A PRIORI BOUNDS FOR SOLUTIONS AND SPECTRA OF ROBIN BOUNDARY VALUE PROBLEMS

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Abstract. We prove global L_p-bounds for weak solutions of Robin boundary value problems which just depend on an upper bound of the volume of the domain rather than on the particular geometry of the domain. We also show that such uniform estimates can not be true in case of the Neumann problem. We also obtain a generalization of the classical Friedrichs inequality giving a uniform positive lower bound on the spectrum of Robin boundary value problems.

1. Introduction

In these notes we give estimates for solutions of the linear Robin boundary value problem

$$\begin{cases} \mathcal{A}(x,\partial)u = f & \text{in } \Omega \\ \dfrac{\partial}{\partial\nu_a}u + b_0 u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\mathcal{A}(x,\partial)$ is uniformly elliptic operator, Ω is a bounded Lipschitz domain, ν_a the outer conormal vector on the boundary $\partial\Omega$ of Ω, $b_0 \geq \beta > 0$ and $f \in L_p$. We shall derive global a priori bounds for weak solutions of (1.1) depending only on the upper bound of the volume $|\Omega|$ of Ω and not on the particular geometry of the domain. Usually, for sufficiently smooth domains, a priori bounds are obtained by means of Sobolev embedding theorems. As the embedding constants are highly dependent on the geometry of $\partial\Omega$ even at a single point, this can not give good control on the norms of the solution of (1.1). Key to obtain the uniform bounds is an inequality due to Maz'ja [12] which generalizes Sobolev's inequality as well as the classical isoperimetric inequality.

We also prove that uniform estimates do not hold in case of the Neumann problem, i.e. the case $b_0 \equiv 0$. This is done by approaching a domain where the Neumann Laplacian does not have compact resolvent by smooth domains.

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We shall also show that there exists a uniform lower bound for the first eigenvalue of the problem

$$
\left\{
\begin{aligned}
\mathcal{A}(x,\partial) &= \lambda u && \text{in } \Omega \\
\frac{\partial}{\partial \nu_a} u + b_0 u &= 0 && \text{on } \partial\Omega\,.
\end{aligned}
\right.
\tag{1.2}
$$

In case the problem is self adjoint we shall prove a lower bound of the spectrum just depending on the width of a strip containing the domain Ω, the ellipticity constant and a lower bound on b_0 and not on the geometry of the boundary. Note also that a lower bound for the first eigenvalue can be obtained from Maz'ja's inequality, but this bound depends on the volume of the domain rather than on the width of a strip. To do this we prove a generalization of Friedrichs inequality using ideas of Friedrichs in case of the Dirichlet problem. As we use Sard's lemma to obtain the inequality we need to assume that the boundary of Ω is of class C^1. To show that the inequality is true for an arbitrary Lipschitz domain we make use of the fact that one can approximate arbitrary bounded Lipschitz domains by smooth domains in such a way that also the surface integrals approach the original one. This is a useful technical lemma because it allows one to prove inequalities for Lipschitz domains provided one knows them for smooth domains with constants independent of the domain.

Although an immediate consequence of Maz'ja's inequality and the variational characterization of the first eigenvalue λ_1 of (1.2) uniform lower bounds for the spectrum of (1.2) seem not to be known for general Lipschitz domains. Lower bounds for the first eigenvalue in the self adjoint case have been proved under various conditions on the geometry of the domain, so for instance by Bossel [4], Sperb [14]–[16]. These results always require restrictions on the curvature of the boundary $\partial\Omega$. Bandle [3] does not consider the problem of lower bounds of λ_1 in the case of Robin boundary conditions saying there exist no lower bounds without imposing further restrictions ([3], Section III.2.5).

One reason for the interest in estimates which hold uniformly on the domain is that they are very useful for studying how solutions vary as the domain is varied. In particular, our ideas could be combined with those in Dancer [5], [6] to study how the number of solutions of nonlinear Robin boundary value problems vary with the domain shape. We will discuss this elsewhere.

2. Global bounds for weak solutions

We prove our estimates for weak solutions of the equation (1.1) for a general second order elliptic operator $\mathcal{A}(x,\partial)$ of the form

$$
\mathcal{A}(x,\partial)u = -\sum_{i,j=1}^{N} \partial_i\bigl(a_{ij}(x)\partial_j u\bigr) + \sum_{i=1}^{N} a_i(x)\partial_i u + a_0 u
\tag{2.1}
$$

where ν_a the outer conormal vector. We assume that $a_{ij} = a_{ji}$, $a_i, a_0 \in L_\infty(\Omega)$ for $i,j = 1,\dots N$, $b_0 \in L_\infty(\partial\Omega)$ with

$$
b_0(x) \geq \beta
\tag{2.2}
$$

for some $\beta > 0$ and that there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi^i\xi^j \geq \alpha|\xi|^2 \tag{2.3}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$. Let us write

$$[a] := [a_{ij}]_{1\leq i,j\leq N} \qquad \text{and} \qquad a := (a_1, \dots, a_N)\,.$$

Define as usual

$$\mathrm{a}_0(u,v) := \int_\Omega ([a]\nabla u|\nabla v)\,dx + \int_\Omega (a|\nabla u)v\,dx + \int_\Omega a_0 uv\,dx$$

and

$$\mathrm{a}(u,v) := \mathrm{a}_0(u,v) + \int_{\partial\Omega} b_0 uv\,d\sigma$$

for all $u, v \in W_2^1(\Omega)$. Recall that a weak (sub, super) solution of (1.1) is a function $u \in W_2^1(\Omega)$ such that

$$\mathrm{a}(u,v)\ (\leq, \geq) = \langle f, v\rangle \tag{2.4}$$

for all $v \in W_2^1(\Omega)$ with $v \geq 0$, where

$$\langle f, v\rangle := \int_\Omega fv\,dx\,.$$

As already pointed out in the introduction the main ingredient for the proof of our a priori bounds is an inequality due to Maz'ja ([12], see [13], Corollary 3.6.3). It asserts that

$$\|u\|_{\frac{N}{N-1}} \leq \frac{\Gamma(1+N/2)^{1/N}}{N\sqrt{\pi}}\big(\|\nabla u\|_1 + \|u\|_{1,\partial\Omega}\big) \tag{2.5}$$

for all for all $u \in W_2^1(\Omega)$. From this inequality we get the classical isoperimetric inequality by setting $u \equiv 1$, and the Sobolev inequality by considering only functions in $\mathring{W}_2^1(\Omega)$. Applying this inequality to u^2 we obtain for any Lipschitz domain Ω with volume $|\Omega| \leq M$

$$\|u\|_{\frac{2N}{N-1}} \leq c(N,M)\big(\|\nabla u\|_2^2 + \|u\|_{2,\partial\Omega}^2\big)^{1/2} \tag{2.6}$$

for all $u \in W_2^1(\Omega)$ where $c(N,M)$ is a constant depending only on N and $M > 0$ ([13], Corollary 4.11.1/2).

We shall prove a priori estimates for (sub, super) solutions of (1.1) for $f \in L_p(\Omega)$ just in terms of

$$\alpha^{-1}\|f\|_p\,, \qquad \delta := \frac{\|a\|_\infty^2}{\alpha^2} + \frac{\|a_0^-\|_\infty}{\alpha}\,, \qquad \gamma := \alpha\beta^{-1} \tag{2.7}$$

as well as $|\Omega|$, N and p. Here, $a_0^- := \sup\{-a_0, 0\}$ denotes the negative part of a_0. Let us start with the case $p < N$.

2.1 Theorem. *Let $M>0$ and $p\in\left(\frac{2N}{N+2},N\right)$ be fixed. Then there exists a constant $C>0$ only dependent on M, p and N such that for any (sub, super) solution u of* (1.1)

$$\|(u^+,u^-)u\|_{\frac{Np}{N-p}}\leq C\max\{1,\gamma\}\left(\alpha^{-1}\|f\|_p+\max\{1,\gamma^\mu\}\delta^{1+\mu}\|(u^+,u^-)u\|_2\right) \tag{2.8}$$

for all $f\in L_p(\Omega)$ whenever $|\Omega|\leq M$, where $\mu:=1$ if $p\in\left(\frac{2N}{N+2},2\right]$ and $\mu:=\frac{N(p-2)}{p}$ if $p>2$. Here, u^+ and u^- are the positive and the negative part of u.

