Gradient-Like Parabolic Semiflows on BUC($\mathbb{R}^N$)

Daniel Daners

School of Mathematics and Statistics
University of Sydney, NSW 2006
Australia
daners@maths.usyd.edu.au

Sandro Merino

Mathematisches Institut
Universität Zürich
Winterthurerstrasse 190
CH-8057 Zürich
Switzerland
merino@math.unizh.ch

June 6, 1997

Abstract

We prove that a class of weighted semilinear reaction diffusion equations on $\mathbf{R}^N$ generates gradient-like semiflows on the Banach space of bounded uniformly continuous functions on $\mathbf{R}^N$. If $N = 1$ we show convergence to a single equilibrium. The key for getting the result is to show the exponential decay of the stationary solutions, which is obtained by means of a decay estimate of the kernel of the underlying linear semigroup.

1 Introduction

We consider a class of semilinear reaction diffusion equations on the whole of $\mathbf{R}^N$. A typical exponent being given by

$$\begin{cases} \partial_t u - \Delta u &= m(x)u - u^3 & \text{in } \mathbf{R}^N \times (0, \infty)\,, \\ u(x, 0) &= u_0(x) & \text{in } \mathbf{R}^N\,. \end{cases} \tag{1.1}$$

The positive and negative parts m^+ and m^- of the bounded weight function m are assumed to have the following properties:

- m^+ vanishes at infinity.
- There exist constants $r, \eta > 0$ such that for all $x \in \mathbf{R}^N$ the estimate $\int_{B(x,r)} m^-(x)\, dx \geq \eta$ holds.

The nonnegative function m^- is called a strongly absorbing potential. We are interested in continuous initial data $u_0(x)$ that do not have any specified decay at infinity but are required to be bounded. More precisely, we choose the phase space for (1.1) as $\mathrm{BUC}(\mathbf{R}^N)$, the Banach space of bounded uniformly continuous real-valued functions on $\mathbf{R}^N$. It is shown in [13] that (1.1) generates a global continuous semiflow $(\varphi, \mathrm{BUC}(\mathbf{R}^N))$ of class $\mathcal{AK}$ on $\mathrm{BUC}(\mathbf{R}^N)$ that consists of classical solutions of (1.1) if m is Hölder continuous.

We remark that in general the canonical Liapunov function associated with (1.1)

$$\mathcal{L}(u(x)) := \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u(x)|^2 dx - \int_{\mathbf{R}^N} \Big(\frac{1}{2}m(x)u(x)^2 - \frac{1}{4}u(x)^4\Big)\, dx \tag{1.2}$$

is of little use for orbits starting in $\mathrm{BUC}(\mathbf{R}^N)$. E.g. if the initial condition $u_0(x)$ is a nonzero constant and if m^+ has compact support, then

$$\mathcal{L}(\varphi(t, u_0)) = \infty$$

for $t \geq 0$. To overcome this difficulty we will make use of the results in [13], where it is shown that the semiflow $(\varphi, \mathrm{BUC}(\mathbf{R}^N))$ possesses a compact global B-attractor $\mathcal{M}$ of finite Hausdorff dimension. Furthermore $\mathcal{M}$ is contained in an order interval $V := [\underline{w}, \overline{w}]$ of the ordered Banach space $\mathrm{BUC}(\mathbf{R}^N)$. Here $\underline{w}$ is the smallest and $\overline{w}$ is the greatest stationary solution of (1.1). The principal part of this paper is devoted to proving that the stationary solutions of (1.1) have exponential decay at infinity. The underlying idea is to write (1.1) as

$$\partial_t u - \Delta u + m^-(x)u \;=\; m^+(x)u - u^3,$$

and to study the kernel $k(t,x,y)$ of the semigroup associated with the generator $\Delta - m^-$. In Section 2 we will prove the estimate

$$0 \;\le\; k(t,x,y) \;\le\; ct^{-\frac{N}{2}}e^{-\eta t}e^{-\frac{|x-y|^2}{ct}}$$

for some constants $c, \eta > 0$. This estimate will then imply the exponential decay at infinity of the stationary solutions of (1.1). As a consequence the elements of the order interval V and, in particular, of the B-attractor $\mathcal{M}$ have exponential decay. This will imply that the Liapunov function is uniformly bounded below on the B-attractor $\mathcal{M}$. As a consequence of the LaSalle invariance principle (e.g. [12, Theorem 2.3]) the ω-limit set $\omega(u_0)$ of each element $u_0 \in \mathrm{BUC}(\mathbf{R}^N)$ is contained in the set of stationary points of the semiflow, i.e., the semiflow induced by (1.1) is gradient-like. In one dimension it turns out that $\omega(u_0)$ consists of a single equilibrium.

