ROBIN BOUNDARY VALUE PROBLEMS ON ARBITRARY DOMAINS

Daniel Daners*

School of Mathematics and Statistics
University of Sydney, NSW 2006
Australia

March 25, 1996

Abstract. We develop a theory of generalised solutions for elliptic boundary value problems subject to Robin boundary conditions on arbitrary domains, which resembles in many ways that of the Dirichlet problem. In particular we establish L_p-L_q-estimates which turn out to be the best possible in that framework. We also discuss consequences to the spectrum of Robin boundary value problems. Finally, we apply the theory to parabolic equations.

1. Introduction

It is well known that the equation $-\Delta u + u = f$ subject to homogeneous Dirichlet or Neumann boundary conditions can be considered on arbitrary bounded domains Ω in $\mathbb{R}^N$. The idea is to introduce a weak formulation and to choose the "right" Hilbert space incorporating the boundary conditions in a generalised sense. One feature of this approach is that if data and domain are smooth enough we are back to classical solutions satisfying the boundary conditions pointwise. Looking at boundary conditions of Robin type such as

$$\frac{\partial}{\partial\nu}u + \beta u = 0 \tag{1.1}$$

(ν being the outer unit normal on $\partial\Omega$ and β a constant) it is well known that Dirichlet and Neumann boundary conditions correspond to two extreme cases, namely $\beta = \infty$ and $\beta = 0$, respectively. However, although introduced quite some time ago by Maz'ja [16], [17], Section 4.11.6, it seems not very well known that there is a weak formulation on arbitrary domains if $\beta \in (0, \infty)$. The difficulty is to give sense to the boundary

1991 *Mathematics Subject Classification.* 35J25, Secondary 35K20, 35D05.

* Supported by a grant of the Australian Research Council

integral appearing in the Dirichlet form as there are problems with traces of functions in Sobolev spaces.

We shall use Maz'ja's approach and develop an L_p-theory for Robin boundary value problems on arbitrary domains and general (non selfadjoint) second order elliptic operators in divergence form with bounded and measurable coefficients. It is known that the Neumann problem does not have any smoothing properties for a general domain as this is equivalent to embedding theorems for Sobolev spaces (see [5], Corollary 3.4), and that the Dirichlet problem always has (e.g. [4], Lemma 1). It turns out that the smoothing properties we prove in this paper lie "half way" between the two, and that they are the best possible in that framework (see Proposition 5.6 below). In a forthcoming paper [6] it will be shown that the L_p-L_q-estimates we obtain can not be improved even if we restrict ourselves to arbitrarily smooth domains if we don't want the constants involved to depend on the geometry of the underlying domain. This follows from domain perturbation results which we shall prove in the above mentioned paper. The whole theory developed in this paper is in fact motivated by applications to domain perturbation of linear and nonlinear elliptic equations subject to Robin boundary conditions. It provides a priori estimates for solutions not depending on the domain geometry, which allows to deal with very singular perturbations of the domains such as cutting holes or adding small pieces. The results also allow to establish a positive lower bound for the first eigenvalue of $-\Delta\varphi = \lambda\varphi$ subject to boundary conditions of the form (1.1) with $\beta > 0$, uniformly with respect to all domains of the same volume. It would be interesting to know whether a Faber-Krahn type inequality holds or not.

We further show that the theory carries over to the corresponding parabolic problem, and that the parabolic problem fits into the framework of semigroup theory and abstract parabolic equations on L_p-spaces for $1 \leq p < \infty$. We also get estimates for the semigroup kernel (heat kernel). It turns out that the usual methods to prove kernel estimates such as described in [27] and [8] either apply directly or can be adapted to our situation very easily.

The outline of the paper is as follows. In Section 2 we give the precise assumptions, introduce notation and state some of our main results. Section 3 deals with the L_2-theory of weak solutions on arbitrary domains as introduced by Maz'ja [16], [17]. We discuss some problems arising with this approach and give examples. In Section 4 we provide global bounds for weak solutions of Robin boundary value problems. It is a slight extension of the results in [5]. These results are the basis to establish an L_p-theory for Robin boundary value problems on arbitrary domains which we develop in Section 5. The final section is concerned with the corresponding parabolic problem.

The paper concludes with an appendix on some facts on forms, the associated operators induced on various Banach spaces and the smoothing properties of their resolvents.

2. Assumptions and Main Results

In this section we give the precise assumptions, fix notation and state some of our main results. For more detailed statements, proofs and further comments we refer to later sections. In particular, we refer to Section 5 for the L_p-theory of elliptic problems and to Section 6 for parabolic problems.

We shall be concerned with the elliptic boundary value problem

$$\begin{aligned} \mathcal{A}u &= f &&\text{in } \Omega \\ \mathcal{B}u &= 0 &&\text{on } \partial\Omega \end{aligned} \tag{2.1}$$

on arbitrary bounded domains $\Omega \subset \mathbb{R}^N$, that is, a bounded open and connected set in $\mathbb{R}^N$. We always assume that $N \geq 2$. For $N = 1$ every bounded domain is smooth and we do not have to prove anything. The differential operators $\mathcal{A}$ and $\mathcal{B}$ are supposed to be of the form

$$\mathcal{A}u := -\sum_{i=1}^N \partial_i\Big(\sum_{j=1}^N a_{ij}(x)\partial_j u + a_i(x)u\Big) + \sum_{i=1}^N b_i(x)\partial_i u + c_0(x)u \tag{2.2}$$

and

$$\mathcal{B}u := \sum_{i=1}^N\Big(\sum_{j=1}^N a_{ij}(x)\partial_j u + a_i(x)u\Big)\nu_i + b_0(x)u\,, \tag{2.3}$$

where $\partial_i := \frac{\partial}{\partial x_i}$ and $\nu := (\nu_1, \dots, \nu_N)$ denotes the outer unit normal on the boundary $\partial\Omega$ of Ω. We also assume that $\mathcal{A}$ is uniformly strongly elliptic, that is, there exists $\alpha_0 > 0$ such that

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2 \tag{2.4}$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$, and that

$$\inf_{x\in\partial\Omega} b_0(x) \geq \beta_0 \tag{2.5}$$

for some $\beta_0 > 0$ (take the essential infimum if b_0 is only measurable). We throughout assume that the coefficients of $(\mathcal{A}, \mathcal{B})$ are real valued L_∞-functions.

We denote by $L_p(X)$ the Lebesgue spaces on a measurable subspace of $X \subset \mathbb{R}^N$ and by $\|\cdot\|_{p,X}$ its norm. If no confusion seems likely we just write L_p and $\|\cdot\|_p$, respectively.

Further, we write $L_{p,\mathrm{loc}}(X)$ for the space of all functions u such that $u|_K \in L_p(K)$ for all $K \subset X$ compact. Moreover, $W_p^k(\Omega)$ is the Sobolev space, which consists of all $u \in L_p$ such that all the distributional derivatives up to the order k lie in L_p. $\mathcal{D}(\Omega)$ is the space of smooth functions with compact support in Ω and $\mathring{W}_2^1(\Omega)$ its closure in $W_2^1(\Omega)$. Finally $C(\Omega)$ and $C^k(\Omega)$ are the spaces of continuous and k-times differentiable functions, respectively. If E, F are Banach spaces, we write $E \hookrightarrow F$ if $E \subset F$ and the natural injection is continuous. We write E' for the topological dual and if $A\colon E \to F$ is a densely defined operator we denote its dual by A'.

Usually, a function $u \in W_2^1(\Omega)$ is called a weak solution of (2.1) if

$$a(u,v) = \langle f, v\rangle \tag{2.6}$$

for all $v \in W_2^1(\Omega)$, where $a(\cdot\,,\cdot)$ is the (generalised) Dirichlet form corresponding to $(\mathcal{A},\mathcal{B})$ which is given by

$$a(u,v) := \int_\Omega \sum_{i=1}^N \Bigl(\sum_{j=1}^N a_{ij}\partial_j u + a_i u\Bigr)\partial_i v + \sum_{i=1}^N b_i v \partial_i u + c_0 uv\,dx + \int_{\partial\Omega} b_0 uv\,d\sigma\,. \tag{2.7}$$

Here, σ is the $(N-1)$-dimensional Hausdorff measure restricted to $\partial\Omega$ (which coincides with the surface measure if $\partial\Omega$ is smooth (e.g. [10], Theorem 3.2.3)). Further,

$$\langle f, u\rangle := \int_\Omega fu\,dx\,. \tag{2.8}$$

It is well known that if data and domain are smooth any sufficiently smooth weak solution satisfies (2.1) pointwise. As $W_2^1(\Omega)$-functions do not have well defined traces on $\partial\Omega$ for an arbitrary domain, and if so, it is not clear whether the boundary integral appearing in (2.7) is finite, we cannot use the space $W_2^1(\Omega)$ when dealing with weak solutions of (2.1) on arbitrary domains. The precise definition of a weak solution will be given in Section 3. As in case of the Dirichlet problem it turns out that (2.8) makes only sense for $f \in L_p(\Omega)$ if p is larger than some $p_0 > 1$. In order to define solutions for inhomogeneities in "lower" L_p-spaces we "extrapolate" the problem by means of duality. This allows to define generalised solutions of (2.1) for all $f \in L_p(\Omega)$ and $p \in [1,\infty]$. One of the main result of this paper is the following theorem.

2.1 Theorem. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain and suppose that $(\mathcal{A},\mathcal{B})$ are as above and that $c_0 \geq 0$. Then, for any $f \in L_p(\Omega)$ $(1 \leq p \leq \infty)$ equation (2.1) has a unique (generalised) solution u which belongs to $L_{m(p)}(\Omega)$, where $m(p) := Np(N-p)^{-1}$ if $p \in (1,N)$, $m(p) = \infty$ if $p > N$ and $m(p) < N(N-1)^{-1}$ arbitrary if $p = 1$.*

Moreover, there exists a constant $c > 0$ depending only on N, p and upper bounds for $\gamma := \max\{\alpha_0^{-1}, \beta_0^{-1}\}$, $\delta := \alpha_0^{-1}\big(\|\mathbf{a}\|_\infty^2 + \|\mathbf{b}\|_\infty^2\big)$ and the volume $|\Omega|$ of Ω such that

$$\|u\|_{m(p)} \leq c\|f\|_p \tag{2.9}$$

for all $f \in L_p(\Omega)$.

We also show that the eigenvalue problem

$$\begin{aligned} \mathcal{A}\varphi &= \lambda\varphi && \text{in } \Omega \\ \mathcal{B}\varphi &= 0 && \text{on } \partial\Omega \end{aligned} \tag{2.10}$$

has discrete spectrum. As a consequence of the above theorem we get the following result.

2.2 Corollary. *Under the assumptions of the above theorem there exists $\lambda_* > 0$ depending only on N and upper bounds for γ, δ and $|\Omega|$ such that*

$$\operatorname{Re}\lambda \geq \lambda_* \tag{2.11}$$

for all eigenvalues λ of (2.10).

In Section 5 we shall rewrite the above results in an abstract form. Theorem 2.1 along with its corollary will be proved in Theorem 5.3. The discreteness of the spectrum of (2.10) is contained in Corollary 5.2.

2.3 Remarks. (a) The existence of a positive lower bound λ_* for the first eigenvalue of (2.10) for all domains of a given volume was first observed in [5]. This even seems to be the case for the simplest selfadjoint problem. Here, we prove this for a slightly more general differential operator than in [5]. If $\mathcal{A} = -\Delta$ and b_0 constant it is an open problem whether a Faber-Krahn type inequality holds, that is, whether λ_* can be chosen to be the first eigenvalue of (2.10) on a ball with volume $|\Omega|$. For partial results on that problem see [23]–[25] or [3]. In all these references it is assumed that the domains satisfy certain geometric conditions such as convexity or restrictions on the curvature of the boundary.

(b) It turns out that the above theorem, its corollary as well as Theorem 2.4 below remain true if we have Dirichlet boundary conditions on a closed subset of $\partial\Omega$ (see Remark 4.3(b) below). $\square$

We finally consider the parabolic problem

$$\begin{aligned} \partial_t u + \mathcal{A}u &= f && \text{in } \Omega \times (0,\infty) \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega \times (0,\infty) \\ u(0,\cdot) &= u_0(\cdot) && \text{in } \Omega\,, \end{aligned} \tag{2.12}$$

where $\partial_t := \partial/\partial t$, and we show that the following results hold.

2.4 Theorem. *Suppose that the same assumptions as in Theorem 2.1 hold but allow c_0 to change sign. Further, let $p, q \in [1, \infty]$, such that $\infty > p \leq q$ and*

$$N\Big(\frac{1}{p} - \frac{1}{q}\Big) < 1 \tag{2.13}$$

Then, for all $u_0 \in L_p$ and $f \in L_\infty\big((0,\infty), L_q\big)$, equation (2.9) has a unique generalised solution $u \in C\big([0,T], L_p\big)$. If $f \in C^\theta\big([0,T], L_q\big)$ for some $\theta \in (0,1)$ then, $u \in C^1\big((0,T], L_p\big)$.