Proof. We only give the proof for subsolutions. The corresponding result for supersolutions follows by replacing u by $-u$. For solutions combine the results for sub- and supersolutions. For $q\geq 1$ and $m>0$ define

$$G_m(s):=\begin{cases} s^{\frac{q+1}{2}} & \text{for } s\leq m\\ m^{\frac{q+1}{2}}+\frac{q+1}{2}m^{\frac{q-1}{2}}(s-m) & \text{for } s\geq m\end{cases}$$

and

$$H_m(s):=\begin{cases} s^{q} & \text{for } s\leq m\\ m^{q}+\frac{q+1}{2}m^{q-1}(s-m) & \text{for } s\geq m\end{cases}$$

Set $w:=G_m(u^+)$ and $v:=H_m(u^+)$. Then, $v,w\in W_2^1(\Omega)$ (c.f [10], Theorem 7.8). An easy calculation then shows that

$$([a]\nabla w|\nabla w)\leq\frac{q+1}{2}([a]\nabla u|\nabla v)\,,\quad (a|\nabla w)w=\frac{q+1}{2}(a|\nabla u)v\quad\text{and}\quad uv\leq w^2\,.$$

Using the uniform ellipticity (2.3) we thus obtain

$$\begin{aligned}\alpha\|\nabla w\|_2^2&\leq\int_\Omega([a]\nabla w|\nabla w)\,dx\leq\frac{q+1}{2}\,\mathrm{a}_0(u,v)-\int_\Omega(a|\nabla w)w\,dx-\frac{q+1}{2}\int_\Omega a_0w^2\,dx\\ &\leq\frac{q+1}{2}\,\mathrm{a}_0(u,v)+\|a\|_\infty\|\nabla w\|_2\|w\|_2+\frac{q+1}{2}\|a_0^-\|_\infty\|w\|_2^2\\ &\leq\frac{q+1}{2}\,\mathrm{a}_0(u,v)+\|a\|_\infty\left(\varepsilon\|\nabla w\|_2^2+\frac{1}{4\varepsilon}\|w\|_2^2\right)+\frac{q+1}{2}\|a_0^-\|_\infty\|w\|_2^2\end{aligned}$$

for arbitrary $\varepsilon>0$. If we set $\varepsilon=\frac{\alpha}{2\|a\|_\infty}$ and take into account the definition of δ we get

$$\|\nabla w\|_2^2\leq\frac{q+1}{2}\left(\frac{2}{\alpha}\,\mathrm{a}_0(u,v)+2\delta\|w\|_2^2\right) \tag{2.9}$$

Observe that from the definition of v and w it follows that

$$w^2\leq\frac{q+1}{2}\,uv\,.$$

Together with (2.2), (2.6) and (2.9) and using the fact that u is a subsolution we have that

$$\begin{aligned}\|w\|_{\frac{2N}{N-1}} &\leq c\max\{1,\gamma\}q\big(\alpha^{-1}\,\mathrm{a}(u,v)+\delta\|w\|_2^2\big)\\ &\leq c\max\{1,\gamma\}q\big(\alpha^{-1}\langle f,v\rangle+\delta\|w\|_2^2\big)\end{aligned}\tag{2.10}$$

where c depends only on N and M. From the definition of v and w it is easy to see that $v\leq w^{\frac{2q}{q+1}}$ and therefore by Hölder's inequality

$$\langle f,v\rangle\leq\|f\|_p\|w^{\frac{2q}{q+1}}\|_{p'}\,.\tag{2.11}$$

Using again Hölder's inequality we get the estimate $\|w\|_2^2\leq\|w^{\frac{2}{q+1}}\|_p\|w^{\frac{2q}{q+1}}\|_{p'}$, and since by definition we have $G_m(s)\leq s^{\frac{q+1}{2}}$ we have that

$$\|w\|_2^2\leq|\Omega|^{\frac{1}{p}-\frac{1}{2}}\|u^+\|_2\|w^{\frac{2q}{q+1}}\|_{p'}\tag{2.12}$$

if $p\in\big(\frac{2N}{N+2},2\big]$. If $p>2$ we have $p'<2<\frac{2N}{(N-1)}$ and we can apply a well known interpolation inequality (c.f. [10], Sect. 7.1) to obtain

$$\|w\|_2^2\leq 2\big(\varepsilon\|w\|^2_{\frac{2N}{N-1}}+\varepsilon^{-\mu}\|w\|^2_{p'}\big)\tag{2.13}$$

with $\mu:=\frac{N(p-2)}{p}$ and $\varepsilon>0$ arbitrary. Moreover, by Cauchy-Schwarz inequality and the fact that $G_m(s)\leq s^{\frac{q+1}{2}}$ we have that

$$\|w\|^2_{p'}\leq\|w^{\frac{2}{q+1}}\|_{p'}\|w^{\frac{2q}{q+1}}\|_{p'}\leq|\Omega|^{\frac{1}{p'}-\frac{1}{2}}\|u^+\|_2\|w^{\frac{2q}{q+1}}\|_{p'}\,.$$

Choosing ε appropriately we get from (2.10) and (2.11)

$$\begin{aligned}\|w^{\frac{2}{q+1}}\|^{q+1}_{(q+1)N'}&=\|w\|_{\frac{2N}{N-1}}\\ &\leq Cq^2\max\{1,\gamma\}\big(\alpha^{-1}\|f\|_p+\max\{1,\gamma^\mu\}\delta^{1+\mu}\|u^+\|_2\big)\|w^{\frac{2}{q+1}}\|^q_{qp'},\end{aligned}\tag{2.14}$$

where N' its the dual exponent to N. The constant C depends only on M,p and N. We obtain the same inequality for $p<2$ with $\mu=0$ using (2.12) instead of (2.13). Set now $q:=\frac{N(p-1)}{(N-p)}$. Dividing the inequality by $\|w^{\frac{2}{q+1}}\|^q_{\frac{Np}{N-p}}$ we finally have

$$\|w^{\frac{2}{q+1}}\|_{\frac{Np}{N-p}}\leq C\max\{1,\gamma\}\big(\alpha^{-1}\|f\|_p+\max\{1,\gamma^\mu\}\delta^{1+\mu}\|u^+\|_2\big)\,,$$

where C depends only on M,p and N. As $G_m(s)\nearrow s^{\frac{q+1}{2}}$ as m tends to infinity we get from the monotone convergence theorem that $\lim_{m\to\infty}\|w^{\frac{2}{q+1}}\|_{\frac{Np}{N-p}}=\|u^+\|_{\frac{Np}{N-p}}$ completing the proof of the theorem. $\square$

Next we prove L_∞-estimates for the solutions uniform in $|\Omega|\leq M$ provided $p>N$. The estimates are obtained by means of Moser iteration techniques in conjunction with Maz'ja's inequality (2.6).

2.2 Theorem. *Let $M > 0$ and $p > N$ be fixed. Then there exists a constant $C > 0$ only dependent on M, p and N such that for any (sub, super) solution u of* (1.1)

$$\|(u^+, u^-)u\|_\infty \leq C\max\{1,\gamma\}\big(\alpha^{-1}\|f\|_p + \max\{1,\gamma^\mu\}\delta^{1+\mu}\|(u^+, u^-)u\|_2\big) \tag{2.15}$$

for all $f \in L_p(\Omega)$ whenever $|\Omega| \leq M$, where $\mu := \frac{N(p-2)}{p}$.