Acknowledgement. We thank E.N. Dancer for valuable discussions, especially on the last part of this paper. This paper was initiated while the second author visited Australia. This visit was made possible by Yihong Du from the Department of Mathematics at the University of New England, Armidale, and we like to thank him for the generous support and the warm hospitality.

2 Kernel Estimates

Suppose that we are given a self adjoint strongly uniformly elliptic operator of second order on $\mathbf{R}^N$ of the form

$$\mathcal{A} \;=\; -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}u\Big) \tag{2.1}$$

with $a_{ij} = a_{ji} \in L_\infty(\mathbf{R}^N)$. By strongly uniformly elliptic we mean as usual that there exists $\alpha_0 > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \;\ge\; \alpha_0|\xi|^2 \tag{2.2}$$

for all $x, \xi \in \mathbf{R}^N$. We consider perturbations of the above operator with a nonnegative potential satisfying a condition introduced in [1].

Definition 2.1 *We say that a potential $V \in L_\infty(\mathbf{R}^N)$ is strongly absorbing if $V \geq 0$ and there exist constants $r, \eta > 0$ such that*

$$\int_{B(x,r)} V(\xi)\, d\xi \geq \eta \tag{2.3}$$

for all $x \in \mathbf{R}^N$. Here, $B(x,r) = \{y \in \mathbf{R}^N : |x-y| < r\}$.

Using form methods one can show that an appropriate restriction of $-\mathcal{A}$ to $L_p(\mathbf{R}^N)$ generates a strongly continuous analytic semigroup on $L_p(\mathbf{R}^N)$ for all $p \in [1, \infty)$. A similar statement is true for the operator $-(\mathcal{A}+V)$. One of the main results in [1] is that if $V \geq 0$ then, the semigroup $T_V(t)$ associated with $-(\mathcal{A}+V)$ is exponentially stable on $L_p(\mathbf{R}^N)$ for some (or equivalently all) $p \in [1, \infty)$ if and only if V is strongly absorbing. By exponentially stable we mean that there exist $\gamma_0, c_0 > 0$ such that

$$\|T_V(t)\|_{\mathcal{L}(L_p,L_p)} \leq c_0 e^{-\gamma_0 t} \tag{2.4}$$

for all $t \geq 0$. Alternative proofs of the result can be found in [2, 5]. The key in the proof is that the kernel $k_0(t;x,y)$ of $T_0(t)$ satisfies upper and lower Gaussian estimate, that is, for some constant $c > 0$

$$c^{-1} t^{-N/2} \exp\Big(-\frac{c|x-y|^2}{t}\Big) \leq k_0(t;x,y) \leq c t^{-N/2} \exp\Big(-\frac{|x-y|^2}{ct}\Big)\,. \tag{2.5}$$

By a simple comparison argument it is clear that for any nonnegative potential the upper estimate for the corresponding semigroup kernel remains valid. For strongly absorbing potentials we shall prove the following better upper estimate.

Theorem 2.2 *Suppose that the above assumptions on $\mathcal{A}$ are satisfied and that V is a strongly absorbing potential. Denote the kernel of the semigroup $T_V(t)$ by $k_V(t;x,y)$. Then there exist constants $c, \eta > 0$ such that*

$$0 \leq k_V(t;x,y) \leq c t^{-N/2} \exp(-\eta t) \exp\Big(-\frac{|x-y|^2}{ct}\Big) \tag{2.6}$$

for all $t > 0$ and $x, y \in \mathbf{R}^N$.

