Almost Periodic Solution of a Competition System with Dominated Infinite Delays

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Abstract. In this paper, we consider an n-species almost periodic Lotka-Volterra competition system with dominated infinite delays. By constructing suitable Lyapunov functionals, we are able to show that, under a set of algebraic conditions, the system has a unique positive almost periodic solution which is globally attractive.

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1. Introduction

In this paper, we consider an almost periodic Lotka-Volterra system

$$\dot{x}_i(t) = x_i(t)\left[b_i(t) - \sum_{j=1}^{n} a_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)\,ds\right], \qquad i = 1, \cdots, n, \quad (1.1)$$

which decrisbes a model of the dynamics of an n-species competition in mathematical ecology. When system (1.1) has delay-independent dominated terms, it takes the following form

$$\dot{x}_i(t) = x_i(t)\left[b_i(t) - a_{ii}(t)x_i(t) - \sum_{j=1, j\neq i}^{n} a_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)\,ds\right], \\ i = 1, \cdots, n. \quad (1.2)$$

Recently, Gopalsamy [3] discussed the system (1.2) with ω-periodic coefficients $b_i, a_{ij}\,(i,j = 1, \cdots, n)$ and proved that, under a set of delay-independent algebraic conditions, the system (1.2) has a unique globally attractive ω-periodic solution. Murakami [10] generalized the discussion to the system (1.2) with almost periodic parameters $b_i, a_{ij}(i,j = 1, \cdots, n)$. By investigating the stability properties of the solutions of system (1.2), Murakami [10] was able to show that (1.2) has an almost periodic solution. We also refer to Hamaya [7] and Hamaya and Yoshizawa [8] for further discussion on the periodic and almost periodic system (1.2) respectively. As one can see easily, when such delay-independent dominated terms are not present, the argument used in Gopalsamy [3], Hamaya [7], Hamaya and Yoshizawa [8] and Murakami [10] can not been used for (1.1). For (1.1), when $n = 1$, the related problem has been done recently, we refer to Gopalsamy et al [5] for periodic case and Gopalsamy and He [4] and Seifert [11] for almost periodic case. We also refer to He and Gopalsamy [9] for the discussion on the periodic system (1.1) with $n = 2$. However, when $n \geq 2$, whether system (1.1) has a unique globally attractive positive almost periodic solution is still an open problem (see also Gopalsamy and He [6] and He and Gopalsamy [9]). It is the purpose of this paper to solve this problem. Motivated by our recent work (see Gopalsamy and He [4,6]), we firstly give an estimates on the uniform upper and lower bounds of positive solutions of (1.1). Then, by constructing some Lyapunov functionals, we obtain a set of algebraic conditions, under which the systems (1.1) has a unique positive almost periodic which is globally attractivie. As in the case $n = 1$ (see Gopalsamy and He [4]), the sufficient conditions are delay-dependent, which characterises the competition systems with delay-dominated terms, while the conditions for (1.2) are often delay-independent (see Gopalsamy [3], Hamaya [7], Hamaya and Yoshizawa [8] and Murakami [10]).

In what follows, we denote by R^n the n-dimensional real Euclidean space and by $|x|$ the norm of $x \in R^n$. Given $x = (x_1, \cdots, x_n) \in R^n$ and $y = (y_1, \cdots, y_n) \in R^n$, we put $x > y$ if $x_i > y_i$ and $x \geq y$ if $x_i \geq y_i$ for all $i \in I = \{1, 2, \cdots, n\}$. R^n_+ will denote the nonnegative cone of R^n. Throughout this paper, we assume that the functions b_i, a_{ij} and K_{ij} in (1.1) are real-valued functions on R and that the following conditions are satisfied:

(H_1) a_{ij} and b_i are continuous, almost periodic functions, and $\inf_{t\in R} a_{ij}(t) \geq 0$ for $i \neq j$, $\inf_{t\in R} a_{ii}(t) > 0$ and $\inf_{t\in R} b_i(t) > 0$ for $i, j \in I$;

(H_2) K_{ij} is nonnegative piecewise continuous, $\int_0^\infty K_{ij}(s)\,ds = 1$, $\int_0^\infty sK_{ij}(s)\,ds < \infty$ and $\int_0^\infty s^2 K_{ij}(s)\,ds < \infty$ for $i, j \in I$.

