DOMAIN PERTURBATION FOR ELLIPTIC EQUATIONS SUBJECT TO ROBIN BOUNDARY CONDITIONS

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Abstract. We prove domain perturbation theorems for linear and nonlinear elliptic equations under Robin boundary conditions. The theory allows very singular perturbation of domains. In particular, it includes cutting holes, parts degenerating to a set of measure zero such as the dumbbell problem, or wildly oscillating boundaries. In the last case we show that the limiting problem is the Dirichlet problem.

1. Introduction

The aim of this paper is to study how the solutions of the linear or nonlinear equation

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega_n \\ \frac{\partial}{\partial\nu}u + \beta_0 u &= 0 && \text{on } \partial\Omega_n \end{aligned} \tag{1.1}$$

behave as the domains Ω_n approach an open bounded set. Here, Δ is the Laplace operator, ν the outer unit normal to the boundary $\partial\Omega$ of Ω and $\beta_0 > 0$ a constant. Unlike in the case of Dirichlet or Neumann boundary conditions the boundary conditions in the limit may be different from the original ones. This depends on how the domains Ω_n approach Ω. We shall prove three perturbation results, each featuring a different behaviour when passing to the limit.

First, we suppose that the boundary of the original problem is only modified in the neighbourhood of a very small set, that is, a set of capacity zero. Examples of that type include rounding off corners, cutting or drilling small holes of any shape. It also includes examples where part of a domain degenerates such as in the dumbbell problem, where the handle shrinks to a line as in Figure 1. The limiting problem to consider is (1.1) on the domain Ω, that is, the original boundary conditions are preserved. The details in the linear case are given in Theorem 3.3.

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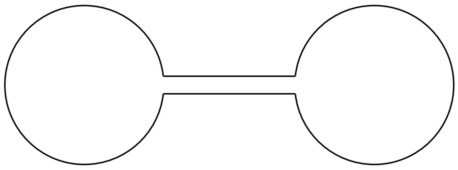

FIGURE 1. A dumbbell with shrinking handle.

Second, we handle the case, where a domain is approached by very rough domains in the sense that $\partial\Omega_n$ is parametrized over $\partial\Omega$ and the surface area of the graph over $\partial\Omega$ goes to infinity locally in measure as n tends to infinity. In this case it turns out that the correct limiting problem is the Dirichlet problem

$$\begin{aligned} -\Delta u = f(u) \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

The precise statement in the linear case is given in Theorem 4.3.

Finally, suppose again that $\partial\Omega_n$ is parametrized over $\partial\Omega$ but assume that the surface measure converges to a bounded measurable function g on $\partial\Omega$ in a rather weak sense to be specified later. Then, the boundary condition of the limiting problem on $\partial\Omega$ is

$$\frac{\partial}{\partial\nu}u + \beta_0 g u = 0. \tag{1.3}$$

For details we refer to Theorem 4.4.

To point out the major ideas, and to keep technicalities within limits, we have refrained from giving the most general perturbation result but handle only the three cases mentioned above which feature different possible limiting problems. We emphasize that they can be combined with different behaviour on different parts of the boundary of Ω (see Remark 5.10). Furthermore, we prove all results for the Laplacian replaced by a (non self-adjiont) elliptic operator in divergence form with real bounded and measurable coefficients, and β_0 a bounded measurable function bounded from below.

One might suspect that, if parts of Ω_n are absorbed into $\partial\Omega$, then the boundary conditions of the limiting problem would change according to the multiplicity (number of local components) of the boundary of $\partial\Omega_n$ absorbed. Theorem 3.3 however shows that this is not at all the case. As an example consider a fold in Ω_n which is absorbed into the boundary of Ω as in Figure 2. Here, the multiplicity of the boundary close to the fold is three. If the neck of that fold shrinks to a point an application of Theorem 3.3 shows that the boundary conditions in the limit are the original ones.

Theorem 3.3 can also be used to show that the assumption that $\partial\Omega_n$ can be parametrized over $\partial\Omega$ in some form is essential to prove the second and third perturbation result. In other words, even if the measure of the boundary of $\partial\Omega_n$ increases locally to infinity in any neighbourhood of $\partial\Omega$ this has no influence on the boundary conditions satisfied in the

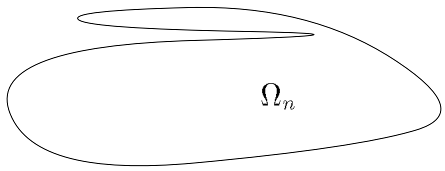

FIGURE 2. A fold absorbed into $\partial\Omega$.

limit. As an example look at a disk with a small piece of large surface wrapped around, which is eventually absorbed into the boundary of the disk as shown in Figure 3. This example is easily modified to show that a suitable parametrization is also necessary for the third result to hold.

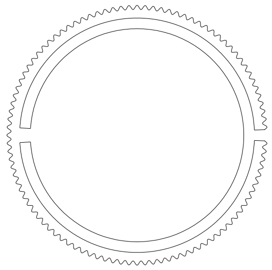

FIGURE 3. A small piece with large surface wrapped around a disk.

Our third result, Theorem 4.4, considerably generalizes earlier results by Stummel [33], Section 2.4. The main improvement is that we do not have to assume that the domains are "uniformly Lipschitz", hence proving results without Stummel's "stability" assumption. Moreover, we can allow exceptional sets, that is, the domains need to be Lipschitz only off a set of capacity zero (see Remark 5.10). We also allow the measure to converge to something else than the original measure on $\partial\Omega$ with changes the coefficient of the boundary operator. This could also be done with his method, as long as the domains are "uniformly Lipschitz" and the "stability" assumption is satisfied.

We also get theorems on the behaviour of the eigenvalues of

$$\begin{aligned} -\Delta\varphi &= \mu_n\varphi && \text{in } \Omega_n \\ \frac{\partial}{\partial\nu}\varphi + \beta_0\varphi &= 0 && \text{on } \partial\Omega_n \end{aligned} \tag{1.4}$$

as n goes to infinity. As opposed to the Neumann problem, it turns out that any finite system of eigenvalues converges to the eigenvalues of

$$\begin{aligned} -\Delta\varphi &= \mu\varphi && \text{in } \Omega \\ \mathcal{B}_\infty &= 0 && \text{on } \partial\Omega\,, \end{aligned} \tag{1.5}$$

where $\mathcal{B}_\infty$ is the boundary operator of the correct limiting problem. As a well known example in [7], Section VI.2.6 shows this is not the case for the Neumann problem in general. A precise analysis on what happens with the eigenvalues of the Neumann problem in case of a dumbbell like domain with handle shrinking to a line as in Figure 1 was done in [2–4,19,20] and references therein. The corresponding problem for Dirichlet boundary conditions was investigated under various conditions on domain convergence in [5, 7, 9–11, 24] to name just a few. The Robin problem turns out to behave much as the Dirichlet problem. The deeper reason for this is that the Robin problem has smoothing properties similar to the Dirichlet problem, independently of the geometry of the domain (see [12]), whereas this is not the case for the Neumann problem. For the special case of cutting holes of a fixed shape into a domain shrinking to a point the behaviour of the eigenvalues of the Robin problem was studied in [29,30,36] (see also the references therein). The first two references deal only with two and three dimensions. Emphasis in all these works is to get the order of convergence of the eigenvalues.

We finally prove results for nonlinear equations. As for the eigenvalue problem the results we get are much closer to those for the Dirichlet problem rather than the Neumann problem. We show that close to a nondegenerate solution of the problem

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega \\ \mathcal{B}_\infty u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.6}$$

there exists a nondegenerate solution of (1.1) for large n. Moreover, it turns out that this solution is unique among the solutions of (1.6) in $L_\infty(\Omega_n)$ in a sufficiently small neighbourhood as shown for the Dirichlet problem in [9], Theorem 1. This is generally not the case for the Neumann problem as the example of a dumbbell with a sufficiently long handle shows (see [17, 18]). Earlier results on Robin boundary value problem with small holes of fixed shape in two and three dimensions were obtained in [31, 34, 35] (see also the references therein) using completely different techniques. The emphasis in these works is in finding the order of convergence whereas we are interested in getting continuity under much more general hypotheses.

As done in [9] our results can be used to construct domains, where a given nonlinear elliptic equation of the form (1.6) has multiple solutions. Note also that under some additional conditions the limiting problem turns out to have a unique positive solution close to a nontrivial nonnegative solution of the limiting problem (see Remark 6.3(c)). Hence, much as in [9] one can get multiple positive solutions for elliptic equations subject to Robin boundary conditions.

An outline of the paper is as follows. In Section 2 we introduce notation, our basic assumptions and give a summary of the results obtained in [12] which are basic for what follows. In the following two sections we handle the linear perturbation theory, in Section 3 the case where the original boundary is not changed much, and in Section 4 the other two cases. Next, in Section 5 we give applications, examples, a more detailed discussion of the results and outline possible generalizations. In the final section we investigate briefly the nonlinear problem.

2. Preliminary Material

In [12] we developed a theory of boundary value problems of the form

$$\begin{aligned} \mathcal{A}u &= f \qquad \text{in } \Omega \\ \mathcal{B}u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

on arbitrary bounded domains Ω. In this section we summarize the results from there we need. We start with our assumptions on the differential operators $\mathcal{A}$ and $\mathcal{B}$ we keep throughout the paper.

2.1. Assumptions. Let us assume that

$$\mathcal{A}u := -\sum_{i=1}^{N} \partial_i \Big(\sum_{j=1}^{N} a_{ij}(x)\partial_j u + a_i(x)u \Big) + \sum_{i=1}^{N} b_i(x)\partial_i u + c_0(x)u \tag{2.2}$$

and

$$\mathcal{B}u := \sum_{i=1}^{N} \Big(\sum_{j=1}^{N} a_{ij}(x)\partial_j u + a_i(x)u \Big)\nu_i + b_0(x)u \tag{2.3}$$

where $\partial_i := \frac{\partial}{\partial x_i}$ and $\nu := (\nu_1, \dots, \nu_N)$ denotes the outer unit normal on the boundary $\partial\Omega$ of Ω. We also assume that $\mathcal{A}$ is uniformly strongly elliptic, that is, there exists $\alpha_0 > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha_0 |\xi|^2 \tag{2.4}$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$. Further, suppose that

$$\inf_{x \in \partial\Omega} b_0(x) \geq \beta_0 \tag{2.5}$$

for some $\beta_0 > 0$ (take the essential infimum if b_0 is only measurable). We throughout assume that the coefficients of $(\mathcal{A}, \mathcal{B})$ are real valued L_∞-functions, and that c_0 is nonnegative.

We denote by $L_p(X)$ the Lebesgue spaces on a measurable set $X \subset \mathbb{R}^N$, and by $\|\cdot\|_{p,X}$ its norm. If no confusion seems likely we just write L_p and $\|\cdot\|_p$, respectively. Further, we write $L_{p,\mathrm{loc}}(X)$ for the space of all functions u such that $u|_K \in L_p(K)$ for all $K \subset X$ compact. Moreover, $W_p^k(\Omega)$ is the Sobolev space, which consists of all $u \in L_p$ such that all the distributional derivatives up to the order k lie in L_p. $\mathcal{D}(\Omega)$ is the space of smooth functions with compact support in Ω and $\mathring{W}_2^1(\Omega)$ its closure in $W_2^1(\Omega)$. Finally, $C(\Omega)$

and $C^k(\Omega)$ are the spaces of continuous and k-times continuously differentiable functions, respectively. If E, F are Banach spaces, we write $E \hookrightarrow F$ if $E \subset F$ and the natural injection is continuous. Further, $\mathcal{L}(E,F)$ is the Banach space of all bounded linear operators from E to F equipped with the usual operator norm, and $\mathcal{L}(E) := \mathcal{L}(E,E)$. The norm in $\mathcal{L}(L_p, L_q)$ we denote by $\|\cdot\|_{p,q}$. If Ω, Ω' are two sets in $\mathbb{R}^N$ we write $\Omega' \subset\subset \Omega$ if the closure of Ω' is contained in the interior of Ω. Further, $\overline{\Omega}$, $\complement\Omega$ and $|\Omega|$ denote the closure, complement and Lebesgue measure of Ω, respectively.

The notion of a weak solution of (2.1) on arbitrary domains (or even any bounded open set) was introduced by Maz'ja ([26], see [27], Section 4.6.11). The underlying Hilbert space $V := V(\Omega)$ is by definition the abstract completion of

$$V_0 := V_0(\Omega) := \{u \in W_2^1(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega) \colon \|u\|_V < \infty\} \tag{2.6}$$

with respect to the norm

$$\|u\|_V := \left(\|\nabla u\|_2^2 + \|u\|_{2,\partial\Omega}^2\right)^{\frac{1}{2}},$$

where $\|\cdot\|_{2,\partial\Omega}$ is the norm in $L_2(\partial\Omega) := L_2(\partial\Omega, \mathcal{H}_{N-1})$, $\mathcal{H}_{N-1}$ being the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$. By definition, u is a weak solution of (2.1) if $u \in V$ satisfies

$$a(u,v) = \langle f, v\rangle \tag{2.7}$$

for all $v \in V$ (or v in a dense subspace of V), where

$$a(u,v) := \int_\Omega \sum_{i=1}^N \Big(\sum_{j=1}^N a_{ij}\partial_j u + a_i u\Big)\partial_i v + \sum_{i=1}^N b_i v \partial_i u + c_0 uv\,dx + \int_{\partial\Omega} b_0 uv\,d\mathcal{H}_{N-1} \tag{2.8}$$

is the generalized Dirichlet form associated with $(\mathcal{A}, \mathcal{B})$ and

$$\langle f, u\rangle := \int_\Omega fu\,dx\,. \tag{2.9}$$

The key to the theory is an inequality due to Maz'ja ([25], see [27], Corollary 4.11.1/2) which says that

$$\|u\|_{\frac{2N}{N-1}} \le c(N, |\Omega|)\|u\|_V \tag{2.10}$$

for all $u \in V$. As $\mathcal{D}(\Omega) \subset V_0$ it follows that V_0 is dense in $L_{\frac{2N}{N-1}}$. Therefore, $L_p \hookrightarrow V'$ for $p \ge 2N(N+1)^{-1}$ and hence the definition makes sense for all $f \in L_p$ for that range of p. By means of a duality theory it is possible to define generalized solutions (which not necessarily are weak solutions) of (2.1) for $f \in L_p(\Omega)$ for all $p \in [1,\infty]$. For the details of that theory as well as for the proof of the following theorem which is essential for our theory of domain perturbation, we refer to [12]. Note that the results remain true for arbitrary open bounded sets as this is true for Maz'ja's L_2-theory it is based on.

2.2. Theorem. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open bounded set. Further, suppose that $(\mathcal{A}, \mathcal{B})$ are as in Assumption 2.1. Then, for every $f \in L_p(\Omega)$ $(1 \leq p \leq \infty)$ equation (2.1) has a unique (generalized) solution u which belongs to $L_{m(p)}(\Omega)$, where $m(p) := Np(N-p)^{-1}$ if $p \in (1, N)$, $m(p) = \infty$ if $p > N$ and $m(p) < N(N-1)^{-1}$ arbitrary if $p = 1$. Moreover, there exists a constant $c > 0$ depending only on N, p and upper bounds for*

$$\gamma := \max\{\alpha_0^{-1}, \beta_0^{-1}\}, \qquad \delta := \alpha_0^{-1}\big(\|\mathbf{a}\|_\infty^2 + \|\mathbf{b}\|_\infty^2\big) \tag{2.11}$$

and $|\Omega|$ such that

$$\|u\|_{m(p)} \leq c\|f\|_p \tag{2.12}$$

for all $f \in L_p(\Omega)$. Furthermore, there exists another constant $c > 0$ depending on the same quantities as above except for p such that

$$\|u\|_V \leq c\|f\|_{\frac{2N}{N+1}} \tag{2.13}$$

for all $f \in L_{\frac{2N}{N+1}}$.