Proof. We only give the proof for subsolutions. The corresponding result for supersolutions follows by replacing u by $-u$ and for solutions combine the estimates for sub- and supersolutions. We proceed in the same way as in the proof of the above theorem to obtain (2.14). As $G_m(s)\nearrow s^{\frac{q+1}{2}}$ as m tends to infinity we get

$$\|u^+\|_{(q+1)N'}^{q+1} \leq Cq^2\max\{1,\gamma\}\big(\alpha^{-1}\|f\|_p + \max\{1,\gamma^\mu\}\delta^{1+\mu}\|u^+\|_2\big)\|u^+\|_{qp'}^q,$$

saying that $u^+ \in L_{qp'}$ implies the stronger inclusion $u^+ \in L_{(q+1)N'}$. Replacing u^+ by

$$\tilde{w} := \frac{u^+}{C\max\{1,\gamma\}\big(\alpha^{-1}\|f\|_p + \max\{1,\gamma^\mu\}\delta^{1+\mu}\|u^+\|_2\big)}$$

we see that $\|\tilde{w}\|_{(q+1)N'}^{q+1} \leq q^2\|\tilde{w}\|_{qp'}^q$ and therefore

$$\|\tilde{w}\|_{(q+1)N'} \leq q^{\frac{2}{q}}\|\tilde{w}\|_{qp'}^{\frac{q}{q+1}} \tag{2.16}$$

for all $q \geq 1$. Set now

$$q_0 := 1 \qquad \text{and} \qquad q_{n+1} := \frac{q_n + 1}{p'}N'$$

for all $n \in \mathbb{N}$. As $p > N$ $(q_n)_{n\in\mathbb{N}}$ is increasing with $\lim_{n\to\infty} q_n = \infty$. Rewriting (2.16) we obtain

$$\|\tilde{w}\|_{q_{n+1}p'} \leq q^{\frac{2}{q_n}}\|\tilde{w}\|_{q_np'}^{(1+\frac{1}{q_n})^{-1}}$$

for all $n \in \mathbb{N}$. Iterating this inequality yields

$$\|\tilde{w}\|_{q_{n+1}p'} \leq q_n^{\frac{2}{q_n}}\prod_{k=0}^{n-1} q_k^{\frac{2}{q_k}\prod_{i=k+1}^n(1+\frac{1}{q_i})^{-1}}\|\tilde{w}\|_{p'}^{\prod_{k=0}^n(1+\frac{1}{q_k})^{-1}}$$

for all $n \in \mathbb{N}$. As $q_k \geq 1$ and $(1+\frac{1}{q_k})^{-1} < 1$ for all $k \in \mathbb{N}$ this simplifies to

$$\|\tilde{w}\|_{q_{n+1}p'} \leq \prod_{k=0}^{n} q_k^{\frac{2}{q_k}}\|\tilde{w}\|_{p'}^{\prod_{k=0}^n(1+\frac{1}{q_k})^{-1}} \tag{2.17}$$

for all $n \in \mathbb{N}$. Next we estimate the product on the right hand side of (2.17). To do this we need estimates of q_n. By definition of q_n it is clear that

$$\frac{N'}{p'} \leq \frac{q_n}{q_{n-1}} \leq \frac{2N'}{p'}$$

for all $n \in \mathbb{N}$, and therefore $\alpha^n \le q_n \le (2\alpha)^n$ for all $n \in \mathbb{N}$, where $\alpha := \frac{N'}{p'} > 1$. Therefore,

$$c_0 := \prod_{k=0}^{n} q_k^{\frac{2}{q_k}} \le (2\alpha)^{2\sum_{k=0}^{n} k\alpha^{-k}} < \infty.$$

From (2.17) we can now conclude that $\|\tilde{w}\|_\infty \le c_0 \|\tilde{w}\|_{p'}^{\theta}$ with $\theta := \prod_{k=0}^{\infty}(1+\frac{1}{q_k})^{-1}$. Obviously, $\theta < 1$ and therefore, using the fact that $\|\tilde{w}\|_{p'} \le |\Omega|^{\frac{1}{p'}} \|\tilde{w}\|_\infty$, we finally arrive at

$$\|\tilde{w}\|_\infty \le \left(c_0 |\Omega|^{\frac{\theta}{p'}}\right)^{\frac{1}{1-\theta}}.$$

As θ and c_0 depend only on N and p we easily obtain (2.15) using the definition of $\tilde{w}$. This completes the proof of the theorem. □

The above Theorems take a very simple form if the problem is self adjoint, i.e. $a = 0$ and a_0 nonnegative. In this case we obtain the following corollary.

2.3 Corollary. *Let $M > 0$ be fixed, $a = 0$ and $a_0 \ge 0$. Then there exists a constant $C > 0$ only dependent on M, p and N such that for any (sub, super) solution u of* (1.1)

$$\|(u^+, u^-)u\|_{m(p)} \le C \max\{\alpha^{-1}, \beta^{-1}\} \|f\|_p$$

for all $f \in L_p(\Omega)$ whenever $|\Omega| \le M$, where $m(p) = \frac{Np}{N-p}$ if $p \in \left(\frac{2N}{N+2}, N\right)$ and $m(p) := \infty$ if $p > N$.

2.4 Remark. Note that in order to prove Theorem 2.1 and 2.2 we only need that (2.4) holds for all $v \in W_2^1(\Omega)$ such that $v \ge 0$ and $uv \ge 0$, because we only deal with test functions satisfying this condition.

Next we prove an L_∞-estimate for (sub, super) solutions which is independent of $\|(u^+, u^-)u\|_2$. To do this we use ideas of Trudinger [17], who proved the corresponding result in case of Dirichlet boundary conditions (see [10], Theorem 8.16).

2.5 Theorem. *Let $M > 0$ and $p > N$ be fixed and a_0 nonnegative. Then there exists a constant $C > 0$ only dependent on M, p, N, $\max\{1, \gamma\}$ and δ such that for any (sub, super) solution u of* (1.1)

$$\|(u^+, u^-)u\|_\infty \le C\alpha^{-1} \|f\|_p \tag{2.18}$$

for all $f \in L_p(\Omega)$ whenever $|\Omega| \le M$.

In order to prove the above theorem we need the following elementary inequalities.

2.6 Lemma. *Let $0 < s < m$. Then, we have that*

$$(a)\quad \log\frac{m}{m-s} \le \frac{s}{m-s} \qquad (b)\quad \left(\log\frac{m}{m-s}\right)^2 \le \frac{s^2}{m(m-s)}.$$

Proof. (a) follows from the fact that

$$\log\frac{m}{m-s} = \log\left(1 + \frac{s}{m-s}\right) \le \frac{s}{m-s}.$$

To prove (b) we set $t := \frac{s}{m}$. It is then easy to see that (b) is equivalent to the fact that

$$\Big(\log\frac{1}{1-t}\Big)^2 \leq \frac{t^2}{1-t}$$

for all $t \in (0,1)$, which is equivalent to $g(t) := t^{-2}(1-t)\big(\log(1-t)\big)^2 \leq 1$ for all $t \in (0,1)$. Using de l'Hôpital's rule we find that $\lim_{t\to 0} g(t) = 1$. As $g \geq 0$ it is now sufficient to show that $g'(t) < 0$ for all $t \in (0,1)$. As

$$g'(t) = -t^{-2}\log(1-t)\big[2 - t^{-1}(t-2)\log(1-t)\big]\,,$$

it remains to show that $t^{-1}(t-2)\log(1-t) > 2$ for all $t \in (0,1)$. Using the Taylor expansion of $\log(1-t)$ we obtain

$$t^{-1}(t-2)\log(1-t) = 2 + \sum_{k=2}^{\infty}\Big(\frac{2}{k+1} - \frac{1}{k}\Big)t^k > 2$$

for all $t \in (0,1)$ completing the proof of (b). $\square$

Now we are in a position to prove Theorem 2.5.

Proof of Theorem 2.5. As in the previous theorems it is sufficient to give the proof for subsolutions. From Theorem 2.2 we already know that $u^+ \in L_\infty$, so that it makes sense to define

$$v := \frac{u^+}{m+k-u^+}$$

where $m := \|u^+\|_\infty$ and $k := \alpha^{-1}\|f\|_p$. It is easy to see that $v \in W_2^1(\Omega)$, so we can use v as a test function in (2.4). By the ellipticity (2.3) of $\mathcal{A}(x,\partial)$ we have that

$$\begin{aligned}\int_\Omega ([a]\nabla u|\nabla v)\,dx &= (m+k)\int_\Omega \frac{(m+k)^2}{(m+k-u^+)^2}([a]\nabla u^+|\nabla u^+)\,dx\\ &\leq \alpha(m+k)\int_\Omega \frac{|\nabla u^+|^2}{(m+k-u^+)^2}\,.dx\end{aligned}$$