Proof To prove (2.6) we make use of (2.4) and the results in [6, Section 3.2]. First of all note that

$$\|T_V(t)\|_{\mathcal{L}(L_p,L_q)} \;\leq\; ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \tag{2.7}$$

whenever $1 \leq p \leq q \leq \infty$. This follows for instance from the upper bound in (2.5), which, as we remarked earlier, is also valid for $k_V(t;x,y)$. By using the semigroup property as well as (2.4) and (2.7) we obtain

$$\begin{aligned}\|T_V(t)\|_{\mathcal{L}(L_2,L_\infty)} \;&\leq\; \|T_V(t/2)\|_{\mathcal{L}(L_2,L_\infty)}\|T_V(t/2)\|_{\mathcal{L}(L_2,L_2)} \\ &\leq\; c_0(t/2)^{-N/4}e^{-\frac{\eta_0}{2}t} \;=\; ct^{-N/4}e^{-\eta t}\end{aligned} \tag{2.8}$$

for some $c, \eta > 0$ and all $t > 0$. To get the "non-diagonal" estimate we modify the proof of the heat kernel estimate associated with an operator of the form (2.1) given in [6]. We shall use the notation from [6]. First of all it follows from [6, Theorem 2.2.4] and (2.8) that for all $u \in W_2^1(\mathbf{R}^N)$ the logarithmic Sobolev inequality

$$\int_{\mathbf{R}^N} u^2 \log_+ u\,dx \;\leq\; \varepsilon a(u,u) + \beta(\varepsilon)\|u\|_2^2 + \|u\|_2^2 \log \|u\|_2 \tag{2.9}$$

holds with

$$\beta(\varepsilon) \;=\; c - \frac{N}{4}\log\varepsilon - \eta\varepsilon \tag{2.10}$$

for some constant $c \in \mathbf{R}$. Here, $a(\cdot\,,\cdot)$ is the Dirichlet form associated with $-\mathcal{A}+V$. Next note that the calculations in [6, Lemma 3.2.1] are not affected by the presence of a nonnegative potential, and so the result and its consequences, not using the explicit form of $\beta(\varepsilon)$ remain valid. In particular, Theorem 3.2.4 remains true. Let us now modify the proof of Theorem 3.2.5. We make the same choice of parameters as there, that is, we set

$$\mu \;=\; \frac{1}{2} \qquad \text{and} \qquad \varepsilon(p) \;=\; \lambda 2^\lambda t p^{-\lambda},$$

where $\lambda > 1$. Then

$$t \;=\; \int_2^\infty p^{-1}\varepsilon(p)\,dp.$$

In order to estimate $M(t)$ we only need to consider one additional term, namely

$$-\int_2^\infty p^{-2}\eta\lambda 2^\lambda t p^{-\lambda}\,dp \;=\; -\frac{\lambda}{2(1+\lambda)}\eta t.$$

Since $\lambda > 1$ this yields

$$M(t) \leq -\frac{N}{4} \log t + \frac{\lambda}{\lambda - 1} \alpha^2 t - \frac{\eta}{4} t + c_\lambda.$$

Choosing λ such that $1 + \delta = \lambda(\lambda - 1)^{-1}$ we get an estimate similar to the one in Theorem 3.2.5, but with the additional exponentially decaying factor $\exp(-\eta t/4)$. This factor does not affect the arguments given in Corollary 3.2.6, Theorem 3.2.7 and Corollary 3.2.8, so we get a bound of the form (2.6). ■

Corollary 2.3 *If the potential V is strongly absorbing then the weak Green's function $g_V(x, y)$ associated with $-(\mathcal{A} + V)$ satisfies an estimate of the form*

$$0 \leq g_V(x, y) \leq c\eta^{\frac{N}{2}-1} \Big(\frac{|x - y|}{\lambda}\Big)^{1-\frac{N}{2}} K_{\frac{N}{2}-1}(\lambda |x - y|) \tag{2.11}$$

for some $c > 0$ and $x \neq y$, where $K_{\frac{N}{2}-1}$ is the Modified Bessel Function, and $\lambda := 2\sqrt{\eta/c}$

Proof First note that $g_V(x, y)$ is given by the Laplace Transform of the kernel $k_V(t; x, y)$. More precisely,

$$g_V(x, y) = \int_0^\infty k_V(t; x, y)\, dt \tag{2.12}$$

for all $x, y \in \mathbf{R}^N$ with $x \neq y$. Hence, by Theorem 2.2 we have

$$0 \leq g_V(x, y) \leq c \int_0^\infty t^{-N/2} \exp(-\eta t) \exp(-\frac{|x - y|^2}{ct})\, dt. \tag{2.13}$$

Recall that the modified Bessel Functions can be represented by

$$K_\nu(\rho) = \frac{1}{2}\Big(\frac{\rho}{2}\Big)^\nu \int_0^\infty s^{-\nu-1} \exp(-\frac{\rho^2}{4s} - s)\, ds \tag{2.14}$$

for all $\nu \in \mathbf{R}$ and $\rho \in \mathbf{R} \setminus \{0\}$ (e.g. [8, page 82, formula (23)]). Hence, by the substitution $s = \eta t$, and setting $\lambda := 2\sqrt{\eta/c}$ we get from (2.13) that (2.11) holds for $x \neq y$. ■