Proof. The a priori estimate (2.12) is one of the main results in [12]. To prove (2.13) note that by a standard argument

$$\|u\|_V^2 \leq 2\gamma\big(a(u,u) + \delta\|u\|_2^2\big) \tag{2.14}$$

for all $u \in V$. By (2.7), Hölder's inequality and (2.10)

$$a(u,u) \leq \|f\|_{\frac{2N}{N+1}}\|u\|_{\frac{2N}{N-1}} \leq c\|f\|_{\frac{2N}{N+1}}\|u\|_V\,.$$

Moreover, by (2.12) and (2.10) the estimate $\|u\|_2^2 \leq \|u\|_{\frac{2N}{N+1}}\|u\|_{\frac{2N}{N-1}} \leq c\|f\|_{\frac{2N}{N+1}}\|u\|_V$ holds. Putting everything together, (2.13) follows. □

If K is a compact subset of $\mathbb{R}^N$ we define its capacity with respect to an open set B_0 by

$$\operatorname{cap}_{B_0}(K) = \inf\{\|\varphi\|^2_{W_2^1(B_0)} \colon \varphi \in \mathcal{D}(B_0) \text{ and } \varphi \geq 1 \text{ on } K\} \tag{2.15}$$

Note that if K has capacity zero with respect to one open bounded set this is true for all bounded open sets containing K. For this reason we just write $\operatorname{cap}(K) = 0$ if $\operatorname{cap}_{B_0}(K) = 0$ for some bounded open set B_0. As a technical lemma we will need the following result on the space V defined above.

2.3. Lemma. *Suppose that Ω is a bounded open set and that $K \subset \overline{\Omega}$ has capacity zero. Then,*

$$V_1 := \{u \in V_0 \colon u = 0 \text{ in a neighbourhood of } K\} \tag{2.16}$$

is dense in V. Moreover, $V(\Omega) = V(\Omega \setminus K)$ if we extend the functions in $V(\Omega \setminus K)$ by zero on K.

Proof. Suppose that B_0 is an open ball containing $\overline{\Omega}$. As $\operatorname{cap}(K) = 0$ there exists a sequence $\varphi_n \in \mathcal{D}(B_0)$ with $\varphi_n = 1$ in a neighbourhood of K and tending to zero in $W_2^1(B_0)$. If $u \in V_0$ is arbitrary we set $u_n := (1 - \varphi_n)u$. Clearly, u_n converges to u in $W_2^1(\Omega)$. Moreover, $u_n|_{\partial\Omega}$ converges to $u|_{\partial\Omega}$ in $L_2(\partial\Omega \setminus K)$. Taking into account that $\mathcal{H}_{N-1}(K) = 0$ (see e.g. [37], Theorem 2.6.16) we have convergence in $L_2(\partial\Omega)$. This shows that $u_n \to u$ in V. Therefore, V_1 is dense in V_0 which in turn is dense in V. This proves the first assertion. Note that the same proof yields that V_1 is dense in $V(\Omega \setminus K)$. As we take closures of essentially the same spaces in the same norm the second assertion follows. □

3. Small Perturbations of the Original Boundary

In this section we show that solutions of the equation

$$\begin{aligned} \mathcal{A}_n u &= f_n && \text{in } \Omega_n \\ \mathcal{B}_n u &= 0 && \text{on } \partial\Omega_n \end{aligned} \tag{3.1}$$

converge to the solution of (2.1) if $\Omega_n \to \Omega$, $(\mathcal{A}_n, \mathcal{B}_n) \to (\mathcal{A}, \mathcal{B})$ and $f_n \to f$ in some sense, provided the boundary of Ω is contained in the boundary of Ω_n except for a small set. We always assume that $\mathcal{A}_n$, $\mathcal{B}_n$ satisfy the following hypotheses.

3.1. Assumptions. Suppose that $\mathcal{A}_n$ and $\mathcal{B}_n$ are of the same structure as (2.2) and (2.3) with real coefficients $a_{ij}^{(n)}, a_i^{(n)}, b_i^{(n)}, c_0^{(n)}$ which are uniformly bounded in $L_\infty(\mathbb{R}^N)$. We also assume that (2.4) and (2.5) are satisfied for a_{ij} replaced by $a_{ij}^{(n)}$ and b_0 replaced by $b_0^{(n)}$, respectively, uniformly with respect to $n \in \mathbb{N}$. Finally, suppose that the coefficients of $\mathcal{A}_n$ converge to the corresponding coefficients of $\mathcal{A}$ almost everywhere, and that $c_0^{(n)}$ is nonnegative for all $n \in \mathbb{N}$.

Throughout, we denote by $a_n(\cdot\,,\cdot)$ the generalized Dirichlet form associated with $(\mathcal{A}_n, \mathcal{B}_n)$ on Ω defined as (2.8).

Convergence of domains in this section is in the following sense.

3.2. Assumptions. Suppose that Ω is an arbitrary open bounded set and that there exists a compact set $K \subset \overline{\Omega}$ of capacity zero such that for any neighbourhood U of K

$$\overline{\Omega}_n \cap \complement(\Omega \cup U) \cap \partial\Omega = \emptyset \quad \text{and} \quad \Omega \cap \complement U \subset \Omega_n \tag{3.2}$$

for $n \in \mathbb{N}$ large enough, and that

$$\lim_{n\to\infty} |\Omega_n \cap \complement\Omega| = 0\,. \tag{3.3}$$

If $v \in L_p(\Omega_n)$ we can always consider v as an element of $L_p(\mathbb{R}^N)$ by continuing it outside Ω_n by zero. We shall always do this in the sequel without further comment. The main result of this section is the following theorem.

3.3. Theorem. *Suppose that Assumption 3.1 and 3.2 hold, and that $b_0^{(n)}$ converges to b_0 in $L_2(\partial\Omega \cap \complement U)$ for all neighbourhoods U of K.*

(i) *Assume that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$ for some $p > 1$ and that u_n is the generalized solution of* (3.1). *Then,*

$$\lim_{n\to\infty} u_n = u \tag{3.4}$$

exists in $L_q(\mathbb{R}^N)$ for all $q \in [1, m(p))$, where $m(p)$ is as in Theorem 2.2, and u is the generalized solution of (2.1).

(ii) *If $f_n \rightharpoonup f$ in $L_p(\mathbb{R}^N)$ for $p > 2N(N+1)^{-1}$, in addition to the assertion in* (i) *for every $\varepsilon > 0$ there exists a neighbourhood U of K such that*

$$\bigl(\|\nabla u_n\|^2_{2,\Omega_n\cap(U\cup\complement\Omega)} + \|u_n\|^2_{2,\partial\Omega_n\cap(U\cup\complement\Omega)}\bigr)^{\frac{1}{2}} \leq \varepsilon\|f_n\|_p \tag{3.5}$$

for all $n \in \mathbb{N}$ large enough. Moreover,

$$\lim_{n\to\infty} \|u_n - u\|_{W_2^1(\Omega\cap\complement U)} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|u_n - u\|_{L_2(\partial\Omega\cap\complement U)} = 0 \tag{3.6}$$

for all neighbourhoods U of K.

We defer the proof to the end of the section and first discuss some consequences. The following example illustrates the generality of the above theorem.

3.4. Example. Assume that Ω is the unit ball in $\mathbb{R}^N$. Take a sequence of bounded domains Ω_n we get from Ω by adding tentacles emanating from one point $x_0 \in \partial\Omega$ towards endpoints with rational coordinates. Then, make sure that (3.2) is satisfied, and that the number of endpoints increases in such a way that $\overline{\Omega}_n$ eventually includes all points with rational coordinates, whereas $|\Omega_n \setminus \Omega|$ becomes arbitrarily small. For this sequence of domains, the above assumptions are clearly satisfied and hence the solutions of the Robin problem on Ω_n converge to the solution of the corresponding problem on Ω.

Let R_n be the solution operator of (3.1). We can consider R_n as an operator from $L_p(\mathbb{R}^N)$ to $L_{m(p)}(\mathbb{R}^N)$ by first restricting a function $f \in L_p(\mathbb{R}^N)$ to Ω_n, solving (3.1), and then extending the solution by zero outside Ω_n. Similarly we can define R to be the solution operator of (2.1). As a consequence of the above Theorem we get the following corollary.

3.5. Corollary. *Suppose the assumptions of the above theorem are satisfied. Then, if $p \in (1, \infty]$ and $q \in [1, m(p))$, we have that*

$$\lim_{n\to\infty} R_n = R \tag{3.7}$$

in $\mathcal{L}\bigl(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N)\bigr)$, that is, in the uniform operator topology.

Proof. Note first that it is sufficient to prove the assertion for $p < \infty$. For $p = \infty$ use an embedding into some L_p for $p \in (N, \infty)$. Suppose that (3.7) is not true. Then there exists $\varepsilon > 0$ and $f_n \in L_p(\mathbb{R}^N)$ with $\|f_n\|_p = 1$ such that $\|R_n f_n - R f_n\|_{p,q} \geq \varepsilon$ for all $n \in \mathbb{N}$.

As f_n is bounded in L_p it has a weakly convergent subsequence which we again denote by f_n converging to f. By Theorem 3.3 it follows that $R_n f_n \to Rf$ and $Rf_n \to Rf$ which contradicts the choice of f_n. □

The above result allow us to prove continuity properties of the eigenvalues of

$$\begin{aligned} \mathcal{A}\varphi &= \lambda\varphi && \text{in } \Omega \\ \mathcal{B}\varphi &= 0 && \text{on } \partial\Omega \end{aligned} \tag{3.8}$$

with respect to the domain. Note, that (3.8) has a discrete spectrum consisting of eigenvalues of finite algebraic multiplicity only (see [12], Corollary 5.2).

3.6. Corollary. *Suppose Σ is a finite system of eigenvalues of* (3.8)*. Then, if U is any open neighbourhood of Σ the corresponding eigenvalue problem on Ω_n has the same number of eigenvalues in U provided n is large enough.*

Proof. First of all note that the eigenvalues of (3.8) are in a one to one correspondence to the eigenvalues of the solution operator of R. A similar statement is true for the eigenvalues of R_n. Now, the assertion follows by combining Corollary 3.5 and the perturbation result in [21], Section IV.3.5. □

3.7. Remark. Together with the bounds from Theorem 2.2 and classical results on interior regularity (e.g. [15], [22]) we have that (3.4) holds in $L_{q,\mathrm{loc}}(\Omega \setminus K)$ provided $q \in [1, \infty)$ satisfies

$$\frac{N}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big) < 1\,. \tag{3.9}$$

If (3.9) holds for $q = \infty$ convergence holds in $C(\Omega \setminus U)$ for any neighbourhood U of K. ($C(\Omega \setminus U)$ is equipped with the topology of uniform convergence on compact subsets.) If $\partial\Omega \setminus K$ is smooth we can replace $\Omega \setminus U$ by $\overline{\Omega} \setminus U$.

To prove Theorem 3.3 we need the following auxiliary results.

3.8. Lemma. *Suppose that $\varphi \in C^\infty(\mathbb{R}^N)$ and that u is the solution of* (2.1) *with $f \in L_p$ for some $p > N$. Then, there exists a constant C depending only on N, p, upper bounds for the L_∞-norms of the coefficients a_{ij}, a_i, b_i of $\mathcal{A}$, $|\Omega|$, as well as upper bounds for γ and δ as defined in* (2.11)*, such that*

$$\|\varphi u\|_V \le C\big(\|\varphi\|_{W_2^1(\Omega)} + \|\varphi\|_\infty^{1/2}\|\varphi\|_{W_2^1(\Omega)}^{1/2} + \|\varphi\|_{2p',\Omega}\big)\|f\|_p\,. \tag{3.10}$$

Proof. An elementary calculation shows that for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\begin{aligned} a(\varphi u, \varphi u) &= a(u, \varphi^2 u) \\ &\quad + \int_\Omega \Big(a_{ij}u^2\partial_j\varphi\partial_i\varphi + (a_{ij} - a_{ji})\varphi u\partial_j\varphi\partial_i u + (b_i - a_i)\varphi u^2\partial_i\varphi\Big)\,dx\,, \end{aligned} \tag{3.11}$$

where we use the summation convention. By Hölders inequality we have that

$$
\begin{aligned}
\int_\Omega a_{ij} u^2 \partial_j\varphi \partial_i\varphi \, dx &\le c_1 \|u\|_\infty^2 \|\nabla\varphi\|_{2,\Omega}^2 \\
\int_\Omega (a_{ij} - a_{ji})\varphi u \partial_j\varphi \partial_i u \, dx &\le c_1 \|\varphi\|_\infty \|\nabla\varphi\|_{2,\Omega} \|u\|_\infty \|\nabla u\|_2 \\
\int_\Omega (b_i - a_i)\varphi u^2 \partial_i\varphi \, dx &\le c_1 \|u\|_\infty^2 \|\varphi\|_{2,\Omega} \|\nabla\varphi\|_{2,\Omega}
\end{aligned}
\tag{3.12}
$$

with c_1 depending only on the L_∞-norms of a_{ij}, a_i, b_i. Furthermore,

$$
\langle f, \varphi^2 u\rangle \le \|f\|_p \|u\|_\infty \|\varphi\|_{2p',\Omega}^2 \tag{3.13}
$$

and

$$
\|u\varphi\|_2^2 \le \|u\|_\infty^2 \|\varphi\|_{2,\Omega}^2 \,. \tag{3.14}
$$

Finally, observe that $u\varphi \in V$, hence by (2.14) we have that

$$
\|\varphi u\|_V^2 \le 2\gamma\big(a(\varphi u, \varphi u) + \delta \|\varphi u\|_2^2\big) \,. \tag{3.15}
$$

As u is a weak solution of (2.1) and $u\varphi^2 \in V$, by definition we obtain

$$
a(u, \varphi^2 u) = \langle f, \varphi^2 u\rangle \,. \tag{3.16}
$$

Using (2.12) and (2.13) to estimate $\|u\|_\infty$ and $\|\nabla u\|_{2,\Omega} \le \|u\|_V$ and putting (3.11)–(3.16) together, the assertion of the lemma follows. □

As a corollary we obtain the following result.

