GLOBAL STABILITY IN N-SPECIES COMPETITION MODELLED BY "PURE-DELAY-TYPE" SYSTEMS-II: NONAUTONOMOUS CASE

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Abstract. Sufficient conditions are obtained for the global extreme stability of the positive solutions of a Lotka-Volterra type n-species nonautonomous competition system modelled by equations of the "pure-delay-type". Estimates of delays are derived for the persistence of stable systems. The existence and global stability of positive periodic and almost periodic solutions of the periodic and almost periodic systems are discussed. The method used is based on the construction of Lyapunov-type functionals for nonautonomous systems; the results obtained include and improve some of the existing results on nonatonomous competition systems both with and without delays.

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1. Introduction.

In our previous article [HG2], we consider the following autonomous systems of Lotka-Volterra equations modelling competition with delays in both the intraspecific and interspecific interactions:

$$\frac{dx_i(t)}{dt} = x_i(t)\Big[r_i - \sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij})\Big], \qquad i = 1, 2, ..., n; \tag{1.1}$$

$$\frac{dx_i(t)}{dt} = x_i(t)\Big[r_i - \sum_{j=1}^{n} a_{ij}\int_0^{\gamma_{ij}} k_{ij}(s)x_j(t-s)\,ds\Big], \qquad i = 1, 2, ..., n; \tag{1.2}$$

$$\frac{dx_i(t)}{dt} = x_i(t)\Big[r_i - \sum_{j=1}^{n} a_{ij}\int_0^{\infty} K_{ij}(s)x_j(t-s)\,ds\Big], \qquad i = 1, 2, ..., n; \tag{1.3}$$

it is assumed that $r_i, a_{ij}, \tau_{ij}, \gamma_{ij}\,(i,j = 1, 2, ..., n)$ are nonnegative real numbers and $k_{ij} : [0, \gamma_{ij}] \to R_+ = [0, \infty)$ and $K_{ij} : R \to R_+\,(i,j = 1, 2, ..., n)$ are piecewise continuous on the respective domains. As we introduced in our previous paper [HG2], such systems are called to be "pure-delay-type" (see also Kolmanovskii et. al. [KTV]). See also Gopalsamy [G1] and Kuang [K] for a review of mathematical results on the theory of the Lotka-Volterra systems with various delays.

It is well know that the dynamics of (1.1)-(1.3) can become quite complicated leading to a delay induced bifurcation to persistent oscillations and even "chaotic" behaviour (for details see Shibata and Saito [SS]). While time delays are realistic components of ecological models, it is often desirable to incorporate temporal inhomogeneity in the models so that the constant parameters r_i, a_{ij} in the above systems become time dependent. The environments of most of the natural populations undergo temporal variation causing changes in the growth characteristics of populations. The temporal inhomogeneous nature of environments can enforce both short term and long term responses of populations. We refer to Vance [V] and Vance and Coddington [VC] for a discussion of the relationship between models described respectively by autonomous and nonautonomous ordinary differential equations. One of the methods of incorporating temporal nonuniformity of the environment in models is to assume that the parameters are either periodic with the same period or almost periodic functions of the time variable. If the various constituent components of the temporally nonuniform environment is periodic with incommensurable (non-integral multiples) periods, then one has to consider the environment to be almost periodic since there is no *apriori* reason to expect the existence of periodic solutions. There have been several investigations of two species Lotka-Volterra competition systems with periodic and almost periodic parameters (for details see Ahmad [A1, A2], Alvarez and Lazer [AL], Cushing [C1, C2], Gopalsamy [G2, G3, G5, G7], He [H], Namba [N], Rosenblatt [R] and Tineo and Alvarez [TA]); but in these investigations no time lags have been considered. Dynamics of population systems with almost periodic parameters and with continuously distributed delays have been considered by Seifert [S1] and Murakami [M]; in these works, however, the results depend crucially on negative stabilising feedback terms which act without any time delay. For the related discussion of "pure-delay-type" periodic and almost periodic systems, we refer to Gopalsamy and He [GH2] and Seifert [S2] for $n = 1$ (with infinite delay) and Gopalsamy and He [GH1] for $n = 2$ and the references cited therein. When $n > 2$, we haven't seen any result on the dynamics of the nonautonomous systems (1.1)-(1.3) of "pure-delay-type".

For systems of ordinary differential equations, LaSalle and Lefschetz [LL] have called the system "extremely stable" if for any pair of solutions $x(t), y(t)$, we have $x(t)-y(t) \to 0$ as $t \to \infty$. The concept of extreme stability has been generalized to functional differential equations by Yoshizawa [Y1, Y2]. In this paper, we study the pure-delay-typed Lotka-Volterra equations (1.1)-(1.2) with variable coefficients $a_{ij}, r_i (i, j = 1, \cdots, n)$. The purpose of this article is to derive sufficient conditions for the extreme stability of positive solutions of the above systems (1.1) and (1.2) with r_i, a_{ij} are the functions of t . Also, when the systems are periodic and almost periodic, we obtain some results on the existence of a globally attractive positive periodic and almost periodic solution, respectively. In all these cases, we give estimates on the size of the region of attraction. These conditions are general, easy to varify and improve several of the existing ones (e.g. Ahmad [A1, A2], Alveraz and Lazer [AL], Gopalsamy [G4, G5, G7] and Gopalsamy and He [GH1]). It should be noted that in the autonomous cases one can easily obtain conditions for the existence of a positive equilibrium while in periodic and almost periodic cases one has to establish the existence of a solution corresponding to dynamic equilibria and this is often a nontrivial task (for instance see Burton [B] and Yoshizawa [Y2]).

Our approach is related to the construction of Lyapunov functionals. Roughly speaking, our results show that the nonautonomous Lotka-Volterra competition systems with delays have the same asymptotical behaviour as the ones without delays provided the sizes of delays are suitable small.

This paper is orginazed as follows. In next section, we shall first modify the idea we used in He and Gopalsamy [HG2] and propose a technique for constructing suitable Lyapunov functionals for the nonautonomous system (1.1) with which one can obtain easily verifiable sufficient conditions for the global extreme stability of the positive solutions of (1.1) with nonatonomous paramaters r_i and $a_{ij}\,(i, j = 1, 2, \cdots, n)$. In particular, we establish explicit estimates for delays which guarantee the global extreme stability of the positive solutions of the systems. In section 3, we shall discuss the existence of positive periodic solution and almost periodic solution by using some known results from Burton [B], Fink [F] and Seifert [S2]. Some discussions on the known results of nondelay systems are also included, from which we shall see that the conditions obtained in this paper are the best in the sense that they are reduced to the conditions in Ahmad [A1, A2], Gopalsamy [G5, G7] and Alvarez and Lazer [AL] when $\tau_{ij} = 0$ and in Gopalsamy [G4, G5, G7] when $\tau_{ii} = 0$. As a special case when $n = 2$, our results include and improve the corresponding results in Gopalsamy and He [GH1]. In Section 4 we indicate briefly the generalization to the nonautonomous systems (1.2) with finite continuously destributed delay. Finally, a discussion is carried out in section

5.

We shall use the following notation in this paper. R^n is the space of n - dimensional vectors and for $x \in R^n$, $|x|$ is any vector norm. Given $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in R^n$, we put $x > y$ if $x_i > y_i$ and $x \geq y$ if $x_i \geq y_i$ for all $i = 1, \cdots, n$. For a given number $\tau > 0$, C (C^+) denotes the space of continuous functions mapping the interval $[-\tau, 0]$ into R^n ($R^n_+ = \{x \in R^n; x > 0\}$). For any $\phi \in C(C^+)$, $\|\phi\| = \sup_{-\tau \leq s \leq 0} |\phi(s)|$. C_H will denote the set of $\phi \in C$ for which $\|\phi\| \leq H$. For any continuous function $x(u)$ define on $-\tau \leq u \leq A$, $A > 0$ and any fixed t, $0 \leq t \leq A$, the symbol x_t will denote the function $x_t(s) = x(t+s), -\tau \leq s \leq 0$.

