DOMAIN PERTURBATION FOR LINEAR AND NONLINEAR PARABOLIC EQUATIONS

DANIEL DANERS*

School of Mathematics and Statistics
University of Sydney, NSW 2006
Australia

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ABSTRACT. We shall develop a general theory for domain perturbation for linear and nonlinear parabolic equations with measurable coefficients subject to Dirichlet boundary conditions. We show how solutions of linear and nonlinear parabolic equations behave as a sequence of domains approaches an open set. Convergence of domains is understood in a very general sense which allows that certain parts of the domains degenerate and are deleted in the limit, or that small sets are removed. We also consider the periodic problem and establish existence and uniqueness results for periodic solutions for the perturbed problem in a neighborhood of a periodic solution of the original equation. The theory can be used for instance to construct domains where a given periodic-parabolic equation has many periodic solutions.

1. INTRODUCTION

The main objective of this paper is to present a general theory on domain perturbation for a wide class of linear and nonlinear parabolic initial value problems subject to Dirichlet boundary conditions. The perturbation of domains may be very singular and includes for instance cutting holes into a domain or joining disjoint domains by narrow strips. The corresponding elliptic theory was developed in Dancer [4], [5]. So far, not much seems to be known for parabolic problems. There are some results for autonomous equations in Rauch and Taylor [18]. Dancer [5] considers nonautonomous equations but explicit time dependence is not allowed in higher order terms, and there are some restrictions on the convergence of the domains we are able to remove here. Moreover, Hale and Vegas [11] study domain perturbation for the autonomous Neumann problem. However, Neumann boundary conditions require different techniques.

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To get a taste of the kind of results proved here we consider the homogeneous heat equation

$$\begin{cases} \partial_t u_n - \Delta u_n = 0 & \text{in } \Omega_n \times (0,\infty) \\ u_n = 0 & \text{on } \partial\Omega_n \times (0,\infty) \\ u_n(\cdot\,,0) = u_{0n}(\cdot) & \text{in } \Omega_n\,, \end{cases} \tag{1.1}$$

where Ω_n is a sequence of bounded domains in $\mathbb{R}^N$ $(N \geq 2)$ approaching an open set Ω as n tends to infinity. For instance, Ω consists of two disjoint balls, and Ω_n is a dumbbell with a small hole in the balls on each side. By shrinking the holes and the handle, we approximate the set Ω as shown in Figure 1 below.

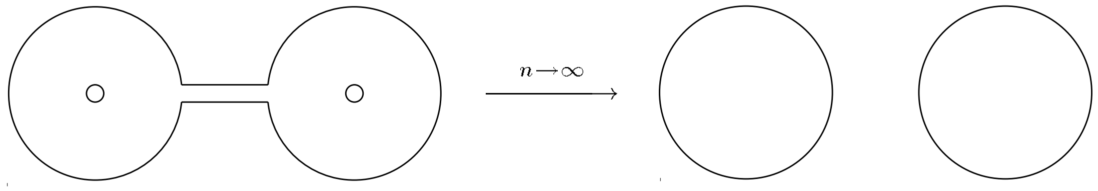

Figure 1

For the general definition of convergence of domains we refer to Section 3. It is well known that (1.1) has a unique weak solution for any $u_{0n} \in L_2(\Omega_n)$. We also may consider (1.1) as an abstract parabolic equation in $L_p(\Omega_n)$ for all $n \in \mathbb{N}$ and $p \in (1,\infty)$, and write the solution by means of the semigroup e^{-tA_n} generated by the L_p-realization of the negative Laplacian subject to Dirichlet boundary conditions. If $p \geq 2$ the solution of the abstract equation is a weak solution of (1.1) and vice versa. For this reason we do not distinguish between the weak solutions and solution of the abstract equation, and if we speak of a solution of (1.1) we always mean a weak solution. Set $\mathrm{p}_n\, u := u|_{\Omega_n}$ for all functions u on $\mathbb{R}^N$ and $\mathrm{e}_n\, u := u$ on Ω_n and $\mathrm{e}_n\, u = 0$ outside Ω_n for any function u on Ω_n. We can define an analytic (but not strongly continuous) semigroup on $L_p(\mathbb{R}^N)$ by setting

$$U_n(t) := \mathrm{e}_n \circ e^{-tA_n} \circ \mathrm{p}_n\ . \tag{1.2}$$

We also define a pseudo resolvent $R_n(\lambda)$ by setting

$$R_n(\lambda) := \mathrm{e}_n \circ (\lambda + A_n)^{-1} \circ \mathrm{p}_n \tag{1.3}$$

whenever λ is in the resolvent set $\varrho(A_n)$ of A_n. In a similar manner, we define $U(\cdot)$ and $R(\cdot)$ by deleting n everywhere. It is well known that one can represent the resolvent by means of the Laplace transform of the semigroup. On the other hand an analytic semigroup has a representation by means of the resolvent as a contour integral over a suitable sector (independent of $n \in \mathbb{N}$) in the complex plane. The facts on semigroups we make use of can be found e.g. in [8], Section XVII.§6. From this, we easily deduce the following proposition.

