_1(s) e^{u(t+s)} \, ds \Big) \Big] [e^{u(t)} - 1]^2. \end{aligned} \tag{4.43}$$

In fact, it follows from (4.41) that

$$I_4(w_t) \le \frac{1}{2}(x^*)^2\Big[\Big(\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}\,dp\,ds\Big)[e^{u(t)}-1]^2 + \int_0^\infty f_1(s)\int_{t-s}^t e^{u(p)}\int_0^\infty f_1(q)[e^{u(p-q)}-1]^2\,dq\,dp\,ds\Big]. \tag{4.44}$$

Let

$$V_3(w_t) = \tfrac{1}{2}(x^*)^2\Big[\int_0^\infty f_1(s)\int_{t-s}^t \Big(\int_\ell^t e^{u(p)}\int_0^\infty f_1(q)[e^{u(p-q)}-1]^2\,dq\,dp\Big)\,d\ell\,ds + T_{f_1}\int_0^\infty f_1(s)\int_{t-s}^t e^{u(p+s)}[e^{u(p)}-1]^2\,dp\,ds\Big]. \tag{4.45}$$

Since $T_{f_1}^* < \infty$, $V_3(w_t) \ge 0$ is well defined and also, with this V_3, one can verify that claim 3 holds.

To the term I_5, we have the following estimates

Claim 4. Let

$$I_5(w_t) = (I_6 + I_7)(w_t)$$

and

$$I_6(w_t) = (I_8 + I_9)(w_t)$$

with

$$I_6(w_t) = ax^*y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}\int_0^\infty f_2(q)[e^{v(p)}-e^{v(p-q)}]\,dq\,dp\,ds, \tag{4.46}$$

$$I_7(w_t) = -ax^*y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}[e^{v(p)}-1]\,dp\,ds, \tag{4.47}$$

$$I_8(w_t) = ax^*y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}\int_p^\infty f_2(q)[e^{v(p)}-e^{v(p-q)}]\,dq\,dp\,ds$$

and

$$I_9(w_t) = ax^*y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}\int_0^p f_2(q)[e^{v(p)}-e^{v(p-q)}]\,dq\,dp\,ds.$$

Then

$$I_8(w_t) \le a|e^{u(t)}-1|\big(\sup_{0\le s\le t}[y(s)+|\phi_2|_\infty]x(s)\big)\int_0^t f_1(s)\int_{t-s}^t\int_p^\infty f_2(q)\,dq\,dp\,ds.$$

Also, there exists a nonnegative functional $V_4(w_t)$ such that

$$(I_9+\dot V_4)(w_t)\le \tfrac{1}{2}ab(x^*)^2y^*\Big\{\quad\Big(\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}\int_0^p f_2(q)\int_{p-q}^p e^{v(\ell)}d\ell\,dq\,dp\,ds\Big)$$
$$+T_{f_1}\Big(\int_0^\infty f_3(p)e^{v(t+p)}A(t+p)\,dp\Big)\Big\}[e^{u(t)}-1]^2 \quad (4.48)$$

with

$$A(t)=\int_0^\infty f_2(s)\int_t^{t+s} e^{u(p)}\,dp\,ds.$$

Indeed, from I_9 and (4.2),

$$\begin{aligned}
I_9(w_t) &= ax^*y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}\int_0^p f_2(q)\int_{p-q}^p e^{v(\ell)}\dot v(\ell)\,d\ell\,dq\,dp\,ds\\
&= ab(x^*)^2y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}\int_0^p f_2(q)\int_{p-q}^p e^{v(\ell)}\\
&\qquad\times\int_0^\infty f_3(r)[e^{u(\ell-r)}-1]dr\,d\ell\,dq\,dp\,ds\\
&\le \frac{1}{2}ab(x^*)^2y^*\Big\{\Big(\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}\int_0^p f_2(q)\int_{p-q}^p e^{v(\ell)}d\ell\,dq\,dp\,ds\Big)[e^{u(t)}-1]^2\\
&\qquad+\int_0^\infty f_1(s)\int_{t-s}^t e^{u(p)}\int_0^p f_2(q)\int_{p-q}^p e^{v(\ell)}\\
&\qquad\quad\times\int_0^\infty f_3(r)[e^{u(\ell-r)}-1]^2\,dr\,d\ell\,dq\,dp\,ds. \qquad (4.49)
\end{aligned}$$