Consequently, define constants $b_i^l, b_i^u, a_{ij}^l, a_{ij}^u\,(i, j \in I)$ by the following:

$$b_i^l = \inf_{t\in R} b_i(t), \quad b_i^u = \sup_{t\in R} b_i(t), \quad a_{ij}^l = \inf_{t\in R} a_{ij}(t), \quad a_{ij}^u = \sup_{t\in R} a_{ij}(t); \quad i, j \in I.$$

Let BC^+ be the set of all bounded nonnegative continuous functions from $R_- = (-\infty, 0]$ into R_+^n satisfying $\phi(0) > 0$. Set $\|\phi\| = \sup_{s\in R_-} \phi(s)$ for $\phi(s) \in BC^+$. We assume the system (1.1) is supplemented with initial condition of the form

$$x(s) = \phi(s) \in BC^+ \qquad \text{for} \qquad s \in R_-. \tag{1.3}$$

2. Uniform upper and lower bounds

In this section, following the idea in Gopalsamy and He [4], we obtain *a priori* upper and lower bounds of the positive solutions of (1.1) and (1.3). One can see that, under the conditions (H_1) and (H_2), the solutions of (1.1) and (1.3) exist for all $t \in [0, r)$ $(r \leq +\infty)$ and remain positive. From the following Lemma 2.1, we know that the solutions of (1.1) and (1.3) are continuable to $t = \infty$. It follows from the positivity of solution $x(t)$ of (1.1) and (1.3) that

$$x'(t) \leq x(t)\left[b_i(t) - a_{ii}(t)\int_0^\infty K_{ii}(s)x_i(t-s)\,ds\right], \qquad i \in I, \tag{2.1}$$

from which, using the same argument as in Theorem 2.1 in Gopalsamy and He [4], we can derive the following estimate on the uniform upper bound of the solutions of (1.1) and (1.3).

Lemma 2.1. *Under assumptions* (H_1) *and* (H_2), *the solutions* $x(t) = (x_1(t), \cdots, x_n(t))$ *of (1.1) and (1.3) satisfy* $x(t) > 0$ *for all* $t \geq 0$, *and furthermore*

$$\limsup_{t\to\infty} x_i(t) \leq M_i \stackrel{def}{=} \frac{b_i^u}{a_{ii}^l \int_0^\infty K_{ii}(s)e^{-b_i^u s}\,ds} \qquad \text{for} \qquad i \in I. \tag{2.2}$$

Using Lemma 2.1 and the idea of Theorem 2.2 in Gopalsamy and He [4], we now have the following estimate on the uniform lower bound of the solutions of (1.1) and (1.3).

Lemma 2.2. *Assume the system (1.1) satisfies* (H_1), (H_2) *and*

(H_3)

$$b_i^l > \sum_{j=1, j\neq i}^{n} a_{ij}^u M_j \tag{2.3}$$

with M_i *defined by (2.2) and* $i, j \in I$;

(H_4) *there exists* $\delta > 0$ *such that*

$$\int_0^\infty K_{ii}(s) \exp\left[-\left(b_i^l - \sum_{j=1}^{n} a_{ij}^u M_j \right) s + \delta s \right] ds < \infty. \tag{2.4}$$

Then the solution $x(t)$ *of (1.1) and (1.3) satisfies*

$$\liminf_{t\to\infty} x_i(t) \geq m_i \overset{def}{=} \frac{b_i^l - \sum_{j=1, j\neq i}^{n} a_{ij}^u M_j}{a_{ii}^u \int_0^\infty K_{ii}(s) \exp[-(b_i^l - \sum_{j=1}^{n} a_{ij}^u M_j) s]\, ds} \quad \text{for} \quad i \in I. \tag{2.5}$$

Proof. The proof is similar to that of Theorem 2.2 in Gopalsamy and He [4] and we indicate it briefly.