3.9. Corollary. *Suppose the assumptions of the above lemma hold and that U, B_0 are open sets in $\mathbb{R}^N$ with $U \subset\subset B_0$. Then, there exists a constant $C > 0$ depending on the quantities listed in Lemma 3.8 such that*

$$
\|\nabla u\|_{2,\Omega\cap U}^2 + \|u\|_{2,\partial\Omega\cap U}^2 \le C\Big(\operatorname{cap}_{B_0}(\overline{U}) + \sqrt{\operatorname{cap}_{B_0}(\overline{U})}\Big)\|f\|_p^2 \,. \tag{3.17}
$$

Proof. If $p > N$ we get from Lemma 3.8 that

$$
\|\varphi u\|_V \le C\big(\|\varphi\|_{W_2^1} + \|\varphi\|_\infty^{1/2} \|\varphi\|_{W_2^1}^{1/2} + \|\varphi\|_{2p'}\big)\|f\|_p
$$

for all $\varphi \in \mathcal{D}(B_0)$ with $0 \le \varphi \le 1$ and $\varphi = 1$ in a neighbourhood of $\overline{U}$. Moreover, as $2p' < 2N(N-2)^{-1}$ Sobolev's inequality (see [15], Theorem 7.10) implies that $\|\varphi\|_{2p'} \le c\|\nabla\varphi\|_2$, where c depends only on N, p and $|B_0|$. Obviously, $\|\nabla u\|_{2,\Omega\cap U}^2 + \|u\|_{2,\partial\Omega\cap U}^2 \le \|\varphi u\|_V^2$ and hence, (3.17) follows by definition of the capacity as given in (2.15). □

Suppose that $\Omega_n \to \Omega$ in the sense of Assumption 3.2. Then, it is easy to see that, for large enough $n \in \mathbb{N}$, we can choose open neighbourhoods U_n of K such that

$$
U_{n+1} \subset\subset U_n \qquad \text{and} \qquad \bigcap_{n\in\mathbb{N}} U_n = K \tag{3.18}
$$

and

$$\overline{\Omega}_n \cap \complement(\Omega \cup U_n) \cap \partial\Omega = \emptyset. \tag{3.19}$$

Indeed, to construct such a sequence of neighbourhoods take any sequence V_k of open neighbourhoods of K such that

$$V_{k+1} \subset\subset V_k \qquad \text{and} \qquad \bigcap_{k\in\mathbb{N}} V_k = K. \tag{3.20}$$

By (3.2), for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$\overline{\Omega}_n \cap \complement(\Omega \cup V_k) \cap \partial\Omega = \emptyset \tag{3.21}$$

for all $n \geq n_k$. Without loss of generality $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Set $U_{n_k} := V_{k-1}$. and choose for $n \in (n_k, n_{k+1})$ an open set U_n such that $U_{n_k} \subset\subset U_{n+1} \subset\subset U_n \subset\subset U_{n_{k-1}}$. Then, by (3.21) we have that

$$\overline{\Omega}_n \cap \complement(\Omega \cup U_n) \cap \partial\Omega \subset \overline{\Omega}_n \cap \complement(\Omega \cup V_k) \cap \partial\Omega = \emptyset$$

for all $n \geq n_k$. Hence, taking into account (3.20) the sequence constructed satisfies (3.18) and (3.19) for $n \geq n_0$. The following lemma helps to separate the solutions u_n of (3.1) to a part on $\Omega \cap \complement U_n$ and another part on $\Omega_n \cap \complement(\Omega \cup U_n)$.

3.10. Lemma. (a) *Under Assumption 3.2 there exist functions $\psi_n \in \mathcal{D}(\mathbb{R}^N)$ with $0 \leq \psi_n \leq 1$ such that $\psi_n = 1$ on $\overline{\Omega} \cap \complement U_{n-1}$, $\psi_n = 0$ on $\overline{U}_n \cup (\overline{\Omega}_n \cap \complement\Omega)$ and*

$$\lim_{n\to\infty} \psi_n|_\Omega = 1 \tag{3.22}$$

in $W_2^1(\Omega)$. Moreover, if $u \in V(\Omega_n)$ then $\psi_n u \in V(\Omega)$.

(b) *Fix $n_0 \in \mathbb{N}$ large enough. Then, there exists a sequence $(\theta_n)_{n>n_0}$ in $\mathcal{D}(\mathbb{R}^N)$ with $0 \leq \theta_n \leq 1$ such that $\theta_n = 0$ on $\overline{\Omega}$, $\theta_n = 1$ on $\overline{\Omega}_n \cap \complement(\Omega \cup U_{n_0-1})$ and*

$$\lim_{n\to\infty} \|\theta_n\|_{W_2^1(\Omega_n)} = 0\,. \tag{3.23}$$

Proof. As $\operatorname{cap}(K) = 0$ there exists a sequence of functions $\varphi_n \in \mathcal{D}(\mathbb{R}^N)$ with $0 \leq \varphi_n \leq 1$, $\operatorname{supp}\varphi_n \subset U_{n-1}$, $\varphi_n = 1$ on $\overline{U}_n$ and

$$\lim_{n\to\infty} \varphi_n = 0 \tag{3.24}$$

in $W_2^1(\mathbb{R}^N)$. Further note that by (3.19) $\overline{\Omega} \cap \overline{\Omega}_n \cap \complement(\Omega \cup U_n) = \emptyset$. Hence, we can choose $\tilde{\psi}_n \in \mathcal{D}(\mathbb{R}^N)$ with $0 \leq \tilde{\psi}_n \leq 1$ such that $\tilde{\psi}_n = 1$ on a neighbourhood of $\overline{\Omega}$ and $\tilde{\psi}_n = 0$ in a neighbourhood of $\overline{\Omega}_n \cap \complement(\Omega \cup U_n)$. Then, using (3.24) we easily see that the functions $\psi_n := (1 - \varphi_n)\tilde{\psi}_n$ have the desired properties. We next show that $\psi_n u \in V(\Omega)$ whenever $u \in V(\Omega_n)$. If $u \in V_0(\Omega_n)$ it easily follows from the fact that $\Omega \cap \complement U_n \subset \Omega_n$ (see (3.2)) and the properties of ψ_n that $\psi_n u \in V_0(\Omega)$ (Recall the definition (2.6) of V_0). Suppose now that $u \in V(\Omega_n)$ is arbitrary. Then there exists a sequence $u_k \in V_0(\Omega_n)$ such that $u_k \to u$ in $V(\Omega_n)$. By what we proved above we have that $\psi_n u_k \in V_0(\Omega)$ for all $k \in \mathbb{N}$. Moreover,

$$\begin{aligned}\|\psi_n(u_k - u)\|_{V(\Omega)} &= \|\psi_n(u_k - u)\|_{V(\Omega_n)} \\ &\leq \|\nabla\psi_n\|_\infty \|u - u_k\|_2 + \|\psi_n\|_\infty \|u_k - u\|_{V(\Omega_n)}.\end{aligned}$$

Using Maz'ja's inequality (2.10) and the assumption on u_k we see that $\psi_n u_k \to \psi_n u$ in $V(\Omega)$ as k goes to infinity. This shows that $\psi_n u \in V(\Omega)$ and hence completes the proof of part (a) of the lemma.

To prove (b) Let $\varphi \in \mathcal{D}(U_{n_0-1})$ be such that $\varphi = 1$ on $\overline{U}_{n_0}$ and $0 \leq \varphi \leq 1$. Further, let $\tilde{\theta}_n \in \mathcal{D}(\mathbb{R}^N)$ be such that $0 \leq \tilde{\theta}_n \leq 1$, $\tilde{\theta}_n = 1$ on $\overline{\Omega}_n \cap \complement(\Omega \cup U_{n_0})$ and $\tilde{\theta}_n = 0$ on $\overline{\Omega}$. This choice of $\tilde{\theta}_n$ is possible because $\overline{\Omega} \cap \overline{\Omega}_n \cap \complement(\Omega \cup U_{n_0}) = \emptyset$ for all $n > n_0$ by (3.19). If we set $\theta_n := \tilde{\theta}_n(1-\varphi)$ it is clear that $|\nabla\theta_n| \leq |\nabla\varphi|$ on Ω_n for all $n > n_0$. As $\operatorname{supp}\theta_n \cap \Omega_n \subset \Omega_n \cap \complement\Omega$, (3.23) follows from (3.3). Hence, θ_n, has the required property. □

We are now in a position to give the proof of Theorem 3.3.

Proof of Theorem 3.3. (i) We first prove (3.4) in case $p > N$. Suppose that ψ_n are the functions from Lemma 3.10. Then, $(\psi_n u_n)_{n\in\mathbb{N}}$ is bounded in $V(\Omega)$. To see this observe that

$$\|\psi_n u_n\|_{V(\Omega)} \leq \|\nabla\psi_n\|_2 \|u_n\|_\infty + \|\psi_n\|_\infty \|u_n\|_{V(\Omega)}.$$

Further note that by Theorem 2.2 and the fact that $\|f_n\|_p$ is a bounded sequence $\|u_n\|_V$ and $\|u_n\|_\infty$ are bounded sequences as well. Our claim now follows since also $\|\nabla\psi_n\|_2$ and $\|\psi_n\|_\infty$ are bounded by Lemma 3.10. Therefore, $(\psi_n u_n)_{n\in\mathbb{N}}$ has a subsequence which we denote again by $\psi_n u_n$ converging weakly to u in V. We now show that u is the solution of (2.1). To do so fix $v \in V_0$ arbitrary with $v = 0$ in a neighbourhood of K. Then, for n large enough $a_n(\psi_n u_n, v) = a_n(u_n, v) = \langle f_n, v\rangle$. Due to the weak convergence of $\psi_n u_n$ in V and f_n in L_p as well as (2.10), we have that

$$\begin{array}{ccccc} a_n(\psi_n u_n, v) & = & a_n(u_n, v) & = & \langle f_n, v\rangle \\ \Big\downarrow{\scriptstyle n\to\infty} & & & & \Big\downarrow{\scriptstyle n\to\infty} \\ a(u,v) & & = & & \langle f, v\rangle \end{array}$$

holds. Here, $a_n(\cdot\,,\cdot)$ is the form corresponding to $(\mathcal{A}_n, \mathcal{B}_n)$ which is defined similarly to (2.8). As $\operatorname{cap}(K) = 0$ the functions in V_0 which are zero in a neighbourhood of K are dense in V by Lemma 2.3. This implies that u is the solution of (2.1). As the solution of (2.1) is unique every subsequence we choose converges to the same function. Hence the whole sequence converges.

Next we show strong convergence in $L_q(\mathbb{R}^N)$ for $q<\infty$. By our choice of ψ_n and a well known interpolation inequality (e.g. [15], Inequality (7.9))we have that

$$\begin{aligned}\|u_n-u\|_q &\le \|u_n-u\|_{q,\Omega}+\|u_n-u\|_{q,\Omega_n\cap\complement\Omega}\\ &\le \|\psi_n u_n-u\|_{q,\Omega}+\|u_n-u\|_\infty|\Omega_n\cap\complement\Omega|^{\frac{1}{q}}\\ &\le \|\psi_n u_n-u\|_{2,\Omega}^{\frac{2}{q}}\|u_n-u\|_\infty^{1-\frac{2}{q}}+\|u_n-u\|_\infty|\Omega_n\cap\complement\Omega|^{\frac{1}{q}}.\end{aligned}\tag{3.25}$$

First observe that V is compactly embedded into $L_2(\Omega)$ ([27], Corollary 4.11.1/3). As we already know that $\psi_n u_n$ tends to u in $V(\Omega)$ weakly this implies that $\psi_n u_n$ converges to u in $L_2(\Omega)$. On the other hand, by Theorem 2.2 we have that $\|u_n-u\|_\infty\le c(\|f_n\|_p+\|f\|_p)$ with a constant c independent of $n\in\mathbb{N}$. Hence, as $\|f_n\|_p$ is uniformly bounded with respect to $n\in\mathbb{N}$ the first term on the right hand side of (3.25) converges to zero as n tends to infinity. Further, by (3.3) the second term on the right hand side of (3.25) can be made arbitrarily small if only n is large enough. Putting this together we conclude that u_n converges in $L_q(\mathbb{R}^N)$ for $q<\infty$.

We next show (3.4) for $p\in(1,N)$. Observe first that the assertions of Corollary 3.5 hold if $p>N$ as the proof just needs what we showed in the previous paragraph. Moreover, it easily follows from Theorem 2.2 that $\|R_n\|_{1,q_0}$ is uniformly bounded with respect to $n\in\mathbb{N}$ for every $q_0<N(N-1)^{-1}$. By the Riesz-Thorin interpolation theorem (e.g. [6], Theorem 1.1.1) it follows that $R_n\to R$ in $\mathcal{L}(L_{p_1},L_{q_1})$ for all $q_1<m(p_1)$, $p_1\in(1,N)$. If now $f_n\rightharpoonup f$ in $L_{p_1}(\Omega)$ we have that

$$\|R_nf_n-Rf\|_{q_1}\le\|R_n-R\|_{p_1,q_1}\|f_n\|_{p_1}+\|R(f_n-f)\|_{p_1}.$$

As R is compact on L_{p_1} (see [12], Proposition 5.5) and $f_n\rightharpoonup f$ weakly the second term on the right hand side tends to zero as n goes to infinity. As moreover, weakly convergent sequences are bounded it follows from the above that the first term converges to zero as well. This shows that (3.4) holds, and that u is the generalized solution of (2.1).

(ii) To prove (3.5) in case $p>2N(N+1)^{-1}$ we again start with the case $p>N$. Fix $\varepsilon\in(0,1)$ arbitrary and choose U_{n_0} such that $\operatorname{cap}(\overline{U}_{n_0-1})<\varepsilon^2$. This is possible because by assumption $\operatorname{cap}(K)=0$. By Corollary 3.9 we have that for some constant $C>0$ independent of $n\in\mathbb{N}$

$$\|\nabla u_n\|^2_{2,\Omega_n\cap U_{n_0-1}}+\|u_n\|^2_{2,\partial\Omega_n\cap U_{n_0-1}}\le C\varepsilon\|f_n\|_p^2\tag{3.26}$$

for all $n\in\mathbb{N}$. Let θ_n be the functions from Lemma 3.10. Then, by using (3.23) and the fact that $0\le\theta_n\le 1$ we see that

$$\|\theta_n\|_{W_2^1(\Omega_n)}+\|\theta_n\|_\infty^{\frac12}\|\theta_n\|_{W_2^1(\Omega_n)}^{\frac12}+\|\theta_n\|_{2p',\Omega_n}<\varepsilon$$

for n large enough. Hence, we see from Lemma 3.8

$$\|\nabla u_n\|^2_{2,\Omega_n\cap\complement(\Omega\cup U_{n_0-1})}+\|u_n\|^2_{2,\partial\Omega_n\cap\complement(\Omega\cup U_{n_0-1})}\le\|\theta_n u_n\|_V^2\le c\varepsilon\|f_n\|_p^2\tag{3.27}$$

provided n is large. Putting together (3.26) and (3.27) we see that for large $n \in \mathbb{N}$

$$\left(\|\nabla u_n\|^2_{2,\Omega_n\cap(U_{n_0-1}\cup\complement\Omega)} + \|u_n\|^2_{2,\partial\Omega_n\cap(U_{n_0-1}\cup\complement\Omega)}\right)^{\frac{1}{2}} \leq \sqrt{(C+c)\varepsilon}\|f_n\|_p\,. \tag{3.28}$$

As $\varepsilon > 0$ was arbitrary, (3.5) follows. To prove (3.5) in case $p \in (2N(N+1)^{-1}, N)$ note that (3.28) implies that the linear operator

$$L_p(\Omega_n) \longrightarrow W^1_2(\Omega_n \cap (U_{n_0-1} \cup \complement\Omega)) \times L_2(\partial\Omega_n \cap (U_{n_0-1} \cup \complement\Omega))$$

mapping f_n onto $(u_n|_{\Omega_n\cap(U_{n_0-1}\cup\complement\Omega)}, u_n|_{\partial\Omega_n\cap(U_{n_0-1}\cup\complement\Omega)})$ is bounded with norm $\sqrt{(C+c)\varepsilon}$ if $p > N$. On the other hand, the same operator is uniformly bounded with respect to $n \in \mathbb{N}$ by (2.13) for $p = 2N(N+1)^{-1}$. Hence, by complex interpolation (3.5) follows (e.g. [6], Theorem 4.1.2 and 5.1.1).