Define

$$\begin{aligned}
V_4(w_t)=\tfrac{1}{2}ab(x^*)^2y^*\quad&\Big\{\int_0^\infty f_1(s)\int_{t-s}^t\int_\alpha^t e^{u(p)}\int_0^p f_2(q)\int_{p-q}^p e^{v(\ell)}\\
&\qquad\times\int_0^\infty f_3(r)[e^{u(\ell-r)}-1]^2\,dr\,d\ell\,dq\,dp\,d\alpha\,ds\\
&+T_{f_1}\int_0^\infty f_2(s)\int_{t-s}^t e^{u(\ell+s)}\int_\ell^t e^{v(p)}\int_0^\infty f_3(q)[e^{u(p-q)}-1]^2\,dq\,dp\,d\ell\,ds\\
&+T_{f_1}\int_0^\infty f_3(s)\int_{t-s}^t e^{v(p+s)}A(p+s)[e^{u(p)}-1]^2\,dp\,ds\Big\}. \qquad (4.50)
\end{aligned}$$

Then $V_4(w_t)\ge 0$ is well defined and one can check that, with this V_4, claim 4 follows.

Now, we turn to the term I_7. Note that there is no negative term in (4.4) to control $[e^{v(t)}-1]^2$. To construct functionals which have the properties of $\dot P_1$ and $\dot P_2$, one need to convert all the delay terms on v to the ones on u. Guiding by this, we first have

$$\begin{aligned}
I_7(w_t) \;&= -ax^*y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t [e^{u(p)}-1][e^{v(p)}-1]\,dp\,ds \\
&\quad -ax^*y^*[e^{u(t)}-1]\int_0^\infty f_1(s)\int_{t-s}^t [e^{v(p)}-1]\,dp\,ds \\
&\quad +ax^*y^*[e^{u(t)}-1]\int_t^\infty f_1(s)\int_{t-s}^t [e^{v(p)}-1]\,dp\,ds \\
&= -ax^*y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t [e^{u(p)}-1][e^{v(p)}-1]\,dp\,ds \\
&\quad -ax^*y^*[e^{u(t)}-1][e^{v(t)}-1]T_{f_1} \\
&\quad +ax^*y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t [e^{v(t)}-e^{v(p)}]\,dp\,ds \\
&\quad +ax^*y^*[e^{u(t)}-1][e^{v(t)}-1]\int_t^\infty sf_1(s)\,ds \\
&= -ax^*y^*[e^{u(t)}-1][e^{v(t)}-1]T_{f_1} \\
&\quad +ax^*y^*[e^{u(t)}-1][e^{v(t)}-1]\int_t^\infty sf_1(s)\,ds \\
&\quad +(I_{10}+I_{11})(w_t), \qquad (4.51)
\end{aligned}$$

where

$$I_{10}(w_t) = -ax^*y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t [e^{u(p)}-1][e^{v(p)}-1]\,dp\,ds \qquad (4.52)$$

and

$$I_{11}(w_t) = ax^*y^*[e^{u(t)}-1]\int_0^t f_1(s)\int_{t-s}^t [e^{v(t)}-e^{v(p)}]\,dp\,ds. \qquad (4.53)$$

Claim 5. Let I_{10} and I_{11} be defined by (4.52) and (4.53), respectively. Then

$$I_7(w_t) = -ax^*y^*T_{f_1}[e^{u(t)}-1][e^{v(t)}-1]+ax^*y^*[e^{u(t)}-1][e^{v(t)}-1]\int_t^\infty sf_1(s)\,ds+(I_{10}+I_{11})(w_t).$$

Furthermore, there exists a nonnegative $V_5(w_t)$ such that

$$\begin{aligned}
(I_{10}+\dot V_5)(w_t) \le \tfrac{1}{2}ax^*y^*\Big[&\left(\int_0^t f_1(s)\int_{t-s}^t |e^{v(p)}-1|\,dp\,ds\right) \\
&+T_{f_1}|e^{v(t)}-1|\Big][e^{u(t)}-1]^2. \qquad (4.54)
\end{aligned}$$

In fact,

$$I_{10}(w_t) \leq \frac{1}{2}ax^*y^* \int_0^t f_1(s) \int_{t-s}^t |e^{v(p)} - 1|([e^{u(t)} - 1]^2 + [e^{u(p)} - 1]^2)\, dp\, ds.$$

Let

$$V_5(w_t) = \frac{1}{2}ax^*y^* \int_0^\infty f_1(s) \int_{t-s}^t \int_q^t |e^{v(p)} - 1|[e^{u(p)} - 1]^2\, dp\, dq\, ds.$$

Then $V_5 \geq 0$ is well defined and claim 5 follows.