Let $x(t) = (x_1(t), \cdots, x_n(t))$ be any solution of (1.1) and (1.3). It follows from Lemma 2.1 and the condition (H_4) that, for $\epsilon < \delta$, there exists a $t_1 > 0$ such that

$$x_i(t) \leq M_i + \epsilon \qquad \text{for} \qquad t \geq t_1 \quad \text{and} \qquad i \in I \tag{2.6}$$

with M_i defined by (2.2). Then, following the positivity of $x(t)$ and (1.1) and (2.6), one can see that, for $t \geq t_1$,

$$\begin{aligned} x_i'(t) &\geq x_i(t)\left[b_i^l - \sum_{j=1}^{n} a_{ij}^u \left(\int_0^{t-t_1} K_{ij}(s) x_j(t-s)\, ds \right.\right. \\ &\qquad\qquad \left.\left. + \int_{t-t_1}^{\infty} K_{ij}(s) x_j(t-s)\, ds \right) \right] \qquad (2.7) \\ &\geq x_i(t)\left[b_i^l - \sum_{j=1}^{n} a_{ij}^u \left([M_j + \epsilon] \int_0^{t-t_1} K_{ij}(s)\, ds \right.\right. \\ &\qquad\qquad \left.\left. + \int_{t-t_1}^{\infty} K_{ij}(s) x_j(t-s)\, ds \right) \right] \\ &= x_i(t) c_i(t) \qquad (2.8) \end{aligned}$$

with

$$c_i(t) = b_i^l - \sum_{j=1}^{n} a_{ij}^u \left([M_j + \epsilon] \int_0^{t-t_1} K_{ij}(s)\, ds + \int_{t-t_1}^{\infty} K_{ij}(s) x_j(t-s)\, ds \right),$$
$$t \geq t_1, \qquad i \in I.$$

By the boundedness of $x(t)$ and assumption (H_2),

$$\lim_{t\to\infty} c_i(t) = C_i \stackrel{def}{=} b_i^l - \sum_{j=1}^n a_{ij}^u (M_j+\epsilon), \qquad i\in I. \tag{2.9}$$

It follows from (2.8) that

$$x_i(t-s) \le x_i(t)\exp[-\int_{t-s}^t c_i(s)\,ds] \qquad \text{for} \qquad t-s>t_1,$$

which, together with (2.7), implies that

$$\begin{aligned} x_i'(t) \ge x_i(t)\Big[b_i^l - a_{ii}^u \Big(\int_0^{t-t_1} K_{ii}(s)\exp\Big[-\int_{t-s}^t c_i(r)\,dr\Big]\,ds\Big) x_i(t) \\ - \sum_{j=1,j\ne i}^n a_{ij}^u (M_j+\epsilon)\int_0^{t-t_1} K_{ij}(s)\,ds \\ - \sum_{j=1}^n a_{ij}^u \int_{t-t_1}^\infty K_{ij}(s) x_j(t-s)\,ds\Big]. \end{aligned} \tag{2.10}$$

Note that

$$\lim_{t\to\infty} \sum_{j=1,j\ne i}^n a_{ij}^u (M_j+\epsilon)\int_0^{t-t_1} K_{ij}(s)\,ds = \sum_{j=1}^n a_{ij}^u (M_j+\epsilon), \qquad i\in I,$$

$$\lim_{t\to\infty}\sum_{j=1}^n a_{ij}^u \int_{t-t_1}^\infty K_{ij}(s)x_j(t-s)\,ds = 0$$

and

$$a_{ii}^u \Big(\int_0^{t-t_1} K_{ii}(s)\exp\Big[-\int_{t-s}^t c_i(r)\,dr\Big]\,ds\Big) \le a_{ii}^u \int_0^\infty K_{ii}(s)e^{-C_i s}e^{\epsilon_1 s}\,ds$$

for $t\ge t_2\ge t_1$ and some ϵ_1 with $0<\epsilon_1<\delta$. Then we can see from (2.10) that

$$x_i'(t) \ge x_i(t)\Big[(b_i^l-\epsilon_2) - \sum_{j=1,j\ne i}^n a_{ij}^u (M_j+\epsilon) - a_{ii}^u \Big(\int_0^\infty K_{ii}(s)e^{-C_i s}e^{\epsilon_1 s}\,ds\Big) x_i(t)\Big] \tag{2.11}$$

for some $\epsilon_2>0$ sufficiently small and all large t. Then, similar to the proof of Theorem 2.2 in Gopalsamy and He [4], one can show from (2.11) that $x_i(t)$ staisfies (2.5). This completes the proof.