Using the definition of v and that u is a subsolution we obtain

$$\begin{aligned}&\alpha\int_\Omega \frac{|\nabla u^+|^2}{(m+k-u^+)^2}\,dx + \frac{1}{2}\frac{\beta}{m+k}\int_{\partial\Omega}\frac{(u^+)^2}{m+k-u^+}\,d\sigma\\ &\quad\leq \frac{1}{m+k}\,\mathrm{a}(u,v) + \frac{\|a\|_\infty}{m+k}\int_\Omega \frac{|u^+||\nabla u^+|}{m+k-u^+}\,dx\\ &\quad\leq \alpha|\Omega|^{\frac{1}{p'}} + \|a\|_\infty\Big(\varepsilon\int_\Omega\frac{|\nabla u^+|^2}{(m+k-u^+)^2}\,dx + \frac{1}{4\varepsilon}|\Omega|^2\Big)\\ &\quad= \frac{\alpha}{2}\int_\Omega \frac{|\nabla u^+|^2}{(m+k-u^+)^2}\,dx + \frac{\alpha}{2}\big(\delta|\Omega|^2 + 2|\Omega|^{\frac{1}{p'}}\big)\end{aligned} \tag{2.19}$$

if we set $\varepsilon := \left(2\sqrt{\delta}\right)^{-1}$. We introduce now an auxiliary function w defined by

$$w := \log \frac{m+k}{m+k-u^+}\,.$$

Then, it is clear that $\nabla w = \frac{\nabla u^+}{m+k-u^+}$, and by Lemma 2.6($b$)

$$\int_{\partial\Omega} w^2\,d\sigma \leq \frac{1}{m+k}\int_{\partial\Omega}\frac{(u^+)^2}{m+k-u^+}\,d\sigma\,.$$

Therefore, by (2.6) and (2.19) we conclude that

$$\|w\|_2 \leq c\max\{1,\gamma\}\left(\delta M^2 + 2M^{\frac{1}{p'}}\right). \tag{2.20}$$

where c merely depends on M and N. In the next step we show that we can apply Theorem 2.2 to estimate the L_∞-norm of w. In order to do this we take

$$v := \frac{\varphi}{m+k-u^+}$$

as a test function in (2.4) where $\varphi \in W_2^1(\Omega)\cap L_\infty(\Omega)$ with $\varphi \geq 0$ and $u\varphi \geq 0$. Then, in the same way as in the proof of [10], Theorem 8.16 we obtain

$$\mathrm{a}(w,\varphi) \leq \mathrm{a}(u,v) \leq \langle f, v\rangle \leq \langle fk^{-1},\varphi\rangle\,.$$

The only difference is that we have to estimate the boundary integral. This is done in an obvious way using Lemma 2.6(a). By a truncation argument we easily see that the above inequality holds for all nonnegative $\varphi \in W_2^1(\Omega)$ such that $u\varphi \geq 0$. Finally note that in this case by definition of w we also have that $w\varphi \geq 0$. Thus, we can apply Theorem 2.2 and Remark 2.4 together with (2.20) to conclude that

$$\|w\|_\infty \leq C\max\{1,\gamma\}\left(1+\max\{1,\gamma^{1+\mu}\}\delta^{1+\mu}(\delta M^2 + 2M^{\frac{1}{p'}})\right). \tag{2.21}$$

The desired result follows now by taking the exponential of (2.21) and the fact that $\|w\|_\infty = k^{-1}(m+k)$. $\square$

We shall discuss now some consequences of the above results. In a first remark we discuss briefly further generalizations.

2.7 Remarks. (a) All the above results are still true (with essentially the same proof) if the boundary conditions are Robin on part of the boundary and Dirichlet on the remainder of the boundary. This is because (2.6) is still valid.

(b) One can get better estimates for other classes of domains satisfying an inequality similar to (2.6) with the $L_{\frac{2N}{N-1}}$-norm replaced by an L_r-norm with $r > \frac{2N}{N-1}$, i.e.

$$\|u\|_r \leq c_0\left(\|\nabla u\|_2^2 + \|u\|_{2,\partial\Omega}^2\right)^{1/2} \tag{2.22}$$

holds for all $u \in W_2^1(\Omega)$. Then, an estimate similar to (2.8) holds for $p < r^* := \frac{r}{r-2}$ and (2.15) if $p > r^*$. More precisely, an inspection of the proofs of Theorem 2.1 and 2.2 shows that for any (sub, super) solution u of (1.1)

$$\|(u^+, u^-)u\|_{m(r,p)} \leq C \max\{1, \gamma\}\bigl(\alpha^{-1}\|f\|_p + \max\{1, \gamma^\mu\}\delta^{1+\mu}\|(u^+, u^-)u\|_2\bigr),$$

where $m(r,p) = \frac{r^* p}{r^* - p}$ if $p \in \bigl(\frac{2N}{N-2}, r^*\bigr)$ and $m(r,p) = \infty$ if $p > r^*$. The constant C depends only on r, p, c_0 and an upper bound for $|\Omega|$, and μ has to be replaced by $\mu := r^* \frac{p-2}{p}$. Classes of domains satisfying (2.22) are characterized in [12] and [13].

To discuss further consequences let us introduce some notation. As $\mathrm{a}\colon W_2^1 \times W_2^1 \to \mathbb{R}$ is a continuous bilinear form it induces an operator $A \in \mathcal{L}\bigl(W_2^1, (W_2^1)'\bigr)$ such that

$$\mathrm{a}(u, v) = \langle Au, v\rangle$$

for all $u, v \in W_2^1$. Moreover it is well known that there exists $\lambda_0 \in \mathbb{R}$ such that $(u, v) \mapsto \mathrm{a}(u, v) + \lambda(u, v)_{L_2}$ is coerzive for all $\lambda \geq \lambda_0$. Therefore by the theorem of Lax-Milgram $\lambda \in \varrho(A)$ whenever $\lambda \geq \lambda_0$. In particular, this implies that A is a closed operator on $(W_2^1)'$ with domain of definition $D(A) = W_2^1$. As Ω is a Lipschitz domain we have from Sobolev embedding theorem that $W_2^1 \hookrightarrow L_p$ for $p \in \bigl[1, \frac{2N}{N-2}\bigr)$ and hence, by duality, $L_p \hookrightarrow (W_2^1)'$ for $p > \frac{2N}{N+2}$. Consequently, the L_p-realization A_p of A, defined by

$$A_p u := Au \qquad \text{for} \qquad D(A) := \{u \in D(A) \cap L_p\colon Au \in L_p\}$$

is a closed operator on L_p for all $p > \frac{2N}{N+2}$. For the sake of completeness we summarize the spectral properties of A and A_p in the following proposition.

2.8 Proposition. *The operator A_p has compact resolvent and $\varrho(A_p) = \varrho(A)$ holds for all $p > \frac{2N}{N+2}$. Furthermore, $(\lambda - A_p)^{-1} L_p \subset L_{m(p)}$, where $m(p) = \frac{Np}{N-p}$ if $p \in \bigl(\frac{2N}{N+2}, N\bigr)$ and $m(p) = \infty$ if $p > N$ and the resolvent is a compact operator from L_p into $L_{m(p)}$ if $p < N$ and from L_p into L_q for all $q < \infty$ if $p > N$.*

Proof. The smoothing properties of the resolvent follow easily from Theorem 2.1 and 2.2. We prove now the last assertion which in particular implies the compactness of the resolvent. In case $p \in \bigl(\frac{2N}{N+2}, 2\bigr]$ the compactness readily follows from the following diagram whenever $\lambda \in \varrho(A)$ since the embeddings are compact.

$$\begin{array}{ccc} (W_2^1)' & \hookleftarrow & L_p \\ \Big\downarrow{\scriptstyle (\lambda - A)^{-1}} & & \Big\downarrow{\scriptstyle (\lambda - A_p)^{-1}} \\ W_2^1 & \hookrightarrow & L_{\frac{Np}{N-p}} \end{array}$$

Compactness in case $p \in (2, N)$ follows from a compactness property of the Riesz-Thorin interpolation. Indeed, let $2 < p < q < N$ be arbitrary. Then, the operator $(\lambda - A_p)^{-1} \in \mathcal{L}(L_p, L_{\frac{pN}{N-p}})$ is obtained by interpolation between the compact operator $(\lambda - A_2)^{-1} \in \mathcal{L}(L_2, L_{\frac{2N}{N-2}})$ and the bounded operator $(\lambda - A_q)^{-1} \in \mathcal{L}(L_q, L_{\frac{qN}{N-q}})$, and therefore compact by a result due to Krasnosel'skiĭ [11]. In a similar way we obtain compactness if $p > N$.