Claim 6. There exists $V_6(w_t)$ such that

$$(I_{11} + \dot{V}_6)(w_t) \leq \tfrac{1}{2}ab(x^*)^2y^* \Big[\quad \left(\int_0^t f_1(s) \int_{t-s}^t \int_p^t e^{v(q)}\, dq\, dp\, ds \right) + T^*_{f_1} \left(\int_0^\infty f_3(s) e^{v(t+s)}\, ds \right) \Big] [e^{u(t)} - 1]^2. \qquad (4.55)$$

To show claim 6, we have from (4.53) and (4.2) that

$$\begin{aligned}
I_{11}(w_t) &= ax^*y^*[e^{u(t)} - 1] \int_0^t f_1(s) \int_{t-s}^t \int_p^t e^{v(\ell)} \dot{v}(\ell)\, d\ell\, dp\, ds \\
&= ab(x^*)^2y^*[e^{u(t)} - 1] \int_0^t f_1(s) \int_{t-s}^t \int_p^t e^{v(\ell)} \\
&\qquad \times \int_0^\infty f_3(q)[e^{u(\ell-q)} - 1]\, dq\, d\ell\, dp\, ds \\
&\leq \frac{1}{2}ab(x^*)^2y^* \Big\{ \left(\int_0^t f_1(s) \int_{t-s}^t \int_p^t e^{v(\ell)}\, d\ell\, dp\, ds \right) [e^{u(t)} - 1]^2 \\
&\qquad + \int_0^\infty f_1(s) \int_{t-s}^t \int_{t-s}^t e^{v(\ell)} \int_0^\infty f_3(q)[e^{u(\ell-q)} - 1]^2\, dq\, d\ell\, dp\, ds \Big\} \\
&= \frac{1}{2}ab(x^*)^2y^* \Big\{ \left(\int_0^t f_1(s) \int_{t-s}^t \int_p^t e^{v(\ell)}\, d\ell\, dp\, ds \right) [e^{u(t)} - 1]^2 \\
&\qquad + \int_0^\infty s f_1(s) \int_0^\infty f_3(q) \int_{t-s}^t e^{v(\ell)}[e^{u(\ell-q)} - 1]^2\, d\ell\, dq\, ds \Big\}. \qquad (4.56)
\end{aligned}$$

Define

$$V_6(w_t) = \tfrac{1}{2}ab(x^*)^2y^* \Big[\quad \int_0^\infty s f_1(s) \int_0^\infty f_3(q) \int_{t-s}^t \int_p^t e^{v(\ell)}[e^{u(\ell-q)} - 1]^2\, d\ell\, dp\, dq\, ds + T^*_{f_1} \int_0^\infty f_3(s) \int_{t-s}^t e^{v(p+s)}[e^{v(p)} - 1]^2\, dp\, ds \Big]. \qquad (4.57)$$

Then $V_6 \geq 0$ is well defined and one can see that claim 6 follows.

We now let

$$V(w_t) = \sum_{i=1}^{6} V_i(w_t).$$

Then it follows from the above claims that $V(w_t) \geq 0$ is well defined and along the solution

$$\begin{aligned}\dot V(w_t) \ &\leq -x^*[e^{u(t)}-1]^2 - ay^*(1+x^*T_{f_1})[e^{u(t)}-1][e^{v(t)}-1] \\ &\quad +D(t)[e^{u(t)}-1]^2 + C(t), \end{aligned} \tag{4.58}$$