It is noticed that, the upper and lower bounds obtained in Lemmas 2.1 and 2.2 depend on the diagonal delay terms only. As a special case of (1.1), it can take the following form with finite discrete delays:

$$\dot{x}_i(t) = x_i(t)\Big[b_i(t) - a_{ii}(t)x_i(t-\tau_i) - \sum_{j=1,j\ne i}^n a_{ij}(t)\int_{-\infty}^t K_{ij}(t-s)x_j(s)\,ds\Big], \\ i=1,\cdots,n, \tag{2.12}$$

where $0 \leq \tau_i \, (i \in I)$ are finite constants. Consequently, from Lemmas 2.1 and 2.2, we have the following bound estimate on the solutions of (2.12), which can also be found in Gopalsamy and He [6]. This result will be used in our later discussion.

Corollary 2.3. *Under assumptions $(H_1) - (H_2)$, if*

$$b_i^l > \sum_{j=1, j\neq i}^{n} a_{ij}^u \frac{b_j^u}{a_{jj}^l} e^{b_j^u \tau_i} \qquad \text{for} \qquad i \in I, \tag{2.13}$$

then the positive solutions $x(t)$ of (2.12) satisfy

$$0 < m_i \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x_i(t) \leq M_i$$

with

$$\begin{aligned} M_i &= \frac{b_i^u}{a_{ii}^l} e^{b_i^u \tau_i} \\ m_i &= \frac{b_i^l - \sum_{j=1, j\neq i}^{n} a_{ij}^u M_j}{a_{ii}^u} \exp[(b_i^l - \sum_{j=1}^{n} a_{ij}^u M_j)\tau_i] \end{aligned} \qquad (i \in I). \tag{2.14}$$

3. Existence and attractivity of almost periodic solution

In this section, we will show that, under a set of algebraic conditions, the system (1.1) has a positive almost periodic solution which is globally attractive.

Lemma 3.1. *Suppose system (1.1) satisfies $(H_1) - (H_4)$. Then there exists a solution $x(t) = (x_1(t), \cdots, x_n(t))$ of (1.1) and (1.3) on R such that $0 < m_i - \epsilon \leq x_i(t) \leq M_i + \epsilon \, (i \in I)$ for $t \in R$ and sufficiently small $\epsilon > 0$, here M_i and $m_i \, (i \in I)$ are defined by (2.2) and (2.5) respectively.*

Lemma 3.1 can be proved by repeating almost same argument as in Lemma 2 in Murakami [10] and Lemm 4 in Seifert [11], so we omit the details.

Theorem 3.2. *Assume $(H_1) - (H_4)$ are satisfied. Suppose that*

$$(M_i a_{ii}^u)^2 \sigma_i < a_{ii}^l \int_0^\infty K_{ii}(s) e^{-b_i^u s} \, ds \qquad (i = 1, \cdots, n)$$

with $\sigma_i = \int_0^\infty s K_{ii}(s) \, ds$ and $E = (e_{ij})_{n\times n}$ is an M-matrix, where

$$e_{ij} = \begin{cases} a_{ii}^l \int_0^\infty K_{ii}(s) e^{-b_i^u s} \, ds - (M_i a_{ii}^u)^2 \sigma_i, & \text{for } i = j \\ -[1 + M_i^2 a_{ii}^u \sigma_i] a_{ij}^u, & \text{for } i \neq j. \end{cases}$$

Then system (1.1) is extreme stable (see Yoshizawa [12, 13]); that is, for any two positive solutions $x(t)$ and $y(t)$ of (1.1) and (1.3), we have

$$\lim_{t\to\infty}[x(t) - y(t)] = 0.$$

Proof. Since the matrix $E = (e_{ij})_{n\times n}$ is an M-matrix, we know that (see [2,6]) there exist $\alpha = (\alpha_1, \cdots, \alpha_n) > 0$ and $\epsilon_0 > 0$ such that

$$\alpha_i(e_{ii} - \epsilon_o) > \sum_{j=1, j\neq i}^{n} \alpha_j(|e_{ji}| + \epsilon_o), \qquad i \in I. \tag{3.1}$$