A proof of the above theorem as well as the precise definition of a generalised solution will be given at the end of Section 6. The idea is to reformulate (2.9) as an abstract parabolic equation in L_p-spaces, where all the standard theory applies. Of course, one can then use the same methods to prove existence and uniqueness of solutions of nonlinear problems. The smoothing properties which for the parabolic problem are reflected by condition (2.13) allows the nonlinearity to have arbitrary growth provided we only take initial values in L_∞.

3. The L_2-Theory

We start this section by recalling Maz'ja's approach to (2.1) on general domains. To do so we define $\tilde{V}$ to be the abstract completion of the space

$$V_0 := V_0(\Omega) := \{u \in W_2^1(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega) \colon \|u\|_V < \infty\}, \tag{3.1}$$

endowed with the norm $\|\cdot\|_V$ given by

$$\|u\|_V := \sqrt{\|\nabla u\|_2^2 + \|u|_{\partial\Omega}\|_{2,\partial\Omega}^2}\,. \tag{3.2}$$

Here, $\|\cdot\|_{2,\partial\Omega}$ is the norm in $L_2(\partial\Omega) := L_2(\partial\Omega, \sigma)$, and σ is the restriction of the $(N-1)$-dimensional Hausdorff measure to $\partial\Omega$. (In Maz'ja's notation this is the space $W_{2,2}^2(\Omega, \partial\Omega)$). The key to the whole theory is the following inequality due to Maz'ja ([15], see [17], Corollary 4.11.1/2). It says that for all $u \in V_0$

$$\|u\|_{\frac{2N}{N-1}} \leq c(N, |\Omega|)\|u\|_V \tag{3.3}$$

holds, where $c(N, |\Omega|) > 0$ is a constant depending only on N and an upper bound for $|\Omega|$. This inequality tells us that the natural embedding

$$j_0 \colon V_0 \to L_{\frac{2N}{N-1}}(\Omega) \tag{3.4}$$

is continuous with norm dominated by $c(N,|\Omega|)$. Hence j_0 has a unique extension $j \in \mathcal{L}(\tilde{V}, L_{\frac{2N}{N-1}})$. Using inequality (3.3) it is easy to check that the form $a(\cdot\,,\cdot)$ is continuous on $V_0 \times V_0$. It has therefore a unique continuous extension to $\tilde{V} \times \tilde{V}$, which we denote again by $a(\cdot\,,\cdot)$. Moreover, using (3.3) and a standard argument it is easy to see that there exist constant $\lambda_0 \in \mathbb{R}$ and $\alpha > 0$ such that

$$a(u,u) + \lambda_0 \|u\|_2 \geq \alpha \|u\|_V^2 \tag{3.5}$$

for all $u \in V_0$ and hence for $u \in \tilde{V}$ by continuity and density. As obviously, $\mathcal{D}(\Omega)$ is a subspace of $\tilde{V}$ the image of j is dense in $L_{\frac{2N}{N-1}}$. Therefore, the dual map $j' \in \mathcal{L}(L_{\frac{2N}{N+1}}, \tilde{V}')$ is an injection, which means that each element of $L_{\frac{2N}{N+1}}$ can be identified in a unique way with an element of $\tilde{V}'$. Therefore, the following definition makes sense.

3.1 Definition. Let $f \in L_p$, with $2N(N+1)^{-1} \leq p \leq \infty$. Then, $u \in \tilde{V}$ is said to be a weak solution of (2.1) if and only if

$$a(u,v) = \langle f, j(v) \rangle \quad (= \langle j'(f), v \rangle_V) \tag{3.6}$$

for all $v \in \tilde{V}$ (or a dense subset thereof), where $\langle \cdot\,,\cdot \rangle_V$ is the duality pairing between $\tilde{V}$ and $\tilde{V}'$, and $\langle \cdot\,,\cdot \rangle$ is defined by (2.8).

3.2 Remarks. (a) Note that if Ω is a Lipschitz domain then due to (3.3), the trace inequality $\|u\|_{W_2^1(\Omega)} \leq c\|u\|_{2,\partial\Omega}$ and the fact that $C^\infty(\overline{\Omega})$ is dense in $W_2^1(\Omega)$ it follows that $\tilde{V} = W_2^1(\Omega)$ up to an equivalent norm. Hence, if Ω is Lipschitz the definition of a weak solution is the same as in the classical theory.

(b) As $\mathcal{D}(\Omega) \subset \tilde{V}$ it follows that a weak solution of (2.1) is a weak solution of the equation

$$\mathcal{A}u = f \qquad \text{in } \Omega$$

in the usual sense, that is, $a_0(u,v) = \langle f, v \rangle$ for all $v \in \mathcal{D}(\Omega)$ where $a_0(u,v)$ is defined by (2.7) but without the boundary integral. Hence, all results on local properties of weak solutions of elliptic equations such as for instance given in [11], Section 8 apply.

(c) If $(\mathcal{A},\mathcal{B})$ is selfadjoint and $c_0 \geq 0$ we can choose $\lambda_0 = 0$ in (3.5) and hence, by the Lax Milgram Theorem (2.1) has a unique weak solution. This is in fact Maz'ja's result in [16], [17]. $\square$

There is one difficulty with the space $\tilde{V}$, which is not mentioned in [16], [17]. By definition of the embedding $j \in \mathcal{L}(\tilde{V}, L_{\frac{2N}{N-1}})$ it is not clear whether it is an injection or not. Injectivity of j is equivalent to the fact that for any sequence (u_n) in V_0 converging to u in $\tilde{V}$ such that $u_n \to 0$ in $L_{\frac{2N}{N-1}}$ implies that $u = 0$ in $\tilde{V}$. Using (3.3) it is clear that $u_n \to 0$ in $W_2^1(\Omega)$ as n tends to infinity, but the question which seems difficult to answer is whether this implies that $u_n|_{\partial\Omega} \to 0$ in $L_2(\partial\Omega)$ for a general domain Ω. The above arguments also show that $u \in \ker j$ if and only if there exists a sequence (u_n) in V_0 such that

$$\lim_{n\to\infty} u_n = 0 \text{ in } W_2^1(\Omega) \quad \text{and} \quad \lim_{n\to\infty} u_n|_{\partial\Omega} = w \text{ in } L_2(\partial\Omega). \tag{3.7}$$

We do not know any example of a domain such that j is not an injection and we suspect that it always is. However, we do not have a proof. For further comments concerning this questions we refer to Remark 3.5. There is a way to get rid of this difficulty by replacing $\tilde{V}$ by a closed subspace of V. To see this observe that by (3.6) any weak solution of (2.1) is orthogonal to the kernel, $\ker j$, of j with respect to the form $a(\cdot\,,\cdot)$ and, moreover, u is a weak solution of (2.1) if and only if (3.6) holds for v in

$$V_a := \{u \in \tilde{V} \colon a(u,v) = 0 \text{ for all } v \in \ker j\}\,. \tag{3.8}$$

If it happens that j is not injective we could replace $\tilde{V}$ by V_a which a priori depends on the operators $(\mathcal{A},\mathcal{B})$. By (3.7) and the definition of $a(\cdot\,,\cdot)$ $u \in V_a$ if and only if

$$\int_{\partial\Omega} b_0 uv\, d\sigma = 0 \tag{3.9}$$

for all $v \in \ker j$ and so V_a is independent of the coefficients of $\mathcal{A}$. It turns out that V_a is also independent of $\mathcal{B}$. The precise statement is contained in the following proposition.

3.3 Proposition. *For all $(\mathcal{A},\mathcal{B})$ satisfying (2.2)–(2.5) we have that $V_a = (\ker j)^\perp =: V$. Furthermore, $\ker j$ is isometric isomorphic to $L_2(S)$, where S is a measurable subset of $\partial\Omega$, and*

$$V = (\ker j)^\perp = \{u \in \tilde{V} \colon u = 0 \text{ on } S\}\,. \tag{3.10}$$

Finally, the function u is a weak solution of (2.1) if and only if $u \in V$ and (3.6) holds for all $v \in V$.

The proof makes use of the following lemma which we prove at the end of this section.

3.4 Lemma. *If $v \in \ker j$ then, $b_0 v \in \ker j$ for all $b_0 \in L_\infty(\partial\Omega)$.*

Note that by definition of $\ker j$ $v = 0$ in Ω, and thus it makes sense to multiply v by a function just defined on $\partial\Omega$.

Proof of Proposition 3.3. To prove the first assertion we have to show that $\int_{\partial\Omega} b_0 uv\, d\sigma = 0$ for all $v \in \ker j$ if and only if $\int_{\partial\Omega} uv\, d\sigma = 0$ for all $v \in \ker v$. But this is easily seen from the above lemma and (2.5) since $b_0^{-1}v$ and $b_0 v$ are in the kernel of j.

Let S be the union of all essential supports of $v \in \ker j$. Then, it follows from the above lemma that all simple functions on S are in $\ker j$, and hence, $\ker j = L_2(S)$. As a simple consequence of this we obtain (3.10). The last assertion is clear from the remarks before Proposition 3.3. $\square$

Next we have some more remarks and examples concerning the space $\tilde{V}$.

3.5 Remarks. (a) Due to (3.3) it turns out that $\tilde{V}$ is isomorphic to the closure of

$$\{(u, u|_{\partial\Omega})\colon u \in V_0\}$$

in $W_2^1(\Omega) \times L_2(\partial\Omega)$. All elements of $\tilde{V}$ have well defined traces on $\partial\Omega$ given by $\lim_{n\to\infty} u_n|_{\partial\Omega}$, where u_n is a sequence in V_0 converging to u in $\tilde{V}$. To keep notation as simple as possible we do not distinguish between u and its trace on $\partial\Omega$ in our notation. However, note that due to the same problem we had with the existence of a kernel for the embedding j we do not know how much the trace can "disconnect" from the function in Ω.

(b) The space

$$V_1 := \{u \in C(\overline{\Omega}) \cap W_2^1(\Omega)\colon \|u\|_V < \infty\}$$

is dense in $\tilde{V}$. Indeed, a careful analysis of the proof that $W_2^1(\Omega) \cap C^\infty(\Omega)$ is dense in $W_2^1(\Omega)$ such as given in [1], Theorem 3.16 reveals that if $u \in C(\overline{\Omega}) \cap W_2^1(\Omega)$ the approximating function ϕ lies in $C(\overline{\Omega}) \cap C^\infty(\Omega) \cap W_2^1(\Omega)$, and has the same boundary values as u. Therefore, V_1 is dense in V_0 and hence in $\tilde{V}$.

(c) The space $\tilde{V}$ is a Banach lattice, that is, if $u \in \tilde{V}$ then the absolute value $|u|$ also lies in $\tilde{V}$ and $\||u|\|_V = \|u\|_V$. To see this note that by [11], Lemma 7.6 $W_2^1(\Omega) \times L_2(\partial\Omega)$ is a Banach lattice. As we know from (a) that $\tilde{V}$ is a closed subspace thereof it is sufficient to show that $|u| \in \tilde{V}_1$ for all $u \in V_1$, where V_1 is the space defined in (b). But this is clear as $C(\overline{\Omega})$ is a lattice too.

(d) Suppose that Ω is a domain with a boundary whose $(N-1)$-dimensional Hausdorff measure is locally infinite, that is, for all $x \in \partial\Omega$ and $\varepsilon > 0$ we have that

$$\sigma\big(B_\varepsilon(x) \cap \partial\Omega\big) = \infty\,. \tag{3.11}$$

Then, $V = \tilde{V} = \mathring{W}_2^1(\Omega)$ up to an equivalent norm. In this case our boundary value problem coincides with the Dirichlet problem. If (3.11) only holds on part of the boundary Dirichlet boundary conditions are satisfied on that part. Examples of such domains are domains with fractional boundaries such as for instance the interior of the well known "snowflake curve" (e.g. [9]).

To prove our claim we first show that $V_0 \subset \mathring{W}_2^1(\Omega)$. To do so let $u \in V \cap C(\overline{\Omega})$ be arbitrary. As u is continuous on $\overline{\Omega}$ it follows from (3.11) that $u|_{\partial\Omega} = 0$. By splitting u in positive and negative part (which by (c) also belong to $V \cap C(\overline{\Omega})$) we can assume without loss of generality that $u \geq 0$. By the uniform continuity of u on $\overline{\Omega}$ it is clear that $u_\varepsilon := \max\{u - \varepsilon, 0\} \in \mathring{W}_2^1(\Omega)$ for all $\varepsilon > 0$. It is then easily seen that u_ε tends to u in V as ε goes to zero. Finally, note that $\|\nabla u\|_2$ is an equivalent norm on $\mathring{W}_2^1(\Omega)$, and thus u_ε also converges in $\mathring{W}_2^1(\Omega)$. Hence, it follows that $V = \mathring{W}_2^1(\Omega)$.

(e) The above remark together with Proposition 3.3 show that wherever the boundary is bad either in the sense that its measure is locally infinite or that it allows j to have a kernel the weak solutions of (2.1) satisfy Dirichlet boundary conditions in some sense. In case of a nontrivial kernel we are not sure whether we really have Dirichlet boundary conditions in the weak sense because we do not know whether we can approximate the solutions by elements of V_0 being zero in a neighbourhood of that part of the boundary.