Let $\dot{x}(t) = D^+ x(t)/Dt$ denote the right-hand derivative of the function $x(t)$ and consider functional differential equation

$$\dot{x}(t) = F(t, x_t), \tag{1.4}$$

where $F(t, \phi) \in R^n$ is defined on $R_+ \times C$ and contiouous in (t, ϕ) . Let $V(t, \phi)$ be a functional in t, ϕ defined for $t \geq 0, \phi \in C_H$ (or $\phi \in C$). The derivative of $V(t, \phi)$ along the solutions of (1.4) will be denoted by $\dot{V}_{(1.4)}$ and is defined to be

$$\dot{V}_{(1.4)} = \limsup_{\delta \to 0^+} \frac{1}{\delta} \{V(t+\delta, x_{t+\delta}(t_o, \phi)) - V(t, x_t(t_o, \phi))\},$$

where $x_t(t_o, \phi)$ is the solution of (1.4) with $x_{t_o}(t_o, \phi) = \phi$. A Lyapunov functional for (1.4) can be defined following the notation of Hale and Verduyn Lunel [HV].

2. Extreme stability.

In this section and next section we will consider the following nonatonomous system

$$\frac{dx_i(t)}{dt} = x_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t) x_j(t - \tau_{ij})\right], \qquad i = 1, 2, ..., n. \tag{2.1}$$

Throughout this paper, we always assume that r_i, a_{ij} are nonnegative continuous functions satisfying

$$0 \leq r_i^\ell \leq r_i(t) \leq r_i^u, \quad 0 \leq a_{ij}^\ell \leq a_{ij}(t) \leq a_{ij}^u, \quad a_{ii}^\ell > 0 \quad (i, j = 1, 2, \cdots, n). \tag{2.2}$$

We also assume that the delays $0 \leq \tau_{ij} \, (i, j = 1, 2, \cdots, n)$ are constants. Since we are interested only in the positive solutions of the systems, we assume the system (2.1) is supplemented with initial conditions of the form

$$x_i(t) = \phi_i(t) \in C([-\tau, 0], R_+), \qquad \phi_i(0) > 0, \quad i = 1, 2, ..., n, \tag{2.3}$$

where $\tau = \max_{1\le i,j\le n}\{\tau_{ij}\}$ and $R_+ = (0,+\infty)$. It is easy to see that all solutions $x(t) = \big(x_1(t), x_2(t), \quad \cdots, x_n(t)\big)^T$ of (2.1) and (2.3) exist and satisfy $x(t) > 0$ for $t \in R_+$. We first note that all the solutions of (2.1) and (2.3) are bounded above, and more preciesly, we have the following estimate on the upper bound of the solutions of (2.1) and (2.3); such bound will be used in the following dicussion.

Lemma 2.1. *Let* $x(t) = \big(x_1(t), x_2(t), \cdots, x_n(t)\big)^T$ *denote any solution of (2.1) and (2.3). Suppose (2.2) holds and* $a_{ii}^\ell > 0, r_i^u > 0\,(i = 1, 2, \cdots, n)$ *. Then all the positive solutions* $x(t)$ *are uniformly ultimately bounded above and satisfy*

$$x_i(t) \le M_i \equiv \frac{r_i^u}{a_{ii}^\ell} e^{r_i^u \tau_{ii}} \qquad \text{for} \quad t \ge t_1 \qquad (i = 1, 2, ..., n) \tag{2.4}$$

for some $t_1 \ge 0$ *, which is independent of the initial conditions.*

The proof of this Lemma follows from Gopalsamy and He [GH1] and also can be found from the proof of Theorem 3.3 in section 3 of this paper.

For the convenient of our discussion, we include the following definition on the uniformly M -matrix introduced in He and Gopalsamy [HG1].

Definition. *Let* $P(t) = [p_{ij}(t)]_{n\times n}\,(t \ge t_o \in R)$ *be an* $n\times n$ *matrix with* $p_{ij}(t) \le 0\,(i \ne j, i, j = 1, \cdots, n)$ *for all* $t \ge t_o$ *.* $P(t)$ *is called an eventually uniform* M *-matrix (with respect to* t *) if there exist positive vectors* $\alpha, \beta \in R^n$ *and* $T_o \ge t_o$ *such that*

$$P^T(t)\alpha \ge \beta \qquad \text{for} \qquad t \ge T_o,$$

where P^T *denote the transpose of* P *.*

We now state our main result of this section.

Theorem 2.2. *Assume that*

$$\limsup_{t\to\infty} M_i b_{ii}(t,\tau_{ii}) c_{ii}(t,\tau_{ii}) < a_{ii}^\ell \qquad (i = 1, 2, ..., n) \tag{2.5}$$

and the matrix $D(t) = (d_{ij}(t))_{n\times n}$ *is an eventually uniform* M *-matrix, where* $M_i\,(i = 1, 2, ..., n)$ *are defined by (2.4),*

$$\begin{aligned}
b_{ii}(t,\tau_{ij}) &= \int_t^{t+\tau_{ii}} a_{ii}(s+\tau_{ij})\,ds,\\
c_{ij}(t,\tau_{jj}) &= a_{ij}(t+\tau_{jj})\exp(R_j(t,\tau_{jj})),\\
R_i(t,\sigma) &= \int_{t-\sigma}^{t} r_i(u)\,du,\\
d_{ij}(t) &= \begin{cases} a_{ii}(t) - M_i b_{ii}(t,\tau_{ii})c_{ii}(t,\tau_{ii}), & \text{for } i = j\\ -[1 + M_i b_{ii}(t,\tau_{ij})]c_{ij}(t,\tau_{jj}), & \text{for } i \ne j. \end{cases}
\end{aligned} \tag{2.6}$$

If the solutions of (2.1) and (2.3) are uniformly persistence, then the system (2.1) is extremely stable in the sense that, for any two positive solutions $x(t), y(t)$ *of (2.1) and (2.3),*

$$\lim_{t\to\infty} |x(t) - y(t)| = 0.$$

Proof. Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ and $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ be any two positive solutions of (2.1) and (2.3). Define $u(t) = (u_1(t), \cdots, u_n(t))$, $v(t) = (v_1(t), \cdots, v_n(t))$ and $w(t) = (w_1(t), \cdots w_n(t))$ by

$$u_i(t) = \ell n[x_i(t)], \quad v_i(t) = \ell n[y_i(t)], \quad w_i(t) = u_i(t) - v_i(t) \quad (i = 1, 2, \cdots, n). \tag{2.7}$$

Then it follows from (2.7) and (2.1) that

$$\frac{d}{dt}[u_i(t) - v_i(t)] = -\sum_{j=1}^{n} a_{ij}(t)\big[e^{u_j(t-\tau_{ij})} - e^{v_j(t-\tau_{ij})}\big] \qquad (i = 1, 2, ..., n). \tag{2.8}$$

Using the mean valued theorem

$$e^p - e^q = \int_0^1 \frac{d}{ds}[e^{ps}e^{q(1-s)}]\, ds = \left[\int_0^1 e^{ps}e^{(1-s)q}\, ds\right][p-q],$$

the equations (2.8) can be rewritten as follows:

$$\begin{aligned}\frac{d}{dt}&[u_i(t) - v_i(t)]\\ &= -\sum_{j=1}^{n} a_{ij}(t)\left[\int_0^1 e^{su_j(t-\tau_{ij})}e^{(1-s)v_j(t-\tau_{ij})}\, ds\right][u_j(t-\tau_{ij}) - v_j(t-\tau_{ij})]\\ &\qquad i = 1, 2, \cdots, n.\end{aligned} \tag{2.9}$$