Clearly there exists an $\epsilon \in (0, \epsilon_o)$ such that

$$b_i^l - \epsilon \le b_i(t) \le b_i^u + \epsilon, \quad 0 < a_{ii}^l - \epsilon \le a_{ii}(t), \quad a_{ij}(t) \le a_{ij}^u + \epsilon$$

for all large t and

$$e_{ii} - \epsilon_o < e_{ii}(\epsilon), \qquad e_{ij} - \epsilon_o < e_{ij}(\epsilon) \tag{3.2}$$

with

$$\left.\begin{aligned} e_{ii}(\epsilon) &= (a_{ii}^l - \epsilon)\int_0^\infty K_{ii}(s)e^{-(b_i^u+\epsilon)s}\,ds - [(a_{ii}^u + \epsilon)(M_i + \epsilon)]^2\sigma_i \\ e_{ij}(\epsilon) &= -(a_{ij}^u + \epsilon)[1 + (M_i + \epsilon)^2(a_{ii}^u + \epsilon)\sigma_i] \qquad i \neq j \end{aligned}\right\}. \tag{3.3}$$

We first know from Lemma 3.1 that there exists a solution, say $y(t) = (y_1(t), \cdots, y_n(t))$ of (1.1) and (1.3) satisfying

$$0 < m_i - \epsilon \le y_i(t) \le M_i + \epsilon \qquad \text{for} \qquad t \in R \qquad \text{and} \qquad i \in I. \tag{3.4}$$

To prove the extreme stability of (1.1), it is enough to show that for any positive solution $x(t) = (x_1(t), \cdots, x_n(t))$ of (1.1) and (1.3) and the solution $y(t)$ satisfying (3.4),

$$\lim_{t\to\infty}[x_i(t) - y_i(t)] = 0, \qquad i \in I. \tag{3.5}$$

Define $u(t) = (u_1(t), \cdots, u_n(t))$, $v(t) = (v_1(t), \cdots, v_n(t))$ and $w(t) = (w_1(t), \cdots, w_n(t))$ as follows:

$$u_i(t) = \ell n[x_i(t)], \qquad v_i(t) = \ell n[y_i(t)], \qquad w_i(t) = u_i(t) - v_i(t), \qquad i \in I. \tag{3.6}$$

Then, from (1.1) and (3.6),

$$\frac{d}{dt}[u_i(t) - v_i(t)] = -\sum_{j=1}^{n} a_{ij}(t)\int_0^\infty K_{ij}(s)y_j(t-s)\left[e^{u_j(t-s)-v_j(t-s)} - 1\right]ds. \tag{3.7}$$

For $i \in I$, equation (3.7) can be written as

$$
\begin{aligned}
\frac{dw_i(t)}{dt} &= -\sum_{j=1}^{n} a_{ij}(t) \int_0^\infty K_{ij}(s) y_j(t-s) \Big[e^{w_j(t-s)} - 1 \Big] \, ds \\
&= -a_{ii}(t) \Big[\int_0^\infty K_{ii}(s) y_i(t-s) \, ds \Big] \Big[e^{w_i(t)} - 1 \Big] \\
&\quad + a_{ii}(t) \int_0^\infty K_{ii}(s) y_i(t-s) \Big(\int_{t-s}^{t} e^{w_i(s_1)} \dot{w}_i(s_1) \, ds_1 \, ds \Big) \\
&\quad - \sum_{j=1, j\neq i}^{n} a_{ij}(t) \int_0^\infty K_{ij}(s) y_j(t-s) \Big[e^{w_j(t-s)} - 1 \Big] \, ds \\
&= -a_{ii}(t) \Big[\int_0^\infty K_{ii}(s) y_i(t-s) \, ds \Big] \Big[e^{w_i(t)} - 1 \Big] \\
&\quad - \sum_{j=1, j\neq i}^{n} a_{ij}(t) \int_0^\infty K_{ij}(s) y_j(t-s) \Big[e^{w_j(t-s)} - 1 \Big] \, ds \\
&\quad - a_{ii}(t) \sum_{j=1}^{n} \int_0^\infty K_{ii}(s) y_i(t-s) \Big(\int_{t-s}^{t} e^{w_i(s_1)} a_{ij}(s_1) \\
&\quad\quad \times \Big(\int_0^\infty K_{ij}(s_2) y_j(s_1 - s_2) \Big[e^{w_j(s_1 - s_2)} - 1 \Big] \, ds_2 \Big) \, ds_1 \, ds \Big). \qquad (3.8)
\end{aligned}
$$