It remains to show (3.6). We again first consider the case $p > N$. We saw in the proof of Part (i) that $\|\psi_n u_n\|_{V(\Omega)}$ is bounded and that $\psi_n u_n$ converges weakly to u in $V(\Omega)$. Part (i) also shows that u_n converges in $L_q(\Omega)$ for all $q \in [1,\infty)$. As $0 \leq \psi_n \leq 1$ converges to 1 in $W^1_2(\Omega)$ it follows that ψ_n converges to 1 in $L_q(\Omega)$ for all $q \in [1,\infty)$, and $\nabla\psi_n$ to 0 in $L_2(\Omega)$. Using formula (3.11), the above observations, and that u_n is a solution of (3.1) which is uniformly bounded in $L_\infty(\Omega)$ (see Theorem 2.2) we conclude that

$$\begin{aligned}\lim_{n\to\infty} a_n(\psi_n u_n, \psi_n u_n) = \lim_{n\to\infty} \Big[&\langle f_n, \psi_n^2 u_n\rangle \\ &+ \int_\Omega \Big(a^{(n)}_{ij} u_n^2 \partial_j\psi_n\partial_i\psi_n + (a^{(n)}_{ij} - a^{(n)}_{ji})\varphi u\partial_j\varphi\partial_i u \\ &\qquad + (b^{(n)}_i - a^{(n)}_i)\psi_n u_n^2\partial_i\psi_n\Big)\,dx\Big] \\ = \langle f, u\rangle = a(u,u).&\end{aligned} \tag{3.29}$$

If we define an equivalent Hilbert norm on $V(\Omega)$ by setting

$$|||u|||^2 := \int_\Omega a_{ij}(x)\partial_i u\partial_j u\,dx + \int_{\partial\Omega} b_0(x)u^2\,d\mathcal{H}_{N-1}$$

it is easily deduced from (3.29) that $|||\psi_n u_n|||$ converges to $|||u|||$ as n tends to infinity. As $V(\Omega)$ is a Hilbert space and we know that $\psi_n u_n$ converges weakly to u in $V(\Omega)$ it follows that $\psi_n u_n$ converges to u in $V(\Omega)$. Hence, (3.6) holds. A similar argument as used to prove (3.5) in case $p < N$ applies here as well, which concludes the proof of the Theorem 3.3. □

4. Changing Large Portions of the Boundary

In this section we study a similar problem as in Section 3 but assume that large portions of the original boundary are changed by the approaching domain. We prove two theorems. The first asserts that if the length of the boundary of the approaching domains Ω_n goes to infinity locally the solutions converge to the solution of the Dirichlet Problem on the limiting domain. The second shows that if the "pull-back" of the measure on $\partial\Omega_n$ onto $\partial\Omega$

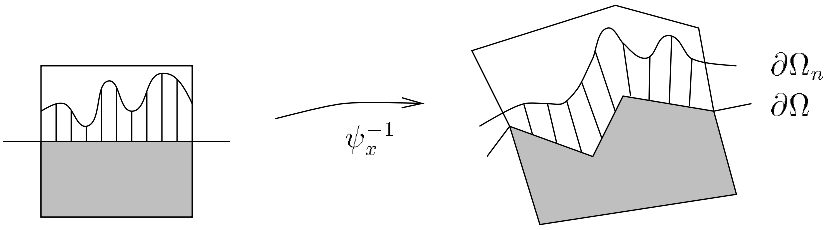

FIGURE 4. Simultaneous parametrization of $\partial\Omega$ and $\partial\Omega_n$.

converges to an L_∞-function on $\partial\Omega$ then, the solutions on Ω_n converge to the solution of a Robin problem on Ω but with a different coefficient on the boundary.

On the operators $\mathcal{A}_n$ and $\mathcal{B}_n$ we again make Assumptions 3.1. Convergence of domains will be in the following sense.

4.1. Assumptions. We first of all assume that $\Omega_n \supset \Omega$ for $n \in \mathbb{N}$ large enough. We further assume that Ω satisfies a strong Lipschitz condition, that is, $\partial\Omega$ is locally the graph of a Lipschitz function in some coordinate system, and Ω lies locally on one side of $\partial\Omega$ (see [27], Section 1.19). We also assume that $\partial\Omega_n$ can be parametrized simultaneously over $\partial\Omega$ by a Lipschitz function, that is, for any $x \in \partial\Omega$ there exists a neighbourhood U_x and a Lipschitz diffeomorphism $\psi_x \colon U_x \to Q_N$ such that

$$\psi_x(U_x \cap \partial\Omega) = Q_{N-1} \times \{0\} \tag{4.1}$$

$$\psi_x(U_x \cap \Omega) = Q_{N-1} \times (-1,0) \tag{4.2}$$

and $\psi_x(U_x \cap \partial\Omega_n)$ is the graph of a Lipschitz function $\rho_{x,n}$ over $Q_{N-1} \times \{0\}$. Here, Q_m denotes the m-fold Cartesian product of $(-1,1)$. The situation is depicted in Figure 4. Finally we assume that

$$\lim_{n\to\infty} |\Omega_n \setminus \Omega| = 0. \tag{4.3}$$

4.2. Remarks. (a) Note that the above assumptions imply that $\partial\Omega_n$ is locally parametrizable over $\partial\Omega$. More precisely, if we keep using the notation from above, the map

$$\phi_{x,n} \colon \partial\Omega_n \cap U_x \to \partial\Omega \cap U_x$$

given by

$$\phi_{x,n} = \psi_x^{-1} \circ P \circ \psi_k \tag{4.4}$$

is a Lipschitz diffeomorphism. Here, $P \colon \mathbb{R}^N \to \mathbb{R}^{N-1} \times \{0\}$ is the natural projection. The inverse of $\phi_{x,n}$ is given by

$$\phi_{x,n}^{-1} = \psi_x^{-1} \circ r_{x,n} \circ \psi_k\,, \tag{4.5}$$

where $r_{x,n}(y,0) := (y, \rho_{x,n}(y))$.

(b) We point out that the above conditions do not imply that Ω_n are strongly Lipschitz (e.g. [27], Section 1.19).

(c) The assumption that $\Omega \subset \Omega_n$ is only for simplicity and can be removed as Remark 5.10(a) shows. Also the assumption that Ω is strongly Lipschitz can be weakened allowing exceptional sets as Remark 5.10(b) shows.

For a Lipschitz function f we denote the m-dimensional Jacobian at x by $J_m f(x)$. The square of the Jacobian $J_m f(x)$ is by definition the sum of the squares of the determinants of the $m \times m$ submatrices of the derivative $Df(x)$ of f at x (see [14], Section 3.2.1). This makes sense because of Rademacher's Theorem which ensures that a Lipschitz function is differentiable almost everywhere (e.g. [14]).

The first result we prove is the following theorem.

4.3. Theorem. *Suppose that Assumption 3.1 and 4.1 hold, and that*

$$\lim_{n\to\infty} \mathcal{H}_{N-1}\bigl(\{z \in \partial\Omega \cap U_x \colon J_{N-1}\phi_{x,n}^{-1}(z) \leq t\}\bigr) = 0 \tag{4.6}$$

for all $x \in \partial\Omega$ and $t \geq 1$, where $\phi_{x,n}^{-1}$ is as in (4.5). Then, if f_n converges to f weakly in $L_p(\Omega)$ for some $p \in (1,\infty)$, the solutions u_n of (3.1) converge to the solution of the Dirichlet problem

$$\begin{aligned} \mathcal{A}u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.7}$$

in $L_q(\Omega)$ for all $q \in [1, m(p))$.

Note that, apart from (2.5), we do not need any other condition on $b_0^{(n)}$ in the above theorem.

4.4. Theorem. *Suppose that Assumption 3.1 and 4.1 hold and that $b_0^{(n)} := b_0|_{\partial\Omega_n}$ for some $b_0 \in C(\mathbb{R}^N)$ with $\inf b_0 > 0$. (Recall that b_0 and $b_0^{(n)}$ are the zero order coefficients of $\mathcal{B}$ and $\mathcal{B}_n$, respectively.) Furthermore, assume that there exists a function $g \in L_\infty(\partial\Omega)$ such that for all $x \in \partial\Omega$*

$$\lim_{n\to\infty} J_{N-1}\phi_{x,n}^{-1}(z) = g \tag{4.8}$$

weakly in $L_q(\partial\Omega \cap U_x)$ for some $q \in (1,\infty)$ or strongly in $L_1(\partial\Omega \cap U_x)$. ($\phi_{x,n}^{-1}$ from (4.5).) Then, if $f_n \rightharpoonup f$ weakly in $L_p(\Omega)$ for some $p \in (1,\infty)$, the solutions u_n of (3.1) converge to the solution of

$$\begin{aligned} \mathcal{A}u &= f && \text{in } \Omega \\ \mathcal{B}u + (g-1)b_0 u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.9}$$

in $L_q(\Omega)$ for all $q \in [1, m(p))$.

Before we prove the two theorems we state a simple consequence on resolvent and spectrum of the boundary value problems.

4.5. Remark. One can show that the conditions (4.6) and (4.8) are independent of the parametrization chosen.

4.6. Corollary. *Under the assumptions of Theorem 4.3 or 4.4 the assertions of Corollary 3.7 hold, where R is the solution operator of the corresponding limiting problem* (4.7) *and* (4.9)*, respectively. Moreover, if we consider the corresponding eigenvalue problems a statement similar to that in Corollary 3.6 holds.*

Proof. The proof is the same as that of Corollary 3.7 and Corollary 3.6, respectively. $\square$

To prove the above theorems we need the following technical lemma.

4.7. Lemma. *Suppose that $u_n \in W_2^1(\Omega_n)$ and that the sequence $\|u_n\|_{W_2^1(\Omega_n)}$ is bounded. Then,*

$$\lim_{n\to\infty} \int_{\partial\Omega\cap U_x} |u_n \circ \phi_{x,n}^{-1} - u_n| \, d\mathcal{H}_{N-1} = 0 \tag{4.10}$$

for all $x \in \partial\Omega$.

Proof. Note first that u_n has a well defined trace on $\partial\Omega \cap U_x$, so that the integrals in (4.10) make sense. To prove our claim we make extensive use of the 'area formula' for Lipschitz functions (see [14], Theorem 3.2.3). Using this formula we get that

$$\begin{aligned}
&\int_{\partial\Omega\cap U_x} |u_n \circ \phi_{x,n}^{-1} - u_n| \, d\mathcal{H}_{N-1} \\
&\quad = \int_{Q_{N-1}} \big|u_n \circ \phi_{x,n}^{-1} \circ \psi_x^{-1}(y,0) - u_n \circ \psi_x^{-1}(y,0)\big| J_{N-1}\psi_x^{-1}(y,0)\, dy \\
&\quad = \int_{Q_{N-1}} \big|u_n \circ \psi_x^{-1}(y,\rho_{x,n}(y)) - u_n \circ \psi_x^{-1}(y,0)\big| J_{N-1}\psi_x^{-1}(y,0)\, dy \qquad (4.11)\\
&\quad = \int_{Q_{N-1}} \Big| \int_0^{\rho_{x,n}(y)} \partial_N u_n \circ \psi_x^{-1}(y,\tau\rho_{x,n}(y))\, d\tau \Big| J_{N-1}\psi_x^{-1}(y,0)\, dy \\
&\quad \le \int_{Q_{N-1}} \int_0^{\rho_{x,n}(y)} \big|\nabla u_n \circ \psi_x^{-1}(y,\tau\rho_{x,n}(y))\big| J_{N-1}\psi_x^{-1}(y,0)\, d\tau\, dy \\
&\quad \le \|D\psi_x^{-1}\|_\infty \|J_{N-1}\psi_x^{-1}\|_\infty \int_{Q_{N-1}} \int_0^{\rho_{x,n}(y)} \big|\nabla u_n\big(\psi_x^{-1}(y,\tau\rho_{x,n}(y))\big)\big|\, d\tau\, dy \\
&\quad \le \frac{\|D\psi_x^{-1}\|_\infty \|J_{N-1}\psi_x^{-1}\|_\infty}{\inf_{z\in Q_{N-1}} J_N\psi_x^{-1}(z)} \int_{\psi_x(U_x\cap(\Omega_n\setminus\Omega))} \big|(\nabla u_n)\circ\psi_x^{-1}(z)\big| J_N\psi_x^{-1}(z)\, dz \qquad (4.12)\\
&\quad = \frac{\|D\psi_x^{-1}\|_\infty \|J_{N-1}\psi_x^{-1}\|_\infty}{\inf_{z\in Q_{N-1}} J_N\psi_x^{-1}(z)} \int_{U_x\cap(\Omega_n\setminus\Omega)} |\nabla u_n(\xi)|\, d\xi\,. \qquad (4.13)
\end{aligned}$$

Here, (4.11) follows from the definition (4.5), (4.12) by Fubini's Theorem, and (4.13) by the area formula. Note also that all the partial derivatives of ψ_x^{-1} are bounded and hence, the derivative $D\psi_x^{-1}$ and all the Jacobians are bounded. As a similar statement is true for ψ_x we also have positive bounds from below for these quantities. Hence, we get from the above calculation and the Cauchy Schwarz inequality that

$$\begin{aligned}\int_{\partial\Omega\cap U_x} |u_n \circ \phi_{x,n}^{-1} - u_n| \, d\mathcal{H}_{N-1} &\leq C \int_{U_x\cap(\Omega_n\setminus\Omega)} |\nabla u_n(\xi)| \, d\xi \\ &\leq C|U_x \cap (\Omega_n \setminus \Omega)| \|u_n\|_{W_2^1(\Omega_n\cap U_x)}\end{aligned} \tag{4.14}$$

for some positive constant C only depending on ψ_x. Now (4.10) follows from (4.3) and the fact that $\|u_n\|_{W_2^1(\Omega_n)}$ is bounded. □

Proof of Theorem 4.3. We first consider the case $p > N$. Then, by Theorem 2.2 we have that

$$\max\bigl\{\|u_n\|_{V(\Omega_n)}, \|u_n\|_\infty\bigr\} \leq C\|f_n\|_p \tag{4.15}$$

for all $n \in \mathbb{N}$ with a constant C not depending on $n \in \mathbb{N}$. As f_n is weakly convergent in L_p the sequence is bounded so that $\|u_n\|_{V(\Omega_n)}$ and $\|u_n\|_\infty$ are bounded. By Maz'ja's inequality (2.10) we conclude that $u_n|_\Omega$ is a bounded sequence in $W_2^1(\Omega)$. Hence, there exists a subsequence we again denote by u_n converging weakly to a function u in $W_2^1(\Omega)$. Therefore, it has a subsequence converging to u in $L_{2,\mathrm{loc}}(\Omega)$. By (4.15) the sequence is bounded in $L_\infty(\Omega)$. Hence, it converges in $L_2(\Omega)$, and by a well known interpolation inequality (e.g. [15], Inequality (7.9)) in $L_q(\Omega)$ for all $q \in [1,\infty)$. As Ω is a Lipschitz domain the trace operator $\gamma_x \in \mathcal{L}\bigl(W_2^1(\Omega\cap U_x), L_2(\partial\Omega\cap U_x)\bigr)$ is well defined and compact (e.g. [28], Théorème 2.6.2). Therefore, $\gamma_x u_n \to \gamma_x u$ in $L_2(\partial\Omega \cap U_x)$. We next show that $\gamma_x u = 0$ for all $x \in \partial\Omega$. As Ω is Lipschitz this implies that $u \in \mathring{W}_2^1(\Omega)$ (e.g. [28], Théorème 2.4.10). To show that $\gamma_x u = 0$ note that by combining (4.15) and Lemma 4.7 we have that

$$\lim_{n\to\infty} \int_{\partial\Omega\cap U_x} |u_n \circ \phi_{x,n}^{-1} - u_n|^2 \, d\mathcal{H}_{N-1} = 0.$$