Denote

$$A_{ij}(t) = a_{ij}(t)\left[\int_0^1 e^{su_j(t-\tau_{ij})}e^{(1-s)v_j(t-\tau_{ij})}\, ds\right] \qquad (i, j = 1, \cdots, n). \tag{2.10}$$

Then (2.7) and (2.9) lead to

$$\frac{dw_i}{dt} = -\sum_{j=1}^{n} A_{ij}(t)w_j(t-\tau_{ij}) \qquad (i = 1, 2, \cdots, n), \tag{2.11}$$

which can be rewritten as

$$\begin{aligned}\frac{d}{dt}w_i(t) &= -A_{ii}(t)w_i(t-\tau_{ii}) - \sum_{j=1,j\neq i}^{n} A_{ij}(t)w_j(t-\tau_{ij}) \\ &= -A_{ii}(t)w_i(t) - \sum_{j=1,j\neq i}^{n} A_{ij}(t)w_j(t-\tau_{ij}) \\ &\quad + A_{ii}(t)\int_{t-\tau_{ii}}^{t} \frac{d}{ds}w_i(s)\,ds \\ &= -A_{ii}(t)w_i(t) - \sum_{j=1,j\neq i}^{n} A_{ij}(t)w_j(t-\tau_{ij}) \\ &\quad - A_{ii}(t)\sum_{j=1}^{n}\int_{t-\tau_{ii}}^{t} A_{ij}(s)w_j(s-\tau_{ij})ds. \end{aligned} \tag{2.12}$$

Let

$$V_{i1}(w_i)(t) = |w_i(t)| \qquad (i=1,2,...,n); \tag{2.13}$$

then it follows from $A_{ij}(t) \geq 0$ that the upper right derivative $\frac{D^+}{Dt}V_{i1}(t)$ of $V_{i1}(t)$ along the solutions of (2.12) is given by

$$\begin{aligned}\frac{D^+}{Dt}V_{i1}(t) \leq -A_{ii}(t)|w_i(t)| &+ \sum_{j=1,j\neq i}^{n} A_{ij}(t)|w_j(t-\tau_{ij})| \\ &+ A_{ii}(t)\sum_{j=1}^{n}\int_{t-\tau_{ii}}^{t} A_{ij}(s)|w_j(s-\tau_{ij})|ds. \end{aligned} \tag{2.14}$$

Now we define

$$V_i(w)(t) = V_{i1}(w)(t) + V_{i2}(w)(t), \qquad i=1,2,...,n, \tag{2.15}$$

with $V_{i1}(w)(t)$ defined by (2.13) and

$$\begin{aligned}V_{i2}(w)(t) = &\sum_{j=1,j\neq i}^{n}\int_{t-\tau_{ij}}^{t} A_{ij}(u+\tau_{ij})|w_j(u)|\,du \\ &+ \sum_{j=1}^{n}\int_{t-\tau_{ii}}^{t} A_{ii}(v+\tau_{ii})\int_{v}^{t} A_{ij}(u)|w_j(u-\tau_{ij})|\,du\,dv \\ &+ \sum_{j=1}^{n}\int_{t-\tau_{ij}}^{t} B_i(u+\tau_{ij})A_{ij}(u+\tau_{ij})|w_j(u)|\,du, \end{aligned} \tag{2.16}$$

where

$$B_i(t) = \int_{t-\tau_{ii}}^{t} A_{ii}(u+\tau_{ii})\,du = \int_{t}^{t+\tau_{ii}} A_{ii}(u)\,du. \tag{2.17}$$

Then $V_i(w)(t) \geq 0$ and we derive from (2.14)-(2.16) that

$$\frac{D^+}{Dt}V_i(t) \leq -A_{ii}(t)|w_i(t)| + \sum_{j=1,j\neq i}^{n} A_{ij}(t+\tau_{ij})|w_j(t)| + \sum_{j=1}^{n} B_i(t+\tau_{ij})A_{ij}(t+\tau_{ij})|w_j(t)|. \tag{2.18}$$

On the other hand, for any psoitive solution $x(t)$ of (2.1), one can see from (2.1) that

$$\frac{d}{dt}x_i(t) \leq r_i(t)x_i(t)$$

which implies

$$x_i(t) \leq x_i(t-\sigma)\exp[R_i(t,\sigma)] \qquad \text{for} \qquad \sigma \in R \tag{2.19}$$

with $R_i(t,\sigma) = \int_{t-\sigma}^{t} r_i(u)\,du$. Using (2.19) and (2.10), we have

$$\begin{aligned}
A_{ij}(t+\tau_{ij}) &= a_{ij}(t+\tau_{ij})\int_0^1 [x_j(t)]^s[y_j(t)]^{1-s}\,ds \\
&\leq a_{ij}(t+\tau_{ij})\int_0^1 [x_j(t-\tau_{jj})e^{R_j(t,\tau_{jj})}]^s[y_j(t-\tau_{jj})e^{R_j(t,\tau_{jj})}]^{1-s}\,ds \\
&= a_{ij}(t+\tau_{ij})\int_0^1 e^{R_j(t,\tau_{jj})}e^{su_j(t-\tau_{jj})+(1-s)v_j(t-\tau_{jj})}\,ds \\
&= \frac{a_{ij}(t+\tau_{ij})}{a_{jj}(t)}e^{R_j(t,\tau_{jj})}A_{jj}(t).
\end{aligned} \tag{2.20}$$

Also, using the ultimate uniform boundedness from above of the solutions of (2.1) and (2.3), Lemma 2.1 and (2.17), we know that there exists a $t_1 > 0$ such that for $t \geq t_1$,

$$B_i(t+\tau_{ij}) = \int_t^{t+\tau_{ii}} A_{ii}(u+\tau_{ij})\,du \leq M_i\int_t^{t+\tau_{ii}} a_{ii}(u+\tau_{ij})\,du. \tag{2.21}$$

Then it follows from

$$b_{ii}(t,\tau_{ij}) = \int_t^{t+\tau_{ii}} a_{ii}(s+\tau_{ij})\,ds, \qquad c_{ij}(t,\tau_{jj}) = a_{ij}(t+\tau_{ij})\exp(R_j(t,\tau_{jj}), \tag{2.22}$$

the definition of $D(t)$ and (2.18) and (2.20)-(2.22) that, for $t \geq t_1$,

$$\begin{aligned}
\frac{D^+}{Dt}V_i(w)(t) &\leq -\frac{A_{ii}(t)}{a_{ii}(t)}[a_{ii}(t) - M_ib_{ii}(t,\tau_{ii})c_{ii}(t,\tau_{ii})]|w_i(t)| \\
&\quad + \sum_{j=1,j\neq i}^{n}[1+M_ib_{ii}(t,\tau_{ij})]c_{ij}(t,\tau_{jj})\frac{A_{jj}(t)}{a_{jj}(t)}|w_j(t)| \\
&= -\sum_{j=1}^{n} d_{ij}(t)\frac{A_{jj}(t)}{a_{jj}(t)}|w_j(t)| \qquad (i=1,\cdots,n).
\end{aligned} \tag{2.23}$$

Since $D(t) = (d_{ij}(t)$ is an eventually uniform M-matrix, there exist $\alpha = (\alpha_1, \cdots, \alpha_n) > 0$, $\beta = (\beta_1, \cdots, \beta_n) > 0$ and $T_o \geq t_1$ such that

$$\sum_{j=1}^{n} \alpha_j d_{ji}(t) \geq \beta_i \qquad \text{for} \qquad t \geq T_o \qquad (i = 1, 2, \cdots, n). \tag{2.24}$$