Let

$$
V_{i1}(w(t)) = |w_i(t)| \qquad (i \in I). \qquad (3.9)
$$

Then, it follows from (3.9) and (3.8) that the upper right derivative $\frac{D^+}{Dt} V_{i1}(w)$ of $V_{i1}(w)$ along the solutions of (3.8) is given by

$$
\begin{aligned}
\frac{D^+}{Dt} V_{i1}(w(t)) &\le -a_{ii}(t) \Big[\int_0^\infty K_{ii}(s) y_i(t-s) \, ds \Big] |e^{w_i(t)} - 1| \\
&\quad + \sum_{j=1, j\neq i}^{n} a_{ij}(t) \int_0^\infty K_{ij}(s) y_j(t-s) |e^{w_j(t-s)} - 1| \, ds \\
&\quad + a_{ii}(t) \sum_{j=1}^{n} \int_0^\infty K_{ii}(s) y_i(t-s) \Big[\int_{t-s}^{t} e^{w_i(s_1)} a_{ij}(s_1) \\
&\quad\quad \times \Big(\int_0^\infty K_{ij}(s_2) y_j(s_1 - s_2) |e^{w_j(s_1 - s_2)} - 1| \, ds_2 \Big) \, ds_1 \, ds \Big]. \qquad (3.10)
\end{aligned}
$$

Let

$$V_{i2}(w)(t) = \sum_{j=1,j\neq i}^{n} \int_0^\infty K_{ij}(s) \int_{t-s}^{t} a_{ij}(s_1+s) y_j(s_1) |e^{w_j(s_1)} - 1| \, ds_1 \, ds$$
$$+ \sum_{j=1}^{n} \int_0^\infty K_{ii}(s) \int_{t-s}^{t} a_{ii}(s_3+s) y_i(s_3) \int_{s_3}^{t} e^{w_i(s_1)} a_{ij}(s_1)$$
$$\times \int_0^\infty K_{ij}(s_2) y_j(s_1-s_2) |e^{w_j(s_1-s_2)} - 1| \, ds_2 \, ds_1 \, ds_3 \, ds. \tag{3.11}$$

Then, from (3.10) and (3.11),

$$\begin{aligned}
&\frac{D^+}{Dt}[V_{i1}+V_{i2}](w(t)) \\
&\quad \le -a_{ii}(t)\Big[\int_0^\infty K_{ii}(s) y_i(t-s)\, ds\Big] |e^{w_i(t)} - 1| \\
&\quad\quad + \sum_{j=1,j\neq i}^{n} \Big(\int_0^\infty K_{ij}(s) a_{ij}(t+s)\Big) y_j(t) |e^{w_j(t)} - 1| \, ds \\
&\quad\quad + \sum_{j=1}^{n} \Big(\int_0^\infty K_{ii}(s) \int_{t-s}^{t} a_{ii}(s_1+s) y_i(s_1)\, ds_1 \, ds\Big) \\
&\quad\quad\quad \times e^{w_i(t)} a_{ij}(t) \int_0^\infty K_{ij}(s) y_j(t-s) |e^{w_j(t-s)} - 1| \, ds.
\end{aligned} \tag{3.12}$$

Denote

$$d_i(t) = \int_0^\infty K_{ii}(s) \int_{t-s}^{t} a_{ii}(s_1+s) y_i(s_1)\, ds_1 \, ds.$$

Note that, for all large t, $e^{w_i(t)} \le M_i + \epsilon \, (i \in I)$ and also, from (3.4),

$$d_i(t) \le (M_i+\epsilon)(a_{ii}^u + \epsilon) \int_0^\infty s K_{ii}(s)\, ds = (M_i+\epsilon)(a_{ii}^u+\epsilon)\sigma_i.$$

It follows from (3.12) and (3.13) that, for all large t,

$$\begin{aligned}
&\frac{D^+}{Dt}[V_{i1}+V_{i2}](w(t)) \\
&\quad \le -a_{ii}(t)\Big[\int_0^\infty K_{ii}(s) y_i(t-s)\, ds\Big] |e^{w_i(t)} - 1| \\
&\quad\quad + \sum_{j=1,j\neq i}^{n} \Big(\int_0^\infty K_{ij}(s) a_{ij}(t+s)\Big) y_j(t) |e^{w_j(t)} - 1| \, ds \\
&\quad\quad + (a_{ii}^u+\epsilon)(M_i+\epsilon)^2 \sigma_i \\
&\quad\quad\quad \times \sum_{j=1}^{n} (a_{ij}^u + \epsilon) \int_0^\infty K_{ij}(s) y_j(t-s) |e^{w_j(t-s)} - 1| \, ds.
\end{aligned} \tag{3.13}$$