(f) Suppose that $K \subset \partial\Omega$ is a compact set of capacity zero and that $\partial\Omega \setminus K$ is locally Lipschitz in the sense that it is locally the graph of a Lipschitz function. In that case every $u \in \tilde{V}$ has locally a trace and for all $x \in \partial\Omega \setminus K$ there exists a ball $B_\varepsilon(x)$ in $\mathbb{R}^N$ and a constant $c(\varepsilon, x)$ such that

$$\Big(\int_{\partial\Omega \cap B_\varepsilon(x)} |u(y)|^2 \, d\sigma\Big)^{\frac{1}{2}} \leq c(\varepsilon, x)\|u\|_{W_2^1(\Omega)} \tag{3.12}$$

for all $u \in V_0$. This implies that the trace of u is in $L_{2,\mathrm{loc}}(\partial\Omega \setminus K)$. Then, it turns out that

$$\tilde{V} = V = \{u \in W_2^1(\Omega) \colon \gamma u \in L_2(\partial\Omega \setminus K)\}\,. \tag{3.13}$$

This characterisation shows that in general the space $\tilde{V}$ is much smaller than $W_2^1(\Omega)$ even if the measure of $\partial\Omega$ is finite and the domain is smooth except for one point. As an example consider a domain with an outward pointing cusp. Using (3.3) it is clear that $\tilde{V} \hookrightarrow W_2^1(\Omega)$. If we had equality the open mapping theorem would imply that the two spaces coincide up to an equivalent norm. This is not possible since $\tilde{V} \hookrightarrow L_{\frac{2N}{N-1}}$ but $W_2^1(\Omega) \not\subset L_p$ for all $p > 2$ if the domain has an exponential cusp (see [1], Theorem 5.32).

To prove (3.13) it is sufficient to prove that

$$V_1 := \{u \in W_2^1(\Omega) \cap C^\infty(\overline{\Omega}) \colon u = 0 \text{ in a neighbourhood of } K\} \tag{3.14}$$

is dense in $W_2^1(\Omega)$ as well as in V_0. It is well known that it is dense in $W_2^1(\Omega)$. As K has zero capacity there exists a sequence $\varphi_k \in \mathcal{D}(\mathbb{R}^N)$ with $0 \leq \varphi_k \leq 1$, $\varphi_k = 1$ in a neighbourhood of K such that $\|\varphi_k\|_{W_2^1(\Omega)} \to 0$ as k tends to infinity. Let $u \in V_0$

be arbitrary and set $u_k := (1-\varphi_k)u$. Then obviously $u_k \in V_1$ and $\nabla u_k \to \nabla u$ in L_2 as k tends to infinity. As K has zero capacity we have that $\sigma(K) = 0$ (e.g. [28], Theorem 2.6.16) and thus φ_k converges to zero σ-almost everywhere on $\partial\Omega$. As $|\varphi_k u| \leq |u|$ on $\partial\Omega$, it follows from the dominated convergence theorem that $u_k \to u$ in $L_2(\partial\Omega)$. This shows that V_1 is dense in V_0 and hence in V_0 which proves our claim.

(g) A sufficient condition making sure that j is injective is that $W_2^1(\Omega)$ allows at least locally a trace operator. More precisely assume that for σ-almost all $x \in \partial\Omega$ (3.12) holds. Then, $u_n|_{\partial\Omega} \to 0$ in $L_2(\partial\Omega \cap B_\varepsilon(0))$ if $u_n \to 0$ in $W_2^1(\Omega)$. From this we conclude that $u = 0$ σ-almost everywhere on $\partial\Omega$, whence $u = 0 \in \tilde{V}$ and our claim follows. $\square$

We conclude this section by proving Lemma 3.4

Proof of Lemma 3.4. In a first step we assume that $b_0 \in C^\infty(\overline{\Omega})$. If $v \in \ker j$ take $v_n \in V_0$ such that $v_n \to v$ in $\tilde{V}$. Clearly $b_0 v_n \in V_0$ and by a simple calculation

$$\|b_0 v_n\|_{W_2^1(\Omega)} \leq (\|b_n\|_\infty + \|\nabla b_0\|_\infty)\|v_n\|_{W_2^1(\Omega)}\,.$$

Thus, as $\|v_n\|_{W_2^1(\Omega)} \to 0$ by (3.7), we conclude that $\|b_0 v_n\|_{W_2^1(\Omega)} \to 0$. Moreover, $\|b_0(v_n - v)\|_{2,\partial\Omega} \leq \|b_0\|_\infty \|v_n - v\|_{2,\partial\Omega} \to 0$ as $n \to \infty$. By (3.7) this implies that $b_0 v \in \ker j$.

In a second step we assume that $b_0 \in C(\overline{\Omega})$. Then, by Tietze's Theorem there exists a continuous extension of b_0 to $\mathbb{R}^N$ which we denote again by b_0. This continuous function can then be approximated by smooth functions b_k uniformly on $\overline{\Omega}$. By the first step $b_k v \in \ker j$ for all $k \in \mathbb{N}$. Moreover, as

$$\|b_k v - b_\ell v\|_{\tilde{V}}^2 = \int_{\partial\Omega} |b_k - b_\ell|^2 v^2\, d\sigma \tag{3.15}$$

for all $k, \ell \in \mathbb{N}$ it turns out that $b_k v$ is a Cauchy sequence in $\tilde{V}$, which converges to $b_0 v$. As $\ker j$ is closed it follows that $b_0 v \in \ker j$.

We next try to approximate $b_0 \in L_\infty(\partial\Omega)$ by continuous functions in a suitable way. To do so first note that the restriction of the $(N-1)$-dimensional Hausdorff measure σ to $\partial\Omega$ is an inner regular Borel measure. Fix $v \in \ker j$ and set for $k \in \dot{\mathbb{N}} := \mathbb{N} \setminus \{0\}$

$$A_k := \{x \in \partial\Omega : |v(x)| \geq k^{-1}\} \qquad \text{and} \qquad A_0 := \{x \in \partial\Omega : |v(x)| \neq 0\}.$$

Then, the union of the A_k is A_0. As $v \in L_2(\partial\Omega)$ it follows that $|A_k| := \sigma(A_k) < \infty$ for all $k \in \dot{\mathbb{N}}$. By Lusin's Theorem (see [10], Theorem 2.3.5) we find for every $k \in \dot{\mathbb{N}}$ a compact set C_k such that

$$|A_k \setminus C_k| < k^{-1} \tag{3.16}$$

and $b_0|_{C_k}$ is continuous. By Tietze's Theorem there exists $b_k \in C(\partial\Omega)$ such that

$$\|b_k\|_\infty \le \|b_0\|_\infty \tag{3.17}$$

and $b_k = b_0$ on C_k. Without loss of generality we can choose C_k in such a way that $C_{k+1} \supset C_k$ for all $k \in \dot{\mathbb{N}}$. We prove now that $b_k \to b_0$ σ-almost everywhere on A_0. Suppose this is not the case. Then, there exists a set $B \subset A_0$ with $|B| > 0$ such that $b_k \not\to b_0$ on B. As the union of A_k is A_0 there exists $k_0 \in \mathbb{N}$ and $\delta > 0$ such that $|B \cap A_k| \ge \delta$ for all $k \ge k_0$. On the other hand, $|B \cap A_k| = |B \cap A_k \setminus C_k| + |B \cap C_k| > \delta$ for $k \ge k_0$, whence $|B \cap C_k| > \delta - k^{-1} > 0$ by (3.16) for k large enough. As $b_0 = b_k$ on C_k this is a contradiction. This shows that $b_k v \to b_0 v$ almost everywhere on $\partial\Omega$. By step two, $b_k v \in \ker j$ for all $k \in \dot{\mathbb{N}}$. Moreover, by (3.17) $|b_k - b_\ell|^2 v^2 \le 2\|b_0\|_\infty v^2$. Hence, it follows from (3.15) and the Dominated Convergence Theorem that $b_k v$ is a Cauchy sequence in $\tilde{V}$ with limit $b_0 v$. As $\ker j$ is closed this concludes the proof of the lemma. □

4. Global Estimates for Weak Solutions

In this section we give global estimates for weak solutions of equation (2.1). The results here slightly generalise the results obtained in [5]. Before giving the details of the proof, which employs a version of Moser iteration techniques, we recall some definitions and state our results.

As usual $u \in V$ is called a (weak) subsolution of (2.1) if

$$a(u, v) \le \langle f, v\rangle \tag{4.1}$$

for all nonnegative $v \in V$, where was defined in Proposition 3.3. If the reverse inequality holds u is called a (weak) supersolution. Further we denote by u^+ and u^- the positive and negative part of u, respectively, that is, $u^\pm := \max\{\pm u, 0\}$. We could also replace V by $\tilde{V}$ in our definition, but then we have to replace v by $j(v)$ on the right hand side of (4.1).

4.1 Theorem. *Suppose* $(\mathcal{A}, \mathcal{B})$ *satisfy* (2.2)–(2.5). *Then, if* $p \in [2, N)$, *there exists a constant* $c > 0$ *depending only on* N, p *and an upper bound for* $|\Omega|$ *such that for any weak solution of* (2.1)

$$\|u\|_{\frac{Np}{N-p}} + \|u\|_{\frac{(N-1)p}{N-p},\partial\Omega} \le c\gamma\big(\|f\|_p + \gamma^\mu \delta^{1+\mu}\|u\|_2\big)\,, \tag{4.2}$$

where

$$\gamma := \max\{\alpha_0^{-1}, \beta_0^{-1}\}, \qquad \delta := \alpha_0^{-1}\big(\|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty\big)^2 + \|c_0^-\|_\infty \tag{4.3}$$

and $\mu := N(p-2)/2p$. *If* $p > N$ *there exists a constant* $c > 0$ *depending on the same quantities as above such that*

$$\|u\|_\infty \leq c\gamma\bigl(\|f\|_p + \gamma^\mu \delta^{1+\mu}\|u\|_2\bigr)\,. \tag{4.4}$$

Moreover, $\|u\|_{\infty,\partial\Omega} \leq \|u\|_\infty$. *Furthermore, if* u *is a sub- or supersolution the above assertions hold for* u *replaced by* u^+ *or* u^-*, respectively.*

The assertion that $\|u\|_{\infty,\partial\Omega} \leq \|u\|_\infty$ is not completely obvious as we do not know how much the trace of a function $u \in V$ can "disconnect" from the function in Ω as already mentioned in Remark 3.5(a). The next theorem contains an L_∞-estimate not containing the L_2-norm.

4.2 Theorem. *Let* $(\mathcal{A}, \mathcal{B})$ *satisfy* (2.2)–(2.5) *with* $c_0 \geq 0$. *Then, if* $p > N$ *there exists a constant* $c > 0$ *depending only on* N, p *and upper bounds for* $|\Omega|$, γ *and* δ *such that for any weak solution* u *of* (2.1)

$$\|u\|_\infty \leq c\|f\|_p\,. \tag{4.5}$$

If u *is a sub- or a supersolution the inequality holds if we replace* u *by* u^+ *and* u^-*, respectively.*

Note that it is sufficient to give a proof of the above theorems for subsolutions. For supersolutions the assertion follows since $-u$ is a subsolution and for solutions by combining the two inequalities.

Proof of Theorem 4.1. We define for every $m > 0$ and $q \geq 2$ the functions $G_{q,m}(\xi) := \min\{\xi^{q-1}, m^{q-1}\}$ and $H_{q,m}(\xi) := \min\{\xi^{\frac{q}{2}}, m^{\frac{q}{2}}\}$ if $\xi \geq 0$ and $G_{q,m}(\xi) = H_{q,m}(\xi) := 0$ for $\xi \leq 0$. It is easy to see that the substitution operators induced by $G_{q,m}$ and $H_{q,m}$ are continuous on $W_2^1(\Omega)$ and on $L_2(\partial\Omega)$ (see also [14]). Hence, it is continuous on V_0 and therefore on V. To simplify notation we set $v := G_{q,m}(u)$ and $w := H_{q,m}(u)$. Then, taking into account [11], Lemma 7.6 we get that

$$\partial_i w \partial_j w = \frac{q^2}{4(q-1)}\partial_i u \partial_j v, \quad w\partial_i w = \frac{q}{2}v\partial_i u = \frac{q}{2(q-1)}u\partial_i v \quad \text{and} \quad w^2 = uv\,.$$

Using this as well as (2.4) and Hölder's inequality and we get that

$$\begin{aligned}\alpha_0\|\nabla w\|_2^2 &\leq \frac{q^2}{4(q-1)}\Bigl(a_0(u,v) - \frac{2(q-1)}{q}\int_\Omega a_i w\partial_i w - \frac{2}{q}\int_\Omega b_i w\partial_i w - \int_\Omega c_0 w^2\Bigr)\\ &\leq q\bigl(a_0(u,v) + 2(\|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty)\|w\|_2\|\nabla w\|_2 + \|c_0^-\|_\infty\|w\|_2^2\bigr)\,,\end{aligned}$$

where $a_0(\cdot\,,\cdot)$ is defined as (2.7) but without the boundary term. By a standard trick we find that

$$\alpha_0\|\nabla w\|_2 \le q\bigl(a_0(u,v)+\delta q\|w\|_2^2\bigr)\,. \tag{4.8}$$

Obviously, we have that

$$\beta_0\|w\|_{2,\partial\Omega} \le q\int_{\partial\Omega} b_0 uv\,d\sigma\,. \tag{4.9}$$