For this reason, for any $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that

$$\int_{\partial\Omega\cap U_x} |u_n|^2 \, d\mathcal{H}_{N-1} \leq \int_{\partial\Omega\cap U_x} |u_n \circ \phi_{x,n}^{-1}|^2 \, d\mathcal{H}_{N-1} + \varepsilon \tag{4.16}$$

for all $n \geq n_0$. We show now that the integral on the right hand side is small if n is large. We do this in two parts. Setting $g_n := J_{N-1}\phi_{x,n}^{-1}$ we first get

$$\begin{aligned}
\int_{[g_n \geq t]} |u_n \circ \phi_{x,n}^{-1}|^2 \, d\mathcal{H}_{N-1} &\leq \frac{1}{t} \int_{[g_n \geq t]} |u_n \circ \phi_{x,n}^{-1}|^2 g_n \, d\mathcal{H}_{N-1} \\
&\leq \frac{1}{t} \int_{U_x \cap \partial\Omega} |u_n \circ \phi_{x,n}^{-1}|^2 g_n \, d\mathcal{H}_{N-1} \\
&= \frac{1}{t} \int_{U_x \cap \partial\Omega_n} |u_n|^2 \, d\mathcal{H}_{N-1} \\
&\leq \frac{1}{t} \int_{U_x \cap \partial\Omega_n} |u_n|^2 \, d\mathcal{H}_{N-1} \leq \frac{1}{t} \|u_n\|^2_{V(\Omega_n)},
\end{aligned}$$

where $[g_n \geq t] := \{x \in \partial\Omega \cap U_x \colon g_n(x) \geq t\}$. Taking into account (4.15) this yields

$$\lim_{t \to \infty} \int_{[g_n \geq t]} |u_n \circ \phi_{x,n}^{-1}|^2 \, d\mathcal{H}_{N-1} = 0 \tag{4.17}$$

uniformly with respect to $n \in \mathbb{N}$. Furthermore, defining $[g_n \leq t]$ similarly to $[g_n \geq t]$ we get

$$\begin{aligned}
\int_{[g_n \leq t]} |u_n \circ \phi_{x,n}^{-1}|^2 \, d\mathcal{H}_{N-1} &\leq \|u_n\|_\infty \int_{[g_n \leq t]} |u_n \circ \phi_{x,n}^{-1}| \, d\mathcal{H}_{N-1} \\
&\leq \|u_n\|_\infty \sqrt{\mathcal{H}_{N-1}([g_n \leq t])} \Big(\int_{U_x \cap \partial\Omega} |u_n \circ \phi_{x,n}^{-1}|^2 \, d\mathcal{H}_{N-1} \Big)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\mathcal{H}_{N-1}([g_n \leq t])}}{\sqrt{\inf g_n}} \|u_n\|_\infty \Big(\int_{U_x \cap \partial\Omega} |u_n \circ \phi_{x,n}^{-1}|^2 g_n \, d\mathcal{H}_{N-1} \Big)^{\frac{1}{2}} \\
&= \frac{\sqrt{\mathcal{H}_{N-1}([g_n \leq t])}}{\sqrt{\inf g_n}} \|u_n\|_\infty \Big(\int_{U_x \cap \partial\Omega_n} |u_n|^2 \, d\mathcal{H}_{N-1} \Big)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\mathcal{H}_{N-1}([g_n \leq t])}}{\sqrt{\inf g_n}} \|u_n\|_\infty \|u_n\|_{V(\Omega_n)}
\end{aligned} \tag{4.18}$$

for all $n \in \mathbb{N}$. By (4.5) we have that

$$g_n(z) = J_{N-1}\phi_{x,n}^{-1}(z) = J_{N-1}\psi_x^{-1}(r_{x,n} \circ \psi_x(z)) J_{N-1} r_{x,n}(\psi_x(z)) J_{N-1}\psi_x(z)$$

for all $n \in \mathbb{N}$. As

$$J_{N-1} r_{x,n}(\psi_x(z)) = \sqrt{1 + |\nabla \rho_{x,n}(\psi_x(z))|^2}$$

we conclude that $\inf_{n \in \mathbb{N}}(\inf g_n) > 0$. Hence, using (4.15) and (4.6) it follows from (4.18) that

$$\lim_{n \to \infty} \int_{[g_n \leq t]} |u_n \circ \phi_{x,n}^{-1}|^2 \, d\mathcal{H}_{N-1} = 0 \tag{4.19}$$

for all $t \geq 1$. By (4.17) we can choose $t > 0$ such that

$$\int_{[g_n \geq t]} |u_n \circ \phi_{x,n}^{-1}|^2 \, d\mathcal{H}_{N-1} \leq \varepsilon$$

for all $n \in \mathbb{N}$. By (4.19) there exists $n_1 \in \mathbb{N}$ such that

$$\int_{[g_n \leq t]} |u_n \circ \phi_{x,n}^{-1}|^2 \, d\mathcal{H}_{N-1} \leq \varepsilon$$

for all $n \geq n_1$. Taking into account (4.16) we see that

$$\int_{U_x \cap \partial\Omega} |u_n|^2 \, d\mathcal{H}_{N-1} \leq 3\varepsilon. \tag{4.20}$$

for all $n \geq \max\{n_0, n_1\}$. As $\varepsilon > 0$ was arbitrary this implies that

$$\lim_{n\to\infty} \int_{U_x \cap \partial\Omega} |u_n|^2 \, d\mathcal{H}_{N-1} = 0.$$

This finishes the proof that $u \in \mathring{W}_2^1(\Omega)$. It remains to show that u is the solution of (4.7). As u_n is a weak solution of (3.1) we have that for any $\varphi \in \mathcal{D}(\Omega)$

$$a_n(u_n, \varphi) = \langle f_n, \varphi \rangle \tag{4.21}$$

for all $n \in \mathbb{N}$, where $a_n(\cdot\,,\cdot)$ is the form corresponding to $(\mathcal{A}_n, \mathcal{B}_n)$. As u_n converges weakly in $W_2^1(\Omega)$ and f_n weakly in L_p it follows that $a(u, \varphi) = \langle f, \varphi \rangle$ for all $\varphi \in \mathcal{D}(\Omega)$, that is, u is a weak solution of (4.7).

For $p < N$ we proceed exactly as in the corresponding part of the proof of Theorem 3.3. This concludes the proof of the theorem. □

Proof of Theorem 4.4. We proceed much as in the proof of Theorem 4.3. Again, we start with the case $p > N$. Then, (4.15) holds with a constant independent of $n \in \mathbb{N}$. Moreover, u_n has a weakly convergent subsequence in $W_2^1(\Omega)$ which we denote again by u_n. As before, we may conclude that u_n converges to u in $L_q(\Omega)$ for all $q \in [1, \infty)$. Moreover, $\gamma_x u_n \to \gamma_x u$ in $L_q(\partial\Omega \cap U_x)$ for all $q \in [1, \infty)$ since $\|\gamma_x u_n\|_\infty \leq \|u_n\|_{\infty, \Omega\cap U_x}$ is uniformly bounded by (4.15), and $\gamma_x u_n$ converges in $L_2(\partial\Omega \cap U_x)$. It remains to show that u satisfies (4.9). To show this let $\varphi \in C^\infty(\mathbb{R}^N)$ be arbitrary. As u_n is a solution of (3.1) equation (4.21) holds for all $n \in \mathbb{N}$. It is well known that the restrictions of functions in $C^\infty(\mathbb{R}^N)$ are dense in $W_2^1(\Omega)$, so we have to prove that $a(u, \varphi) = \langle f, \varphi \rangle$ only for this set of test functions.

We first show that the integrals over Ω_n in $a_n(u_n, \varphi)$ and $\langle f_n, \varphi \rangle$ converge to the corresponding integrals on Ω for all $\varphi \in C^\infty(\mathbb{R}^N)$. We just look at the highest order term, the other terms are handled in a similar manner. Then,

$$\begin{aligned}
\Big|\int_{\Omega_n} a_{ij}^{(n)}\partial_i u_n \partial_j\varphi\,dx - \int_\Omega a_{ij}\partial_i u\partial_j\varphi\,dx\Big| \\
&\leq \Big|\int_\Omega a_{ij}^{(n)}\partial_i u_n\partial_j\varphi - a_{ij}\partial_i u\partial_j\varphi\,dx\Big| + \Big|\int_{\Omega_n\setminus\Omega} a_{ij}^{(n)}\partial_i u_n\partial_j\varphi\,dx\Big| \\
&\leq \Big|\int_\Omega a_{ij}^{(n)}\partial_i u_n\partial_j\varphi - a_{ij}\partial_i u\partial_j\varphi\,dx\Big| + C\|\nabla u_n\|_2\|\nabla\varphi\|_{2,\Omega_n\setminus\Omega}
\end{aligned}$$

for all $n \in \mathbb{N}$. From the remarks in the previous paragraph it easily follows that the right hand side of the above inequality converges to zero.

Next we look at the boundary integral. It remains to prove that

$$\lim_{n\to\infty}\int_{\partial\Omega_n} b_0^{(n)}u_n\varphi\,d\mathcal{H}_{N-1} = \int_{\partial\Omega} b_0 g u\varphi\,d\mathcal{H}_{N-1}\,. \tag{4.22}$$

Obviously, it is sufficient to prove the above statement locally. Using the area formula we get that

$$\begin{aligned}
\int_{\partial\Omega_n\cap U_x} u_n\varphi b_0^{(n)}\,d\mathcal{H}_{N-1} - \int_{\partial\Omega\cap U_x} u\varphi b_0 g\,d\mathcal{H}_{N-1} \\
&= \int_{\partial\Omega\cap U_x} (u_n\varphi b_0)\circ\phi_{x,n}^{-1}J_{N-1}\phi_{x,n}^{-1} - u\varphi b_0 g\,d\mathcal{H}_{N-1} \\
&= \int_{\partial\Omega\cap U_x} \big((u_n\varphi b_0)\circ\phi_{x,n}^{-1} - u\varphi b_0\big)g\,d\mathcal{H}_{N-1} \\
&\quad + \int_{\partial\Omega\cap U_x} (u_n\varphi b_0)\circ\phi_{x,n}^{-1}\big(J_{N-1}\phi_{x,n}^{-1} - g\big)\,d\mathcal{H}_{N-1}.
\end{aligned} \tag{4.23}$$

To show that the first integral in (4.23) goes to zero it is sufficient to show that

$$\lim_{n\to\infty}(u_n\varphi b_0)\circ\phi_{x,n}^{-1} = u\varphi b_0 \tag{4.24}$$

in $L_1(\partial\Omega\cap U_x)$. Taking into account (4.15) and applying Lemma 4.7 to $u_n\varphi$ we may conclude that

$$\lim_{n\to\infty}(u_n\varphi)\circ\phi_{x,n}^{-1} = u\varphi \tag{4.25}$$

holds in $L_q(\partial\Omega\cap U_x)$ for all $q\in[1,\infty)$. Next we prove that

$$\lim_{n\to\infty} b_0\circ\phi_{x,n}^{-1} = b_0 \tag{4.26}$$

in $L_q(\partial\Omega\cap U_x)$ for all $q\in[1,\infty)$. By definition (4.5) of $\phi_{x,n}^{-1}$ and the area formula for the surface integral we get that

$$\int_{\partial\Omega\cap U_x} b_0\circ\phi_{x,n}^{-1}\,d\mathcal{H}_{N-1} = \int_{Q_{N-1}} b_0\circ\psi_x^{-1}(y,\rho_{x,n}(y))J_{N-1}\psi_x^{-1}(y,0)\,dy\,.$$

As the Jacobian $J_{N-1}\psi_x^{-1}(y,0)$ is bounded (4.26) follows if we can show that

$$\lim_{n\to\infty} b_0 \circ \psi_x^{-1}(\cdot,\rho_{x,n}(\cdot)) = b_0 \circ \psi_x^{-1} \tag{4.27}$$

in $L_q(Q_{N-1})$ for all $q \in (1,\infty)$. Note that $J_N\psi_x^{-1}$ is bounded and therefore by the area formula and (4.3) $\rho_{x,n}$ converges to zero in $L_1(Q_{N-1})$. As $0 \le \rho_{x,n} \le 1$ convergence takes place in $L_q(Q_{N-1})$ for all $q \in (1,\infty)$. Further, observe that $b_0 \circ \psi_x^{-1}$ is a bounded continuous function on Q_N. Hence, by [1], Theorem 3.1 and 3.7 it follows that (4.27) and hence, (4.26) holds. Putting together (4.25) and (4.26), and taking into account that $b_0 \circ \phi_{x,n}^{-1}$ is uniformly bounded we conclude that (4.24) holds in $L_q(\partial\Omega \cap U_x)$ for all $q \in [1,\infty)$. This shows that the first integral in (4.23) converges to zero. Using this it is quite obvious from assumption (4.8) that also the second integral converges to zero as n goes to infinity. This finishes the proof of (4.22) if $p > N$. In case $p \in (1,N)$ we proceed exactly as in the corresponding part of the proof of Theorem 3.3 □

5. Applications, Examples and Remarks

In this section we collect applications, examples and possible generalizations of the results obtained in Sections 3 and 4. We divide the the section into four parts. First we give an application of Theorem 3.3 announced in [12]. Then we give an example where Theorem 4.3 applies. Next, we illustrate Theorem 4.4 and finally, we discuss possible generalizations of our results.

An application of Theorem 3.3. It is well known that if Ω is a smooth bounded domain and the coefficients of $\mathcal{A}$, $\mathcal{B}$ are sufficiently regular, then, for any $f \in L_p(\Omega)$ the boundary value problem (2.1) has a unique solution in $W_p^2(\Omega)$, which satisfies the a priori estimate $\|u\|_{W_p^2(\Omega)} \le C(\Omega)\|f\|_p$ for all $f \in L_p(\Omega)$. By the Sobolev embedding theorem this implies that

$$\|u\|_q \le c_q(\Omega)\|f\|_p\,, \tag{5.1}$$

where $q = Np(N-2p)^{-1}$ if $p \in (1,N/2)$ and $q = \infty$ if $p > N/2$. Suppose that $c_q(\Omega)$ is the best possible constant such that (5.1) holds for all $f \in L_p(\Omega)$.

In [12], Theorem 5.6 we showed that there exists a plane domain Ω, which is smooth except for one point such that $m(p)$ in (2.12) cannot be replaced by any $q > m(p)$. As announced in [12] we show that Theorem 2.2 is optimal even if we restrict to smooth bounded domains. This means that though a better estimate holds for smooth domains the constant involved depends on the particular geometry of the domain rather than the volume. The precise result is as follows.

5.1. Theorem. *There exists a sequence of smooth domains Ω_n such that for any choice of $q > m(p)$*

$$\lim_{n\to\infty} c_q(\Omega_n) = \infty. \tag{5.2}$$

Proof. Consider a domain Ω having the properties of that one in [12], Theorem 5.6. For any $n \in \mathbb{N}$ modify it near the point where it is nonsmooth to get a sequence of smooth domains Ω_n approaching Ω in the sense of Assumption 3.2. Suppose that (5.1) holds for some $p \in (1, N)$ and $q > m(p)$, and that $c_q(\Omega_n) \leq c < \infty$ for all $n \in \mathbb{N}$. If $f \in L_p(\mathbb{R}^N)$ is arbitrary set $f_n := f|_{\Omega_n}$. Then, $f_n \to f$ in $L_p(\mathbb{R}^N)$ and it follows from Theorem 3.3 that $u_n \to u$, in $L_r(\mathbb{R}^N)$ as n tends to infinity for all $r \in [1, m(p))$, and u is the solution of (2.1). On the other hand, it follows from (5.1) and our assumption on Ω_n that $\|u\|_q \leq c\|f\|_p$. Hence, by a well known interpolation inequality (e.g. [15], Inequality (7.9)) $u_n \to u$ in $L_r(\mathbb{R}^N)$ for all $r \in [1, q)$. Together with the fact that $f_n \to f$ in $L_p(\mathbb{R}^N)$ and the assumption that $c_q(\Omega_n) \leq c$ we conclude from (5.1) that $\|u\|_r \leq c\|f\|_p$ for all $n \in \mathbb{N}$, $f \in L_p(\Omega)$ and $r \in [1, q)$. In particular, this holds for some $r > m(p)$. As this is a contradiction to our choice of Ω (5.2) must hold. □

An Example illustrating Theorem 4.3.