Now we can define

$$V(w)(t) = \sum_{i=1}^{n} \alpha_i V_i(w)(t). \tag{2.25}$$

Then (2.23)-(2.25) lead to

$$\begin{aligned} \frac{D^+}{Dt} V(w)(t) &\leq -\sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} d_{ij}(t) \frac{A_{jj}(t)}{a_{jj}(t)} |w_j(t)| \\ &= -\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \alpha_j d_{ji}(t) \right) \frac{A_{ii}(t)}{a_{ii}(t)} |w_i(t)| \\ &\leq -\sum_{i=1}^{n} \beta_i \frac{A_{ii}(t)}{a_{ii}(t)} |w_i(t)| \qquad \text{for} \quad t \geq T_o. \end{aligned} \tag{2.26}$$

Since the uniform persistence of the positive solutions of (2.1), there exist $T_1 \geq T_o$ and $m_i > 0$ such that $A_{ii}(t) \geq m_i$ for $t \geq T_1$ and $i = 1, \cdots, n$. Let $\gamma = \min_{1 \leq i \leq n}\{\gamma_i\}$ with $\gamma_i = \beta_i m_i / a_{ii}^u$. Then (2.26) leads to

$$\frac{D^+}{Dt} V(w)(t) \leq -\gamma \sum_{i=1}^{n} |w_i(t)| \qquad \text{for} \quad t \geq T_1,$$

which, together with the uniform boundedness, implies the extreme stability of the system (2.1). This complests the proof.

The uniform persistence of Lotka-Volterra system, both with and without delays, has been studied extensively recently. The following result come from Theorem 8.5.1 in Kuang ([K].

Poposition 2.3. *Let* $A^u = (a_{ij}^u)_{n\times n}$ *and* $r^\ell = (r_1^\ell, \cdots, r_n^\ell)^T$. *Assume that*

(i) A^u *is nonsingular and* $A^u x = r^\ell$ *has a positive solution;*
(ii) $(A^u)^{-1}$ *is an* M *-matrix.*

Then the system (2.1) is uniformly persistence.

One can see from the Position 2.3 that the delays are harmless in some sense, this is an advertagement of this type results. However, it does not indicate the explicit estimates on the persistent region and this is also an importement part to understand the asymptotical behaviour of the systems. The following Lemma gives us an explicit estimation on the lower bound of the positive solutions.

Lemma 2.4. *Suppose*

$$r_i^\ell > \sum_{j=1,j\neq i}^{n} a_{ij}^u M_j \qquad (i=1,\cdots,n) \tag{2.27}$$

with M_i *defined by (2.4). Then all the positive solutions of (2.1) are uniformly persistence and satisfy*

$$x_i(t) \geq m_i := \frac{r_i^\ell - \sum_{j=1,j\neq i}^{n} a_{ij}^u M_j}{a_{ii}^u} \exp\left(\left[r_i^\ell - \sum_{j=1,j\neq i}^{n} a_{ij}^u M_j\right]\tau_{ii}\right) \tag{2.28}$$

for $t \geq t_2$ *and* $i=1,\cdots,n$ *, where* $t_2 \geq 0$ *is some costant which is independent of the initial conditions.*

The proof of Lemma 2.4 can be seen from Gopalsamy and He [GH1] and also from the proof of Theorem 3.3 in the next section.

One can easily derive the following Corollary from Theorem 2.2, Proposition 2.3 and Lemma 2.4.

Corollary 2.5. *Suppose that*

$$M_i (a_{ii}^u)^2 e^{r_i^u \tau_{ii}} \tau_{ii} < a_{ii}^\ell \qquad (i=1,\cdots,n) \tag{2.29}$$

and $E=(e_{ij})_{n\times n}$ *is an* M *-matrix, where*

$$e_{ij} = \begin{cases} a_{ii}^\ell - M_i (a_{ii}^u)^2 e^{r_i^u \tau_{ii}} \tau_{ii}, & \text{for } i=j \\ -[1 + M_i a_{ii}^u \tau_{ii}] a_{ij}^u e^{r_j^u \tau_{jj}}, & \text{for } i\neq j. \end{cases}$$

If either the assumptions of Proposition 2.3 or (2.27) hold, then the system (2.1) is extremely stable.

Remark 2.1. Let a_{ij}, r_i be constants and $\tau_{ii}=0\,(i=1,\cdots,n)$. For $n=2$, one can check that the assumptions of Corollary 2.5 become

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}$$

which is also necessary condition for the existence of a globally attractive positive equilibrium for the following autonomous system

$$\dot{x}_i(t) = x_i[r_i - a_{ii}x_i(t) - \sum_{j=1,j\neq i}^{n} a_{ij}x_j(t-\tau_{ij}] \qquad (i=1,\cdots,n) \tag{2.30}$$

when $n = 2$. For $n \geq 2$, the conditions of Corollary 2.5 are satisfied if

$$r_i > \sum_{j=1, j \neq i}^{n} \frac{a_{ij}}{a_{jj}} r_j \qquad (i = 1, \cdots, n). \tag{2.31}$$

Indeed, in this case, $M_i = r_i/a_{ii}$ and (2.27) reducsd to (2.31), which implies that $E = (e_{ij})_{n \times n}$ with $e_{ii} = a_{ii}$ for $i = j$, $e_{ij} = -a_{ij}$ for $i \neq j \, (i, j = 1, \cdots, n)$ is an M -matrix (see [AK] and [HG2]). It is well known (see [HG2]) that the condition (2.31) is also a sufficient condition for the existence of a global asymptotical stable positive equilibrium for (2.30). It is in this sense that our conditions are the best.

Remark 2.2. Let $n = 2$ and $\tau_{ij} = 0$. Then the sufficient conditions in Corollary 2.5 become

$$\frac{a_{11}^{\ell}}{a_{21}^{u}} > \frac{r_1^u}{r_2^{\ell}} \geq \frac{r_1^{\ell}}{r_2^u} > \frac{a_{12}^u}{a_{22}^{\ell}},$$

which has been derived by Gopalsamy [G4, G5, G7], Ahmad [A1, A2] and Alvarez and Lazer [AL]. When $n > 2$ and $\tau_{ii} = 0$, one can see from Remark 2.1 that the conditions of Corollary 2.5 become

$$r_i^{\ell} > \sum_{j=1, j \neq i}^{n} \frac{a_{ij}^u}{a_{jj}^{\ell}} r_j^u, \qquad (i = 1, 2, \cdots, n),$$

which generalizes the results in [A1, A2] and [AL] from $n = 2$ to $n > 2$, and also improves the results in [G7] for $n \geq 2$.

3. Positive periodic and almost periodic solutions.

In this section, we concern the existence and stability of positive periodic and almost periodic solutions of the periodic and almost periodic systems (2.1), respectively. We first modify the difinitions of uniform boundedness in Burton [B] since we only concern the existence of "positive" periodic and almost periodic solutions of (2.1).

Definition 3.1. Solutions of (2.1) and (2.3) are said to be uniformly bounded above and below by positive constant vectors at $t = 0$ if for any $\delta^{\ell}, \delta^u \in R_+^n$ with $0 < \delta^{\ell} < \delta^u$, there exist $m^{\ell}(\delta^{\ell}, \delta^u)$, $m^u(\delta^{\ell}, \delta^u) \in R_+^n$ such that for any $\phi \in C^+(0) = \{\phi : [-\tau, 0] \to R_+^n, |\phi \text{ is continuous}\}$ with $\delta^{\ell} < \phi(s) < \delta^u \, (s \in [-\tau, 0])$ we have $m^{\ell}(\delta^{\ell}, \delta^u) < x(t, \phi) < m^u(\delta^{\ell}, \delta^u)$ for $t > 0$.