Now we show the p-independence of the spectrum of A_p. To do that note first that by Sobolev embedding theorem $(\lambda - A)^{-1} W_2^1 \subset L_p$ if $p \in \left(\frac{2N}{N+2}, \frac{2N}{N-2}\right)$. Then, a bootstrapping argument using the smoothing properties of the resolvent shows that all the eigenfunctions of A are in L_∞. By the compactness of the resolvent we know that the spectrum consists only of eigenvalues of finite multiplicity and therefore we see that $\varrho(A) = \varrho(A_p)$ for all $p > \frac{2N}{N+2}$. $\square$

Finally, note that Theorem 2.5 tells us that $0 \in \varrho(A)$ provided a_0 is nonnegative, and that

$$\|A_p^{-1}\|_{\mathcal{L}(L_p, L_\infty)} \leq C\alpha^{-1} \tag{2.23}$$

holds for all $p > N$ with a constant C depending merely on p and an upper bound for $|\Omega|, \delta$ and γ.

3. Neumann Problems

In this section we show that uniform estimates for weak solutions as in the case of Dirichlet or Robin boundary condition can not be true for Neumann boundary conditions. As the Laplacian under Neumann boundary conditions is not invertible we consider the following problem:

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \dfrac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega \,, \end{cases} \tag{3.1}$$

where Ω is an arbitrary bounded domain in $\mathbb{R}^N$. By a weak solution we mean as usual a function $u \in W_2^1(\Omega)$ such that

$$\int_\Omega (\nabla u | \nabla v) + uv \, dx = \int_\Omega fv \, dx \tag{3.2}$$

for all test functions $v \in W_2^1(\Omega)$. Taking the u as a test function we easily see that

$$\|u\|_2 \leq \|f\|_2 \,, \tag{3.3}$$

which means that we have a uniform L_2-bound for the solution. We shall see that there is no uniform L_p-bound for all $p > 2$. To show this we need an approximation theorem for solutions defined in different domains. Let $(\Omega_n)_{n \in \mathbb{N}}$ be a sequence of domains such that $\Omega_n \subset \Omega$, $\Omega_n \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcup_{n=1}^\infty \Omega_n = \Omega$. In this case we say that $\Omega_n \nearrow \Omega$. If Ω_1, Ω_2 are open bounded subsets of $\mathbb{R}^N$ we write $\Omega_1 \subset\subset \Omega_2$ if $\overline{\Omega}_1 \subset \Omega_2$.

3.1 Proposition. *Suppose that $\Omega_n \nearrow \Omega$ and that u_n is the solution of (3.1) with Ω replaced by Ω_n. Then,*

$$\lim_{n\to\infty} u_n|_{\Omega'} = u|_{\Omega'} \tag{3.4}$$

in $W_2^1(\Omega')$ for all $\Omega' \subset\subset \Omega$, where u is the solution of (3.1).

Proof. Note first that it follows easily from (3.2) using u_n as a test function and (3.3) that

$$\|u_n\|_{W_2^1(\Omega_n)} \leq \|f\|_2 \,. \tag{3.5}$$

Let $(\Omega'_n)_{\in\mathbb{N}}$ be a sequence of domains such that $\Omega'_n \subset\subset \Omega_n$ and $\bigcup_{n=1}^\infty \Omega'_n = \Omega$. Then, by interior regularity of the weak solution u_n (c.f. [10], Theorem 8.8) for any $n \in \mathbb{N}$ we find a constant $c > 0$ such that

$$\|u_k\|_{W_2^2(\Omega'_n)} \leq c\big(\|u_k\|_{W_2^1(\Omega'_n)} + \|f\|_2\big) \leq 2c\|f\|_2$$

for all $k \geq n+1$, where we used (3.5) for the last inequality. This means that the sequence $(u_k)_{k>n}$ is bounded in $W_2^2(\Omega'_n)$. Thus, we can find a subsequence converging strongly in $W_2^1(\Omega'_n)$ to w_n. By a diagonalization argument we can make sure that $w_{n+1}|_{\Omega'_n} = w_n$ for all $n \in \mathbb{N}$. As $\bigcup_{n=1}^\infty \Omega'_n = \Omega$ we can define a function $w \in W_{2,loc}^1(\Omega)$ in such a way that $w|_{\Omega'_n} = w_n$ for all $n \in \mathbb{N}$. From (3.5) we easily deduce that $w \in W_2^1(\Omega)$ with $\|w\|_{W_2^1(\Omega)} \leq \|f\|_2$. From the construction of w it is clear that $\lim_{n\to\infty} u_n|_{\Omega'} = w|_{\Omega'}$ in $W_2^1(\Omega')$ for all $\Omega' \subset\subset \Omega$. If we can show that $u = w$ this proves (3.4). By uniqueness of the solution it is therefore sufficient to show that w is a solution of (3.1). To this end let $\varphi \in W_2^1(\Omega)$ be arbitrary. Then, given $\varepsilon > 0$ we find $n_0 \in \mathbb{N}$ such that

$$\int_{\Omega\setminus\Omega_n} |\nabla\varphi|^2 + \varphi^2 \, dx < \varepsilon^2$$

for all $n \geq n_0$. From Cauchy-Schwarz inequality and (3.5) we easily deduce that

$$\int_{\Omega_n\setminus\Omega_{n_0}} (\nabla u_n|\nabla\varphi) + u_n\varphi \, dx < \varepsilon\|f\|_2 \,.$$

Hence,

$$\int_{\Omega_n} (\nabla u_n|\nabla\varphi) + u_n\varphi \, dx \leq \int_{\Omega_{n_0}} (\nabla u_n|\nabla\varphi) + u_n\varphi \, dx + \varepsilon\|f\|_2 \,.$$

A similar argument shows that

$$\int_{\Omega\setminus\Omega_{n_0}} (\nabla w|\nabla\varphi) + w\varphi \, dx \leq \varepsilon\|f\|_2 \,.$$

As $\varepsilon > 0$ was arbitrary we obtain by combining the above results that

$$\lim_{n\to\infty} \int_{\Omega_n} (\nabla u_n | \nabla \varphi) + w\varphi \, dx = \int_\Omega (\nabla w | \nabla \varphi) + w\varphi \, dx \, .$$

On the other hand, it is clear that

$$\lim_{n\to\infty} \int_{\Omega_n} f\varphi \, dx = \int_\Omega f\varphi \, dx \, .$$

which shows that w is the weak solution of (3.1) completing the proof of the proposition. $\square$

3.2 Proposition. *If Ω is a bounded domain such that the embedding of $W_2^1(\Omega)$ into $L_2(\Omega)$ is not compact, then*

$$R := (-\Delta_N + \mathrm{I}\,)^{-1} \notin \mathcal{L}\big(L_2(\Omega), L_p(\Omega)\big)$$

for all $p > 2$. Here, Δ_N denotes the Laplacian under Neumann boundary conditions.

For an example of a bounded domain where the embedding of $W_2^1(\Omega)$ into $L_2(\Omega)$ is not compact see e.g. [7], Theorem V.4.21.

Proof. Suppose that $R \in \mathcal{L}\big(L_2(\Omega), L_p(\Omega)\big)$ for some $p > 2$. Then, using Hölder's inequality we see that, given $\varepsilon > 0$, there exists a domain Ω' with $\overline{\Omega}' \subset \Omega$ such that

$$\|Rf\|_{2,\Omega\setminus\Omega'} \le \varepsilon \tag{3.6}$$

for all $f \in \mathbb{B} := \{g \in L_2(\Omega) \colon \|g\|_2 \le 1\}$. (We choose Ω' such that $\Omega \setminus \Omega'$ has small measure.) On the other hand, we know that $\{Rf \colon f \in \mathbb{B}\}$ is a bounded set in $W_2^1(\Omega)$ and therefore, $\{Rf|_{\Omega'} \colon f \in \mathbb{B}\}$ is relatively compact in $L_2(\Omega')$. This, together with (3.6) easily implies the compactness of R as an operator on $L_2(\Omega)$, which means that $-\Delta_N$ has compact resolvent on $L_2(\Omega)$. But this is not possible because this is equivalent of $W_2^1(\Omega)$ being compactly embedded in $L_2(\Omega)$. Therefore our claim follows. $\square$

3.3 Remark. One can use interpolation theory and duality to show that

$$(-\Delta + \mathrm{I}\,)^{-1} \notin \mathcal{L}\big(L_p(\Omega), L_q(\Omega)\big)$$

for all $1 < p < q \le \infty$ if Ω satisfies the condition of the above Proposition.