Combining the two estimates and taking into account (3.3) we get that

$$\|w\|^2_{\frac{2d}{d-2}} \le c(N,|\Omega|)^2\|w\|_V^2 \le c(N,|\Omega|)^2\gamma q\Bigl(a(u,v)+\delta q\|w\|_2^2\Bigr) \tag{4.10}$$

where $d := 2N$. Using Hölder's inequality, (3.3) and the fact that u is a subsolution of (2.1) we have that

$$\|w\|^2_{\frac{2d}{d-2}} \le c(N,|\Omega|)^2\|w\|_V^2 \le c(N,|\Omega|)^2\gamma q^2\Bigl(\|f\|_p\|v\|_{p'}+\delta\|w\|_2^2\Bigr)\,. \tag{4.11}$$

Furthermore, if $p\ge 2$ then $r := 4p(3p-2)^{-1}\le 2$, and by a well known interpolation inequality we obtain

$$\|w\|_2^2 \le \|w\|^{2(1-\theta)}_{\frac{2d}{d-2}}\|w\|_r^{2\theta}$$

with θ satisfying $2^{-1}=\theta r^{-1}+(1-\theta)(d-2)(2d)^{-1}$. Applying Young's inequality (see [11], Section 7.1) and (3.3) this implies that for all $\varepsilon>0$

$$\|w\|_2^2 \le \varepsilon c(N,|\Omega|)^2\|w\|_V^2+\varepsilon^{-\mu}\|w\|_r^2$$

with $\mu := d(p-2)(4p)^{-1}$. Furthermore, by Hölder's inequality

$$\|w\|_r^2 = \|w^{\frac{2}{q}+\frac{2}{q}(q-1)}\|_{\frac{r}{2}} \le \|u^+\|_2\|w^{\frac{2}{q}(q-1)}\|_{p'} = \|u^+\|_2\|w^{\frac{2}{q}}\|^{q-1}_{p'(q-1)}.$$

Note also that $v = w^{\frac{2}{q}(q-1)}$. Combining all this and setting $\varepsilon := (2c(N,|\Omega|)^2\gamma\delta q^2)^{-1}$ and $\bar c := 2c(N,|\Omega|)^2\gamma$ we conclude from (4.11) that

$$\|w^{\frac{2}{q}}\|^q_{\frac{dq}{d-2}} \le c(N,|\Omega|)^2\|w\|_V^2 \le \bar c q^{2(1+\mu)}\Bigl(\|f\|_p+\bar c^{\mu}\delta^{1+\mu}\|u^+\|_2\Bigr)\|w^{\frac{q}{2}}\|^{q-1}_{p'(q-1)}\,. \tag{4.12}$$

In case $p\in[2,d/2)$ we set $q := p(d-2)(d-2p)^{-1}$ and divide inequality (4.12) by $\|u\|^{q-1}_{\frac{dp}{d-2p}}$. As $w^{\frac{2}{q}}=\min\{u^+,m\}$ we obtain

$$\|u^+\|_{\frac{dp}{d-2p}} \le C\bar c\bigl(\|f\|_p+\bar c^{\mu}\delta^{1+\mu}\|u^+\|_2\bigr)$$

by letting m to infinity. Note that C just depends on d and p. To show that the L_∞-estimate holds for $p > N = d/2$ we set $q_0 := 2$ and $q_{n+1} := 1 + \eta q_n$ for all $n \in \mathbb{N}$, where

$$\eta := \frac{d(p-1)}{p(d-2)}.$$

Moreover, set $\bar{u} := u^+ M^{-1}$ with $M := \bar{c}\big(\|f\|_p + \bar{c}^\mu \delta^{1+\mu}\|u\|_2\big)$. Then (4.12) yields

$$\|\bar{u}\|_{\frac{dq_{n+1}}{d-2}} \le q_{n+1}^{2\frac{1+\mu}{q_{n+1}}} \|\bar{u}\|_{\frac{dq_n}{d-2}}^{1-\frac{1}{q_{n+1}}},$$

from which we get by induction that

$$\|\bar{u}\|_{\frac{dq_{n+1}}{d-2}} \le q_{n+1}^{2\frac{1+\mu}{q_{n+1}}} \prod_{k=1}^{n} q_k^{2\frac{1+\mu}{q_k}\prod_{i=k+1}^{n}(1-\frac{1}{q_i})} \|\bar{u}\|_{\frac{2d}{d-2}}^{\prod_{k=1}^{n+1}(1-\frac{1}{q_k})} \tag{4.13}$$

Observe now that $\eta \le q_{n+1}/q_n \le 2\eta$ and therefore, $\eta^n \le q_n \le (2\eta)^n$. Together with (4.13) we see that

$$\|\bar{u}\|_{\frac{dq_{n+1}}{d-2}} \le (2\eta)^{2\frac{1+\mu}{q_{n+1}}\sum_{k=0}^{\infty} k\alpha^{-1}} \|\bar{u}\|_{\frac{2d}{d-2}}^{\prod_{k=1}^{n+1}(1-\frac{1}{(2\eta)^k})}.$$

As $\eta > 1$ for the range of p under consideration we can let n to infinity to get

$$\|\bar{u}\|_\infty \le C\|\bar{u}\|_{\frac{2d}{d-2}}^{\theta} \tag{4.14}$$

for some $\theta \in (0,1)$ depending only on d and p. As $|\Omega|$ this implies that

$$\|\bar{u}\|_\infty \le \big(C|\Omega|^{\theta(\frac{1}{2}-\frac{1}{d})}\big)^{1-\theta} \tag{4.15}$$

which implies (4.4).

It remains to prove the estimate on the boundary. By definition of $\|\cdot\|_V$ and setting $q := p(N-1)(N-p)^{-1}$ it follows from (4.12) that $\|\bar{u}\|^q_{\frac{p(N-1)}{N-p},\partial\Omega} \le \|\bar{u}\|^{q-1}_{\frac{Np}{N-p}}$. Using the estimate for $\|\bar{u}\|_{\frac{dp}{d-2p}}$ we conclude that (4.2) holds. To prove that $\|u\|_{\infty,\partial\Omega} \le \|u\|_\infty$ note that as a consequence of (4.12) we have that $\|\bar{u}\|_{q,\partial\Omega} \le \big(q^{\frac{1}{q}}\big)^{2(1+\mu)}\|\bar{u}\|_\infty^{1-\frac{1}{q}}$ for all $q \ge 2$ and our assertion follows by letting q to infinity as the constants in front of $\|\bar{u}\|_\infty$ tend to one. This proves Theorem 4.1. $\square$

Note that in the proof of Theorem 4.1 in case of a subsolution we just needed that (4.1) holds for all $v \in V$ such that $uv \ge 0$. This is essential in the proof of Theorem 4.2 which follows.

Proof of Theorem 4.2. To prove (4.5) we use a modification of a test function argument due to Trudinger [26] (see [11], proof of Theorem 8.6). We set

$$v := \frac{u^+}{m+k-u^+} \qquad \text{and} \qquad w := \log \frac{m+k}{m+k-u^+}\,,$$

where $\|u^+\|_\infty$ and $k := \|f\|_p$. We already know that $u \in L_\infty$ so the above definition makes sense. To estimate the boundary integral $\int_{\partial\Omega} w^2\,d\sigma$ we need the following inequality, which holds for all $0 \leq s \leq r$:

$$\Bigl(\log \frac{r}{r-s}\Bigr)^2 \leq \frac{s^2}{r(r-s)}\,. \tag{4.16}$$

To prove the inequality we set $t := s/r$. It is then easy to see that (4.16) is equivalent to the fact that

$$\Bigl(\log \frac{1}{1-t}\Bigr)^2 \leq \frac{t^2}{1-t}$$

for all $t \in (0,1)$, which in turn is equivalent to $g(t) := t^{-2}(1-t)\bigl(\log(1-t)\bigr)^2 \leq 1$ for all $t \in (0,1)$. Using de l'Hôpital's rule we find that $\lim_{t\to 0} g(t) = 1$. As $g \geq 0$ it is now sufficient to show that $g'(t) < 0$ for all $t \in (0,1)$. As

$$g'(t) = -t^{-2}\bigl(2 - t^{-1}(t-2)\log(1-t)\bigr)\log(1-t)\,,$$

it remains to show that $t^{-1}(t-2)\log(1-t) > 2$ for all $t \in (0,1)$. Using the Taylor expansion of $\log(1-t)$ we obtain

$$t^{-1}(t-2)\log(1-t) = 2 + \sum_{k=2}^{\infty}\Bigl(\frac{2}{k+1} - \frac{1}{k}\Bigr)t^k > 2$$

for all $t \in (0,1)$ completing the proof of (4.16). Hence, setting $r := m+k$ we conclude that

$$\int_{\partial\Omega} w^2\,d\sigma \leq \frac{1}{m+k}\int_{\partial\Omega} \frac{(u^+)^2}{m+k-u^+}\,d\sigma\,. \tag{4.17}$$

Now we estimate $\|w\|_2$ in the same way as in the proof of [11] Theorem 8.16 using (4.17) for the boundary integral. This yields

$$\|w\|_2 \leq C\,, \tag{4.18}$$

where C just depends on N and upper bounds for δ, γ and $|\Omega|$. Set now $v := \varphi(m+k-u^+)^{-1}$. Again using the same arguments as in the proof of [11] Theorem 8.16 and estimating the boundary integral in an obvious way we see that w satisfies

$$a(w,\varphi) \leq a(u,v) \leq \langle f, v\rangle \leq \langle f k^{-1}, \varphi\rangle$$

for all nonnegative $\varphi \in V$ satisfying $u\varphi \geq 0$. As $w\varphi \geq 0$ if $u\varphi \geq 0$, we may apply Theorem 4.1 to estimate $\|w\|_\infty$ in terms of $\|k^{-1}f\|_p$ and $\|w\|_2$. Using (4.18) and taking exponentials we finally obtain (4.5). □

4.3 Remarks. (a) Inequality (4.2) tells us that the solutions of (2.1) with $f \in L_p$ lie in $L_{\frac{Np}{N-p}}(\Omega) \times L_{\frac{p(N-1)}{N-p}}(\partial\Omega)$, that is, they lie in the Lebesgue spaces the Sobolev spaces $W_p^1(\Omega)$ and its trace space $W_p^{1-\frac{1}{p}}(\partial\Omega)$ inject into. Note that even not in case of a polygonal domain and smooth f we can expect the solution of the Robin problem to ly in $W_p^1(\Omega)$ for all $p > 2$. Concerning this, see [12], Theorem 4.4.3.7, where the $S_{j,m}$ turn out to be not in $W_p^1(\Omega)$ for p large enough.

(b) The above results remain true if we have Dirichlet boundary conditions on a closed subset Γ_0 of $\partial\Omega$. To see that we replace the space V_0 introduced in Section 3 by

$$V_{0,\Gamma_0} := \{u \in V_0 \colon \operatorname{supp} u \in \overline{\Omega} \setminus \Gamma_0\}.$$

Then, (3.3) still holds for all $u \in V_{0,\Gamma_0}$ and hence, for all u in the completion V_{Γ_0} of V_{0,Γ_0} with respect to the Norm $\|\cdot\|_V$. As the proof of the above theorems are based on this inequality everything remains true.

(c) If we restrict the class of domains we can get better estimates than those in Theorem 4.1. For instance if we take domains in the class $\mathcal{K}_{\alpha,1}$, $\alpha < 1$, introduced by Maz'ja (see [17], Section 3.6.1), which are determined by the same constant $\mathcal{E}$ it turns out that $V \hookrightarrow L_{\frac{2d}{d-2}}$ for $d := 2(1-\alpha)^{-1}$ (replace u by u^2 in [17], Theorem 3.6.3). In the above proofs we just have to replace $d = 2N$ by the new value of d and in the assertion N by $d/2$. The classes of domains mentioned above allow to deal with nonsmooth domains having cusps of restricted sharpness. The sharpness of the cusp determines the class $\mathcal{K}_{\alpha,1}$ ([17], §3.6). Above we only considered the class $\mathcal{K}_{1-\frac{1}{N},1}$ where all bounded domains belong to with the same constant $\mathcal{E}$ which is given by the isoperimetric constant ([17], Example 3.6.2/1).

(d) By using (3.3) we obviously can weaken the assumptions on the coefficients a_i, b_i and c_0. It is sufficient to assume that $a_i, b_i \in L_r$ for some $r > 2N$ (r may vary from one coefficient to another) eand $c_0 \in L_s$ for some $s > N$. We just have to replace the L_∞-norms in δ by the corresponding weaker norms. □

5. The L_p-Theory

In this section we prove smoothing properties of problem (2.1) making use of the a priori estimates established in the previous section, the L_2-theory in Section 2 and the abstract results in the Appendix. In particular we give a proof of Theorem 2.1.