Now, let

$$V_i(w)(t) = [V_{i1} + V_{i2} + V_{i3}](w(t)), \qquad i \in I \tag{3.14}$$

with

$$\begin{aligned} V_{i3}(w(t)) = {} & (a_{ii}^u + \epsilon)(M_i + \epsilon)^2 \sigma_i \\ & \times \sum_{j=1}^{n} (a_{ij}^u + \epsilon) \int_0^\infty K_{ij}(s) \int_{t-s}^{t} y_j(s_1) |e^{w_j(s_1)} - 1| \, ds_1 \, ds. \end{aligned}$$

Then, one can derive from (3.14) and (3.13) that, for all large t,

$$\begin{aligned} \frac{D^+}{Dt} V_i(w)(t) \leq & - a_{ii}(t) \bigg[\int_0^\infty K_{ii}(s) y_i(t-s) \, ds \bigg] |e^{w_i(t)} - 1| \\ & + \sum_{j=1, j \neq i}^{n} \bigg[\int_0^\infty K_{ij}(s) a_{ij}(t+s) \, ds \bigg] y_j(t) |e^{w_j(t)} - 1| \\ & + (a_{ii}^u + \epsilon)(M_i + \epsilon)^2 \sigma_i \sum_{j=1}^{n} (a_{ij}^u + \epsilon) y_j(t) |e^{w_j(t)} - 1|. \end{aligned} \tag{3.15}$$

On the other hand, one have from (1.1) that

$$y_i'(t) \leq b_i(t) y_i(t),$$

which implies

$$y_i(t) \leq y_i(t-s) e^{\int_{t-s}^{t} b_i(s_1)\, ds_1} \qquad (i \in I). \tag{3.16}$$

From (3.16), for all large t,

$$\begin{aligned} -a_{ii}(t) \bigg[\int_0^\infty K_{ii}(s) y_i(t-s) \, ds \bigg] & \leq -a_{ii}(t) \bigg[\int_0^\infty K_{ii}(s) e^{-\int_{t-s}^{t} b_i(s_1)\, ds_1} \, ds \bigg] y_i(t) \\ & \leq -(a_{ii}^l - \epsilon) \int_0^\infty K_{ii}(s) e^{-(b_i^u + \epsilon)s} \, ds y_i(t). \end{aligned} \tag{3.17}$$

Note that $y_i(t)|e^{w_i(t)} - 1| = |x_i(t) - y_i(t)|$. Therefore, from (3.15) and (3.17), for all large t,

$$\begin{aligned} \frac{D^+}{Dt} V_i(w)(t) \leq & - (a_{ii}^l - \epsilon) \bigg[\int_0^\infty K_{ii}(s) e^{-(b_i^u + \epsilon)s} \, ds \bigg] |x_i(t) - y_i(t)| \\ & + \sum_{j=1, j \neq i}^{n} (a_{ij}^u + \epsilon) |x_j(t) - y_j(t)| \\ & + (a_{ii}^u + \epsilon)(M_i + \epsilon)^2 \sigma_i \sum_{j=1}^{n} (a_{ij}^u + \epsilon) |x_j(t) - y_j(t)| \\ = & - \sum_{j=1}^{n} e_{ij}(\epsilon) |x_j(t) - y_j(t)|, \end{aligned} \tag{3.18}$$

where $e_{ij}(\epsilon)$ is defined by (3.3). Now, let

$$V(w(t)) = \sum_{j=1}^{n} \alpha_i V_i(w(t)).$$

Then, from (3.18), (3.2) and (3.1), for all large t,

$$\begin{aligned}
\frac{D^+}{Dt} V(w(t)) &\leq -\sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} e_{ij}(\epsilon)|x_j(t) - y_j(t)| \\
&= -\sum_{i=1}^{n} \left[\sum_{j=1}^{n} \alpha_j e_{ji}(\epsilon)\right] |x_i(t) - y_i(t)| \\
&\leq -\sum_{i=1}^{n} \left[\alpha_i(e_{ii} - \epsilon_o) - \sum_{j=1, j\neq i}^{n} \alpha_j(|e_{ji}| + \epsilon_o)\right] |x_i(t) - y_i(t)| \\
&= -\sum_{i=1}^{n} \beta_i |x_i(t) - y_i(t)|, \qquad (3.19)
\end{aligned}$$