As a corollary we obtain our result that the solution of (3.1) does not satisfy a uniform estimate for all smooth domains with a given bound on the volume.

3.4 Corollary. *Let $M > 0$ be given. Then, the map $R := (-\Delta_N + \mathrm{I}\,)^{-1}$ can not satisfy a uniform bound with respect to all smooth domains with $|\Omega| \le M$ as a map from $L_2(\Omega)$ to $L_p(\Omega)$ whenever $p > 2$.*

Proof. As the embedding constant L_q into L_p depends just on the volume of the domain and the exponents p and q it obviously enough to prove the assertion for p arbitrarily close to 2. Choose Ω as in the previous proposition and suppose that $(\Omega_n)_{n\in\mathbb{N}}$ is a sequence of smooth domains such that $\Omega_n\nearrow\Omega$. Suppose that u_n is the solution of a (3.1) with Ω replaced by Ω_n and u the solution of (3.1) with the same $f\in L_2(\Omega)$. Then, from Proposition 3.1 we conclude that $\lim_{n\to\infty}\|u_n\|_{p,\Omega'}=\|u\|_{p,\Omega'}$ for $p>2$ sufficiently close to 2 and $\Omega'\subset\subset\Omega$ (Here we used Sobolev embedding theorem). Therefore, if $\|R\|_{\mathcal{L}(L_2(\Omega_n),L_p(\Omega_n))}$ was uniformly bounded with respect to $n\in\mathbb{N}$ for some $p>2$ this would imply that the same is true for the domain Ω. But this contradicts Proposition 3.2. $\square$

3.5 Remarks. (a) By interpolation and duality it is easily seen that there is no uniform bound for $(-\Delta_N+\mathrm{I})^{-1}$ as a map from $L_p(\Omega)$ to $L_q(\Omega)$ whenever $1<p<q\leq\infty$.

(b) On the other hand, if we restrict ourselves to consider just domains in a class for which

$$\|u\|_r\leq c\|u\|_{W_2^1} \tag{3.7}$$

holds for some $r>2$ and a constant c independent of the domain in the given class we can obtain uniform bounds on $(-\Delta_N+\mathrm{I})^{-1}$ as a map from $L_p(\Omega)$ to $L_q(\Omega)$ for suitable $q>p$. Indeed, proceeding similar as in the proof of Theorem 2.1 we can show that

$$(-\Delta_N+\mathrm{I})^{-1}\in\mathcal{L}\big(L_p(\Omega),L_{\frac{r^*p}{r^*-p}}(\Omega)\big) \tag{3.8}$$

with a norm depending only on c in (3.7), r and p provided $p\in\big(\frac{2N}{N+2},r^*\big)$. In case $p>r^*$ we can prove using the same iteration method as in the proof of Theorem 2.2 that

$$(-\Delta_N+\mathrm{I})^{-1}\in\mathcal{L}\big(L_p(\Omega),L_\infty(\Omega)\big) \tag{3.9}$$

with norm depending on the same quantities as above. Here, as in Remark 2.7(b) we have set $r^*:=\frac{r}{r-2}$. Classes of domains satisfying (3.7) are for instance domains satisfying the cone condition with the same fixed cone (e.g. [1], Lemma V.5.10). In that case $r=\frac{2N}{N-2}$ for $n\geq3$ and arbitrary $r<\infty$ if $N=2$. More general classes are considered in [13], where s is not necessarily the Sobolev exponent.

4. Lower bounds for the spectrum

In this section we shall first prove the existence of a uniform lower bound for the first eigenvalue of (1.2), where $\mathcal{A}(x,t)$ satisfies the hypotheses of Section 2 and a_0 is nonnegative. We shall show then, that in case of a self adjoint problem (i.e. $a\equiv0$) the eigenvalues of (1.2) are uniformly bounded away from zero by a constant depending only on the width of a strip containing the domain and a lower bound for α and β.

Note that it follows from the weak maximum principle (c.f. [10], Theorem 8.1 and 8.19) and the fact that $b_0>0$ that $\lambda_1>0$. It is clear that

$$\lambda_1^{-1}=\operatorname{spr}(A_p)\leq\|A_p\|_{\mathcal{L}(L_p)}\,.$$

Therefore, by (2.23) and the fact that $\varrho(A)=\varrho(A_p)$ by Proposition 2.8 we have the following result.

4.1 Theorem. *Suppose that a_0 is nonnegative. Then there exist a constant $\lambda_0 > 0$ depending only on upper bounds of $|\Omega|$, δ and γ such that the first eigenvalue λ_1 of (1.2) satisfies $\lambda_1 \geq \alpha\lambda_0$. Here, γ and δ are defined as in (2.7).*

Let us now assume that $a \equiv 0$. In that case, using the the variational characterization of the first eigenvalue of (1.2)

$$\lambda_1 = \inf_{\substack{u \in W_2^1(\Omega) \\ u \neq 0}} \frac{\mathrm{a}(u,u)}{\|u\|_2^2} \tag{4.1}$$

we prove now a more explicit estimate for the first eigenvalue. In order to do this we need an inequality of the form

$$\|u\|_2 \leq c\big(\|\nabla u\|_2^2 + \|u\|_{2,\partial\Omega}^2\big)^{\frac{1}{2}} \tag{4.2}$$

for all $u \in W_2^1(\Omega)$. This is of course well known. The proof of (4.1) is usually given by localization at the boundary. Therefore, the constant c depends on the parametrizations and hence on the geometry of the domain. Alternatively one could use (2.2) to prove the inequality estimating the $L_{\frac{2N}{N-1}}$-norm by the weaker L_2-norm. The constant then depends only on N and upper bounds for $|\Omega|$ and $\max\{\alpha^{-1}, \beta^{-1}\}$. We shall generalize the classical proof of Friedrichs inequality ([9]) in case of the Dirichlet problem for smooth domains, and use an approximation argument in the case of a Lipschitz domain in order to show that c in fact does only depend on the width of a strip containing Ω and a lower bound for b_0.

4.2 Theorem. *Let Ω be a Lipschitz domain contained in a strip of width $d_0 > 0$. Then,*

$$\|u\|_2^2 \leq 2d_0\big(d_0\|\nabla u\|_2^2 + \|u\|_{2,\partial\Omega}^2\big) \tag{4.3}$$

for all $u \in W_2^1(\Omega)$.

Proof. We prove the assertion in case Ω is of class C^1. In the general case the result is obtained by approximating a Lipschitz domain by smooth domains in such a way that also the boundary integrals approximate the original ones. For details we refer to the appendix.

Let Ω be a bounded domain contained in a strip of width d_0. By changing the coordinates we may assume without loss of generality that the strip containing Ω is parallel to the hyperplane F spanned by the first $N-1$ vectors of the natural basis of $\mathbb{R}^N$. Denote by P the orthogonal projection onto F. As Ω is of class C^1 the restriction $P_{\partial\Omega}$ of P to $\partial\Omega$ is a C^1-map from $\partial\Omega$ to F. By Sard's Theorem (e.g. [2], Theorem 4.3.1), the set of critical values $\mathcal{S}$ of $P_{\partial\Omega}$ has Lebesgue measure zero in F. Clearly, $\mathcal{S}$ is a closed set in F. Its complement, the set $\mathcal{O}$ of regular values of $P_{\partial\Omega}$, is an open set. Moreover, the cardinality of $P_{\partial\Omega}^{-1}(\hat{x})$ is finite for all $\hat{x} \in \mathcal{O}$ and locally constant (e.g. [2], Prop. 4.1.5 and Cor. 4.1.6). Hence, for any connected component V of $\mathcal{O}$ the cardinality of $P_{\partial\Omega}^{-1}(\hat{x})$ is constant for all $\hat{x} \in V$. Therefore we can find finitely many functions $\varphi_i, \psi_i \in C^1(V, \partial\Omega)$, $i = 1, \dots, k$ such that $\varphi_i < \psi_i$ for all $i = 1, \dots, k$ and $\partial\Omega \cap V \times \mathbb{R}$ is the disjoint union of the graphs of these functions. Let now $x = (\hat{x}, x_N)$ be an arbitrary point in $\Omega_V := \Omega \cap V \times \mathbb{R}$. Then

$$\varphi_i(\hat{x}) < x_N < \psi_i(\hat{x})$$

holds for some $i \in \{1, \dots, k\}$. If $u \in C^\infty(\overline{\Omega})$ is arbitrary then

$$u(x) = u\big(\hat{x}, \varphi_i(\hat{x})\big) + \int_{\varphi_i(\hat{x})}^{x_N} \partial_\xi u(\hat{x}, \xi)\, d\xi\,.$$