5.2. Example. We construct an example where Theorem 4.3 applies. The idea is to approach the unit disk Ω in the plane by domains Ω_n with rapidly oscillating boundaries. These domains are defined to be the interior of the curves given in polar coordinates by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \bigl(1 + \frac{1}{n}(1 + \cos(n^2\pi t))\bigr) \begin{pmatrix} \cos(\pi t) \\ \sin(\pi t) \end{pmatrix}, \tag{5.3}$$

where $t \in [-1, 1]$. For $n = 5, 10$ the domains look as in Figure 5. All these domains

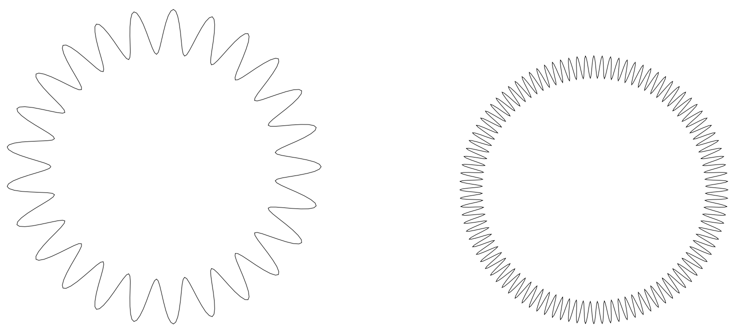

FIGURE 5. Approaching a disk by domains with wildly oscillating boundaries.

obviously contain the open unit disk. It is easy to check that this sequence satisfies the assumptions of Theorem 4.3. Indeed, $\psi^{-1}\colon Q_2 \to \mathbb{R}^2$ is given by

$$\psi^{-1}(t,s) = (s+1)\begin{pmatrix}\cos(\pi t)\\ \sin(\pi t)\end{pmatrix} \tag{5.4}$$

and $\rho_n\colon (-1,1) \to [0,1)$ by

$$\rho_n(t) := \frac{1}{n}(1+\cos(n^2\pi t))$$

for $n \geq 2$. As $\rho_n(t)$ converges to zero uniformly with respect to $t \in (-1,1)$ (4.3) is clearly satisfied. Finally we have to show that the set where the Jacobian

$$g_n(t) := \sqrt{1+|\rho_n'(t)|^2} = \sqrt{1+|n\pi\sin(n^2\pi t)|^2} \leq M \tag{5.5}$$

has small measure as n goes to infinity. To see this note first of all that

$$\sqrt{1+|n\pi\sin(n^2\pi t)|^2} \geq n|\sin(n^2\pi t)|.$$

Hence, it is sufficient to show that for any $M \geq 1$ the measure of the sets where

$$|\sin(n^2\pi t)| \leq \frac{M}{n}$$

is going to zero as n tends to infinity. As the sawtooth function given by connecting the zeros of $|\sin(n^2\pi t)|$ with the points where this function attains one lies below $|\sin(n^2\pi t)|$ we get that

$$\begin{aligned}\bigl|\{t\in(-1,1)\colon g_n(t)\leq M\}\bigr| &\leq \bigl|\{t\in(-1,1)\colon |\sin(n^2\pi t)|\leq \frac{M}{n}\}\bigr|\\ &\leq n^2\frac{1}{n^2}\frac{M}{n} = \frac{M}{n}\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, the assertion follows.

Examples illustrating Theorem 4.4.

5.3. Example. Suppose Ω is the open disk and that ψ^{-1} is given by (5.4). Assume that ρ_n is a sawtooth function as in Figure 6 with teeth of slope $\pm\alpha$ and height decreasing to zero as n goes to infinity. Then, the Jacobian g_n is given by $g_n(t) = \sqrt{1+\alpha^2}$. Hence, the limiting problem in that case is (4.9) with $g = \sqrt{1+\alpha^2}$.

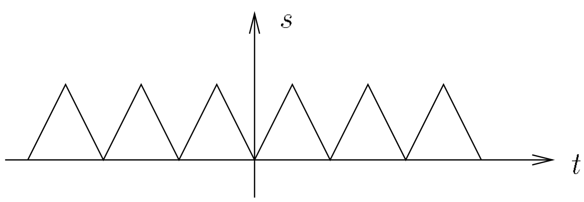

FIGURE 6. Sawtooth function.

We next construct an example very similar to the above but where the Jacobians merely converge weakly. To do so we need some properties of sequences of periodic functions. Suppose that $g_n \in C(\mathbb{R})$ is a sequence of function of period τ_n and that τ_n converges to zero as n goes to infinity. Further assume that $\sup_{n\in\mathbb{N}} \|g_n\|_\infty < \infty$ and that

$$\lim_{n\to\infty} \frac{1}{\tau_n} \int_0^{\tau_n} g_n(s)\,ds = \bar{g} \tag{5.6}$$

exists. Then, the following results holds.

5.4. Lemma. *Under the above assumptions*

$$\lim_{n\to\infty} \frac{1}{b-a} \int_a^b g_n(s)\,ds = \bar{g} \tag{5.7}$$

for all $a < b$. Moreover, if this limit exists for some $a < b$ it holds for all $a < b$ and (5.6) is true.

Proof. Given $a < b$ we write can write $b - a = k\tau_n + \sigma_n$ with $k \in \mathbb{N}$ and $\sigma \in [0, \tau_n)$. Then, as $\tau_n \to 0$ it follows that also $\sigma_n \to \infty$ as n goes to infinity. Then, by using the periodicity of g_n we get that

$$\begin{aligned}\frac{1}{b-a} \int_a^b g_n(s)\,ds &= \frac{k}{b-a} \int_0^{\tau_n} g_n(s)\,ds + \frac{1}{b-a} \int_0^{\sigma_n} g_n(s)\,ds \\ &= \frac{1}{\tau_n} \int_0^{\tau_n} g_n(s)\,ds + \frac{1}{b-a}\Big(\int_0^{\sigma_n} g_n(s)\,ds - \frac{\sigma_n}{\tau_n} \int_0^{\tau_n} g_n(s)\,ds\Big).\end{aligned} \tag{5.8}$$

As σ_n goes to zero the last two integrals go to zero as n goes to infinity and so the first assertion of the lemma follows. The same calculation also reveals that if (5.7) holds for some $a < b$ then (5.6) follows, and hence by what we proved already (5.7) holds for all $a < b$. This concludes the proof of the lemma. $\square$

5.5. Proposition. *Suppose the above assumptions hold and that I is a bounded interval. Then, $g_n \rightharpoonup \bar{g}$ weakly in $L_p(I)$ for all $p \in (1, \infty)$.*

Proof. As $\mathcal{D}(I)$ is dense in $L_p(I)$ for $p \in (1,\infty)$ it is sufficient to prove that for all $\varphi \in \mathcal{D}(I)$

$$\lim_{n\to\infty} \int_I (g_n(s) - \bar{g})\varphi(s)\,ds = 0. \tag{5.9}$$

By partial integration we get that

$$\int_I (g_n(s) - \bar{g})\varphi(s)\,ds = -\int_I G_n(s)\varphi'(s)\,ds\,,$$

where

$$G_n(t) := \int_0^t g_n(s) - \bar{g}\,ds.$$

We can rewrite $G_n(t)$ in the form

$$G_n(t) = t\Big(\frac{1}{t}\int_0^t g_n(s)\,ds - \bar{g}\Big)$$

and conclude from Lemma 5.4 that $G_n(t) \to 0$ for all $t \in I$. As moreover $G_n\varphi'$ is uniformly bounded (5.9) follows by the dominated convergence theorem. $\square$

5.6. Example. Let again ψ^{-1} be defined by (5.4) and Ω the unit disk. Further suppose that a_n and b_n are sequences tending to infinity such that

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \alpha \tag{5.10}$$

exists. We define

$$\rho_n(t) := \frac{1}{b_n}\big(1+\cos(a_n\pi t)\big)$$

and Ω_n the domain given by the interior of the curve $\psi^{-1}(t,\rho_n(t))$, where $t \in [-1,1]$. As before, the relevant Jacobian is given by

$$g_n(t) = \sqrt{1+|\rho_n'(t)|^2} = \sqrt{1+|\pi\frac{a_n}{b_n}\sin(a_n\pi t)|^2}.$$

This is a periodic function with period $\tau_n = 1/a_n$. Moreover, $\|g_n\|_\infty$ is a bounded sequence. By a change of variable we see that

$$\frac{1}{\tau_n}\int_0^{\tau_n} g_n(s)\,ds = \int_0^1 \sqrt{1+|\pi\frac{a_n}{b_n}\sin(\pi s)|^2}\,ds\,.$$

Hence, taking into account (5.10) we see that

$$\lim_{n\to\infty}\frac{1}{\tau_n}\int_0^{\tau_n} g_n(s)\,ds = \int_0^1 \sqrt{1+|\pi\alpha\sin(\pi s)|^2}\,ds =: g\,.$$

It follows from Proposition 5.5 that g_n converges to g weakly in $L_p((-1,1))$ for $p \in (1,\infty)$. Hence, we may apply Theorem 4.4 to this situation and conclude that the limiting problem is (4.9). Note that the functions g_n do not converge pointwise. Note also that, if $\alpha = \infty$ in (5.10) then similar arguments as in Example 5.2 apply and we get that the limiting problem is (4.7).

We finally give an example where $\partial\Omega_n$ approaches Ω uniformly, the Jacobian g_n converges to one in $L_1(\partial\Omega)$ but $\|g_n\|_\infty$ goes to infinity.

5.7. Example. Suppose that ψ and Ω the open unit disk. Suppose that C_0 is the interval $[0,1]$. Define recursively C_{n+1} by removing the open middle third of each interval in C_n. Then, $C = \cap_{n\in\mathbb{N}} C_n$ is a Cantor like set, which has measure zero in $\mathbb{R}$ but nonzero capacity in $\mathbb{R}^2$. Indeed, note that $\mathcal{H}_{\frac{\log 2}{\log 3}}(C) > 0$ is finite (e.g. [14], Section 2.10.28). As for any set of capacity zero in $\mathbb{R}^2$ the ε-dimensional Hausdorff measure is zero for any $\varepsilon > 0$

(e.g. [37], Theorem 2.6.16), C must have nonzero capacity. Define ρ_n to be zero outside C_n, and piecewise linear on each of the intervals in C_n so that ρ_n is zero on the boundary of each interval and $2^{-(n+1)}n^{-1}$ at the mid point. Part of the graphs of the functions ρ_3, ρ_4 and ρ_5 are shown in Figure 7. Obviously, ρ_n converges to zero uniformly. Moreover, the generalized derivative of ρ_n satisfies $|\rho_n'| = (3/2)^n n^{-1}$ on C_n, whence, $|\rho_n| \to \infty$ on C. Furthermore, as the measure of C_n is $(2/3)^n$ we get that $\|\rho_n'\|_1 = 1/n$ which converges to zero. Hence, we conclude that

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} \sqrt{1 + |\rho_n'|^2} = 1$$

in $L_1((-1,1))$. Suppose that u_n is the solution of (3.1) on the domain Ω_n given by the interior of the curve $\psi^{-1}(t, \rho_n(t))$, where $t \in [-1, 1]$. Then, by Theorem 4.4 u_n converges to the solution of (2.1) as n goes to infinity. In this case all the domains Ω_n are Lipschitz, but not uniformly, and as the limiting problem we still get the original problem on Ω. This is an example where the results in [33] do not apply.

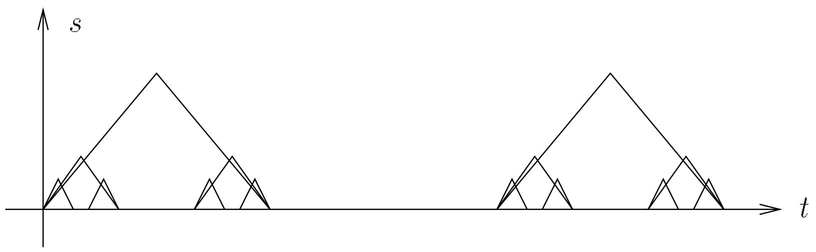

FIGURE 7

5.8. Example. Even if the Jacobian $\sqrt{1 + |\rho_n'|^2}$ does converge neither strongly in L_1 nor weakly in L_p we still might get that the solutions of (3.1) converge to the solution of (2.1). To see this let ψ be as in the above examples and ρ_n given by zero in $[-1, -n^{-1}] \cup [n^{-1}, 1]$ and $1 - n|t|$ in $[-n^{-1}, n^{-1}]$. In this case Theorem 4.4 does not apply, but Theorem 3.3 does. On the other hand, in Example 5.7, we could use Theorem 4.4, but as the Cantor set C has positive capacity in $\mathbb{R}^2$ we can not apply Theorem 3.3.

Remarks and possible generalizations. We start with some remarks concerning Theorem 4.3.

5.9. Remarks. (a) In Theorem 4.3 we do not need that the $\rho_{x,n}$ are Lipschitz if $N = 2$. The only reason why we assume it is Lipschitz is because we make use of the "area formula" for the parametrization $\phi_{x,n}^{-1}$ of $\partial\Omega_n$ over $\partial\Omega$. In two dimensions it turns out that this formula holds for absolutely continuous functions, so it is sufficient to assume that $\rho_{x,n}$ is absolutely continuous. To see that the area formula holds for such functions we have to combine [16], Theorem 18.24 and Corollary 20.5, [32], Theorem 8.5 as well as [13], Lemma 5.4.

(b) We conjecture that the assumption in Theorem 4.3 that the functions $\rho_{x,n}$ are Lipschitz is not necessary and comes only from the fact that we use the area formula in the proof. One should be able to replace that condition by the assumption that the Hausdorff measure of the graph of $\rho_{x,n}$ becomes locally infinite everywhere as n approaches infinity. (We stress that the local parametrization is essential as the remark in the introduction shows.) As an example consider the extreme case where the Hausdorff measure of the graph of $\rho_{x,n}$ is infinite for all $n \in \mathbb{N}$. According to [12], Remark 3.5(d) the solution of (3.1) coincides with that of the Dirichlet problem on Ω_n. Applying general results on domain perturbation for the Dirichlet problem such as those in [9] we conclude that the solutions converge to the solution of the Dirichlet problem (4.7).

Next we discuss possible generalizations.

5.10. Remarks. (a) As pointed out in Remark 4.2 the hypotheses in Assumption 4.1 that $\Omega \subset \Omega_n$ can be removed. What we need is that for all compact subsets $K \subset \Omega$ we have $K \subset \Omega_n$ for n large enough, and that the measure of the symmetric differences of Ω and Ω_n tends to zero as n goes to infinity. The idea is to pull back the measure locally from $\partial\Omega_n$ to a translate of $\partial\Omega$ inside Ω rather than $\partial\Omega$.