where

$$\begin{aligned} D(t) = \ & B(t) + \frac{1}{2}(x^*)^2\Big[\Big(\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}\,dp\,ds\Big) + T_{f_1}\Big(\int_0^\infty f_1(s)e^{u(t+s)}\,ds\Big)\Big] \\ & +\frac{1}{2}ab(x^*)^2y^*\Big[\Big(\int_0^t f_1(s)\int_{t-s}^t e^{u(p)}\int_0^p f_2(q)\int_{p-q}^p e^{v(\ell)}d\ell\,dq\,dp\,ds\Big) \\ & \qquad +T_{f_1}\Big(\int_0^\infty f_3(s)e^{v(t+s)}\int_0^\infty f_2(p)\int_{t+s}^{t+s+p} e^{u(q)}\,dq\,dp\,ds\Big)\Big] \\ & +\frac{1}{2}ax^*y^*\Big[\Big(\int_0^t f_1(s)\int_{t-s}^t |e^{v(p)}-1|\,dp\,ds\Big) + T_{f_1}|e^{v(t)}-1|\Big] \\ & +\frac{1}{2}ab(x^*)^2y^*\Big[\Big(\int_0^t f_1(s)\int_{t-s}^t\int_p^t e^{v(q)}\,dq\,dp\,ds\Big) \\ & \qquad +T^*_{f_1}\Big(\int_0^\infty f_3(s)e^{v(t+s)}\,ds\Big)\Big] \end{aligned} \tag{4.59}$$

and

$$\begin{aligned} C(t) \ &= \frac{1}{x^*}|x(t)-x^*|[x(t)+|\phi_1|_\infty]\int_t^\infty f_1(s)\,ds \\ &\quad +\frac{1}{x^*}|x(t)-x^*|[y(t)+\|\phi_2|_\infty]\int_t^\infty f_2(s)\,ds \\ &\quad +a|e^{u(t)}-1|\big(\sup_{0\leq s\leq t}[y(s)+|\phi_2|_\infty]x(s)\big)\int_0^t f_1(s)\int_{t-s}^t\int_p^\infty f_2(q)\,dq\,dp\,ds \\ &\quad +a|x(t)-x^*||y(t)-y^*|\int_t^\infty sf_1(s)\,ds. \end{aligned} \tag{4.60}$$

From the boundedness of the solution and the assumptions, one can see that $C(t) \in L_1[0,\infty)$.

Estimation of $D(t)$: First of all, from (4.23), there exists a $T_1 > T$ such that

$$B(t) \leq abx^*[T_{f_2}(M_2+\epsilon)+\epsilon] \qquad \text{for} \qquad t \geq T_1.$$

Secondly, for $t > T_1$, similar to the derivation of (4.21), we have

$$\int_0^t f_1(s) \int_{t-s}^t e^{u(p)}\, dp\, ds \le \frac{1}{x^*} M_1 T_{f_1} + e_2(\epsilon)$$

with some $e_2(\epsilon)$ satisfying $e_2(\epsilon) \to 0$ as $\epsilon \to 0$ and

$$\int_0^\infty f_1(s) e^{u(t+s)}\, ds \le \frac{1}{x^*}(M_1 + \epsilon).$$

On the other hand, from $\dot{y}(t) \ge -r_2 y(t)$, we know that $y(p) \ge y(\ell) e^{-r_2(p-\ell)}$ for $p \ge \ell > 0$, that is,

$$e^{v(\ell)} \le e^{v(p)} e^{r_2(p-\ell)} \qquad \text{for} \qquad p \ge \ell \ge 0.$$

Using this,

$$\begin{aligned}
&\int_0^t f_1(s) \int_{t-s}^t e^{u(p)} \int_0^p f_2(q) \int_{p-q}^p e^{v(\ell)} d\ell\, dq\, dp\, ds \\
&\quad \le \frac{1}{r_2} \int_0^t f_1(s) \int_{t-s}^t e^{u(p)} \int_0^p f_2(q) e^{v(p)} [e^{r_2 q} - 1]\, dq\, dp\, ds \\
&\quad \le \frac{1}{r_2} \int_0^t f_1(s) \int_{t-s}^t e^{u(p)} e^{v(p)} \int_0^\infty f_2(q) [e^{r_2 q} - 1]\, dq\, dp\, ds \\
&\quad \le T_{f_2}^* \int_0^t f_1(s) \int_{t-s}^t e^{u(p)} e^{v(p)}\, dp\, ds
\end{aligned} \tag{4.61}$$