Suppose that V is as in Proposition 3.3 and $a(\cdot\,,\cdot)$ is the form defined by (2.7). We already proved in Section 2 that $a(\cdot\,,\cdot)$ extends to a bounded form on V, and that it satisfies (3.5) for some $\alpha > 0$ and $\lambda_0 \in \mathbb{R}$. As $\mathcal{D}(\Omega) \subset V$ clearly $V \cap L_p$ is dense in $L_p(\Omega)$ for all $p \in [1,\infty)$. If we put $H := L_2(\Omega)$ we are therefore in a setting which allows us to apply the abstract results in the Appendix. Define the adjoint form $a^\sharp(\cdot\,,\cdot)$ by $(A.11)$. Note that this form corresponds to the formal adjoint boundary value problem $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$ to $(\mathcal{A}, \mathcal{B})$ given by

$$\mathcal{A}^\sharp u := -\sum_{i=1}^N \partial_i \Big(\sum_{j=1}^N a_{ji}(x)\partial_j u + b_i(x)u \Big) + \sum_{i=1}^N a_i(x)\partial_i u + c_0(x)u \tag{5.1}$$

and

$$\mathcal{B}^\sharp u := \sum_{i=1}^N \Big(\sum_{j=1}^N a_{ji}(x)\partial_j u + b_i(x)u \Big)\nu_i + b_0(x)u\,. \tag{5.2}$$

Denote by A and $A^\sharp$ the operators induced on V' by the corresponding forms $a(\cdot\,,\cdot)$ and $a^\sharp(\cdot\,,\cdot)$, respectively. Note also that $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$ has the same structure as $(\mathcal{A}, \mathcal{B})$. Hence, A and $A^\sharp$ satisfy $(A.6)$ with the same constants (see proof of Theorem 4.1). Suppose that A_p and $A_p^\sharp$ are defined as in Part III of the Appendix.

5.1 Theorem. *All assertion of Theorem* A.9 *hold with $d = 2N$. In particular,*

$$(\lambda + A)^{-1}(L_p) \subset L_{m(p)} \tag{5.3}$$

for all $\lambda \in \varrho(-A)$, and $m(p)$ is as in Theorem 2.1. Moreover, A_p has compact resolvent for all $p \in [1,\infty]$ and for all $\lambda \geq \lambda_0$ $(\lambda + A)^{-1}$ is irreducible.

Proof. Due to the results in Section 4 we can apply Theorem A.9 with $d = 2N$. Compactness of the resolvent follows since V is compactly embedded into L_2 ([17], Corollary 4.11.1/3). Positivity of $(\lambda + A)^{-1}$ follows by a simple test function argument (take the negative part of $u := (\lambda + A)^{-1} f$ as a test function and conclude that it is zero). By a density argument it turns out that $(\lambda + A_p)^{-1}$ is positive for all $p \in [0,\infty]$. To show irreducibility it is sufficient to prove that there exists $k \in \mathbb{N}$ such that for all nonnegative $f \in L_p$ with $f \neq 0$ $(\lambda + A_p)^{-k} f$ is strictly positive almost everywhere in Ω, that is, for any such f, $(\lambda + A_p)^{-k} f$ is in the quasi interior of the positive cone in L_p (see [22], Section III.8). Using the smoothing properties of the resolvent we find $n \in \mathbb{N}$ such that $(\lambda + A_p)^{-n}(L_1) \subset L_\infty$. Clearly, for all $0 \neq f \in L_p$ nonnegative $0 \neq v := (\lambda + A_p)^{-n} f \geq 0$. As $v \in L_\infty \cap W_2^1(\Omega)$ is a strict supersolution of $(\lambda + \mathcal{A})w = 0$ in Ω in the usual sense the Harnack inequality (cf. [11], Theorem 8.18) implies that $(\lambda + A_p)^{-1} v > 0$ almost everywhere. Setting $k = n + 1$ our claim follows. $\square$

5.2 Corollary. *The spectrum of A_p consists of eigenvalues of finite algebraic multiplicity and the eigenfunctions lie in L_∞. Moreover, the spectral bound* $\operatorname{spb}(A_p) = \inf\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A_p)\}$ *is independent of $p \in [1,\infty]$. It is an algebraically simple eigenvalue and the only one having positive eigenfunction.*

Proof. That the spectrum consists of eigenvalues of finite algebraic multiplicity follows since A_p has compact resolvent. The assertion on the spectral bound is a consequence of the abstract theory on compact irreducible operators (e.g. [22], Section V.5). □

Clearly, a solution u of the equation

$$A_p u = f \tag{5.4}$$

in $L_p(\Omega)$ is a weak solution of (2.1) and vice versa whenever $p \geq 2N(N+1)^{-1}$. For smaller p this is not the case in general. In either case we call a solution u of (5.4) a generalised solution of (2.1). We next establish solvability of (5.4) under the additional assumption that the zero order coefficient c_0 of $\mathcal{A}$ is nonnegative. The following theorem is a reformulation of Theorem 2.1

5.3 Theorem. *Let $c_0 \geq 0$ in (2.2). Then, $0 \in \varrho(A)$ and there exists a constant $c > 0$ depending only on the quantities listed in Theorem 2.1 such that*

$$\|A_p^{-1}\|_{p,m(p)} \leq c \tag{5.5}$$

for all $p \in [1,\infty]$, where $m(p)$ is as in Theorem 2.1. Moreover, there exists $\lambda_* > 0$ depending only on N and upper bounds for γ, δ and $|\Omega|$ such that $\lambda_1 \geq \lambda_*$, where λ_1 is the first eigenvalue of $A\varphi = \lambda\varphi$.

Proof. The first assertion follows from the results in Section 4 and Theorem A.10. For the remaining assertion observe that the spectral radius of A_p^{-1} is given by λ_1^{-1} whence, $\lambda_1^{-1} \leq \|A_p^{-1}\|_{\mathcal{L}(L_p)} \leq c|\Omega|^{\frac{1}{p}-\frac{1}{m(p)}}$. Taking the minimum over all p we get the existence of $\lambda_* \geq 0$. □

5.4 Remarks. (a) Note that one could get an estimate of the form (5.5) from (4.2) and by using the fact that $0 \in \varrho(A_2)$. However, using this argument we have no control over the constant c. To have more information we used the bound for $p > N$, duality and interpolation. We also mention that in case of the Dirichlet problem considered in [11], Section 8 the corresponding estimate for small p is not included. By the same proof but this time with $d = N$ one can get the corresponding estimates for the Dirichlet problem as well.

(b) Obviously, Remark 4.3(b)–(d) apply here as well. □

The next proposition will be used to show that without restrictions on the domains the mapping property (5.3) cannot be improved, that is, $m(p)$ is the largest possible exponent in general.

5.5 Proposition. *Let $p \in (1, N)$ and suppose that $(\lambda + A)^{-1}(L_p) \subset L_q$ for some $q < Np(N-2p)^{-1}$ if $p < N/2$ and $q \leq \infty$ if $p > N/2$. Then,*

$$(\lambda + A_p)^{-1} \in \mathcal{L}(L_p, L_r) \tag{5.6}$$

is compact for all $r \in [1, q)$.

Proof. Note first that by our hypotheses $D(A_p) \subset L_q$. As $L_q \hookrightarrow L_p$ it follows from the closed graph theorem that $D(A_p) \hookrightarrow L_q$ and hence

$$(\lambda + A_p)^{-1} \in \mathcal{L}(L_p, L_q)\,. \tag{5.7}$$

Note also that we can assume without loss of generality that $p \geq 2N(N+1)^{-1}$ as in the other case we consider the formal adjoint which has compact resolvent if and only if its dual has compact resolvent. To prove the assertion of the proposition it is sufficient to show that if $f_n \rightharpoonup 0$ weakly in L_p then, $u_n := (\lambda + A_p)^{-1} f_n \to 0$ strongly in L_r for $r \in (1, q)$. As bounded operators are weakly continuous we also have that $u_n \rightharpoonup 0$ in L_q. Recall that by Remark 3.2(b) u_n is a weak solution of

$$\lambda u_n + \mathcal{A} u_n = f_n$$

in Ω in the usual sense. If we fix $\Omega' \subset\subset \Omega$ (that is, the closure of Ω' is contained in the interior of Ω) by interior regularity of weak solutions (e.g. [11], Theorem 9.11) there exists a constant c depending on Ω' such that

$$\|u_n\|_{W_p^2(\Omega')} \leq c\big(\|u_n\|_p + \|f_n\|_p\big)$$

for all $n \in \mathbb{N}$. As weakly convergent sequences are bounded it follows from (5.7) that u_n is a bounded sequence in L_q. Hence, the above inequality shows that u_n is bounded in $W_p^2(\Omega')$. By a well known embedding theorem we have that $W_p^2(\Omega) \hookrightarrow L_q$ compactly if we choose Ω' sufficiently smooth. As we already know that $u_n \rightharpoonup 0$ in $L_q(\Omega)$ it follows that $u_n \to 0$ in $L_q(\Omega')$ strongly. Given $\varepsilon > 0$ we choose $\Omega' \subset\subset \Omega$ sufficiently smooth such that $|\Omega \setminus \Omega'| < \varepsilon$. By Hölder's inequality we can write

$$\|u_n\|_r \leq |\Omega \setminus \Omega'|^{\frac{1}{r} - \frac{1}{q}} \|u_n\|_q + \|u_n\|_{q,\Omega'} \leq \varepsilon^{\frac{1}{r} - \frac{1}{q}} \|u_n\|_q + \|u_n\|_{q,\Omega'}\,.$$

By the above considerations the last term tends to zero as n goes to infinity. As $\|u_n\|_q$ is uniformly bounded and ε was arbitrary this shows that u_n tends to zero in L_r as n goes to infinity. This completes the proof of the proposition. □

The following proposition shows that (5.3) is optimal.

5.6 Proposition. *There exists a bounded domain Ω with smooth boundary except for one point and* (5.3) *fails for any $q > m(p)$, where $p \in (1, N)$.*

Proof. First note that there exists a domain Ω such that the embedding

$$V \hookrightarrow L_{\frac{2N}{N-1}} . \tag{5.8}$$

is not compact (see [17], pages 259/60). This domain can easily be modified in such a way that its boundary is of class C^∞ except for one point. The proof of the proposition is complete if we can show that (5.8) is compact if (5.7) holds for some $p \in (1, p)$ and $q > m(p)$. In case $p = 2N(N+1)^{-1}$ we conclude from Proposition 5.5 that $(\lambda + A)^{-1} \in \mathcal{L}(L_{\frac{2N}{N+1}}, L_{\frac{2N}{(N-1)}})$ is compact. Setting $E := L_{\frac{2N}{N-1}}$ it follows from Proposition A.5 that (5.8) is a compact embedding. If $p > (<)\ 2N(N+1)^{-1}$ we choose $r > (<)\ 2N(N+1)^{-1}$. By Theorem 5.3 we have that $(\lambda + A_r)^{-1} \in \mathcal{L}(L_r, L_{m(r)})$ and hence by interpolation $(\lambda + A)^{-1} \in \mathcal{L}(L_{\frac{2N}{N+1}}, L_s)$ for some $s > 2N(N-1)^{-1}$ and we are back to the situation before. This concludes the proof of the proposition. □

5.7 Remark. Together with the domain perturbation results in [6] the above proposition can be used to show that the estimate (5.5) can not be improved even if we restrict ourselves to smooth domains without making the upper bound c dependent on the geometry rather than the volume of the domain. □

6. Parabolic Problems

In this section we shall briefly discuss some consequences on parabolic problems of the form (2.9). We start off with the homogeneous abstract Cauchy problem

$$\begin{aligned} \dot{u} + A_p u &= 0 \qquad \text{in } (0, \infty) \\ u(0) &= u_0 , \end{aligned} \tag{6.1}$$

where $\dot{u} := du/dt$. By well known abstract results $-A_2$ generates a strongly continuous analytic semigroup $T_2(t) := e^{-tA_2}$ on $L_2(\Omega)$ (see e.g. [7], Proposition XVII.6/3) and hence, (6.1) has a unique solution for all $u_0 \in L_2$ which is given by $u(t) = e^{-tA_2} u_0$. We shall show that $T_2(t)$ acts on all the L_p-spaces for all $t > 0$. To do so note that by a well known smoothing property of analytic semigroups

$$T_2(t)(L_2) \subset \bigcap_{k=0}^{\infty} D(A_2^k) \tag{6.2}$$

for all $t > 0$. It follows from (5.3) that $\bigcap_{k=1}^{\infty} D(A_2^k) \hookrightarrow L_\infty$ for k large enough and hence, therefore,

$$T_2(t) \in \mathcal{L}(L_2, L_\infty) \tag{6.3}$$

for all $t > 0$. In particular this shows that T_2 acts on L_p for all $p \in [2, \infty]$ which allows us to define $T_p(t) := T_2(t)|_{L_p}$ for that range of p. Obviously, we can do the same thing with the semigroup $T_2^\sharp$ generated by $-A_2^\sharp$. Moreover, as $A_2' = A_2^\sharp$, we have that $T_2' = T_2^\sharp$ (see e.g. [20], Corollary 1.10.6). Hence, it makes sense to set

$$T_p(t) := \big(T_{p'}^\sharp(t)\big)' \tag{6.4}$$

for all $p \in (1, 2]$, where $p^{-1} + (p')^{-1}$. Due to the above remarks we have for all $u, v \in L_2$ and $t > 0$

$$|\langle T_2(t)u, v\rangle| = |\langle u, T_2^\sharp(t)v\rangle| \leq \|u\|_1 \|v\|_2 \|T_2^\sharp(t)\|_{2,\infty}\,,$$

whence,

$$\|T_2(t)u\|_2 \leq \|T_2^\sharp(t)\|_{2,\infty} \|u\|_1\,.$$

For this reason, $T_2(t)$ has a unique continuous extension to L_1 which we denote by $T_1(t)$. A simple density argument shows that $T_1(\cdot)$ is a semigroup on L_1. The main result of this section is the following theorem. The proof works much as in the case of the Dirichlet problem but with other exponents.