An outline of the proof is as follows. As usual, we just need to consider the case $p > N$, the other case follows by interpolation as in the proof of Theorem 4.3. We first show that the solutions u_n of (3.1), if extended by zero outside Ω_n, converge to a solution of $\mathcal{A}u = f$ in Ω. Fix any compact set $K \subset \Omega$. Then, by (4.15) u_n is bounded in $W_2^1(K)$. Hence there exists a weakly convergent subsequence which converges strongly in $L_2(K)$. By a diagonalizing procedure we can construct a subsequence of u_n denoted again by u_n converging in $L_{2,\mathrm{loc}}(\Omega)$. As we have the a priori estimate (4.15) convergence takes place in $L_2(\mathbb{R}^N)$. Furthermore, it is easy to see that $a_n(u_n, \varphi) \to a(u, \varphi)$ and $\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle$ for all $\varphi \in \mathcal{D}(\Omega)$, which shows that $a(u, \varphi) = \langle f, \varphi\rangle$, that is, u is a (weak) solution of $\mathcal{A}u = f$ in Ω. It remains to show what boundary conditions are satisfied.

We first outline the necessary modifications of the proof of Theorem 4.3. We use the same notation as in Section 4. The assumptions on Ω_n imply that for any $\eta > 0$ $\psi_x(\Omega_n \cap U_x) \subset Q_{N-1} \times [-1, -\eta]$ and hence, $\rho_{x,n} \geq -\eta$ for n large enough. Set $V_{x,\eta} := \psi_x^{-1}(Q_{N-1} \times [-1, -\eta])$ and $\Gamma_{x,\eta} := \psi_x^{-1}(Q_{N-1} \times \{-\eta\})$. Define $\phi_{x,\eta,n}$ in the same way as $\phi_{x,n}$ in (4.4) replacing P by the projection P_η onto $Q_{N-1} \times \{-\eta\}$. The same arguments leading to (4.14) show that for any $\varepsilon > 0$ there exists $\eta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{\Gamma_{x,\eta}} |u_n \circ \phi_{x,\eta,n}^{-1} - u_n|\, d\mathcal{H}_{N-1} \leq \varepsilon \quad \text{and} \quad \Big|\int_{\partial\Omega\cap U_x} u\, d\mathcal{H}_{N-1} - \int_{\Gamma_{x,\eta}} u\, d\mathcal{H}_{N-1}\Big| \leq \varepsilon$$

for all $n \geq n_0$. Moreover, by the same argument as in the proof of Theorem 4.3 we know that $u_n \to u$ in $L_q(\Gamma_{x,\eta})$ for all $q \in [1, \infty)$. Using the above information as well as (4.15) we conclude that for a suitable choice of $\eta > 0$

$$\int_{\partial\Omega\cap U_x} |u|^2\, d\mathcal{H}_{N-1} \leq \int_{\Gamma_{x,\eta}} |u_n \circ \phi_{x,\eta,n}^{-1}|^2\, d\mathcal{H}_{N-1} + 3\varepsilon$$

for n large enough. Finally, note that the same arguments as used in the proof of Theorem 4.3 show that

$$\lim_{n\to\infty} \int_{\Gamma_{x,\eta}} |u_n \circ \phi_{x,\eta,n}^{-1}|^2\, d\mathcal{H}_{N-1} = 0.$$

Indeed, note that obviously, (4.6) holds if we replace $\phi_{x,n}$ by $\phi_{x,\eta,n}$. Therefore, $\|u\|_{2,\partial\Omega\cap U_x}$ is as small as we like, which means that $\gamma_x u = 0$. This completes the proof of Theorem 4.3 in the more general case.

To see that Theorem 4.4 remains true observe that we first of all have to find a new argument to prove (4.25). We denote by $\theta_{x,\eta}$ the Lipschitz diffeomorphism between $\partial\Omega\cap U_x$ and $\Gamma_{x,\eta}$ given by $\psi_x^{-1} \circ P_\eta \circ \psi_x$. Then, we can write

$$\begin{aligned}
&\Big|\int_{\partial\Omega\cap U_x} (u_n\varphi)\circ\phi_{x,n}^{-1} - u\phi\, d\mathcal{H}_{N-1}\Big| \\
&\qquad = \Big|\int_{\Gamma_{x,\eta}} \big((u_n\varphi)\circ\phi_{x,n}^{-1}\circ\theta_{x,\eta} - (u\phi)\circ\theta_{x,\eta}\big) J_{N-1}\theta_{x,\eta}\, d\mathcal{H}_{N-1}\Big| \\
&\qquad \leq \|J_{N-1}\theta_{x,\eta}\|_\infty \Big[\int_{\Gamma_{x,\eta}} \big|(u_n\varphi)\circ\phi_{x,n}^{-1}\circ\theta_{x,\eta} - u_n\phi\big|\, d\mathcal{H}_{N-1} \\
&\qquad\quad + \int_{\Gamma_{x,\eta}} |u_n - u||\varphi|\, d\mathcal{H}_{N-1} \\
&\qquad\quad + \int_{\Gamma_{x,\eta}} |u\phi - (u\phi)\circ\theta_{x,\eta}|\, d\mathcal{H}_{N-1}\Big] \\
&\qquad = \|J_{N-1}\theta_{x,\eta}\|_\infty [J_{1,n} + J_{2,n} + J_3]
\end{aligned}$$

The same argument leading to (4.14) shows that, given $\varepsilon > 0$, there exists $\eta > 0$ and $n_0 \in \mathbb{N}$ such that $J_{1,n} + J_3 \leq \varepsilon$ for all $n \geq n_0$. Here we also used (4.15). Furthermore, as in the proof of Theorem 4.4 we know that $u_n \to u$ in $L_q(\Gamma_{x,\eta})$ for all $q \in [1,\infty)$. Hence, (4.25) holds in $L_1(\Gamma_{x,\eta})$. By (4.15) convergence takes place in $L_q(\Gamma_{x,\eta})$ for all $q \in [1,\infty)$.

Finally, it remains to show that the integrals in $a_n(u_n,\varphi)$ over Ω_n converge to the corresponding integrals over Ω. We just show convergence of the top order terms as the others are very similar. Suppose that Ω' is a compact subset of Ω, and that $\varphi \in C^\infty(\mathbb{R}^N)$. Then, for large enough $n \in \mathbb{N}$

$$\begin{aligned}
\Big|\int_{\Omega_n} a_{ij}^{(n)}\partial_i u_n \partial_j\varphi\,dx - \int_{\Omega} a_{ij}\partial_i u\partial_j\varphi\,dx\Big| \\
&\leq \Big|\int_{\Omega'} a_{ij}^{(n)}\partial_i u_n\partial_j\varphi - a_{ij}\partial_i u\partial_j\varphi\,dx\Big| \\
&\quad + \Big|\int_{\Omega_n\setminus\Omega'} a_{ij}^{(n)}\partial_i u_n\partial_j\varphi\,dx\Big| + \Big|\int_{\Omega\setminus\Omega'} a_{ij}\partial_i u\partial_j\varphi\,dx\Big| \\
&\leq \Big|\int_{\Omega'} a_{ij}^{(n)}\partial_i u_n\partial_j\varphi - a_{ij}\partial_i u\partial_j\varphi\,dx\Big| \\
&\quad + C\big(\|\nabla u_n\|_2\|\nabla\varphi\|_{2,\Omega_n\setminus\Omega'} + \|\nabla u\|_2\|\nabla\varphi\|_{2,\Omega\setminus\Omega'}\big)\,.
\end{aligned}$$

We already showed in the second last paragraph that the integral over Ω' converges to zero. Further, as $\|\nabla u_n\|_2$ is bounded by Theorem 2.2 and our choice of Ω' was arbitrary it follows that the other terms can be made as small as we want. This completes the proof.

(b) In both Theorem 4.3 and Theorem 4.4, we only need that Ω is Lipschitz off a set $K \subset \partial\Omega$ of capacity zero. In that case the trace of a function $u \in W_2^1(\Omega)$ is still defined locally on $\partial\Omega\setminus K$. As we only use local properties of traces, and all of our arguments are of local nature the proofs go through with minor modifications. For Theorem 4.3 observe that $\gamma u = 0$ on $\partial\Omega\setminus K$ still implies that $u \in \mathring{W}_2^1(\Omega)$ since K has capacity zero. For Theorem 4.4 we use the fact that the restrictions to Ω of functions in the set

$$\big\{\varphi \in C^\infty(\mathbb{R}^N)\colon \varphi = 0 \text{ in a neighbourhood of } K\big\}$$

is dense in V.

(c) Similar arguments as above show that we could combine the features of the three perturbation results, Theorem 3.3, 4.3 and 4.4. More precisely, suppose that there exists a set $K \subset \overline{\Omega}$ of capacity zero such that for all $x \in \partial\Omega\setminus K$ there exists a Lipschitz diffeomorphism ψ_x satisfying (4.1) and (4.2), that for all neighbourhoods U of K the set $\overline{\Omega}_n \cap \complement(\Omega\cup U)\cap\partial\Omega$ is empty for n large enough, and that (3.3) holds. Further, assume that (4.6) holds on $\Gamma_0 \subset \partial\Omega$ and (4.8) holds on $\Gamma_1 := \partial\Omega\setminus(K\cup\Gamma_0)$. If $f_n \to f$ weakly in $L_p(\mathbb{R}^N)$ then, the solutions u_n of (3.1) converges to the solution of

$$\begin{aligned}
\mathcal{A}u &= f && \text{in } \Omega\\
u &= 0 && \text{on } \Gamma_0\\
\mathcal{B}u + (g-1)b_0 u &= 0 && \text{on } \Gamma_1\,.
\end{aligned}$$

in $L_q(\mathbb{R}^N)$ for all $q \in [1, m(p))$.

6. Nonlinear Problems

In this section we prove approximation results for nonlinear problems. Depending on the assumptions on the sequence of domains Ω_n approaching the open set Ω we consider a different limiting problem which is

$$\begin{aligned} \mathcal{A}u &= f(x,u) && \text{in } \Omega \\ \mathcal{B}_\infty u &= 0 && \text{on } \partial\Omega\,, \end{aligned} \tag{6.1}$$

where the boundary operator $\mathcal{B}_\infty$ is given by

$$\mathcal{B}_\infty u := \begin{cases} \mathcal{B}u & \text{under the assumptions of Theorem 3.3,} \\ u & \text{under the assumptions of Theorem 4.3,} \\ \mathcal{B}u + (g-1)b_0 u & \text{under the assumptions of Theorem 4.4.} \end{cases} \tag{6.2}$$

The operators $\mathcal{A}$ and $\mathcal{B}$ are as in Assumption 2.1. We also assume that Ω_n and Ω are contained in a sufficiently large closed ball $B \subset \mathbb{R}^N$. Finally, we assume that $f \in C^1(B\times\mathbb{R})$, and that for all $(x,\xi) \in B \times \mathbb{R}$

$$|\partial_\xi f(x,\xi)| \leq c_1\bigl(1 + |\xi|^{\beta-1}\bigr) \tag{6.3}$$

for some constants $c_1 > 0$ and $\beta \geq 1$. By a solution of (6.1) we always mean a weak solution, that is, $u \in V$ such that

$$a(u,v) = \langle f(u), v\rangle$$

holds for all $v \in V$, where $a(\cdot\,,\cdot)$ and V are defined as in Section 2. In case of the Dirichlet problem we replace V by $\overset{\circ}{W}{}^1_2(\Omega)$. If the zero order coefficients of $\mathcal{A}$ and $\mathcal{A}_n$ is not positive we can always achieve this by replacing these operators by $\mathcal{A} + \lambda$ and $\mathcal{A}_n + \lambda$ and $f(x,u)$ by $\tilde{f}(x,u) = f(x,u) + \lambda u$ for λ large enough. Hence the assumption on the zero order coefficients is no restriction on the operators. As usual, a nondegenerate solution u_0 of (6.1) is a solution such that the linearised problem

$$\begin{aligned} \mathcal{A}v - \partial_\xi f(x,u_0(x))v &= 0 && \text{in } \Omega \\ \mathcal{B}_\infty v &= 0 && \text{on } \partial\Omega\,, \end{aligned} \tag{6.4}$$

has only the trivial solution. The main result of this section is the following theorem.

6.1. Theorem. *Suppose that either the assumptions of Theorem 3.3, 4.3 or 4.4 hold, and that u_0 is a nondegenerate solution of* (6.1).

(i) *If f satisfies* (6.3) *and $u_0 \in L_p(\Omega)$ for some $p > \max\bigl\{N(\beta-1), 2N\beta(N+1)^{-1}\bigr\}$ then, close to u_0 in $L_p(B)$, the problem*

$$\begin{aligned} \mathcal{A}_n u &= f(x,u) && \textit{in } \Omega_n \\ \mathcal{B}_n u &= 0 && \textit{on } \partial\Omega_n\,, \end{aligned} \tag{6.5}$$

has a unique solution $u_n \in L_p(\Omega_n)$ if only n is large enough. In that case $u_n, u_0 \in L_\infty(B)$ with $\|u_n\|_\infty$ being uniformly bounded, and u_n is close to u_0 in $L_q(B)$ for all $q \in [1,\infty)$.

(ii) *Suppose that f does not satisfy the growth condition* (6.3) *and that $u_0 \in L_\infty(\Omega)$. Then, for n large enough there exists a solution $u_n \in L_\infty(\Omega_n)$ of* (6.5) *which is close to u_0*

in $L_q(B)$ for all $q \in [1,\infty)$. Moreover, it is unique among those solutions close to u_0 in $L_q(B)$ satisfying $\|u_n\|_\infty \le K$ for all $n \in \mathbb{N}$, where $K > \|u_0\|_\infty$ is large.

(iii) *Suppose that u_n are solutions of* (6.5) *with $\|u_n\|_\infty$ a bounded sequence. Then, for large n, the u_n are close to some solution of* (6.1) *in $L_q(B)$ for all $q \in [1,\infty)$.*

In all cases considered in the above theorem we found a sequence of solutions u_n of (6.5) converging to a solution u of (6.1) in $L_q(B)$ for all $n \in \mathbb{N}$. Moreover, we always had $\|u_n\|_\infty \le K$ for all $n \in \mathbb{N}$. The following theorem shows that in that case the spectrum of the corresponding linearisations behave well. In particular this shows that if the solution of (6.1) is stable (unstable) for the corresponding parabolic problem, then the same is true for (6.5) for n large.

6.2. Theorem. *Suppose that we are in the situation described above and that Σ is a finite system of eigenvalues of*

$$\begin{aligned} \mathcal{A}v - \partial_\xi f(x,u_0(x))v &= \mu v \qquad && \text{in } \Omega \\ \mathcal{B}_\infty v &= 0 && \text{on } \partial\Omega\,, \end{aligned} \tag{6.6}$$

If U an open set containing Σ then,

$$\begin{aligned} \mathcal{A}_n v - \partial_\xi f(x,u_n(x))v &= \mu_n v \qquad && \text{in } \Omega_n \\ \mathcal{B}_n v &= 0 && \text{on } \partial\Omega_n\,, \end{aligned} \tag{6.7}$$

has, counting multiplicity, the same number of eigenvalues in U as (6.6).

Proof. It follows from (6.3) that the superposition operator induced by $\partial_\xi f$ is continuous from $L_{\beta q}(B)$ into $L_q(B)$ for all β sufficiently large. Hence, as $u_n \to u_0$ in $L_{\beta q}(B)$ for all $q \in [1,\infty)$ by part (i), or (ii) of Theorem 6.1 we conclude that $\partial_\xi f(\cdot,u_n(\cdot))$ converges to $\partial_\xi f(\cdot,u_0(\cdot))$ in $L_q(B)$ for all $q \in [1,\infty)$. Given that, continuity of the spectrum follows from Corollary 3.6 and 4.6, respectively by replacing $\mathcal{A}$ by $\mathcal{A} - \partial_\xi f(\cdot,u_0(\cdot))$, and $\mathcal{A}_n$ by $\mathcal{A}_n - \partial_\xi f(\cdot,u_n(\cdot))$. □

6.3. Remarks. (a) If $1 < \beta < (N+1)(N-1)^{-1}$ the assumptions on p in part (i) of Theorem 6.1 are automatically satisfied for every solution of (6.1). To see this note that by (2.10) any solution lies in $L_{\frac{2N}{N-1}}(\Omega)$. It is easy to check that

$$2N(N-1)^{-1} > \max\{N(\beta-1), 2N\beta(N+1)^{-1}\}\,,$$

which proves our claim.