Remark 2.4 *Using asymptotic expansions of the Modified Bessel Functions (see [8, Section 7.2]) we get that*

$$0 \leq g_V(x,y) \leq \begin{cases} c\exp(-|x-y|/c) & \text{if } N = 1 \\ c\log(|x-y|/c)\exp(-|x-y|/c) & \text{if } N = 2 \\ c|x-y|^{-N+2}\exp(-|x-y|/c) & \text{if } N \geq 3 \end{cases} \tag{2.15}$$

for some constant $c > 0$. In particular, we see that the Green's Function of $-\mathcal{A} + V$ decreases exponentially in space if V is strongly absorbing.

Remark 2.5 *To prove* (2.11) *we only used an estimate of the form* (2.6). *Therefore, if we have a potential $V = V_1 - V_2$ with V_1 strongly absorbing and $V_2 \in L_\infty(\mathbf{R}^N)$ and $\|V_2\|_\infty$ small enough we still get an estimate of the form* (2.6) *and* (2.11).

3 Nonlinear Elliptic Equations

Consider the nonlinear elliptic equation

$$\mathcal{A}u = f(x,u), \quad x \in \mathbf{R}^N. \tag{3.1}$$

For simplicity we will only discuss properties of nonnegative solutions of (3.1). But under further restrictions on the nonlinearity f the arguments remain valid for arbitrary solutions (compare [13, Example 3.3b)]). Following [13, Section 3] we impose the following restrictions on the nonlinearity $f(x,\xi)$.

$f(\cdot,\xi) \in \mathrm{BUC}^\mu(\mathbf{R}^N)$, for some $\mu \in (1,2)$, uniformly for ξ in bounded subsets of $\mathbf{R}$. (3.2)

The derivative $\partial_\xi f$ exists and is uniformly Lipschitz continuous on sets of the form $\mathbf{R}^N \times B$, where B is a bounded subset of $\mathbf{R}$. (3.3)

There exists a constant $M \geq 0$ such that $f(x,c) \leq 0$ but $f(\cdot,c) \not\equiv 0$ for $c \geq M$ and $x \in \mathbf{R}^N$. (3.4)

f is of the form $f(x,\xi) = h(x,\xi)\xi$. Moreover, for each $u \in X^+$ the positive part $h^+(\cdot\,,u(\cdot))$ of $h(\cdot\,,u(\cdot))$ is in $C_0(\mathbf{R}^N)$, whereas the negative part $h^-(\cdot\,,u(\cdot))$ is strongly absorbing. (3.5)

Under the assumptions considered in the introduction the nonlinearity in (1.1) satisfies all the above properties. We shall prove the following theorem.

Theorem 3.1 *There exists an exponentially decaying function $\psi \in C_0(\mathbf{R}^N)$ such that any solution $u \in L_\infty(\mathbf{R}^N)$ satisfies $|u(x)| \leq \psi(x)$ for all $x \in \mathbf{R}^N$.*

Proof Suppose that $u \in L_\infty(\mathbf{R}^N)$ is a nonnegative solution of (3.1) and that $\varepsilon > 0$ is given. Setting

$$m_1(x) := h^-(x, u(x)) - \varepsilon, \quad m_2(x) := h^+(x, u(x)) - \varepsilon,$$

we may rewrite (3.1) as

$$\mathcal{A}u + m_1 u = m_2 u. \tag{3.6}$$

Furthermore note that by (3.5) the potential $m_1 + \varepsilon$ is strongly absorbing. Also note that

$$m_2(x) \leq (h(x, u(x)) - \varepsilon)^+.$$

Therefore, by the results of the previous section, we see that

$$|u(x)| \leq \int_{\mathbf{R}^N} g_{m_1}(x, y)(h(x, u(x)) - \varepsilon)^+(y)\, dy \|u\|_\infty$$

for all $x \in \mathbf{R}^N$. If we choose $\varepsilon > 0$ small enough and take into account Corollary 2.3 and Remark 2.5 we see that $g_{m_1}(x, y)$ decays exponentially as $|x - y|$ goes to infinity. Due to (3.5) the function $(h^+(x, u(x)) - \varepsilon)^+$ has compact support and thus

$$\tilde{\psi}(x) := \int_{\mathbf{R}^N} g_{m_1}(x, y)(h(x, u(x)) - \varepsilon)^+(y)\, dy \tag{3.7}$$

decays exponentially as $|x|$ goes to infinity. Next observe that due to (3.4) all nonnegative solutions of (3.1) are bounded by the same constant M. Hence, $\psi := M\tilde{\psi}$ has the required properties. This completes the proof of the theorem. ■