with

$$T_{f_2}^* = \frac{1}{r_2} \int_0^\infty f_2(s) [e^{r_2 s} - 1]\, ds.$$

Similarly, for large t,

$$\int_0^t f_1(s) \int_{t-s}^t e^{u(p)} e^{v(p)}\, dp\, ds \le \frac{1}{x^* y^*} [(M_1 + \epsilon)(M_2 + \epsilon) T_{f_1} + \epsilon]$$

and hence

$$\int_0^t f_1(s) \int_{t-s}^t e^{u(p)} \int_0^p f_2(q) \int_{p-q}^p e^{v(\ell)} d\ell\, dq\, dp\, ds \le \frac{M_1 M_2}{x^* y^*} T_{f_2}^* T_{f_1} + e_3(\epsilon)$$

with $e_3(\epsilon) \to 0$ as $\epsilon \to 0$;

$$\int_0^\infty f_3(s) e^{v(t+s)} \int_0^\infty f_2(p) \int_{t+s}^{t+s+p} e^{u(q)}\, dq\, dp\, ds \le \frac{1}{x^* y^*}(M_1 + \epsilon)(M_2 + \epsilon) T_{f_2},$$

$$\int_0^t f_1(s)\int_{t-s}^t |e^{v(p)}-1|\,dp\,ds$$
$$\le \frac{1}{y^*}\max\{y^*T_{f_1},(M_2+\epsilon)T_{f_1}+\epsilon\} \le \frac{1}{y^*}[(M_2+\epsilon)T_{f_1}+\epsilon], \tag{4.62}$$

$$\begin{aligned}
&\int_0^t f_1(s)\int_{t-s}^t\int_p^t e^{v(q)}\,dq\,dp\,ds\\
&= \Big[\int_0^{t-T} f_1(s)\int_{t-s}^t\int_p^t e^{v(q)}\,dq\,dp\,ds + \int_{t-T}^t f_1(s)\int_{t-s}^t\int_p^t e^{v(q)}\,dq\,dp\,ds\Big]\\
&\le \frac{1}{y^*}\Big[(M_2+\epsilon)\int_0^{t-T} f_1(s)\int_{t-s}^t (t-p)\,dp\,ds\\
&\quad +\big(\sup_{s\in[0,t]} y(s)\big)\int_{t-T}^t f_1(s)\int_{t-s}^t (t-p)\,dp\,ds\Big]\\
&= \frac{1}{y^*}\big[(M_2+\epsilon)\int_0^{t-T}\frac{s^2}{2}f_1(s)\,ds + \big(\sup_{s\in[0,t]} y(s)\big)\int_{t-T}^t \frac{s^2}{2}f_1(s)\,ds\big]\\
&\le \frac{1}{2y^*}[(M_2+\epsilon)T^*_{f_1} + \big(\sup_{s\in[0,t]} y(s)\big)\epsilon]\\
&= \frac{1}{2y^*}M_2T^*_{f_1} + e_4(\epsilon)
\end{aligned} \tag{4.63}$$

with $e_4(\epsilon)\to 0$ as $\epsilon\to 0$;

$$\int_0^\infty f_3(s)e^{v(t+s)}\,ds \le \frac{M_2+\epsilon}{y^*}.$$

Based on the above estimations, we conclude that for all large t

$$D(t) \le D + E(\epsilon)$$

with

$$\begin{aligned}
D &= x^*\Big[abM_2T_{f_2} + M_1T_{f_1} + \frac{1}{2}abM_1M_2(T_{f_2}+T^*_{f_2})T_{f_1}\\
&\qquad +aM_2T_{f_1} + \frac{3}{4}abx^*M_2T^*_{f_1}\Big]
\end{aligned} \tag{4.64}$$

and some $E(\epsilon)$ satisfying $E(\epsilon)\to 0$ as $\epsilon\to 0$. Therefore, from (4.58),

$$\begin{aligned}
\dot V(w_t) &\le -x^*[e^{u(t)}-1]^2 - ay^*(1+x^*T_{f_1})[e^{u(t)}-1][e^{v(t)}-1]\\
&\qquad +[D+E(\epsilon)][e^{u(t)}-1]^2 + C(t),
\end{aligned} \tag{4.65}$$