Applying Cauchy's inequality we obtain

$$|u(x)| \leq \big|u\big(\hat{x}, \varphi_i(\hat{x})\big)\big| + d_0^{\frac{1}{2}} \Big(\int_{\varphi_i(\hat{x})}^{\psi_i(\hat{x})} |\partial_\xi u(\hat{x}, \xi)|^2\, d\xi\Big)^{\frac{1}{2}},$$

where d_0 is the width of the strip containing Ω. An elementary inequality yields

$$|u(x)|^2 \leq 2\Big(\big|u\big(\hat{x}, \varphi_i(\hat{x})\big)\big|^2 + d_0 \int_{\varphi_i(\hat{x})}^{\psi_i(\hat{x})} |\partial_\xi u(\hat{x}, \xi)|^2\, d\xi\Big)\,.$$

Integrating over Ω_V we obtain

$$\begin{aligned}
\int_{\Omega_V} |u(x)|^2 \leq 2\Big(&\int_V \sum_{i=1}^k \int_{\varphi_i(\hat{x})}^{\psi_i(\hat{x})} \big|u\big(\hat{x}, \varphi_i(\hat{x})\big)\big|^2\, dx_N\, d\hat{x} \\
&+ d_0 \int_V \sum_{i=1}^k \int_{\varphi_i(\hat{x})}^{\psi_i(\hat{x})} \int_{\varphi_i(\hat{x})}^{\psi_i(\hat{x})} |\partial_\xi u(\hat{x}, \xi)|^2\, d\xi\, dx_N\, d\hat{x}\Big) \\
\leq 2d_0 \Big(&\sum_{i=1}^k \int_V \big|u\big(\hat{x}, \varphi_i(\hat{x})\big)\big|^2 \sqrt{1 + |\nabla\varphi_i(\hat{x})|^2}\, d\hat{x} \\
&+ d_0 \sum_{i=1}^k \int_V \int_{\varphi_i(\hat{x})}^{\psi_i(\hat{x})} |\partial_\xi u(\hat{x}, \xi)|^2\, d\xi\, d\hat{x}\Big) \\
= 2d_0 \Big(&\sum_{i=1}^k \|u\|^2_{L_2(\varphi_i(V))} + d_0 \|\nabla u\|^2_{L_2(\Omega_V)}\Big)\,.
\end{aligned}$$

Using that the Lebesgue measure of $\mathcal{S}$ is zero, and adding up over all connected components of $\mathcal{O}$ we see that (4.3) is true for all $u \in C^\infty(\overline{\Omega})$. By density of $C^\infty(\overline{\Omega})$ in $W_2^1(\Omega)$ this is true for all $u \in W_2^1(\Omega)$. $\square$

As an easy consequence of the above Result and (4.1) we obtain the following estimate for the first eigenvalue of (1.2).

4.3 Corollary. *Suppose Ω satisfies the same conditions as in the above Theorem. Then, the first eigenvalue λ_1 of (1.2) satisfies*

$$\lambda_1 \geq \frac{1}{2d_0 \max\{d_0\alpha^{-1}, \beta^{-1}\}}.$$

Finally we remark that an optimal estimate (i.e. an isoperimetric inequality) for domains contained in a strip are proved in Sperb [16] provided the mean curvature on the boundary is nonnegative.

Appendix

In this appendix we shall prove a technical lemma on the approximation of Lipschitz domains by smooth domains. We shall prove the following result.

A.1 Proposition. *Let Ω be a bounded Lipschitz domain. Then there exists a sequence (Ω_n) of domains of class C^∞ such that for any open set O containing $\overline{\Omega}$ and any compact subset K of Ω we have that*

$$K \subset \Omega_n \subset O \tag{A.1}$$

for all $n \in \mathbb{N}$ large enough. Moreover, for any $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\lim_{n\to\infty} \int_{\Omega_n} \varphi(x)\, dx = \int_\Omega \varphi(x)\, dx \tag{A.2}$$

and

$$\lim_{n\to\infty} \int_{\partial\Omega_n} \varphi(x)\, d\sigma = \int_{\partial\Omega} \varphi(x)\, d\sigma\,. \tag{A.3}$$

Of course the only difficulty is to prove that we can choose the approximation in such a way that $(A.3)$ holds. The proof will be given in a sequence of lemmas. We always assume that U and V are open sets in $\mathbb{R}^k$. The proof of the following result on differentiability is elementary and left to the reader.

A.2 Lemma. *Let $\varphi \in C(U,V)$ be a homeomorphism and $f\colon U \to \mathbb{R}^N$, $g\colon V \to \mathbb{R}^N$ be such that $f \circ \varphi = g$. Suppose that φ and g are differentiable in $x_0 \in U$ and that $D\varphi(x_0)$ is invertible. Then, f is differentiable in $\varphi(x_0)$ and*

$$Df\big(\varphi(x_0)\big) = Dg(x_0)\big[D\varphi(x_0)\big]^{-1}.$$

Taking g the identity and $f = \varphi^{-1}$ we obtain the following corollary.

A.3 Corollary. *(a) Let $\varphi \in C(U,V)$ be a homeomorphism. If φ is differentiable in x_0 and $D\varphi(x_0)$ is invertible then φ^{-1} is differentiable in $\varphi(x_0)$ and*

$$D\varphi^{-1}\big(\varphi(x_0)\big) = \big[D\varphi(x_0)\big]^{-1}.$$

(b) Suppose in addition that φ and φ^{-1} are locally Lipschitz. Then φ is differentiable in x_0 if and only if φ^{-1} is differentiable in $\varphi(x_0)$.

The second part follows from the first because $D\varphi(x)$ is nonsingular whenever φ is differentiable in x if φ and φ^{-1} are locally Lipschitz.

A.4 Lemma. *Let $g_i \in C^{1-}(\mathbb{R}^{N-1})$ (i=1,2) and set*

$$G_i(x) := T_i \begin{bmatrix} x \\ g_i(x) \end{bmatrix} \tag{A.4}$$

for all $x \in R^{N-1}$, where $T_i \in SO(N)$. Suppose that $U_i \in \mathbb{R}^N$ are open sets such that

$$M := G_1(U_1) = G_2(U_2).$$

Further assume that $(\psi_\varepsilon)_{\varepsilon>0}$ is a mollifier and $\alpha \in \mathcal{D}(\mathbb{R}^{N-1})$ with $0 \le \alpha \le 1$. Define

$$g_{1\varepsilon} := \alpha(g_1 * \psi_\varepsilon) + (1-\alpha)g_1 \qquad \text{and} \qquad G_{1\varepsilon}(x) := T_1 \begin{bmatrix} x \\ g_{1\varepsilon}(x) \end{bmatrix}. \tag{A.5}$$

Then, for all ε small enough there exists a function $g_{1\varepsilon} \in C^{1-}(\mathbb{R}^{N-1})$ and an open set $U_{2\varepsilon} \subset \mathbb{R}^{N-1}$ such that

$$M_\varepsilon := G_{1\varepsilon}(U_1) = G_{2\varepsilon}(U_{2\varepsilon}),$$

where $G_{2\varepsilon} := T_2 \begin{bmatrix} x \\ g_{2\varepsilon}(x) \end{bmatrix}$.

The above lemma asserts that if M is a graph of a Lipschitz function in two different directions then also the approximation M_ε is a graph in the same directions.