4 Convergence to the Set of Equilibria

In this section we study the semilinear parabolic evolution problem in the Banach space $X := \mathrm{BUC}(\mathbf{R}^N)$. For simplicity we restrict our attention to nonnegative initial conditions u_0 in the positive cone X^+ of X.

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u & = & f(x,u) & \text{in } \mathbf{R}^N \times (0,\infty), \\ u(\cdot,0) & = & u_0 & \text{in } \mathbf{R}^N. \end{array}\right. \tag{4.1}$$

As in the previous section we require that f has the properties (3.2)–(3.5). Under these assumptions the existence of a global semiflow (φ, X^+) that consists of classical solutions of (4.1) as well as the existence of the compact global B-attractor $\mathcal{M} \subset X^+$ is shown in [13, Theorem 3.1]. The proof of [13, Theorem 3.1] also shows that the attractor $\mathcal{M}$ is contained in an order interval V that is invariant under the flow. V is of the form

$$V := [0, \overline{w}] \subset C_0(\mathbf{R}^N), \tag{4.2}$$

where $\overline{w}$ is a steady state of (4.1). Making use of the results of the previous sections we will verify that the functional

$$\mathcal{L}(u) := \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 \, dx - \int_{\mathbf{R}^N} F(x,u)\, dx \tag{4.3}$$

is a Liapunov function, where for $(x,\xi) \in \mathbf{R}^N \times \mathbf{R}$ we have set

$$F(x,\xi) := \int_0^\xi f(x,\eta)\, d\eta.$$

We will see that the functional $\mathcal{L}$ is well-defined on the B-attractor $\mathcal{M}$. This will be sufficient to prove that the ω-limit set of each point in X^+ is contained in the set of equilibria of the semiflow (φ, X^+). To prove that $\mathcal{L}$ is a Liapunov function we show some regularity properties of solutions of (4.1).

Lemma 4.1 *Let $u_0 \in L_2(\mathbf{R}^N) \cap \mathrm{BUC}(\mathbf{R}^N)$. Then, for some $\eta > 0$, the solution u of* (4.1) *lies in*

$$C^1([\varepsilon, T], W_2^1(\mathbf{R}^N)) \cap \mathrm{BUC}^{3+\eta,(3+\eta)/2}(\mathbf{R}^N \times [\varepsilon, T])$$

for all $\varepsilon > 0$ and $T > \varepsilon$.

Proof Fix $u_0 \in L_2(\mathbf{R}^N) \cap \mathrm{BUC}(\mathbf{R}^N)$, $\varepsilon > 0$ and $T > \varepsilon$. We first show Hölder regularity. As a consequence of the classical regularity theorems for parabolic equations with Hölder continuous coefficients and the abstract theory of parabolic evolution equations we conclude that, for some $\eta > 0$, the solution u of (4.1) lies in $\mathrm{BUC}^{2+\eta,1+\eta/2}(\mathbf{R}^N \times [\varepsilon, T])$ (see [4, Theorem 25.2]). Hence, by assumption (3.2) this implies that

$$m(x,t) := h(\cdot, u(t,\cdot)) \in \mathrm{BUC}^{1+\eta}(\mathbf{R}^N \times [\varepsilon, T]), \tag{4.4}$$

where we possibly need to decrease η. Applying [11, Theorem IV.5.1] we get the claimed Hölder regularity. For the Sobolev regularity we study the equation

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u - m(x,t)u & = & 0 & \text{in } \mathbf{R}^N \times (0,T), \\ u(\cdot,0) & = & u_0 & \text{in } \mathbf{R}^N. \end{array} \right. \tag{4.5}$$

whose solution lies in $C((0,T], W_2^1(\mathbf{R}^N))$. To see this define the multiplication operator $M(t)u(x) := m(x,t)u(x)$. Then,

$$M(\cdot) \in C^\eta([0,T], \mathcal{L}(L_2) \cap \mathcal{L}(W_2^1)). \tag{4.6}$$