Let W be defined by (4.28). Then, from the discussion on W in the proof of Theorem 4.1, $\dot{W}$ satisfies (4.29) for all large t. Define

$$L(w_t) = bx^*V(w_t) + ay^*(1 + x^*T_{f_1})W(w_t).$$

Then it turns out from (4.65) and (4.29) that for all large t

$$\dot{L}(w_t) \leq -b[\gamma - h(\epsilon)][x(t) - x^*]^2 + bx^*C(t),$$

with

$$\gamma = 1 - \frac{D}{x^*} - abM_2(1 + x^*T_{f_1})e^{\beta}T_{f_3}$$

and some $h(\epsilon) \geq 0$ satisfying $h(\epsilon) \to 0$ as $\epsilon \to 0$. Since (4.33), it is found that $\gamma > 0$. Note that $C(t) \in L_1[0, \infty)$. Then, using the argument in the appendix and the last part of the proof in Theorem 4.1, one can show that $(x(t), y(t)) \to (x^*, y^*)$ as $t \to \infty$. This completes the proof.

Similar to the Remark 2 and Corollary 4.1, we have the following result on the global attractivity, which shows that small delays do not change the stability of the system.

Corollary 4.2. *Let $T^*_{f_i}\,(i = 1, 2, 3)$ be defined by Theorem 4.2. Assume $T^*_{f_i} < \infty\,(i = 1, 2, 3)$. Then the positive equilibrium of the system (3.1) is globally attractive provided $T_{f_i}\,(i = 1, 2, 3)$ and $T^*_{f_1}$ are sufficiently small.*

5. Remarks

In our previous paper [18], it was shown that the approach of perturbation and degenerate Lyapunov functionals is very effective to the study of the global attractivity of the well-known prey-predator system with finite discrete delays. In this paper, we have developed this approach very successfully to the system with infinite delays. With the knowledge of the corresponding ODE system, one can treat the delay systems as perturbations of the ODE systems. Depending on the appearance of the delays and the properties of the Lyapunov function of the unperturbed ODE system, the perturbated system can be a neutral differential system, like what we have done in this paper. The way we construct the required degenerate Lyapunov functionals is complicated, however, we do have some tractable ideas to follow. This approach could be used to many well-studied delay systems.

For the system (3.1), by analysing its associated characteristic equation, Cushing [6] proved that the positive equilibrium of (3.1) is locally asymptotically stable for *"small delays"*. As we known such analysis on the characteristic equations can be very complicated and, after all, the analysis only lead to the existence of smallness of the delays in most of the cases. However, one can use the ideas of this paper and [18] to discuss the local stability and get some explicit estimates on the smallness of the delays, instead of the existence only. As we mentioned early, the estimate of the upper bound M_2 depends on the solutions, in general. If an uniform ultimate upper bound M_2 can be found, our results would lead to some more explicit conditions on the global attractivity in terms of delays, which is very importment in applications.

Appendix A.

The following proposition has been used in the proof of Theorem 4.1.

Proposition. Under the assumptions of Theorem 4.1, $[y(t)-y^*]^2 \in L_1[T_2,\infty)$.

Proof. Consider a function

$$I(u,v)=\int_0^{u+v}[e^s-1]\,ds.$$

Then $I(u,v)\geq 0$ for any $u,v\in\mathbb{R}$. From (4.8), along the solution $w(t)=(u(t),v(t))$,

$$\begin{aligned}\dot I(w_t) &= [e^{u(t)+v(t)}-1][\dot u(t)+\dot v(t)]\\ &= [e^{u(t)+v(t)}-1]\Big\{-x^*[e^{u(t)}-1]-ay^*\int_0^\infty f_2(s)[e^{v(t-s)}-1]\,ds\\ &\qquad +bx^*\int_0^\infty f_3(s)[e^{u(t-s)}-1]\,ds\Big\}\\ &=(I_1+I_2+I_3)(w_t),\end{aligned}\tag{A.1}$$

where

$$I_1(w_t)=-x^*[e^{u(t)+v(t)}-1][e^{u(t)}-1],$$

$$I_2(w_t)=bx^*[e^{u(t)+v(t)}-1]\int_0^\infty f_3(s)[e^{u(t-s)}-1]\,ds$$

and

$$I_3(w_t)=[e^{u(t)+v(t)}-1]\Big[-ay^*\int_0^\infty f_2(s)[e^{v(t-s)}-1]\,ds\Big].$$