6.1 Theorem. *For $1 \leq p \leq \infty$ $T_p(\cdot)$ is a positive irreducible analytic semigroup of compact operators on L_p. For $1 \leq p < \infty$ it is strongly continuous at zero and its generator is $-A_p$. Further, $T_\infty(\cdot)$ is weak* continuous at zero. Moreover, for $1 \leq p \leq \infty$*

$$\|T_p(t)\|_{p,p} \leq e^{\delta t} \tag{6.5}$$

and

$$\|T_1(t)\|_{1,\infty} \leq c(1 + \delta t)^{-N} t^{-N} e^{\delta t}\,, \tag{6.6}$$

where δ is as in (4.3) and $c > 0$ a constant just depending on N and upper bounds for γ and $|\Omega|$. Finally, $\big(T_p(t)\big)' = T_{p'}^\sharp(t)$ for all $t \geq 0$ and $1 \leq p < \infty$.

Proof. We first prove (6.5) and (6.6). To do so we set for all $q \geq 2$ $u_q := \operatorname{sign} u|u|^{q-1}$. Applying (4.10) to u and $-u$, using the definition of A and letting m to infinity we get that

$$\||u|^{\frac{q}{2}}\|^q_{\frac{qN}{N-1}} \leq 2c(N, \Omega|)^2 q\gamma \langle Au + q\delta u, u_q\rangle \tag{6.7}$$

holds whenever $u \in V$ and the right hand side of (6.7) is finite. We are now in a position to apply Theorem II.3.5 in [27] with $n = 2N$, which implies that

$$\|T_2(t)\|_{2,\infty} \leq c(1 + \delta t)^{\frac{N}{2}} t^{-\frac{N}{2}} e^{\delta t} \tag{6.8}$$

with a constant $c > 0$ depending only on N and upper bounds for γ and $|\Omega|$. As the same holds for the formal adjoint problem we obtain the same estimate for $\|T_1(t)\|_{1,2}$ by duality. Writing $T_1(t) = T_2(t/2)T_1(t/2)$ and combining the two estimates (6.6) follows. For $p \geq 2$ (6.5) is a simple consequence of an intermediate step in the proof of Theorem II.3.5 in [27], and for $p \leq 2$ it follows by duality since the same estimate holds for the formal adjoint problem.

That $\big(T_p(t)\big)' = T_{p'}(t)$ holds for $p \in (1,\infty)$ is just a reformulation of (6.4). For $p = 1$ the assertion follows by a density argument by using that $T_1(t)$ is a bounded extension of $T_p(t)$ for all $t > 0$.

The remaining assertions are clear for $p = 2$. In particular, compactness and irreducibility of $T_2(t)$ for $t > 0$ follows since the resolvent of its generator $-A_2$ has the same property by Theorem 5.1 (see [18], Section C-III.3 for the irreducibility). For $p \neq 2$ the assertion follows from the above by writing for $\varepsilon > 0$ arbitrary and $t > \varepsilon$

$$T_p(t) = T_2(\varepsilon/2)T_2(t-\varepsilon)T_p(\varepsilon/2)$$

and taking into account the smoothing properties of T_2 and the embeddings for L_p-spaces.

Strong continuity of $T_p(\cdot)$ in zero for $1 \leq p \leq 2$ follows since $T_2(\cdot)$ has that property, L_2 is dense in L_p and (6.5). Next we show that $-A_p$ is the generator of T_p for $1 \leq p \leq 2$. For $p = 2$ the result is well known. As by the results Theorem 5.1 A_p is the closure of A_2 it is sufficient to show that $D(A_2)$ is a core for the generator $-B_p$ of T_p. If so, A_p and B_p are closures of A_∞ and hence must coincide. By a well known theorem (e.g. [18], Proposition I.1.9) we have to show that $D(A_2)$ is dense in L_p, that it is invariant under $T_p(t)$, and that A_2 is a restriction of B_p. Density of $D(A_2)$ in L_p follows from Theorem 5.1, invariance from (6.6) and (6.2), and $A_2 \subset A_1$ since A_2 is the generator on L_2, the density of L_2 in L_1 and the bound (6.5). Hence, $-A_p$ is the generator of T_p for $p \in [1,2)$. The same is true for the formal adjoint problem and hence, by Theorem 5.1 and [20], Theorem 1.10.6 for $\in (2,\infty)$. This shows that $-A_p$ is the generator of T_p for all $p \in [1,\infty)$. $\square$

6.2 Remarks. (a) It is easy to see that a solution u of (6.1) is a weak solution of the parabolic problem

$$\partial_t u + \mathcal{A}u = 0$$

in $\Omega \times (0,\infty)$ in the usual sense. Hence, all the "interior regularity" results for parabolic equations apply to our situation.

(b) The comments in Remark 4.3(b)–(d) apply to the parabolic situation as well.

(c) As $T_1(t) \in \mathcal{L}(L_1, L_\infty)$ it has a representation of the form

$$T_1(t)u(x) = \int_\Omega k_t(x,y)u(y)\,dy$$

where $k_t(\cdot\,,\cdot) \in L_\infty(\Omega) \otimes L_1(\Omega)$ is called the heat kernel. It is also well known that the estimate (6.6) leads to an estimate of the kernel of the form

$$|k_t(x,y)| \le c(1+\delta t)^N t^{-N} e^{\delta t}\,.$$

Moreover, $k_t(x,y) \to \delta(x-y)$ weakly as $t \to 0$. (see e.g. [21], Appendix C.1.) By the positivity of the semigroup we have that $k(x,y,t) > 0$ for all $x, y \in \Omega$ and $t > 0$. As we have the same representation formula for all $p \in [1,\infty]$ we denote the semigroup simply by $T(\cdot)$. □

For selfadjoint operators we can prove that the heat kernel has upper Gaussian estimates with N replaced by $2N$. More precisely, we have the following results, whose proof is analogous to that given in [8] for the Dirichlet or Neumann problem.

6.3 Theorem. *Suppose that $(\mathcal{A},\mathcal{B})$ is selfadjoint with $c_0 \ge 0$, that is, $a_{ij} = a_{ji}$ and $a_i = b_i = 0$ for $i,j = 1,\dots,N$. Then, for all $x,y \in \Omega$ and $t > 0$*

$$0 \le k_t(x,t) \le Ct^{-N} e^{-\frac{|x-y|^2}{Ct}}\,, \tag{6.9}$$

where C depends on the same quantities as c in (6.6).

Proof. The proof is a simple modification of the proof for the Dirichlet problem. First of all note that by (6.6) (with $\delta = 0$) and the abstract Lemma 2.2.3 in [8] a logarithmic Sobolev inequality with $\beta(\varepsilon) = c - (N/2)\log\varepsilon$ and a constant c depending only on the quantities listed in the theorem holds for all $0 \le u \in V \cap L_\infty \cap L_1$. Note also that $V_0 \cap V$ is a form core for $a(\cdot\,,\cdot)$. Now we can repeat the calculations in [8], Section 3.3 with minor modifications including the boundary term of the form $a(\cdot\,,\cdot)$. □

6.4 Remarks. (a) With the aid of (6.9) one can determine the angle of the semigroups T_p. It turns out to be $\pi/2$. The proof is a modification of the proof in Ouhabaz [19]. Indeed, note that by (6.9) and [8], Theorem 3.4.8 we get an estimate for the complex heat kernel with the only difference that we have to replace N by $2N$. Then, (2.2) in [19] holds, also with N replaced by $2N$ (Note that we only integrate over a set of finite measure). Hence, the proof of Theorem 2.4 in the above mentioned paper works in our case.

(b) By using the Laplace transform we get from (6.9) the estimate

$$0 \le g(x,y) \le \frac{C^N \Gamma(N-1)}{|x-y|^{2(N-1)}}$$

for the kernel $g(\cdot\,,\cdot)$ of A^{-1}, where C is the same as in (6.9) and $\Gamma(\cdot)$ the Gamma function. □

Using the above results it is easy to deal with the abstract parabolic equation

$$\begin{aligned} \dot u + A_p u &= f(t) \qquad \text{in } (0,\infty) \\ u(0) &= u_0 \,. \end{aligned} \tag{6.10}$$

As usual we call

$$u(t) := T(t)u_0 + \int_0^t T(t-s)f(s)\,ds \tag{6.11}$$

a mild solution of (6.10) (e.g. [20]). If u is a mild solution of (6.10) we say that u is a generalised solution of (2.9). Using the results above it is now easy to prove Theorem 2.4.

Proof of Theorem 2.4. By (6.5), (6.6) and interpolation it follows that for all $1 \le q \le p \le \infty$

$$\|T(t)\|_{q,p} \le c(1+\delta t)^{N(\frac{1}{q}-\frac{1}{p})} t^{-N(\frac{1}{q}-\frac{1}{p})} e^{\delta t}$$

Using this, (6.11) obviously exists and hence (2.9) has a generalised solution. If f is Hölder continuous it easily follows that u is differentiable in t for $t > 0$. □

Appendix: Abstract Regularity Results

The purpose of this appendix is do provide some results on L_p-regularity of solutions to abstract linear variational problems. Suppose that (X,m) is a measure space and set $H := L_2 := L_2(X,m)$. Further, assume that V is a Hilbert space which is continuously and densely embedded in H. Identifying H with its dual we have that

$$V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'.$$

The standard inner product in H is as usual given by

$$(u|v) := \int_X uv\,dm\,.$$

We further suppose that $\langle\cdot\,,\cdot\rangle$ is the duality pairing on V induced by $(\cdot \mid \cdot)$, that is, $\langle u,v\rangle = (u|v)$ holds for all $u \in H \subset V'$ and $v \in V$. The duality pairing $\langle\cdot\,,\cdot\rangle$ is well defined as V is dense in H. Finally, we assume that

$$a(\cdot\,,\cdot)\colon V \times V \to \mathbb{R} \tag{A.1}$$

is a continuous sesquilinear form and that there exists $\alpha > 0$ and $\lambda_0 \in \mathbb{R}$ such that

$$a(u,u) + \lambda_0 \|u\|_H^2 \geq \alpha \|u\|_V . \tag{A.2}$$

It is well known that $a(\cdot\,,\cdot)$ induces a an operator $A \in \mathcal{L}(V, V')$ satisfying

$$a(u,v) = \langle Au, v\rangle \tag{A.3}$$

for all $u, v \in V$. We study properties of solutions of the equation

$$Au = f . \tag{A.4}$$

Clearly, u is a solution of (A.4) if and only if u satisfies

$$a(u,v) = \langle f, v\rangle \tag{A.5}$$

for all $v \in V$. We divide this appendix into three parts. In a first part we establish a priori estimates for solutions of (A.4). The method we use is an abstract version of the well known Moser iteration technique. The results are the "elliptic" counterpart to the results in the "parabolic" case given e.g. in [27], Theorem II.3.5. In a second part we look at properties of maximal restrictions of operators to Banach spaces and study duality. In the third part we apply and extend these results in case of L_p-spaces. Particularly in part three many results we prove are probably more or less folklore, but as we do not now of any explicit reference, we include the exact statements detailed proofs.

I. A priori estimates. In this section we prove a priori estimates for solutions of (A.4). The idea is to provide a general L_p-regularity result which bases on an inequality which is not hard to verify for concrete problems such as elliptic boundary value problems. We assume that there exist constants $d > 2$, $\bar{c}, \delta \geq 0$ and a nondecreasing functions $c_1, c_2 \colon [2,\infty) \to [1,\infty)$ of polynomial growth such that

$$\| |u|^{\frac{q}{2}} \|_{\frac{2d}{d-2}}^2 \leq \bar{c} c_1(q) \Big(\big\langle Au, \operatorname{sign} u |u|^{q-1} \big\rangle + \delta c_2(q) \| |u|^{\frac{q}{2}} \|_2^2 \Big) , \tag{A.6}$$

for all $u \in D(A)$ and $q \geq 2$ for which the right hand side makes sense.

A.1 Theorem. *Suppose that $m(X) < \infty$ and that u is the solution of the equation $Au = f$ with $f \in L_p$ and that* (A.6) *holds. Then, if $p \in [2, d/2)$*

$$\|u\|_{\frac{dp}{d-2p}} \leq C \bar{c} \big(\|f\|_p + \bar{c}^{\mu} \delta^{1+\mu} \|u\|_2 \big) , \tag{A.7}$$

where $C>0$ only depends on d and p. If $p>d/2$

$$\|u\|_\infty \leq C\bar{c}\big(\|f\|_p + \bar{c}^\mu \delta^{1+\mu}\|u\|_2\big)\,, \tag{A.8}$$

where $C>0$ depends on d,p and an upper bound for $m(X)$.