where $\beta_i = \alpha_i[e_{ii} - \epsilon_o] - \sum_{j=1,j\neq i}^{n} \alpha_j(|e_{ji}| + \epsilon_o) > 0$ (by (3.1)). Note that $V(w(t)) \geq 0$. Integrating (3.19) from t_1 (large enough) to $t \geq t_1$, we have

$$\sum_{i=1}^{n} \beta_i \int_{t_1}^{t} |x_i(t) - y_i(t)|\, ds < \infty,$$

from which, one can conclude that (see, also, Gopalsamy [3] and Gopalsamy and He [4]) (3.5) holds. This completes the proof.

Now consider the system

$$x_i'(t) = x_i(t)\left[b_i^*(t) - \sum_{j=1}^{n} a_{ij}^*(t) \int_0^{\infty} K_{ij}(s) x_j(t-s)\, ds\right], \qquad i \in I, \qquad (3.20)$$

where K_{ij} as in (1.1) and for some sequence $\{t_n\}$,

$$b_i(t+t_n) \to b_i^*(t), \qquad a_{ij}(t+t_n) \to a_{ij}(t) \qquad \text{as} \qquad n \to \infty,$$

uniformly for $t \in R$. Then, using our Theorem 3.2 and the arguments of Lemma 6 and Theorem in Seifert [11], one can conclude the following result on the existence and global attractivity of positive almost periodic solution of (1.1). For more details, we refer to Seifert [11] (see, also, Fink [1] and Yoshizawa [13, 14]) and we omit the proof.

Theorem 3.3. *Under the assumptions of Theorem 3.2, system (1.1) has a positive almost periodic solution, which is globally attractive.*

We conclude this paper with the following remark. The conditions of Theorem 3.2 depend only on the size of diagonal delays, which are measured by $\sigma_i\,(i \in I)$. For (2.12), which is a special case of (1.1), $\sigma_i = \tau_i$ and the conditions of Theorem 3.2 become (2.13), $(M_i a_{ii}^u)^2 \tau_i < a_{ii}^l e^{-b_i^u \tau_i}$ and that $E^* = (e_{ij}^*)_{n\times n}$ is an M-matrix with $e_{ii}^* = a_{ii}^l e^{-b_i^u \tau_i} - (M_i a_{ii}^u)^2 \tau_i$ and $e_{ij}^* = -[1 + M_i^2 a_{ii}^u \tau_i] a_{ij}^u$ for $i \neq j\, i,j \in I$, where $M_i\,(i \in I)$ are defined by (2.14). In particularly, when $\tau_i = 0\,(i \in I)$, system (2.12) becomes (1.2) and the corresponding conditions become

$$b_i^l > \sum_{j=1, j\neq i}^{n} a_{ij}^u \frac{b_j^u}{a_{jj}^l} \qquad i \in I \tag{3.21}$$

and that $E = (e_{ij})_{n\times n}$ with $e_{ii} = a_{ii}^l$ and $e_{ij} = -a_{ij}^u\,(i \neq j)$ is an M-matrix. Using the properties of M-matrix (see Gopalsamy and He [6]), one can verify that condition (3.21) implies that E is an M-matrix. In fact, in addition to (3.21), under the condition

$$a_{ii}^l > \sum_{j=1, j\neq i}^{n} a_{ji}^u \qquad i \in I, \tag{3.22}$$

the existence of a strictly positive almost periodic solution has been shown by Gopalsamy [3] for the periodic case in (1.2) and by Murakami [10] for the almost periodic case in (1.2). It is shown by Hamaya and Yoshozawa [8] that, for the almost periodic system (1.2), the condition (3.22) is not necessary. Therefore, when (1.1) takes the form (1.2), our conditions are reduced to the one for (1.2). It is in this sense that our result is a significant generalization of the known results.

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