Definition 3.2. Solutions of (2.1) and (2.3) are said to be uniformly ultimately bounded above and below by positive constant vectors at $t = 0$ if there exist $m^{\ell}, m^u \in R_+^n$ with $m^{\ell} < m^u$ such that for any $\delta^{\ell}, \delta^u \in R_+^n$ with $0 <$

$\delta^\ell < \delta^u$, there is a $t_1(\delta^\ell, \delta^u) > 0$ such that for any $\phi \in C^+(0) = \{\phi : [-\tau, 0] \to R^n_+ \,|\phi \text{ is continuous}\}$ with $\delta^\ell < \phi(s) < \delta^u \, (s \in [-\tau, 0])$ we have $m^\ell < x(t, \phi) < m^u$ for $t > t_1(\delta^\ell, \delta^u)$.

Theorem 3.3. *Suppose the functions* $a_{ij}(t), r_i(t) \, (i, j = 1, \cdots, n)$ *in (2.1) satisfy*

$$r_i^\ell > \sum_{j=1, j\neq i}^{n} a_{ij}^u \frac{r_j^u}{a_{jj}^\ell} e^{r_j^u \tau_{jj}} \qquad (i = 1, \cdots, n). \tag{3.1}$$

Then all the positive solutions of the system (2.1) are uniformly bounded and uniformly ultimately bounded above and below by positive constant vectors.

Proof. For any $\phi \in C^+(0)$, let $x(t) = x(t, \phi)$ be the solution of (2.1) with $x(s) = \phi(s)$ for $s \in [-\tau, 0]$. We first prove that the solutions of (2.1) and (2.3) are uniformly bounded above and below by positive constant vectors. We show this in two steps. Let $\delta^\ell, \delta^u \in R^n$ with $0 < \delta^\ell < \delta^u$ and $\phi \in C^+(0)$ with $\delta^\ell < \phi(s) < \delta^u$ for $s \in [-\tau, 0]$; that is $\delta_i^\ell < \phi_i(s) < \delta_i^u$ for $i = 1, \cdots, n$ and $s \in [-\tau, 0]$. Then $x(t) = x(t, \phi) = (x_1(t), \cdots, x_n(t)) > 0$ exists for $t \geq 0$.

Step 1. For $i \in \{1, 2, \cdots, n\}$, consider the solution $x_i(t)$ on an interval $[0, L_i]$ for any positive constant $L_i > \tau_{ii}$. Let $x_i(t_{L_i}^u) = \max_{s\in[0,L_i]} x_i(s)$. Then either $t_{L_i}^u \in [0, \tau_{ii}]$ or $t_{L_i}^u \in (\tau_{ii}, L]$. If $t_{L_i}^u \in [0, \tau_{ii}]$, then it follows from (2.1) that

$$\dot{x}_i(t) \leq x_i(t)[r_i(t) - a_{ii}(t) x_i(t - \tau_{ii})], \tag{3.2}$$

which implies

$$x_i(t_{L_i}^u) \leq x_i(0) \exp\left[\int_0^{t_{L_i}^u} r_i^u \, ds\right] \leq x_i(0) e^{r_i^u \tau_{ii}} < \delta_i^u e^{r_i^u \tau_{ii}}. \tag{3.3}$$

If $t_{L_i}^u \in (\tau_{ii}, L_i]$, then from (3.2) we have

$$0 = \dot{x}_i(t_{L_i}^u) \leq x_i(t_{L_i}^u)[r_i^u - a_{ii}^\ell x_i(t_{L_i}^u - \tau_{ii})],$$

which implies $x_i(t_{L_i}^u - \tau_{ii}) \leq r_i^u / a_{ii}^\ell$. Integrating (3.2) on $[t_{L_i}^u - \tau_{ii}, t_{L_i}^u]$, it leads to

$$x_i(t_{L_i}^u) \leq x_i(t_{L_i}^u - \tau_{ii}) \exp \int_{t^u - L_i^u}^{t_{L_i}^u} r_i^u \, ds \leq \frac{r_i^u}{a_{ii}^\ell} e^{r_i^u \tau_{ii}} := M_i^u. \tag{3.4}$$

Hence $x_i(t) \leq \max\{\delta_i^u e^{r_i^u \tau_{ii}}, M_i^u\} := B_i^u(\delta^u)$ for $t \in [0, L_i]$. Obviousely, $B_i^u(\delta^u)$ is independent of L_i . Let $L_i \to \infty$, we obtain $x_i(t) \leq B_i(\delta^u) < B_i^u(\delta) + 1 := m_i^u(\delta^u)$ for all $t \geq 0$. This means that all the solutions of (2.1) and (2.3) are uniformly bounded above by positive constant vector $m^u = (m_1^u, \cdots, m_n^u)$.

Step 2. For $i \in \{1, 2, \cdots, n\}$, consider the solution $x_i(t)$ on an interval $[0, L_i]$ for any positive constant $L_i > 2\tau$. Let $x_i(t^\ell_{L_i}) = \min_{s\in[0,L_i]} x_i(s)$. Then either $t^\ell_{L_i} \in [0, 2\tau]$ or $t^\ell_{L_i} \in (2\tau, L]$. If $t^\ell_{L_i} \in [0,\, 2\tau]$, then it follows from (2.1) and step 1 that

$$\dot{x}_i(t) \geq x_i(t)\Big[r^\ell_i - \sum_{j=1}^{n} a^u_{ij} x_i(t - \tau_{ii})\Big] \geq x_i(t)\Big[r^\ell_i - \sum_{j=1}^{n} a^u_{ij} m^u_j\Big]. \tag{3.5}$$

Integrating (3.5) on $[0, t^\ell_{L_i}]$, one can see that

$$\begin{aligned} x_i(t^\ell_{L_i}) &\geq x_i(0) \exp\Big[\int_0^{t^\ell_{L_i}} [r^\ell_i - \sum_{j=1}^{n} a^u_{ij} m^u_j]\, ds\Big] \\ &= x(0) \exp\Big([r^\ell_i - \sum_{j=1}^{n} a^u_{ij} m^u_j] t^\ell_{L_i}\Big). \end{aligned} \tag{3.6}$$

From (3.6), it is easy to see that $x_i(t^\ell_{L_i}) \geq x_i(0) > \delta^\ell_i$ if $r^\ell_i - \sum_{j=1}^n a^u_{ij} m^u_j \geq 0$ and

$$x_i(t^\ell_{L_i}) \geq x_i(0) \exp(2\tau[r^\ell_i - \sum_{j=1}^{n} a^u_{ij} m^u_j]) \geq \delta^\ell_i \exp(2\tau[r^\ell_i - \sum_{j=1}^{n} a^u_{ij} m^u_j]) := B^\ell_i(\delta^\ell_i,\, \delta^u)$$

if $r^\ell_i - \sum_{j=1}^n a^u_{ij} M^u_j \leq 0$. Obviousely, $x_i(t^\ell_{L_i}) \geq B^\ell_i(\delta^\ell_i,\, \delta^u)$ in both cases. If $t^\ell_{L_i} \in (2\tau, L_i]$, it is noticed from (3.4) and the following analysis that $x_i(t) \leq M^u_i$ for $t \geq \tau_{ii}$, then (3.5) implies

$$0 = \dot{x}_i(t^\ell_{L_i}) \geq x_i(t^\ell_{L_i})\Big[\Big(r^\ell_i - \sum_{j=1, j\neq i}^{n} a^u_{ij} M^u_j\Big) - a^u_{ii} x_i(t^\ell_{L_i} - \tau_{ii})\Big]. \tag{3.7}$$