Thus, taking into account that Δ generates an analytic semigroup on L_2 we find that the above claim is true (This follows e.g. from [4, Theorem 5.10 and Corollary 5.4] noting that $D(\Delta) = W_2^2(\mathbf{R}^N)$ and $[L_2, W_2^2]_{1/2} = W_2^1$). This implies that $u(\varepsilon) \in W_2^1(\mathbf{R}^N)$. As Δ also generates a strongly continuous analytic semigroup on W_2^1 it follows from (4.6) by the same argument that $u \in C^1([\varepsilon, T], W_2^1(\mathbf{R}^N))$. This concludes the proof of the lemma. ■

Lemma 4.2 *For $u_0 \in V$ the function $\mathcal{L}(\varphi(t,u_0))$ is well defined for $t > 0$. Moreover, it is bounded from below on V, and*

$$\frac{d}{dt}\mathcal{L}(\varphi(t,u_0)) < 0,$$

unless u_0 is an equilibrium of (φ, X^+).

Proof Let u_0 be an arbitrary element in V. We first show that $\mathcal{L}(\varphi(t,u_0))$ is finite for $t > 0$. By Theorem 3.1 the function $\overline{w}$ in (4.2) is exponentially decreasing, and since V is invariant

$$0 \le u(t,x) := \varphi(t,u_0)(x) \le Ce^{-\alpha|x|}, \quad x \in \mathbf{R}^N, \tag{4.7}$$

for some constants $C, \alpha > 0$ independent of t and u_0. In particular, this implies that $V \subset L_2(\mathbf{R}^N)$. It follows now from Lemma 4.1 that the gradient term in (4.3) is finite for all $t > 0$. We next show the finiteness of the other term. Using (4.7) we conclude that

$$|F(x, u(x,t))| \leq \int_0^{u(x,t)} |f(x,\xi)|\, d\xi \leq \int_0^{Ce^{-\alpha|x|}} |f(x,\xi)|\, d\xi.$$

The assumption (3.5) implies that $|f(x,\xi)| \leq c_1 < \infty$ for $(x,\xi) \in \mathbf{R}^N \times [0, C]$. We thus find that

$$|F(x, u(x,t))| \leq c_1 e^{-\alpha|x|}. \tag{4.8}$$

This shows that

$$\Big|\int_{\mathbf{R}^N} F(x, u(x,t))dx\Big| \leq c_0 < \infty$$

with $c_0 > 0$ independent of $u_0 \in V$, whence (4.3) is defined and bounded from below on V. This proves the first assertion. Next we show differentiability of $\mathcal{L}(u(t))$ with respect to t. By Lemma 4.1 and the fact that the inner product on $W_2^1(\mathbf{R}^N)$ is differentiable we get that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^N} |\nabla u(t)|^2\, dx = \int_{\mathbf{R}^N} (\nabla u|\nabla u_t)\, dx \tag{4.9}$$

Regarding the other term we already know the uniform bound (4.8). On the other hand,

$$\frac{d}{dt}F(x, u(x,t)) = f(x, u(x,t))u_t(x,t) = m(x,t)u(x,t)u_t(x,t)$$

is bounded by $\|m(t)u_t(t)\|_\infty |u(x,t)|$, and by (4.7) and Lemma 4.1 the derivative of $F(x, u(x,t))$ is bounded uniformly with respect to t in compact subsets of $(0,\infty)$ by an integrable function. Hence, we can interchange the order of integration and differentiation to get

$$\frac{d}{dt}\int_{\mathbf{R}^N} F(x, u(x,t))\, dx = \int_{\mathbf{R}^N} f(x, u(x,t))u_t\, dx.$$

Together with (4.9) we arrive at

$$\frac{d}{dt}\mathcal{L}(u(t)) = \int_{\mathbf{R}^N} (\nabla u|\nabla u_t)\, dx - \int_{\mathbf{R}^N} f(x,u)u_t\, dx. \tag{4.10}$$

Using that $\operatorname{div}[u_t \nabla u] = u_t \Delta u + (\nabla u | \nabla u_t)$ and applying the Divergence Theorem in the form

$$f \in W_1^1(\mathbf{R}^N, \mathbf{R}^N) \quad \Longrightarrow \quad \int_{\mathbf{R}^N} \operatorname{div} f(x)\, dx = 0$$

we obtain

$$\int_{\mathbf{R}^N} (\nabla u | \nabla u_t)\, dx = -\int_{\mathbf{R}^N} u_t \Delta u\, dx.$$

Finally, using that u is a solution of (4.1) we conclude from (4.10) that

$$\frac{d}{dt}\mathcal{L}(\varphi(t, u_0)) = -\int_{\mathbf{R}^N} u_t^2\, dx.$$

The last term is clearly negative if u is not stationary, which finishes the proof. ■

As a consequence of [12, Theorem 2.3] and the previous lemma we obtain the following result.