Proof. The proof of the above theorem is almost identical to part of the proof of Theorem 4.1. In this case we set $v := \operatorname{sign} u|u|^{q-1}$ and $w := |u|^{\frac{q}{2}}$. Then, if f is a solution of (A.4) we get the estimates (4.11) and (4.12) for $\|w\|^2_{\frac{2d}{d-2}}$ with q^2 replaced by $c_3(q) := \max\{c_1(q), c_2(q)\}$. If $p \geq d/2$ we proceed exactly as in the proof of Theorem 4.1. In case $p \in [2, d/2)$ we have to be a little more careful as we do not know a priori that $w \in L_{p'(q-1)}$ and hence we can not just set $q := \bar{q} := p(d-2)(d-2p)^{-1}$ and divide inequality (4.12) by $\|u\|^{\bar{q}-1}_{\frac{dp}{d-2p}}$ to get the result. The main problem therefore is to show that u has a finite $L_{\frac{dp}{d-2p}}$-norm. To do so we define q_n and η as in case of the L_∞-estimate. As for that range of p $q_n < \bar{q}$ and $\eta < 1$ we have that

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} \eta^n + \sum_{k=0}^{n} \eta^k = (1-\eta)^{-1} = \bar{q}\,.$$

Using these facts we deduce from (4.13) that

$$\|w\|_{\frac{dq_{n+1}}{d-2}} \leq c_3(\bar{q})^{\frac{1+\mu}{2}\sum_{k=0}^\infty (1-\frac{1}{\bar{q}})^k} \|w\|^{(1-\frac{1}{\bar{q}})^n}_{\frac{2d}{d-2}} \leq c_3(\bar{q})^{\frac{1+\mu}{2}\bar{q}} \|w\|^{(1-\frac{1}{\bar{q}})^n}_{\frac{2d}{d-2}}$$

for all $n \in \mathbb{N}$. Letting n to infinity we obtain

$$\|w\|_{\frac{dp}{d-2p}} = \|w\|_{\frac{d\bar{q}}{d-2}} \leq c_3(\bar{q})^{\frac{1+\mu}{2}\bar{q}}$$

This completes the proof in case $p \in [2, d/2)$. □

A.2 Remarks. With a slight modification of the proof the above theorem remains true if we omit the assumption that $m(X) < \infty$. In that case we have to assume that $f \in L_2 \cap L_p$, and we get that for $p > d/2$

$$\|u\|_\infty \leq C\bar{c}^{1+\mu}\big(\|f\|_p + \delta^{1+\mu}\|u\|_2\big) + C\|u\|_2\,, \tag{A.9}$$

where $C>0$ just on d and p. To see this note first that the only place we used that the volume is finite is in (4.15). So we only have to prove $(A.9)$. To do so note that (4.14) implies that $\|u\|_\infty \leq M^{1-\theta}\|w\|^\theta_{\frac{2d}{d-2}}$ with M defined as in the proof of Theorem 4.1. By Young's inequality we get that $\|u\|_\infty \leq CM + C\|u\|_{\frac{2d}{d-2}}$. Finally, by a well known interpolation inequality we have that

$$\|u\|_{\frac{2d}{d-2}} \leq 2^{-1}\|u\|_\infty + 2^{\frac{2}{d-2}}\|u\|_2$$

which completes the proof of (A.9). Note that (A.8) can not be true on unbounded domains. As a counterexample consider the equation $-\Delta u = f$ in $\mathbb{R}^N$. In that case $\Delta = 0$ and (A.8) would imply that $(-\delta)^{-1}$ is bounded from L_p to L_∞, which is not true. □

II. Maximal restrictions and duality. Suppose that V, H, the form $a(\cdot\,,\cdot)$ and the operator A are as before. Due to $(A.2)$ and the Lax-Milgram Theorem $\lambda + A$ has a bounded inverse for all $\lambda \geq \lambda_0$. As $V \overset{d}{\hookrightarrow} V'$ we may consider A as a densely defined operator on V' with domain of definition V. It follows from the boundedness of $a(\cdot\,,\cdot)$ and $(A.2)$ that the graph norm $\|u\|_{D(A)} = \|Au\|_{V'} + \|u\|_{V'}$ is equivalent to the norm on V. As $A \in \mathcal{L}(V, V')$ it follows that A is a closed operator on V'. If F is a Banach space with $F \hookrightarrow V'$ the F-realisation A_F of A defined by

$$A_F u := Au \qquad \text{for all} \qquad u \in D(A_F) := \{v \in V \cap F \colon Av \in F\}$$

is a closed operator on F. The operators A_F are also called the maximal restriction of A to F.

A.3 Lemma. *Let $F \overset{d}{\hookrightarrow} V'$ and $(\lambda + A)^{-1}(F) \subset F$ for some $\lambda \in \varrho(-A)$, then $D(A_F)$ is dense in V.*

Proof. First note that under the present assumptions $\varrho(A) \subset \varrho(A_F)$ so in particular $\lambda_0 \in \varrho(-A_F)$. It is sufficient to show that the orthogonal complement of $D(A_F)$ with respect to the bilinear form $a(\cdot\,,\cdot) + \lambda_0(\cdot \mid \cdot)_H$ is zero. To see this note that $0 = a(u, v) + \lambda_0(u|v)_H = \langle(\lambda_0 + A_F)u, v\rangle$ for all $u \in D(A_F)$ implies that $v = 0$ as the range of $\lambda_0 + A_F$ is F which is dense in V'. $\square$

The following proposition which turns out to be useful later is a consequence of a result due to Arendt [2].

A.4 Proposition. *Suppose that $F \hookrightarrow V'$, that $(\lambda + A)^{-1}(F) \subset F$ and $(\lambda + A)^{-k}(V') \subset F$ for some $\lambda \in \varrho(-A)$ and $k \in \mathbb{N}$. Then, $\varrho(A) = \varrho(A_F)$ and $(\lambda + A_F)^{-1} = (\lambda + A)^{-1}|_F$ for all $\lambda \in \varrho(-A)$.*

A.5 Proposition. *Suppose that E is a Banach space with $V \overset{d}{\hookrightarrow} E$. Then, the embedding $V \hookrightarrow E$ is compact if and only if $(\lambda_0 + A_{E'})^{-1} \in \mathcal{L}(E', E)$ is compact.*

Proof. That the condition is necessary is clear. To prove the reverse we can assume that $\lambda_0 = 0$ in (5.2) by replacing A by $\lambda_0 + A$. Doing so we get that for all $u \in E' \hookrightarrow V'$

$$\alpha\|A^{-1}u\|_V^2 \leq a(A^{-1}u, A^{-1}u) = \langle u, A^{-1}u\rangle \leq \|u\|_{E'}\|A^{-1}u\|_E\,. \tag{A.9}$$

Suppose that $B \subset E'$ is bounded and that w_n is a sequence in $A^{-1}(B)$, that is, $w_n = A^{-1}u_n$ for some $u_n \in B$. As A^{-1} is compact as an operator to E by hypotheses it follows that w_n has a convergent subsequence in E. Applying $(A.9)$ to $u = u_k - u_\ell$ we get that

$$\|w_k - w_\ell\|_V^2 \leq \alpha^{-1}\|u_k - u_\ell\|_{E'}\|w_k - w_\ell\|_E$$

for all $k,\ell\in\mathbb{N}$. As B is bounded in E' this shows that this subsequence also converges in V. Consequently, $A^{-1}(B)$ is relatively compact in V, whence $A^{-1}\in\mathcal{L}(E',V)$ is compact. Therefore, the embedding $E'\hookrightarrow V'$ given by AA^{-1} is compact which is equivalent to $V\hookrightarrow E$ being compact. This concludes the proof of the proposition. □

Next we define extensions of A to Banach spaces which are not necessarily subspaces of V'. To do this let us consider the dual form $a^\sharp(\cdot\,,\cdot)\colon V\times V\to\mathbb{R}$ given by

$$a^\sharp(u,v):=a(v,u) \tag{A.11}$$

for all $u,v\in V$. Denote by $A^\sharp$ the closed operator induced by $a^\sharp(\cdot\,,\cdot)$ on V' and let F be a Banach space such that $F'\overset{d}{\hookrightarrow}V'$ and $(\lambda+A^\sharp)^{-1}(F')\subset F'$. Assume in addition that $D(A^\sharp_{F'})$ is dense in F'. Then, we can define

$$A_F:=(A^\sharp_{F'})', \tag{A.12}$$

where the domain of definition of A_F is given by

$$D(A_F)=\left\{u\in F\colon v\mapsto\langle A^\sharp_{F'}v,u\rangle \text{ is continuous on } D(A^\sharp_{F'})\subset F'\right\}.$$

Then the following proposition holds.

A.6 Proposition. *Suppose F and A_F are defined as above. Moreover, let E be a Banach space such that $E\hookrightarrow F$, $E\overset{d}{\hookrightarrow}V'$ and $(\lambda+A)^{-1}(E)\subset E$ for some $\lambda\in\varrho(-A)$. Then, $(A_F)_E=A_E$, that is, the E-realisation of A_F coincides with A_E. In particular, A_F is an extension of A_E.*

Proof. Let us assume without loss of generality that $0\in\varrho(A)\cap\varrho(A^\sharp)$ by replacing A abd $A^\sharp$ by λ_0+A and $\lambda_0+A^\sharp$, respectively. By our assumptions it is clear that $0\in\varrho(A_E)\cap\varrho(A^\sharp_{F'})$. Let $u\in D(A_E)$. Then, for all $v\in D(A^\sharp_{F'})$

$$|\langle A^\sharp_{F'}v,u\rangle|=|a^\sharp(v,u)|=|a(u,v)|=|\langle Au,v\rangle|\leq\|Au\|_F\|v\|_{F'}\leq c\|A_Eu\|_E\|v\|_{F'},$$

where we used that $E\hookrightarrow F$. Hence, $u\in D(A_F)$. The above calculation also shows that $\langle A_Fu,v\rangle=\langle A_Eu,v\rangle$ for all $v\in D(A^\sharp_{F'})$, and therefore by density of $D(A^\sharp_{F'})$ in F' we conclude that $A_Fu=A_Eu$ for all $u\in D(A_E)$, that is, A_F is an extension of A_E. Next assume that $u\in D((A_F)_E)$, that is, $u\in D(A_F)\cap E$ such that $A_Fu\in E$. As $0\in\varrho(A_E)$ and $A_Fu\in E\hookrightarrow V'$ there exists $w\in D(A_E)\subset V$ such that $A_Fu=A_Ew\in E$. Hence, by the definition of A_F

$$\langle A^\sharp_{F'}v,u\rangle=\langle A_Fu,v\rangle=\langle A_Ew,v\rangle=a(w,v)=a^\sharp(v,w)=\langle A^\sharp_{F'}v,w\rangle$$

for all $v \in D(A^\sharp_{F'})$. As the range of $A^\sharp_{F'}$ is F' (recall that $0 \in \varrho(A^\sharp_{F'})$) it follows that $u = w \in V \cap E$ and $A_F u = A_E u \in E$ which completes the proof of the proposition. □

A.7 Remarks. (a) Taking $E = F = V'$ in the above proposition we get that $(A^\sharp)' = A$ which is of course well known. In particular, we have that $\varrho(A) = \varrho(A^\sharp)$.

(b) Let F be as in the above proposition, and suppose that $(\lambda + A^\sharp)^{-k}(V') \subset F'$ for some $\lambda \in \varrho(-A^\sharp)$ and $k \in \mathbb{N}$. Then, by definition $(A.12)$ and Proposition A.4 we get that $\varrho(A_F) = \varrho(A^\sharp_{F'}) = \varrho(A^\sharp) = \varrho(A)$. □

III. Smoothing properties of $(\lambda + A)^{-1}$. In what follows we throughout assume that $m(X) < \infty$ and that in addition to the assumptions given earlier

$$V \cap L_p \quad \text{is dense in} \quad L_p \tag{A.13}$$

for all $p \in [2, \infty)$. We further suppose that the operators A and $A^\sharp$ induced by the forms $a(\cdot\,,\cdot)$ and $a^\sharp(\cdot\,,\cdot)$ satisfy (A.6) with the same constants. Observe that by (A.6) with $q = 2$, (A.3) and the continuity of $a(\cdot\,,\cdot)$ on $V \times V$ we get that $\|u\|_{\frac{2d}{d-2}} \le c_1 \|u\|_V$, that is,

$$V \overset{d}{\hookrightarrow} L_{\frac{2d}{d-2}}\,. \tag{A.14}$$

Hence, by duality and the embeddings for L_p-spaces we conclude that

$$L_p \overset{d}{\hookrightarrow} V' \tag{A.15}$$

for all $p \in [2d(d+2)^{-1}, \infty]$.

A.7 Lemma. *For all $p \in [2, \infty]$ we have that*

$$(\lambda_0 + A)^{-1} \in \mathcal{L}(L_p, L_{m(p)})\,, \tag{A.16}$$

where $m(p) = dp(d - 2p)^{-1}$ if $p < d/2$ and $m(p) = \infty$ if $p > d/2$, and λ_0 from (A.2). Moreover, there exists $k \in \mathbb{N}$ only depending on d such that

$$(\lambda_0 + A)^{-k}(V') \subset L_\infty\,. \tag{A.17}$$

The assertion is the same if we replace A by $A^\sharp$.