It follows from (3.1) that $r^\ell_i - \sum_{j=1,j\neq i}^n a^u_{ij} M^u_j > 0$ and hence (3.7) implies $x_i(t^\ell_{L_i} - \tau_{ii}) \geq [r^\ell_i - \sum_{j=1,j\neq i}^n a^u_{ij} M^u_j]/a^u_{ii}$. Integrating (3.5) on $[t^\ell_{L_i} - \tau_{ii}, t^\ell_{L_i}]$, it leads to

$$\begin{aligned} x_i(t^\ell_{L_i}) &\geq x_i(t^\ell_{L_i} - \tau_{ii}) \exp \int_{t^\ell_{L_i} - \tau_{ii}}^{t^\ell_{L_i}} \Big[r^\ell_i - \sum_{j=1}^{n} a^u_{ij} M^u_j\Big]\, ds \\ &\geq \frac{r^\ell_i - \sum_{j=1,j\neq i}^n a^u_{ij} M^u_j}{a^u_{ii}} \exp\Big(\Big[r^\ell_i - \sum_{j=1}^{n} a^u_{ij} M^u_j\Big]\tau_{ii}\Big) := M^\ell_i. \end{aligned} \tag{3.8}$$

Hence $x_i(t) \geq \min\{B^\ell_i(\delta^\ell_i, \delta^u),\, M^\ell_i\} := C^\ell_i(\delta^\ell,\, \delta^u)$ for $t \in [0, L_i]$. Obviousely, $C^\ell_i(\delta^\ell,\, \delta^u)$ is independent of L_i. Let $L_i \to \infty$, we obtain $x_i(t) \geq C_i(\delta^\ell,\, \delta^u) > C^\ell_i(\delta^\ell,\, \delta^u) - \beta := m^\ell_i(\delta^\ell,\, \delta^u) > 0$ for some $\beta > 0$ and all $t \geq 0$. This means that all the solutions of (2.1) and (2.3) are uniformly bounded below by positive constant vector $m^\ell = (m^\ell_1, \cdots, m^\ell_n)$.

Step 3. Now we show that the solutions of (2.1) and (2.3) are uniformly ultimately bounded above and below by positive constant vectors at $t = 0$. Let $\delta^\ell,\, \delta^u \in R^n$ be any two positive constant vectors with $0 < \delta^\ell < \delta^u$ and let $x(t) = x(t,\phi) = (x_1(t), \cdots, x_n(t))$ be any solution of (2.1) with $\delta^\ell < \phi(s) < \delta^u\, (s \in [-\tau, 0])$. For $i \in \{1, \cdots, n\}$, consider the solution $x_i(t)$ on an interval $[2\tau, L]$ for any $L > 2\tau$. Let $x_i(t)$ has an maximum say at $t_L^u \in [2\tau, L]$. Then it follows from (3.2) that

$$0 = \dot{x}_i(t_L^u) \leq x_i(t_L^u)[r_i^u - a_{ii}^\ell x_i(t_L^u - \tau_{ii})]$$

which implies $x_i(t_L^u - \tau_{ii}) \leq r_i^u / a_{ii}^\ell$. Integrating (3.2) on $[t_L^u - \tau_{ii}, t_L^u]$, we obtain

$$x_i(t_L^u) \leq x_i(t_L^u - \tau_{ii}) \exp\left(\int_{t_L^u - \tau_{ii}}^{t_L^u} r_i^u \, ds \right) \leq \frac{r_i^u}{a_{ii}^\ell} e^{r_i^u \tau_{ii}} = M_i^u.$$

Hence $x_i(t) \leq M_i^u$ for $t \in [2\tau, L]$. Since M_i^u is independent of L and δ^u, δ^ℓ , letting $L \to \infty$ and then we deduce $x_i(t) \leq M_i^u$ for $t \geq 2\tau$, which implies that the solutions of (2.1) and (2.3) are uniformly ultimately bounded above by positive constant vector.

The last part we have to show is that all the solutions of (2.1) and (2.3) are uniformly ultimately bounded below by positive constant vector. This can be seen from the argument used in steps 2 and 3, so we omit the details. Hence the proof is completed.

Remark 3.1. The proof of Theorem 3.3 also gives explicit estimates on the upper and lower uniform bounds for the positive solutions of (2.1), which have been given in Lemmas 2.1 and 2.5.

Now we consider the periodic system (2.1), that is; we assume that functions $a_{ij}, r_i\, (i = 1, \cdots, n)$ in the system (2.1) are periodic functions with period $\omega > 0$. The existence of positive periodic solutions of the system is established by employing Theorem 4.2.2 in Burton [B].

Theorem 3.4. *Assume* $a_{ij}, r_i\, (i, j = 1, 2, \cdots, n)$ *are* ω *-periodic functions. If (3.1) holds, then the system (2.1) has a positive periodic solution with period* ω *.*

Proof. Denote $f^T(t, x_t) = (f_1(t, x_t), \cdots, f_n(t, x_t))$ with $f_i(t, x_t) = r_i(t) - \sum_{j=1}^n a_{ij}(t) x_j(t - \tau_{ij})\, (i = 1, \cdots, n)$. Then (2.1) can be written in the form

$$\dot{x}_i(t) = x_i(t) f_i(t, x_t). \tag{3.9}$$

To show the existence of a positive periodic solution of (2.1), we consider the solutions of (2.1) with initial functions $\phi(s) > 0$ for $s \in [-\tau, 0]$. Let $x(t) = x(t,\phi)$

be the solution of (3.9) corresponding to such a $\phi \in C^+(0)$ and define $y(t) = (y_1(t), \cdots, y_n(t))$ with $y_i(t) = \ell n x_i(t) \, (i = 1, \cdots, n)$. Then $y(t)$ satisfies

$$\dot{y}_i(t) = f_i(t, e^{y_t}) := F_i(t, y_t) \qquad \text{for} \qquad t \geq 0 \tag{3.10}$$

and $y(s) = \psi(s)$ for $s \in [-\tau, 0]$ with $\psi(s) = (\psi_1(s), \cdots, \psi_n(s))$ and $\psi_i(s) = \ell n \phi_i(s) \, (i = 1, \cdots, n)$. Obviousely $\psi \in C(0)$. Then it follows from Theorem 3.3 and $y(t) = \ell n x(t)$ that the solutions of (3.10) are uniform bounded and uniform ultimate bounded (in the sense of Burton's definitions, see Burton [B], p. 248). And also, by the periodicity of $r_i, a_{ij} \, (i, j = 1, \cdots, n)$, if $y(t)$ is a periodic solution of (3.10) with period $\omega > 0$, then $y(t+\omega)$ is also an ω -periodic solution of (3.10). Using Burton's Theorem 4.2.2 (see Burton [B]), we can conclude that (3.10) has an ω -periodic solution, say $\tilde{y}(t)$. Let $\tilde{x}(t) = exp(\tilde{y}(t))$, then obviousely $\tilde{x}(t)$ is a positive ω -periodic solution of (2.1). This completes the proof.

From Theorems 3.2, 3.3, 2.2 and Corollary 2.5, one can derive the following Corollaries.

Corollary 3.5. *Suppose* $a_{ij}, r_i \, (i, j = 1, \cdots, n)$ *are* ω *-periodic functions and (2.5) and (3.1) are satisfied. If the matrix* $D(t)$ *defined by Theorem 2.2 is an eventually uniform* M *-matrix, then the system (2.1) has a unique positive* ω *-periodic solution which is globally attractive.*

Corollary 3.6. *Suppose* $a_{ij}, r_i \, (i, j = 1, \cdots, n)$ *are* ω *-periodic functions and (2.29) and (3.1) are satisfied. If the matrix* E *defined by Corollary 2.5 is an* M *-matrix, then the system (2.1) has a unique positive* ω *-periodic solution which is globally attractive.*

Remark 3.2. Let $\tau_{ii} = 0$ in (3.1). Then we have

$$r_i^\ell > \sum_{j=1, j \neq i}^{n} \frac{a_{ij}^u}{a_{jj}^\ell} r_j^u \qquad (i = 1, 2, \cdots, n). \tag{3.11}$$

As pointed in Remark 2.1, (3.11) also implies the matrix E defined by Corollary 2.5 is an M -matrix. Therefore, by Corollary 3.6, the following periodic system

$$\dot{x}_i(t) = x_i(t) \left[r_i(t) - a_{ii}(t) - \sum_{j=1, j \neq i}^{n} a_{ij}(t) x_j(t - \tau_{ij}) \right] \qquad i = 1, \cdots, n \tag{3.12}$$

has a unique positive periodic solution which is globally asymptotically stable under the condition (3.12). This improves the main results in [G5], [A-L] and [A1].