Proof. Denote by P the projection of $\mathbb{R}^N$ onto the subspace $\mathbb{R}^{N-1}$ spanned by the first $N-1$ vectors of the natural basis of $\mathbb{R}^N$. By our assumptions it is clear that $\varphi := PT_2^{-1}G_1 \in C^{1-}(U_1, U_2)$ is bijective with $\varphi^{-1} = PT_1^{-1}G_2 \in C^{1-}(U_2, U_1)$. From Lemma A.1 and Corollary A.2 we infer that g_1 is differentiable in $x \in U_1$ iff and only if g_2 is differentiable in $\varphi(x) \in U_2$. By Rademacher's theorem (e.g. [8], Theorem 3.1.6) g_1, g_2 are differentiable almost everywhere. Hence, the unit normal on M is defined almost everywhere and given by

$$\nu(x) = \frac{1}{\sqrt{1+|\nabla g_1(x)|^2}} T_1 \begin{bmatrix} \nabla g_1(x) \\ -1 \end{bmatrix} = \frac{\pm 1}{\sqrt{1+|\nabla g_2(\varphi(x))|^2}} T_2 \begin{bmatrix} \nabla g_2(\varphi(x)) \\ -1 \end{bmatrix}.$$

By changing the orientation we may assume without loss of generality that the positive sign holds. From this we conclude that for almost all $x \in U_1$

$$(\nu(x)|T_2 e_N) = \frac{1}{\sqrt{1+|\nabla g_2(\varphi(x))|^2}} \ge \delta$$

for some constant $\delta > 0$. Hence,

$$\nu(x) \in C_\delta := \left\{ v \in \mathbb{R}^N (v|T_2 e_N) \ge \delta \right\}.$$

for almost all $x \in U_1$, where $e_N := (0, \dots 0, 1) \in \mathbb{R}^N$. We shall show that

$$\nu_\varepsilon(x) \in C_{\frac{\delta}{2}} \tag{A.6}$$

for almost all $x \in U_1$ and ε small enough. Here $\nu_\varepsilon(x)$ is the normal unit vector on $M_\varepsilon := G_{1\varepsilon}(U_1)$. To show this it is sufficient to show that $\nabla g_{1\varepsilon}$ lies in a suitable small neighborhood of the closed convex hull of $\{\nabla g_1(x)\colon x \in U_1\}$. As $\nabla g_{1\varepsilon}$ is given by

$$\nabla g_{1\varepsilon} = \big[\alpha(\nabla g_1 * \psi_\varepsilon) + (1-\alpha)\nabla g_1\big] + \big[\nabla\alpha(g_1 * \psi_\varepsilon - g_1)\big]$$

this follows easily. Indeed, $\nabla g_1 * \psi_\varepsilon$ lies in the convex hull and hence the term in the first brackets. The last term tends to zero uniformly as ε approaches zero. The next lemma shows that $(A.6)$ implies the assertion of the lemma. □

A.5 Lemma. *Let $g \in C^{1-}(U)$. Denote the graph of g by M and by ν the outer unit normal on M. Suppose z is such that*

$$(\nu|z) \geq \delta > 0 \tag{A.7}$$

almost everywhere. Then, M is the graph of a Lipschitz function over the hyperplane E normal to z.

Proof. E is the image of $\mathbb{R}^{N-1}$ under an orthogonal transformation $T \in O(N)$. Define

$$\varphi(x) := PT^{-1}\begin{bmatrix} x \\ g(x) \end{bmatrix}$$

for all $x \in U$, where P is the projection of $\mathbb{R}^N$ onto $\mathbb{R}^{N-1}$ along the space spanned by e_N. Our assertion follows if we are able to show that φ is injective with Lipschitz inverse. In that case we can define $\tilde{g}(y) := g\big(\varphi^{-1}(y)\big)$.

In a first step we consider the case $N = 2$. Then, an elementary calculation shows that

$$(\nu(x)|z) = -\frac{\varphi'(x)}{\sqrt{1 + |g'(x)|^2}}\,,$$

where

$$\nu(x) := \frac{1}{\sqrt{1+|g'(x)|^2}}\begin{bmatrix} g'(x) \\ -1 \end{bmatrix}$$

is the unit normal vector on M, which is defined almost everywhere. Using the assumption on ν, this implies that for almost all $x \in U$

$$-\sqrt{1+|g'(x)|^2} \leq \varphi'(x) \leq -\delta\sqrt{1+|g'(x)|^2}$$

and therefore φ is a strictly decreasing continuous function. Thus φ is injective and φ^{-1} is Lipschitz with Lipschitz constant smaller than δ^{-1}.

In a second step we consider the case $N > 2$ and prove that M is a graph over E. Let F be the subspace of $\mathbb{R}^N$ spanned by z and e_N. By the continuity of g it is sufficient to prove that $M_a := M \cap (a+F)$ is a graph over $E_a := E \cap (a+F)$ for almost all $a \in F^\perp$. As $\mathrm{dom}(D\varphi)$ is U up to a set of measure zero, a similar assertion holds for $D_a := \mathrm{dom}(D\varphi) \cap (a+F)$ for almost all $a \in F^\perp$ (This follows from Fubini's Theorem).

For all $x \in D_a$ the normal on M_a in $(x, g(x))$ is up to a scalar factor the projection of the normal on M onto $a + F$. From the first step we can conclude now that M_a is the graph of a function over E_a for almost every $a \in F^\perp$. This shows that φ is injective. It remains to show the Lipschitz continuity of φ^{-1}. To this end note that φ is differentiable almost everywhere. It follows from $(A.7)$ and the definition of φ that $D\varphi(x)$ is injective and hence invertible whenever the derivative exists. Moreover, by Cor. A.3 and $(A.7)$ the norm of $D\varphi^{-1}\big(\varphi(x)\big)$ is uniformly bounded proving that φ^{-1} is Lipschitz. This completes the proof of the lemma. □

Of course, the proof of the above lemma is obvious from the implicit function theorem if $g \in C^1(U)$. We are now ready to give the proof of Proposition A.1

Proof of Proposition A.1. Let Ω be a Lipschitz domain. Then there exist $g_i \in C^{1-}(\mathbb{R}^N)$, $T_i \in SO(N)$ and open sets $U_i \subset \mathbb{R}^{N-1}$ $(i = 1\dots, m)$ such that

$$\partial\Omega = \bigcup_{i=1}^m G_i(U_i)\,, \tag{A.8}$$

where G_i is defined as in $(A.4)$. Choose $V_1 \subset \mathbb{R}^{N-1}$ in such a way that its closure is contained in U_1 and $(A.8)$ still holds when we replace U_1 by V_1. Take $\alpha \in \mathcal{D}(U_1)$ with $0 \le \alpha \le 1$ and $\alpha \equiv 1$ on V_1, and define $g_{1\varepsilon}$ and $G_{1\varepsilon}$ as in $(A.5)$. Replacing g_1 by $g_{1\varepsilon}$ we obtain a new domain Ω_ε. By our choice of α the part $G_{1\varepsilon}(V_1)$ of $\partial\Omega_\varepsilon$ is of class C^∞. Moreover, as

$$\lim_{\varepsilon\to 0} g_{1\varepsilon} = g_1$$

uniformly,

$$\lim_{\varepsilon\to 0} \nabla g_{1\varepsilon} = \nabla g_1$$

almost everywhere, and $\|g_{1\varepsilon}\|_\infty \le \|\nabla g_1\|_\infty + \|\nabla\alpha\|_\infty \|g_1\|_\infty$ we easily see that $(A.1)$–$(A.3)$ hold as ε approaches zero. By Lemma A.4 we can find Lipschitz functions $g_{i\varepsilon}$ $(i = 2, \dots m)$ and open sets $U_{i\varepsilon} \subset \mathbb{R}^{N-1}$ such that

$$\partial\Omega_\varepsilon = \bigcup_{i=1}^m G_{i\varepsilon}(U_{i\varepsilon})\,,$$

where $G_{i\varepsilon}$ are defined as in Lemma A.4. Fix now $\varepsilon = \varepsilon^{(1)} > 0$ small enough. Repeating the above construction using $g_{2\varepsilon^{(1)}}$ instead of g_1 we obtain a sequence $\Omega_{\varepsilon^{(1)}\varepsilon}$ having the required properties. Note that in each step we have to shrink slightly the open sets $U_{i\varepsilon}$. After m steps we find domains $\Omega_{\varepsilon^{(1)}\dots\varepsilon^{(m)}}$ of class C^∞. Finally we choose sequences $(\varepsilon_n^{(i)})_{n\in\mathbb{N}}$ $(i = 1\dots m)$ tending to zero as n tends to infinity, and set $\Omega_n := \Omega_{\varepsilon_n^{(1)}\dots\varepsilon_n^{(m)}}$. Then, the Ω_n satisfy the claimed properties, and the proof of the proposition is complete. □

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