Proof. Clearly $\lambda_0 \in \varrho(-A) = \varrho(-A^\sharp)$ and $(\lambda_0 + A)^{-1}(V') \subset V \hookrightarrow L_2$. Hence, there exists a constant $c > 0$ such that $\|u\|_2 \le c\|f\|_p$ for $p \ge 2$, where $u := (\lambda_0 + A)^{-1} f$. Using (A.7) and (A.8) with A replaced by $A + \lambda_0$ the first assertion follows. For the second assertion we again use that $(\lambda_0 + A)^{-1}(V') \subset V \hookrightarrow L_2$ and iterate the first result a finite number of times depending on d only. □

For $p \geq 2$ we denote by A_p and $A_p^\sharp$ the L_p-realisation of A and $A^\sharp$, respectively. Recall that $(L_p)' = L_{p'}$ with equal norms for all $p \in [1,\infty)$, where as usual $p^{-1} + (p')^{-1} = 1$. Thus, if $p \in (1,2]$, $(L_p)' \overset{d}{\hookrightarrow} V'$, and by the above lemma $(\lambda + A^\sharp)^{-1}(Lp') \subset L_{p'}$. Hence, by assumption (A.13) and Lemma A.3 $D(A_{p'}^\sharp)$ is dense in $L_{p'}$. Therefore, for all $p \in (1,2)$ we can define

$$A_p := (A_{p'}^\sharp)', \tag{A.18}$$

for $p \in (1,2]$. As a consequence of Proposition A.6 A_p is an extension of A_q for all $q > p$. On the other hand, A_p is the closure of A_q for such p, q. To see this it is sufficient to show that $D(A_\infty)$ is dense in $D(A_p)$ for all $p > 1$. Indeed, as L_∞ is dense in L_p there exists a sequence v_n in L_∞ approaching $(\lambda_0 + A)u \in L_p$. Set $u_n := (\lambda_0 + A_\infty)^{-1} v_n$. As $(\lambda_0 + A_\infty)^{-1}|_{L_p} = (\lambda_0 + A_p)^{-1}$ this shows the assertion.

Note that $D(A_\infty^\sharp)$ is not dense in L_∞ in general, and that $(L_\infty)' \neq L_1$. For this reason we can not define A_1 to be the dual of $A_\infty^\sharp$. Instead, we define A_1 to be the closure of A_p for some arbitrary $p > 1$. The next lemma shows that this is possible.

A.8 Lemma. *For $p > 1$ A_p is closable in L_1 and the closure is independent of p. If we denote its closure by A_1 we have that $(A_1)' = A_\infty^\sharp$.*

Proof. Let $p \in (1,\infty)$ be arbitrary and suppose that $u_n \in L_1 \cap D(A_p)$ is a sequence such that $A_p u_n \to g$ in L_1 and $u_n \to 0$ in L_1 as n tends to infinity. For $w \in D(A_\infty^\sharp) \subset D(A_{p'}^\sharp)$ we have by (A.18) that $\langle A_p u_n, w\rangle = \langle u_n, A_{p'}^\sharp w\rangle$. As $A_{p'}^\sharp w \in L_\infty$ it follows that the last term tends to zero, whereas the first tends to $\langle g, w\rangle$ as n tends to infinity. We know from (A.15) and Lemma A.3 that $D(A_\infty^\sharp)$ is dense in V and therefore dense in L_q for $q \leq 2d(d-2)^{-1}$. Hence, we can approximate every function $v \in L_\infty$ pointwise by a sequence $w_n \in D(A_\infty^\sharp)$ with $\|w_n\|_\infty$ uniformly bounded and the Dominated Convergence Theorem yields $0 = \langle g, w_n\rangle \to \langle g, w\rangle = 0$, which implies that $g = 0$. This shows that A_p is closable. The closure is independent of p because A_p is the closure of A_∞ for all $p > 1$.

It remains to show that $(A_1)' = A_\infty^\sharp$. Let $u \in D(A_\infty^\sharp)$, that is, $u \in V \cap L_\infty$ and $A^\sharp u \in L_\infty$, and suppose that $v \in D(A_p) \subset D(A_1)$ for some $p \in (1,\infty)$. Using (A.18) we obtain

$$|\langle u, A_1 v\rangle| = |\langle u, A_p v\rangle| = |\langle A_{p'}^\sharp u, v\rangle| \leq \|A_\infty^\sharp u\|_\infty \|v\|_1 \,.$$

As $D(A_p)$ is dense in L_1 this implies that $u \in D((A_1)')$ and $(A_1)' u = A_\infty^\sharp u$. Suppose now that $u \in D((A_1)')$. Therefore, for all $v \in D(A_p)$ we have that

$$|\langle u, A_p v\rangle| = |\langle u, A_1 v\rangle| \leq c\|v\|_1 \leq c_1 \|u\|_p$$

for some constants $c, c_1 > 0$ independent of $v \in D(A_p)$. Hence, $u \in D((A_p)') = D(A^\sharp_{p'})$. By (A.18) we conclude that $(A_1)'u = (A_p)'u = A^\sharp_{p'}u = A^\sharp u$. As $u \in L_\infty$ and $(A_1)'u \in L_\infty$ this implies that $u \in D(A^\sharp_\infty)$. This concludes the proof of the lemma. □

A.9 Theorem. *Let A_p and $A^\sharp_p$ defined as above for all $p \in [1,\infty]$. Then, the following assertions hold.*

(a) *For $p \in [1,\infty]$ A_p are closed. It is densely defined if $p < \infty$. Moreover, A_p is the closure of A_q, and A_q the L_q-realisation of A_p whenever $1 \leq p \leq q \leq \infty$.*

(b) *$A'_p = A^\sharp_{p'}$ for all $p \in [1,\infty)$.*

(c) *The spectrum of A_p is independent of $p \in [1,\infty]$. Moreover, if V is compactly embedded in L_2, then A_p has compact resolvent for all $p \in [1,\infty]$, and if A_p has compact resolvent for some $p \in [1,\infty]$ this is true for all and $V \hookrightarrow L_2$ compactly.*

(d) *For $p \in [1,\infty]$ and $\lambda \in \varrho(-A)$ (A.16) holds, where $m(p) = dp(d-2p)^{-1}$ if $p \in (1, d/2)$, $m(p) = \infty$ if $p > d/2$, and $m(p) \in [1, d(d-2)^{-1})$ arbitrary if $p = 1$. Furthermore,*

$$\|(\lambda + A_p)^{-1}\|_{p,q} = \|(\lambda + A^\sharp_{q'})^{-1}\|_{q',p'} \tag{A.19}$$

whenever $p, q \in [1,\infty]$ and one of the norms is finite.

Proof. (a) is clear from Lemma A.8 and the preceding discussion, and (b) follows from definition (A.18) and Lemma A.8. Independence of the spectrum of A_p from $p \in (1,\infty]$ is a consequence of the definition of A_p and Remark A.7.(b). That $\varrho(A_1)$ is the same follows by duality from (b). Compactness of the resolvent in case $p = 2$ is clear since $V \hookrightarrow L_2$ compactly. Exactly in the same way as in the proof of Theorem 6.1 one can show that $-A_p$ generates a C_0-semigroup of compact operators on L_p for all $p \in [1,\infty)$. Hence, it follows from [20], Theorem 2.3.3 that A_p has compact resolvent for all $p \in [1,\infty)$. By duality, also A_∞ has compact resolvent. If A_p has compact resolvent for some p this implies that it is also true for $p = 2$ and by (b) for $A^\sharp_2$. Hence, it follows from Proposition A.5 that $V \hookrightarrow L_2$ compactly. This proves (c). To prove (d) note that $[(\lambda + A_p)^{-1}]' = (\lambda + A^\sharp_{p'})^{-1}$ for all $p \in [1,\infty)$. As an operator and its dual have the same norm this implies (A.19). The remaining assertion follows from this together with (A.16). □

A.10 Theorem. *Suppose that $V \hookrightarrow L_2$ compactly and that for all $p > d/2$ there exists a constant $M_p > 0$ such that for the solution of* (A.4)

$$\|u\|_\infty \leq M_p \|f\|_p \tag{A.20}$$

Further suppose that a similar statement holds for the solution of the adjoint problem with the same constants M_p. Then, $0 \in \varrho(A)$ and there exists a constant $c > 0$ depending only on $\bar{c}, d, p, \delta$ and M_d such that

$$\|A_p^{-1}\|_{p,m(p)} \leq c \tag{A.21}$$

for all $p \in [1, \infty]$, where $m(p)$ is as in Theorem A.9.

Proof. We first show that $0 \in \varrho(A)$. To do so note that u is a solution of $(A.4)$ with $f \in L_p$ if and only if u is a solution of

$$u - \lambda(\lambda + A_p)^{-1}u = (\lambda + A_p)^{-1}f \tag{A.22}$$

for some $\lambda \in \varrho(-A)$. As A_p has compact resolvent by Theorem A.9 $\mathrm{I} - \lambda(\lambda + A_p)^{-1}$ is Fredholm of index zero. On the other hand, we conclude from (A.20) that if $p > d/2$ (A.4) and hence (A.22) has at most one solution. By the Fredholm alternative it follows that (A.22) and hence (A.4) has a unique solution for all $f \in L_p$. It also follows from (A.20) that $0 \in \varrho(A_p)$ for this range of p and that (A.21) holds. By Theorem A.9 $0 \in \varrho(A) = \varrho(A_p)$ for all $p \in [1, \infty]$. A similar statement with the same constants holds if we replace A by $A^\sharp$. It follows in particular that $A^{-1}, (A^\sharp)^{-1} \in \mathcal{L}(L_d, L_\infty)$. By Theorem A.9 $A_1^{-1} \in \mathcal{L}(L_1, L_{\frac{d}{d-1}})$ with norm equal to $\|(A^\sharp)^{-1}\|_{d,\infty}$. Hence, by interpolation $A^{-1} \in \mathcal{L}(L_{\frac{2d}{d+2}}, L_2)$ with a norm depending on the same quantities. This means that the solution u of (A.4) with $f \in L_p$ for $p \geq 2d(d+2)^{-1}$ satisfies an a priori estimate $\|u\|_2 \leq c\|f\|_p$ with a constant $c > 0$ as mentioned above. If we plug this estimate into (A.7) we get that (A.21) holds for $p \geq 2$. As the same assertion holds for the formal dual we get the remaining cases by duality using Theorem A.9. $\square$

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] W. Arendt, *Gaussian estimates and interpolation of the spectrum in L_p*, Differential and Integral Equations **7** (1994), 1153–1168.

[3] M-H. Bossel, *Membranes élastiquement liées inhomogènes ou sur une surface: une nouvelle extension du théorème isopérimérique de Raleigh-Faber-Krahn*, ZAMP **39** (1988), 733–742.

[4] E.N. Dancer, *The effect of domain shape on the number of positive solutions of certain nonlinear equations*, Journal of Differential Equations **74** (1988), 120–156.

[5] E.N. Dancer and D. Daners, *A priori estimates for solutions and spectra of Robin boundary value problems* (1994) (Preprint, University of Sydney).

[6] E.N. Dancer and D. Daners, *Domain perturbation for Robin boundary value problems* (in preparation).

[7] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 5, Springer, Berlin, 1992.

[8] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989.

[9] K.J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, Chichester, 1990.

[10] H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1969.

[11] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.

[12] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, London, 1985.

[13] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.

[14] M. Marcus and V.J. Mizel, *Every superposition operator mapping one Sobolev Space into another is continuous*, Journal of Functional Analysis **33** (1979), 217–229.

[15] V.G. Maz'ja, *Classes of regions and embedding theorems for function spaces*, Soviet Math. Dokl. **1** (1960), 882–885.

[16] V.G. Maz'ja, *Zur Theorie Sobolewscher Räume*, Teubner-Texte zur Mathematik, Teubner, Leipzig, 1981.

[17] V.G. Maz'ja, *Sobolev Spaces*, Springer, Berlin, 1985.

[18] R. Nagel et al., *One-parameter semigroups of positive operators*, LNM 1184, Springer, Berlin, 1986.

[19] E. Ouhabaz, *Gaussian estimates and holomorphy of semigroups*, Proc. Amer. Math. Soc. **123** (1995), 1465–1474.

[20] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer, New York, 1983.

[21] D.W. Robinson, *Elliptic Operators and Lie Groups*, Oxford University Press, Oxford, 1991.

[22] H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer, Berlin, 1974.

[23] R.P. Sperb, *Untere und obere Schranken für den tiefsten Eigenwert der elastisch gestützten Membran*, ZAMP **23** (1972), 231–244.

[24] R.P. Sperb, *Maximum Principles and Their Applications*, Academic Press, New York, 1981.

[25] R.P. Sperb, *An isoperimetric inequality for the first eigenvalue of the Laplacian under Robin boundary conditions*, General Inequalities 6 (W. Walter, ed.), International Series of Numerical Mathematics 103, Birkhäuser, Basel, 1992, pp. 361–367.

[26] N.S. Trudinger, *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa **27** (1973), 265–308.

[27] N.Th. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, Cambridge, 1992.

[28] W.P. Ziemer, *Weakly Differentiable Functions*, Springer, New York, 1989.