Now we turn to the system (2.1) with almost periodic coefficient functions $a_{ij}, r_i\,(i,j=1,\cdots,n)$. Consider the system

$$\dot{x}_i(t) = x_i(t)\left[r_i^*(t) - \sum_{j=1}^{n} a_{ij}^*(t)x_j(t-\tau_{ij})\right] \qquad i=1,\cdots,n, \tag{3.13}$$

where , for some sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$,

$$a_{ij}(t+t_n) \to a_{ij}^*(t), \quad r_i(t+t_n) \to r_i^*(t) \qquad \text{as} \quad n \to \infty,$$

uniformly for $t \in R$. Note that a_{ij}^* , $r_i^*\,(i,j=1,2,\cdots,n)$ are also almost periodic.

Following the argument used in Lemma 2 in Murakami [M] or in Lemma 4 in Seifert [S2], we can obtain the following Lemma.

Lemma 3.7. *Suppose (2.27) holds. Then there exists a solution* $x^*(t) = (x_1^*(t),\cdots, x_n^*(t))$ *of (3.13) on* R *such that* $m_i \le x_i^*(t) \le M_i$ *for* $i=1,\cdots,n$ *and* $t \in R$ *, where* M_i *and* m_i *are defined by (2.4) and (2.28), respectively.*

Observing that a_{ij}^* and r_i^* have the same bounds as a_{ij} and $r_i\,(i,j=1,\cdots,n)$, one can use the Lyapunov functionals constructed in Theorem 2.2 and the argument used in Lemma 5 in Seifert [S2] to derive the following result on the uniqueness of the solution of (3.13).

Lemma 3.8. *Suppose (3.1) and (2.29) hold. If the matrix* E *defined by Corollary 2.5 is an* M *-matrix, then there is exactly one solution* $x^*(t)$ *of (3.13) on* R *such that* $0 \le x_i^* \le M_i$ *for* $i=1,\cdots,n$ *and* $t \in R$ *.*

Using Lemmas 3.7 and 3.8 to (2.1), we derive the following result on the existence and stability of positive almost periodic solution for (2.1).

Theorem 3.9. *Let* $a_{ij}, r_i\,(i,j=1,\cdots,n)$ *are almost periodic function in* t *. Then, under the conditions of Lemma 3.8, there exists a unique positive almost periodic solution* $x^*(t)$ *of (2.1) on* R *which is globally attractive.*

The proof on the existence of almost periodic solution of (2.1) can be formed in exactly the same way as Theorem in Seifert [S2] or Theorem 10.1 in Fink [F]. The global attractivity follows from Corollary 2.5.

Remark 3.3. By Theorem 3.9, when $n=2$ and $\tau_{ij}=0\,(i,j=1,\cdots,n)$, one can derive the main results in [A2]; when $n \ge 2$ and $\tau_{ii}=0\,(i=1,\cdots,n)$, we improve the results in [G7].

Remark 3.4. The results we obained in this paper improve the main results in [GH1] when $n=2$ with constant delays.

4. Systems with finite distributed delays.

In this section, we briefly indicate the generalization of our discussion on (2.1) to the following system

$$\frac{dx_i(t)}{dt} = x_i(t)\Big[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)\int_0^{\gamma_{ij}} k_{ij}(s)x_j(t-s)\,ds\Big], \quad i=1,\cdots,n. \tag{4.1}$$

For $i,j=1,\cdots,n$, we assume that $\gamma_{ij}\ge 0$ are finite constants and $k_{ij}(t)\ge 0$ are piecewise continuous functions on $[-\gamma,0]$ with $\gamma=\max_{1\le i,j\le n}\{\gamma_{ij}\}$ and satisfy the normalisation conditions $\int_0^{\gamma_{ij}} k_{ij}(s)\,ds = 1$. We shall assume that the system (4.1) is supplemented with initial conditions of the form

$$x_i(t) = \phi_i(t) \in C^+([-\gamma,0],R_+), \qquad i=1,\cdots,n. \tag{4.2}$$

Similar to Lemma 2.1, we have the following ultimate bound estimation on the positive solutions of (4.1).

Lemma 4.1. *Let* $x(t)=(x_1(t),\cdots,x_n(t))^T\in R^n$ *denote any solution of (4.1) and (4.2). Suppose (2.2) holds and* $a_{ii}^{\ell}>0, r_i^u>0\,(i=1,\cdots,n)$ *. Then the solutions of (4.1) and (4.2) are eventually uniformly bounded above and satisfy*

$$x_i(t)\le M_i := \frac{r_i^u}{a_{ii}^{\ell}}e^{r_i^u\gamma_{ii}} \qquad \text{for} \quad t\ge t_1 \qquad (i=1,\cdots,n) \tag{4.3}$$

for some $t_1\ge 0$ *.*

Now let $x(t)=(x_1(t),\cdots,x_n(t))^T$ and $y(t)=(y_1(t),\cdots,y_n(t))^T$ be any two positive solutions of (4.1) and (4.2). Define $u(t)$, $v(t)$ and $w(t)$ as in (2.7). Then

$$\frac{d}{dt}[u_i(t)-v_i(t)] = -\sum_{j=1}^{n} a_{ij}(t)\int_0^{\gamma_{ij}} k_{ij}(s)b_j(t-s)[u_j(t-s)-v_j(t-s)]\,ds, \tag{4.4}$$

where

$$b_j(t) = \int_0^1 e^{s_1u_j(t)+(1-s_1)v_j(t)}\,ds_1 \qquad (j=1,\cdots,n). \tag{4.5}$$

For $i \in \{1, \cdots, n\}$, equation (4.4) can be written as

$$
\begin{aligned}
\frac{dw_i(t)}{dt} &= -\sum_{j=1}^{n} a_{ij}(t) \int_0^{\gamma_{ij}} k_{ij}(s) b_j(t-s) w_j(t-s)\, ds \\
&= -a_{ii}(t) \Big[\int_0^{\gamma_{ii}} k_{ii}(s) b_i(t-s)\, ds \Big] w_i(t) \\
&\quad + a_{ii}(t) \int_0^{\gamma_{ii}} k_{ii}(s) b_i(t-s) \Big(\int_{t-s}^{t} \dot{w}_i(s_1)\, ds_1\, ds \Big) \\
&\quad - \sum_{j=1, j\neq i}^{n} a_{ij}(t) \int_0^{\gamma_{ij}} k_{ij}(s) b_j(t-s) w_j(t-s)\, ds \\
&= -a_{ii}(t) \Big[\int_0^{\gamma_{ii}} k_{ii}(s) b_i(t-s)\, ds \Big] w_i(t) \\
&\quad - a_{ii}(t) \sum_{j=1}^{n} \int_0^{\gamma_{ii}} k_{ii}(s) b_i(t-s) \int_{t-s}^{t} a_{ij}(s_1) \\
&\qquad \times \int_0^{\gamma_{ij}} k_{ij}(s_2) b_j(s_1 - s_2) w_j(s_1 - s_2)\, ds_2\, ds_1\, ds \\
&\quad - \sum_{j=1, j\neq i}^{n} a_{ij}(t) \int_0^{\gamma_{ij}} k_{ij}(s) b_j(t-s) w_j(t-s)\, ds. \qquad (4.6)
\end{aligned}
$$