Note that

$$[e^{u(t)+v(t)}-1]=e^{v(t)}[e^{u(t)}-1]+[e^{v(t)}-1].$$

Hence

$$\begin{aligned} I_1(w_t) &= -x^* e^{v(t)}[e^{u(t)}-1]^2 - x^*[e^{u(t)}-1][e^{v(t)}-1] \\ &\leq \frac{x^*}{2}\left(\frac{1}{e_1}[e^{u(t)}-1]^2 + e_1[e^{v(t)}-1]^2\right) \end{aligned} \tag{A.2}$$

for some constant $e_1 > 0$.

For I_2, we have for all lage t

$$\begin{aligned} I_2(w_t) &= bx^*\left(e^{v(t)}[e^{u(t)}-1] + [e^{v(t)}-1]\right)\int_0^\infty f_3(s)[e^{u(t-s)}-1]\,ds \\ &\leq \frac{bx^*(M_2+\epsilon)}{2y^*}\left([e^{u(t)}-1]^2 + \int_0^\infty f_3(s)[e^{u(t-s)}-1]^2\,ds\right) \\ &\quad + \frac{bx^*}{2}\left(e_2[e^{v(t)}-1]^2 + \frac{1}{e_2}\int_0^\infty f_3(s)[e^{u(t-s)}-1]^2\,ds\right) \end{aligned} \tag{A.3}$$

for some constant $e_2 > 0$. Define

$$I_4(w_t) = \frac{bx^*}{2}[\frac{1}{e_2} + \frac{M_2+\epsilon}{y^*}]\int_0^\infty f_3(s)\int_{t-s}^t [e^{u(p)}-1]^2\,dp\,ds.$$

Then

$$(I_2 + \dot{I}_4)(w_t) \leq \frac{bx^*}{2}\left(e_2[e^{v(t)}-1]^2 + (\frac{2(M_2+\epsilon)}{y^*} + \frac{1}{e_2})[e^{u(t)}-1]^2\right). \tag{A.4}$$

For I_3, since

$$\begin{aligned} -ay^* &\int_0^\infty f_2(s)[e^{v(t-s)}-1]\,ds \\ &= -ay^*[e^{v(t)}-1] + ay^*\int_0^\infty f_2(s)[e^{v(t)} - e^{v(t-s)}]\,ds \\ &= -ay^*[e^{v(t)}-1] + ay^*\int_t^\infty f_2(s)[e^{v(t)} - e^{v(t-s)}]\,ds \\ &\qquad + ay^*\int_0^t f_2(s)\int_{t-s}^t e^{v(p)}\dot{v}(p)\,dp\,ds, \end{aligned} \tag{A.5}$$

then, using the above idea, we have

$$\begin{aligned} -ay^* &\ [e^{u(t)+v(t)}-1][e^{v(t)}-1] \\ &\leq -ay^*[e^{v(t)}-1]^2 + \frac{a(M_2+\epsilon)}{2}\big(e_3[e^{v(t)}-1]^2 + \frac{1}{e_3}[e^{u(t)}-1]^2\big) \end{aligned} \tag{A.6}$$

for some $e_3 > 0$,

$$\begin{aligned} ay^*[e^{u(t)+v(t)} - 1] \quad & \int_t^\infty f_2(s)[e^{v(t)} - e^{v(t-s)}]\, ds \\ & \le \frac{a}{x^*y^*}|x(t)y(t) - x^*y^*|[y(t) + |\phi_2|_\infty] \int_t^\infty f_2(s)\, ds = I_5(t), \quad \text{(A.7)} \end{aligned}$$