Theorem 4.3 *The ω-limit set $\omega(u_0)$ of each point $u_0 \in X^+$ is contained in the set of stationary points of the semiflow (φ, X^+).*

Proof The global B-Attractor $\mathcal{M}$ is contained in the order interval V. Hence the function $\mathcal{L}$ is well-defined on $\mathcal{M}$. The result is now obtained similarly as in [12, Theorems 2.3 and 3.2]. Instead of using the continuity of the Liapunov function we make use of the fact that $\mathcal{L}$ is bounded below uniformly on V. Hence by the previous Lemma and the invariance of V

$$\lim_{t \to \infty} \mathcal{L}(\varphi(t, u_0))$$

exists for each u_0 in V. This is the only nontrivial modification needed in [12, Theorem 2.3] to match the present situation. We also note that the semiflow (φ, X^+) is of class $\mathcal{AK}$ and that as a consequence the ω-limit set $\omega(u_0)$ is not empty and compact. ■

Remark 4.4 *A class of problems with polynomial nonlinearities satisfying the conditions* (3.2)–(3.5) *is given in [13, Example 3.3.a)].*

We note that the theorem above does not imply that $\omega(u_0)$ consists of a single stationary point of the semiflow. However, we will see in the next section that this is true in the scalar case $N = 1$. In [3] and [14], it is shown that the corresponding result is true for compact intervals

5 Convergence to a Single Equilibrium

In this section we shall employ a result due to Hale and Raugel [10] to show that in one dimension any trajectory converges to a single equilibrium. Suppose that u_0 is any equilibrium of (4.1). We need show that zero is an eigenvalue of

$$\Delta\varphi + f_u(x, u_0(x))\varphi \;=\; \lambda\varphi \qquad \text{in } \mathbf{R}^N \tag{5.1}$$

of algebraic multiplicity at most one, and that there exist no purely imaginary eigenvectors.

Lemma 5.1 *For all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \geq 0$ the operator*

$$\Delta + f_u(\cdot\,, u_0(\cdot)) - \lambda \tag{5.2}$$

is Fredholm of index zero in $\mathrm{BUC}(\mathbf{R}^N)$, $C_0(\mathbf{R}^N)$ and $L_2(\mathbf{R}^N)$. Moreover, all eigenvalues λ with $\operatorname{Re}\lambda \geq 0$ of (5.2) are real. They are algebraically simple in one dimension.

Proof Throughout the proof X denotes one of the spaces $L_2(\mathbf{R}^N)$, $C_0(\mathbf{R}^N)$ or $\mathrm{BUC}(\mathbf{R}^N)$. By assumption (3.5) we have

$$m(x) \;:=\; f_u(x, u_0) \;=\; h(x, u_0) + h_u(x, u_0)u_0.$$

As u_0 is exponentially decaying, and again using (3.5) it follows that m^- is strongly absorbing and $m^+ \in C_0(\mathbf{R}^N)$. Due to Corollary 2.3 and Remark 2.5, we can write $m = -V + g$ where g having compact support, and $[\operatorname{Re} z \geq -\omega_0] \subset \varrho(\Delta - V)$ for some $\omega_0 > 0$ for all spaces under consideration. By interior regularity (e.g. [9, Theorem 8.22]) it follows that for all $u \in L_\infty(\mathbf{R}^N)$ the function $v := (\Delta - V)^{-1}u$ is Hölder continuous with a Hölder constant independent on $\|u\|_\infty \leq M$. Also, $\|\nabla v\|_2$ is bounded uniformly uniformly in $\|u\|_2 \leq \infty$. Hence, embedding theorems for bounded domains imply that

$$v \;\to\; g(\Delta - V)^{-1}v$$

is compact on X. Hence, due to [7, Theorem IX.2.1] the first assertion of the lemma follows.