Define $V_{i1}(w_i)$ as in (2.13), then from (4.6),

$$
\begin{aligned}
\frac{D^+}{Dt} V_{i1}(w_i(t)) \leq &- a_{ii}(t) \Big[\int_0^{\gamma_{ii}} k_{ii}(s) b_i(t-s)\, ds \Big] |w_i(t)| \\
&+ a_{ii}(t) \sum_{j=1}^{n} \int_0^{\gamma_{ii}} k_{ii}(s) b_i(t-s) \int_{t-s}^{t} a_{ij}(s_1) \\
&\qquad \times \int_0^{\gamma_{ij}} k_{ij}(s_2) b_j(s_1 - s_2) |w_j(s_1 - s_2)|\, ds_2\, ds_1\, ds \\
&+ \sum_{j=1, j\neq i}^{n} a_{ij}(t) \int_0^{\gamma_{ij}} k_{ij}(s) b_j(t-s) |w_j(t-s)|\, ds. \qquad (4.7)
\end{aligned}
$$

Let

$$
V_i(w)(t) = V_{i1}(w_i)(t) + V_{i2}(w)(t), \qquad (4.8)
$$

where

$$V_{i2}(w)(t) = \sum_{j=1,j\neq i}^{n} \int_0^{\gamma_{ij}} k_{ij}(s) \int_{t-s}^{t} a_{ij}(s_1+s) b_j(s_1)|w_j(s_1)|\, ds_1\, ds$$
$$+ \sum_{j=1}^{n} \int_0^{\gamma_{ii}} k_{ii}(s) \int_{t-s}^{t} a_{ii}(s_3+s) b_i(s_3) \int_{s_3}^{t} a_{ij}(s_1)$$
$$\times \int_0^{\gamma_{ij}} k_{ij}(s_2) b_j(s_1-s_2)|w_j(s_1-s_2)|\, ds_2\, ds_1\, ds_3\, ds$$
$$+ \sum_{j=1}^{n} \int_0^{\gamma_{ij}} k_{ij}(s) \int_{t-s}^{t} a_{ij}(s_1+s) c_i(s_1+s) b_j(s_1)|w_j(s_1)|\, ds_1\, ds \tag{4.9}$$

with

$$c_i(t) = \int_0^{\gamma_{ii}} k_{ii}(s) \int_{t-s}^{t} a_{ii}(s_1+s) b_i(s_1)\, ds_1\, ds. \tag{4.10}$$

It can be varified that

$$\frac{D^+}{Dt} V_i(w)(t) \leq - a_{ii}(t)\Big[\int_0^{\gamma_{ii}} k_{ii}(s) b_i(t-s)\, ds\Big] |w_i(t)|$$
$$+ \sum_{j=1,j\neq i}^{n} \Big[\int_0^{\gamma_{ij}} k_{ij}(s) a_{ij}(t+s)\, ds\Big] b_j(t)|w_j(t)|$$
$$+ \sum_{j=1}^{n} \Big[\int_0^{\gamma_{ij}} k_{ij}(s) a_{ij}(t+s) c_i(t+s)\, ds\Big] b_j(t)|w_j(t)|. \tag{4.11}$$

Follwong from (2.19), we have

$$b_i(t-s) \geq b_i(t) e^{-R_i(t,s)}, \qquad c_i(t) \leq M_i d_i(t) \tag{4.12}$$

with $d_i(t) = \int_0^{\gamma_{ii}} k_{ii}(s) \int_{t-s}^{t} a_{ii}(s_1+s)\, ds_1\, ds$. From (4.11) and (4.12),

$$\frac{D^+}{Dt} V_i(w)(t) \leq$$
$$- \Big(\int_0^{\gamma_{ii}} k_{ii}(s) \Big[a_{ii}(t) e^{-R_i(t,s)} - M_i a_{ii}(t+s) d_i(t+s)\Big]\, ds \Big) b_i(t)|w_i(t)|$$
$$+ \sum_{j=1,j\neq i}^{n} \Big(\int_0^{\gamma_{ij}} k_{ij}(s) a_{ij}(t+s)[1 + M_i d_i(t+s)]\, ds \Big) b_j(t)|w_j(t)|$$
$$= - \sum_{j=1}^{n} \Big[\int_0^{\gamma_{ij}} k_{ij}(s) e_{ij}(t,s)\, ds \Big] b_j(t)|w_j(t)|, \tag{4.13}$$

where

$$e_{ij}(t,s) = \begin{cases} a_{ii}(t) e^{-R_i(t,s)} - M_i a_{ii}(t+s) d_i(t+s) & \text{for } i = j \\ -a_{ij}(t+s)[1 + M_i d_i(t+s)], & \text{for } i \neq j. \end{cases} \tag{4.14}$$

Based on above analysis, following the proof of Theorem 2.2, one can obtain the following result on the extreme stability of the system (4.1).

Theorem 4.2. *Assmue*

$$\lim_{t\to\infty}\int_0^{\gamma_{ii}} k_{ii}(s)e_{ii}(t,s)\,ds > 0 \qquad (i=1,\cdots,n)$$

and the matrix $F(t)=(f_{ij}(t))_{n\times n}$ *is an eventually* M *-matrix with* $f_{ij}(t)=\int_0^{\gamma_{ij}} k_{ij}(s)e_{ij}(t,s)\,ds$ *and* $e_{ij}(t,s)\,(i,j=1,\cdots,n)$ *defined by (4.14). If the solutions of (4.1) and (4.2) are uniformly persistence, then the system (4.1) is extremely stable.*

We also have a similar result to Lemma 2.4 for the system (4.1) and therefore, using Theorem 4.2, one can derive the following Corollary.

Corollary 4.3. *Suppose that*

$$M_i(a_{ii}^u)^2 e^{r_i^u\gamma_{ii}}\gamma_{ii} < a_{ii}^\ell \qquad (i=1,\cdots,n)$$

and $G=(g_{ij})_{n\times n}$ *is an* M *-matrix, where*

$$g_{ij}=\begin{cases} a_{ii}^\ell e^{-r_i^u\gamma_{ii}} - M_i(a_{ii}^u)^2\gamma_{ii}, & \text{for } i=j\\ -[1+M_i a_{ii}^u\gamma_{ii}]a_{ij}^u, & \text{for } i\neq j.\end{cases}$$

If either the system (4.1) is uniformly persistent or (2.27) hold with M_i *be defined by (4.3), then the system (4.1) is extremely stable.*

Remark 4.1. As in Section 3, one can show that, for the system (4.1), all the positive solutions of (4.1) and (4.2) are uniformly bounded and uniformly ultimately bounded above and below by positive constant vectors under (2.27) with M_i defined by (4.3) and therefore (4.1) has a positive periodic solution when the coefficial functions are periodic in t . We can also discuss the existence of positive almost periodic solution for the almost periodic system (4.1).

5. Discussion.

In this paper, we study the nonautonomous Lotka-Volterra competition systems of "pure-delay-type" and obtain sufficient conditions for the extreme stabilities of the system (2.1). Some explicit estimates on the uniform persistence region and allowable size of delays for the persistence of global attractivity are indicated. When the systems are periodic and almost periodic, we derive the existence and global attractivity of positive periodic and almost periodic solutions, respectively. All our results can be treated as the generalization of the correspong nondelay cases. The conditions are easily checked.

We also show that the analysis on (2.1) can be generalized to the system (4.1) with finite distributed delays. We remark that it is a very interesting problem to generalize our discussion further to the nonautonomous system (1.3). We refer to Gopalsamy and He [GH2] and Seifert [S2] for the case when $n=1$. In the case of $n\geq 2$, since the delays are not finite, there are some difficulties to overcome and we leave this as our further study.

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