$$\begin{aligned} ay^* \quad & [e^{u(t)+v(t)} - 1] \int_0^t f_2(s) \int_{t-s}^t e^{v(p)} \dot{v}(p)\, dp\, ds \\ & = abx^*y^*\Big(e^{v(t)}[e^{u(t)} - 1] + [e^{v(t)} - 1] \Big) \\ & \qquad \times \int_0^t f_2(s) \int_{t-s}^t e^{v(p)} \int_0^\infty f_3(q)[e^{u(p-q)} - 1]\, dq\, dp\, ds \\ & \le \frac{abx^*(M_2 + \epsilon)}{2} \Big\{ \Big(\int_0^t f_2(s) \int_{t-s}^t e^{v(p)}\, dp\, ds \Big) [e^{u(t)} - 1]^2 \\ & \qquad + \int_0^\infty f_2(s) \int_{t-s}^t e^{v(p)} \int_0^\infty f_3(q)[e^{u(p-q)} - 1]^2\, dq\, dp\, ds \end{aligned}$$

$$\begin{aligned} + \frac{abx^*y^*}{2} & \Big\{ e_4 \Big(\int_0^t f_2(s) \int_{t-s}^t e^{v(p)}\, dp\, ds \Big) [e^{v(t)} - 1]^2 \\ & + \frac{1}{e_4} \int_0^\infty f_2(s) \int_{t-s}^t e^{v(p)} \int_0^\infty f_3(q)[e^{u(p-q)} - 1]^2\, dq\, dp\, ds \Big\} = I_6(w_t) \quad \text{(A.8)} \end{aligned}$$

for some constant $e_4 > 0$. From the boundedness of the solution, we know that $I_5(t) \in L_1[0, \infty)$. Similar to the discussion in (4.17) and (4.18), one can show that there exists a nonnegative functional $I_7(w_t)$ such that

$$\begin{aligned} (I_6 + \dot{I}_7)(w_t) \quad & \le \frac{abx^*(M_2 + \epsilon)}{2} \Big\{ \Big(\int_0^t f_2(s) \int_{t-s}^t e^{v(p)}\, dp\, ds \Big) \\ & \qquad + T_{f_2} \Big(\int_0^\infty f_3(s) e^{v(t+s)}\, ds \Big) \Big\} [e^{u(t)} - 1]^2 \\ & \quad + \frac{abx^*y^*}{2} \Big\{ e_4 \Big(\int_0^t f_2(s) \int_{t-s}^t e^{v(p)}\, dp\, ds \Big) [e^{v(t)} - 1]^2 \\ & \qquad + \frac{1}{e_4} T_{f_2} \Big(\int_0^\infty f_3(s) e^{v(t+s)}\, ds \Big) [e^{u(t)} - 1]^2 \Big\}. \quad \text{(A.9)} \end{aligned}$$

By (4.21) and (4.22), we have from (A.9) that

$$\begin{aligned}(I_6 + \dot{T}_7)(w_t) \ &\leq \frac{abx^*(M_2+\epsilon)}{y^*}[T_{f_2}(M_2+\epsilon)+\epsilon][e^{u(t)}-1]^2 \\ &+\frac{abx^*y^*}{2}(e_4[T_{f_2}(M_2+\epsilon)+\epsilon][e^{v(t)}-1]^2 \\ &+\frac{1}{e_4}(M_2+\epsilon)T_{f_2}[e^{u(t)}-1]^2. \end{aligned} \tag{A.10}$$

Therefore, from (A.1), (A.2), (A.4), (A.6)-(A.9) and (A.10),

$$\begin{aligned}\tfrac{d}{dt}(I_1+I_4+I_7)(w_t) \ &\leq \Big(-ay^* + \frac{1}{2}[x^*e_1 + bx^*e_2 + a(M_2+\epsilon)e_3 \\ &+abx^*[(M_2+\epsilon)T_{f_2}+\epsilon]e_4\Big)[e^{v(t)}-1]^2 \\ &+E(e_1,e_2,e_3,e_4)[e^{u(t)}-1]^2 + I_5(t) \end{aligned} \tag{A.11}$$

with $E(e_1, e_2, e_3, e_4) \geq 0$. One can select $e_i\,(i = 1, 2, 3, 4)$ small enough such that the coefficient of $[e^{v(t)}-1]^2$ in (A.11) is negative. Then, by the nonnegativity of I_1, I_4 and I_7 and the fact that $I_5(t)$, $[e^{u(t)}-1]^2 \in L_1[T_2, \infty)$, we conclude that $[e^{v(t)}-1]^2 = \frac{1}{(y^*)^2}[y(t)-y^*]^2 \in L_1[T_2, \infty)$. This completes the proof.

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