Note that the operator under consideration is selfadjoint in $L_2(\mathbf{R}^N)$, and therefore has real spectrum. From the above we in particular conclude that any point in the spectrum with positive real part is an eigenvalue. If we show that each eigenvalue in BUC is in L_2 it follows that this part of the spectrum is real. Observe that if $\lambda \in \mathbf{C}$ then the heat kernel associated with $-\Delta + V + \lambda$ is $k_V(t;x,y)e^{\lambda t}$ and thus satisfies the estimate

$$|k_{(V+\lambda)}(t;x,y)| \leq |k_{(V+\mathrm{Re}\,\lambda)}(t;x,y)|$$

If now $\mathrm{Re}\,\lambda \geq 0$ we conclude as in the proof of Theorem 3.1 that any eigenvector to λ decays exponentially and thus belongs to $L_2(\mathbf{R}^N)$. This proves the second part of the lemma.

Suppose now that the dimension is one. We first show that all eigenvalues with $\mathrm{Re}\,\lambda \geq 0$ must be geometrically simple. Suppose to the contrary that (5.1) has two decaying solutions φ_1, φ_2. Then the Wronskian for the associated first order system of ordinary differential equations is

$$W(x) = \varphi_1(x)\varphi_2'(x) - \varphi_2(x)\varphi_1'(x) = \text{constant}.$$

for all $x \in \mathbf{R}$. Note that by Lemma 4.1 the derivatives φ_1' and φ_2' are bounded. If φ_1, φ_2 are both decaying also $W(x)$ has to decay, which is impossible. Hence λ is geometrically simple. As algebraic and geometric multiplicity coincide for selfadjoint operators any eigenvalue $\lambda \geq 0$ is algebraically simple in L_2. The same is true in BUC and C_0. Indeed, suppose an eigenvalue $\lambda \geq 0$ has higher algebraic multiplicity. Then there exists $u \in$ BUC such that $(\Delta - m - \lambda)^2 u = 0$ and $v = (\Delta - m - \lambda)u \neq 0$. Hence v is an eigenvalue of $\Delta + m$, and, according to what we proved in the previous paragraph, $v \in L_2$. This means that λ has a higher multiplicity in L_2 which is a contradiction. ■

Theorem 5.2 *Suppose that $N = 1$. Then, any solution of* (4.1) *with nonnegative $u_0 \in$* BUC($\mathbf{R}$) *converges to an equilibrium in* BUC($\mathbf{R}$).

Proof From Theorem 4.3 we know that all solutions converge to the set of equilibria. Using Lemma 5.1 it is easy to see that all assumptions of [10, Theorem 2.4] satisfied. Hence, the assertion of the Theorem follows. ■

References

[1] W. Arendt and C.J.K Batty, *Exponential stability of a diffusion equation with absorption*, Differential and Integral Equations **6** (1993), 427–448.

[2] C.J.K. Batty, *Asymptotic stability of Schrödinger semigroups: path integral methods*, Math. Ann. **292** (1992), 457–492.

[3] P. Brunovský, P. Poláčik and B. Sandstede, *Convergence in general periodic parabolic equations in one space dimension*, Nonlinear Analysis TMA, **18** (1992), 209–215.

[4] D. Daners and P. Koch Medina, *Abstract evolution equations, periodic problems and applications*, Pitman Research Notes in Mathematics Series, vol. 279, Longman Scientific & Technical, Harlow, Essex, 1992.

[5] D. Daners and P. Koch Medina, *Exponential stability, change of stability and eigenvalue problems for linear time-periodic parabolic equations on* $\mathbf{R}^N$, Differential and Integral Equations **7** (1994), 1265–1284.

[6] E.B. Davies, *Heat kernels and spectral theory*, Cambridge University Press, Cambridge, 1989.

[7] D.E. Edmunds and W.D. Evans, *Spectral theory and differential operators*, Clarendon Press, Oxford, 1987.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher transcendental functions*, vol. II, McGraw-Hill, New York, 1953.

[9] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, Berlin, 1983.

[10] J.K. Hale and G. Raugel, *Convergence in gradient-like systems with applications to PDE*, Z. Angew. Math. Phys. **43** (1992), 63–124.

[11] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc., Providence, Rhode Island, 1968.

[12] O. Ladyzhenskaya, *Attractors for semigroups and evolution equations*, Cambridge University Press, Cambridge, 1991.

[13] S. Merino, *On the existence of the compact global attractor for semilinear reaction diffusion systems on* $\mathbf{R}^N$, J. Differential Equations **132** (1996), 87–106.

[14] T.I. Zelenyak, *Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable*, Differentsial'nye Uravneniya **4** (1968), no. 1, 35–45.