(b) All the generalizations in Remark 5.10 apply to the nonlinear situation as well. We just have to use the right limiting problem.

(c) Suppose that u_0 is a nonnegative solution of (6.1), that $f(x,0) \ge 0$ and that $f(x,0) > 0$, or $u_0 > 0$ on all of Ω, or the principal eigenvalue of

$$\begin{aligned} \mathcal{A}\varphi - \partial_\xi f(x,0)\varphi &= \lambda\varphi \qquad && \text{in } \Omega' \\ \mathcal{B}_\infty\varphi &= 0 && \text{on } \partial\Omega'\,. \end{aligned}$$

with $\Omega' := \{x \in \Omega \colon u_0(x) = 0\}$ is strictly positive. (Note that the Harnack Inequality (e.g. [15], Theorem 8.18) ensures that Ω' is a union of components of Ω and thus is open.) Then the nearby solution of (6.5), whose existence we proved in Theorem 6.1, is non-negative. The proof uses the main result in [8] and is essentially the same given in [9], Theorem 2.

(d) We could weaken the notion of solution in Theorem 6.1 and look at generalized solutions rather than weak solutions. By a generalized solution we mean $u \in L_p$ with $g := f(\cdot, u(\cdot)) \in L_q$ for some $q > 1$, such that u is a generalized solution of (2.1) with f replaced by g. Then, in Theorem 6.1(i) we only need to assume that $p > \max\{N(\beta-1), \beta\}$ to make sure that the assertions hold. This comes from the fact that also Proposition 6.4 below remains true under this hypothesis as an analysis of its proof shows, and that the proof of Theorem 6.1 essentially uses abstract arguments based on the assertions of the linear theory and Proposition 6.4. Note however, that in any case the generalized solution we consider in Theorem 6.1 is a weak solution since by Proposition 6.4 $f(\cdot, u(\cdot)) \in L_q$ for some $q > 2N(N+1)^{-1}$.

As in the linear case domain independent a priori estimates play a central role to prove perturbation theorems. They are contained in the following proposition.

6.4. Proposition. *Suppose that $u \in V$ is a weak solution of (6.1) in $L_p(\Omega)$ with $p > \max\bigl\{N(\beta-1), 2N\beta(N+1)^{-1}\bigr\}$. Then, $u \in L_\infty(\Omega)$ and*

$$\|u\|_\infty \leq b\bigl(\|u\|_p\bigr),$$

where $b \colon \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function depending only on the quantities listed in Theorem 2.2 as well as β and c_1 from (6.3) and $\|f(\cdot,0)\|_{L_\infty(\Omega)}$.

Proof. Note first that (6.3) implies that there exists a constant c_2 depending only on β, c_1 and $\|f(\cdot,0)\|_{L_\infty(\Omega)}$ such that

$$|f(x,\xi)| \leq c_2(1 + |\xi|^\beta) \tag{6.8}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}$. This implies that the substitution operator induced by f maps $L_q(\Omega)$ into $L_{q/\beta}$ for all $q \geq \beta$, and that

$$\|f(\cdot, u(\cdot))\|_{q/\beta} \leq c_2\bigl(|\Omega| + \|u\|_q^\beta\bigr) \tag{6.9}$$

for all $u \in L_q(\Omega)$. In particular, this implies that $f((\cdot, u(\cdot)) \in L_{\frac{2N}{N+1}}(\Omega)$ if $u \in L_p(\Omega)$ with $p \geq 2N\beta(N+1)^{-1}$. Suppose that $u \in L_p(\Omega)$ is a solution of (6.1). If $p > N\beta$ it follows from (2.12) and (6.9) that

$$\|u\|_\infty \leq C\bigl(|\Omega| + \|u\|_p^\beta\bigr) \tag{6.10}$$

and we are done. If $p < N\beta$ let $\eta := N(N\beta - p)^{-1}$. Then, using (2.12) and (6.9) we see that

$$\|u\|_{\eta p} \leq C\big(|\Omega| + \|u\|_p^\beta\big) \tag{6.11}$$

By our assumption $\eta > 1$. We can therefore chose $k \in \mathbb{N}$ minimal such that $\eta^{k+1} > N\beta$. Substituting p by ηp and using (2.12) again we have that

$$\|u\|_{\eta^j p} \leq C\big(|\Omega| + \|u\|_{\eta^{j-1}p}^\beta\big) \tag{6.12}$$

for $j = 1, \dots, k$ with C only depending on the quantities listed in the theorem. We now obtain the L_∞-bound for u by induction starting with (6.11), using (6.12) and finally applying (6.10) with p replaced with $\eta^k p$. It is now obvious how to define the function b having the required properties. □

We next prove Theorem 6.1. The main ideas of the proof go back to those in [9], Theorem 1. Here, we follow very closely the proof given in [11], Theorem 7.1 where we got a similar result for the periodic-parabolic Dirichlet problem. Contrary to [9], the proof is completely based on the results for linear problems established in Section 3 and 4.

Proof of Theorem 6.1. (i) That the solution u_n, if it exists, lies in $L_\infty(\Omega)$, and that $\|u_n\|_\infty$ is bounded is clear from Proposition 6.4. If u_n is close to u_0 in $L_p(B)$ it therefore follows by a well known interpolation inequality (e.g. [15], Inequality (7.9)) that u_n is close to u_0 in $L_q(B)$ for all $[p, \infty)$ and hence for all $q \in [1, \infty)$.

We next prove the assertion on uniqueness. To do so assume to the contrary that there exist two sequences u_n and v_n with $u_n \neq v_n$ solving (6.5), and tending to u_0 in $L_p(B)$ as n goes to infinity. Setting

$$w_n := \frac{u_n - v_n}{\|u_n - v_n\|_2}\,,$$

a simple calculation shows that w_n solves

$$\begin{aligned} \mathcal{A} w_n &= d_n w_n && \text{in } \Omega_n \\ \mathcal{B} w_n &= 0 && \text{on } \partial\Omega_n\,, \end{aligned} \tag{6.13}$$

where d_n is given by

$$d_n(x) = \int_0^1 \partial_\xi f\big(x, u_n(x) + \tau(u_n(x) - v_n(x))\big)\,d\tau\,.$$

Our assumption yields

$$\lim_{n\to\infty} u_n + \tau(u_n - v_n) = u_0 \tag{6.14}$$

in $L_p(B)$ for all $\tau \in [0, 1]$. By Proposition 6.4 there exists a constant $C > 0$ independent of $n \in \mathbb{N}$ such that

$$\|u_n + \tau(u_n - v_n)\|_\infty \leq C \tag{6.15}$$

for all $n \in \mathbb{N}$ and $\tau \in [0,1]$. Therefore, we can replace $\partial_\xi f$ by an appropriate truncation and conclude from (6.14) and [1], Theorem 3.1 and 3.7 that

$$\lim_{n\to\infty} \partial_\xi f(u_n(\cdot) + \tau(u_n(\cdot) - v_n(\cdot))) = \partial_\xi f(\cdot, u_0(\cdot))$$

in $L_p(\Omega)$ for all $\tau \in [0,T]$. With the above information it is easily seen that

$$\lim_{n\to\infty} d_n = \partial_\xi f(\cdot, u_0(\cdot)) \tag{6.16}$$

in $L_p(B)$. As $\|w_n\|_2 = 1$ for all $n \in \mathbb{N}$ there exists a subsequence of w_n which we denote again by w_n converging to w weakly in $L_2(\Omega)$. Using (6.15) and (6.16) we get that

$$\lim_{n\to\infty} d_n w_n = \partial_\xi f(\cdot, u_0(\cdot))w \tag{6.17}$$

weakly in $L_2(B)$. By Theorem 3.3, 4.3 or 4.4, respectively, we get that w_n converges to a solution v of

$$\begin{aligned} \mathcal{A}v &= \partial_\xi f(x, u_0(x))w && \text{in } \Omega \\ \mathcal{B}_\infty v &= 0 && \text{on } \partial\Omega \end{aligned}$$

in $L_2(B)$. Since we know that w_n converges weakly to w we have that $v = w$ and thus w is a solution of (6.4). As u_0 is a nondegenerate solution of (6.1) we have that $w = 0$. On the other hand, $\|w_n\|_2 = 1$ for all $n \in \mathbb{N}$ and hence, $\|w\|_2 = 1$ since convergence takes place in $L_2(B)$. As this is a contradiction the uniqueness follows.

To prove existence we introduce the operators

$$Q(v) := R \circ F(v) \qquad \text{and} \qquad Q_n(v) := R_n \circ F(v)$$

where R_n is the solution operator of the problem (3.1) and R that of

$$\begin{aligned} \mathcal{A}v &= g && \text{in } \Omega \\ \mathcal{B}_\infty v &= 0 && \text{on } \partial\Omega\,. \end{aligned}$$

The map F is the substitution operator induced by f, that is,

$$F(u)(x) := f(x, u(x))$$

for all functions $u\colon B \to \mathbb{R}$. By assumption (6.3) and [1], Theorem 3.1 and 3.12 it follows that $F \in C^1(L_p(B), L_{p/\beta}(B))$. Moreover, by (6.9) it maps bounded sets in $L_p(B)$ onto bounded sets of $L_{p/\beta}(B)$. Further, as $R, R_n \in \mathcal{L}(L_{p/\beta}(B), L_r(B))$ are compact for all $r < m(p/\beta)$ it follows that Q, Q_n are completely continuous maps from $L_p(B)$ into $L_r(B)$. By the assumptions on p we can choose $r = p$, and so $Q, Q_n \in C^1(L_p(B), L_p(B))$ are completely continuous. Hence, the Leray-Schauder degree is well defined (e.g. [23]). Note also that u_n is a solution of (6.5) if and only if it is a fixed point of Q_n. A similar statement holds for Q.

As u_0 is nondegenerate the map $I - Q'(u_0)$ has a trivial kernel. If $B_\varepsilon(u_0)$ is any ball in $L_p(B)$ with center u_0 and radius $\varepsilon > 0$ whose closure does not contain a solution of (6.1) apart from u_0 we therefore have that

$$\deg(I - Q, 0, B_\varepsilon(u_0)) = \pm 1 \tag{6.18}$$

(e.g. [23], Theorem 8.1.1 and 5.2.3). We shall use a homotopy argument to see that

$$\deg(I - Q_n, 0, B_\varepsilon(u_0)) = \deg(I - Q, 0, B_\varepsilon(u_0)) \tag{6.19}$$

for n large enough. Then, by the solution property of the degree (e.g. [23], Theorem 4.3.2) and (6.18) it follows that Q_n has a fixed point in $B_\varepsilon(u_0)$. This shows that (6.5) has a solution in $L_p(\Omega_n)$. By the homotopy invariance of the degree (e.g. [23], Theorem 4.3.4) (6.19) follows if we can show that

$$v \neq \theta Q(v) + (1 - \theta)Q_n(v) \tag{6.20}$$

for all $v \in L_p(B)$ with $\|v - u_0\|_p = \varepsilon$, $\theta \in [0, 1]$ and $n \in \mathbb{N}$ large enough. Assume to the contrary that there exist a sequences $\theta_{n_k} \in [0, 1]$ and u_{n_k} with $\|u_{n_k} - u_0\|_p = \varepsilon$ such that

$$u_{n_k} = \theta_{n_k} Q(u_{n_k}) + (1 - \theta_{n_k})Q_{n_k}(u_{n_k}) \tag{6.21}$$

for all $k \in \mathbb{N}$. Replacing u_{n_k} and θ_{n_k} by a subsequence we can assume without loss of generality that $u_{n_k} \rightharpoonup v$ weakly in $L_p(B)$ and $\theta_{n_k} \to \theta_0$ in $[0, 1]$. As $F \in C(L_p(B), L_{p/\beta}(B))$ is bounded by (6.9) we can select a further subsequence which we denote in the same way such that $F(u_{n_k}) \rightharpoonup f_0$ weakly in $L_{p/\beta}(B)$. By Theorem 3.3, 4.3 or 4.4, respectively, and the definition of Q, Q_n it follows that

$$\lim_{k\to\infty} Q(u_{n_k}) = \lim_{k\to\infty} Q_{n_k}(u_{n_k}) = R(f_0) \tag{6.22}$$

Since θ_{n_k} converges this, together with (6.21), implies in particular that u_{n_k} converges strongly in $L_p(B)$ to v. By continuity of F we conclude that $f_0 = F(v)$. Hence, by (6.22) and (6.21) we conclude that

$$v = \lim_{k\to\infty} u_{n_k} = \lim_{k\to\infty} \theta_{n_k} Q(u_{n_k}) + (1 - \theta_{n_k})Q_{n_k}(u_{n_k}) = Q(v)\,.$$

Hence, v is a fixed point of Q. But this is not possible by our choice of $B_\varepsilon(u_0)$. Therefore, (6.19) must hold. This finishes the proof of part (i) of Theorem 6.1.

(ii) Replace f by $\tilde f \in C^1(B\times\mathbb{R})$ such that $\tilde f$ and $\partial_\xi \tilde f$ are bounded and $f(x, \xi) = \tilde f(x, \xi)$ for $(x, \xi) \in B\times[-M, M]$, where $M > \|u_0\|_\infty$ needs to be determined later. As $\|u_0\|_\infty \le M$, the function u_0 is obviously a nondegenerate solution of (6.1) with f replaced by $\tilde f$. Moreover, since the assumptions of part (i) are satisfied there exist solutions $u_n \in L_\infty(\Omega_n)$ of (6.5) with f replaced by $\tilde f$. We are done if we can show that $\|u_n\|_\infty \le M$ for the right choice of M, and n large enough. Theorem 2.2 yields for $p > N$

$$\|u_n\|_\infty \le c\|\tilde f(\cdot, u_n(\cdot))\|_p$$

for a constant independent of $n \in \mathbb{N}$. By Part (i) u_n converges to u_0 in L_q for all $q < \infty$ and hence, $\tilde{f}(\cdot, u_n(\cdot))$ tends to $\tilde{f}(\cdot, u_0(\cdot))$ in L_p for all $p \in (N, \infty)$ as n goes to infinity. Therefore, choosing $M := c\|\tilde{f}(\cdot, u_0(\cdot))\|_p + 1$ for some fixed $p \in (N, \infty)$ it follows from the above facts that $\|u_n\|_\infty \leq M$ for n large enough showing the existence of a solution of (6.5) near u_0. The assertion on the uniqueness of u_n follows by applying part (i) and by truncating f as above.

(iii) Suppose that Q_n, Q are as above. Then, $Q_n(u_n) = u_n$ for all $n \in \mathbb{N}$. As u_n is bounded in $L_\infty(B)$ by assumption there exists a weakly convergent subsequence u_{n_k} in $L_p(B)$ for $p < \infty$ with limit v. By an argument similar to that used in the last part of the proof of (i) we see that

$$\lim_{k\to\infty} Q_{n_k}(u_{n_k}) = Q(v) = v$$

in $L_p(B)$. Hence, v is a solution of (6.1) and u_{n_k} is close to it for large k. This completes the proof of Theorem 6.1. □

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