1.1 Proposition. *Let $1 < p < \infty$. The following are equivalent:*

(a) $\lim_{n\to\infty} U_n(t) = U(t)$ *in* $\mathcal{L}(L_p(\Omega))$ *uniformly with respect to* t *in compact subintervals of* $(0,\infty)$,

(b) $\lim_{n\to\infty} R_n(\lambda) = R(\lambda)$ *in* $\mathcal{L}((L_p(\Omega))$ *for all* λ *in a sector* $\{z \in \mathbb{C}\colon |\arg z| \le \vartheta\}$ *of the complex plane with angle* $\vartheta > \pi/2$,

It is shown implicitly in [4], Remark 1.5 that the convergence in (b) of the above proposition holds in the strong operator topology. It is in fact not hard to show that it holds in the uniform operator topology. We shall carry out the easy argument in the proof of Corollary 3.3 below. This allows us to prove the following theorem.

1.2 Theorem. *Suppose that* $\mathrm{e}_n(u_{0n})|_\Omega$ *converges to* u_0 *weakly in* $L_2(\Omega)$. *Let* u_n *be the solution of* (1.1), *and* u *the solution of equation* (1.1) *with* n *deleted. Then,*

$$\lim_{n\to\infty} u_n = u \tag{1.4}$$

holds in $C((0,T], L_q)$ *for all* $q \in (1,\infty)$ *and* $T > 0$ *finite.*

Proof. The assertion follows if we show that each term on the right hand side of the inequality

$$\|u_n(t) - u(t)\|_q \le \|U_n(t-\delta)\|_{2,q}\|U(\delta)(u_{0n} - u_0)\|_2 + \|U_n(t) - U(t)\|_{2,q}\|u_0\|_2$$

converges to zero uniformly with respect to $t \in [\delta, T]$ for all $\delta > 0$ and $T > \delta$. Here, $\|\cdot\|_p$ denotes the L_p-norm and $\|\cdot\|_{p,q}$ the norm in $\mathcal{L}(L_p, L_q)$. Convergence of the first term follows if we can show that the the family $\mathrm{e}_n \circ e^{-\delta A_n} u_{0n}$ is relatively compact in $L_2(\mathbb{R}^N)$. To see this note first that

$$\|e^{-\delta A_n} u_{0n}\|_{W_2^1} \le c|(A_n e^{-\delta A_n} u_{0n} | e^{-\delta A_n} u_{0n})| \le c\delta^{-1}\|u_{0n}\|_2\,,$$

where c is a constant independent of $n \in \mathbb{N}$ and $(\cdot\,|\,\cdot)$ the L_2-scalar product. By Rellich's theorem, the desired compactness follows. Convergence of the second term follows from Theorem 1.1 since

$$\begin{aligned}\|U_n(t) - U(t)\|_{2,q} \le \|U_n(t-\delta') &- U(t-\delta')\|_{q,q}\|U(\delta')\|_{2,q}\\ &+ \|U(t-\delta')\|_{2,q}\|U_n(\delta') - U(\delta')\|_{2,2}\end{aligned}$$

for all $\delta' \in (0,\delta)$. $\square$

The results proved so far already generalize and considerably simplify similar results in Dancer [5], Section 5, and Rauch and Taylor [18]. We can drop the condition that the projections p_n converge strongly in $\mathcal{L}(L_2(\mathbb{R}^N))$, which was assumed in [5]. For an example of a situation where the projections do not converge we refer to Remark 3.2(b) below. It is straightforward to prove the same results replacing the Laplacian by a general divergence form elliptic operator with bounded measurable coefficients. Consider now the nonlinear heat equation

$$
\begin{cases} \partial_t u_n - \Delta u_n = f(t, u_n) & \text{in } \Omega_n \times (0, \infty) \\ u_n = 0 & \text{on } \partial\Omega_n \times (0, \infty) \\ u_n(\cdot\,, 0) = u_{0n}(\cdot) & \text{in } \Omega_n\,, \end{cases} \tag{1.5}
$$

where f is a C^1-function on $(0, \infty) \times \mathbb{R}$. Here we can ask the same question as for the homogeneous case. In fact, using the variation of constants formula and continuous dependence of fixed points obtained by the contraction mapping principle we are able to prove the following theorem. The details of the proof are the same as in the general case and are given in Section 6.

1.3 Theorem. *Suppose the assumptions of Theorem 1.2 hold, that u_n is the solution of* (1.5)*, and that u is the solution of the corresponding equation with n deleted. Then* (1.4) *holds in $C\big((0, T], L_q\big)$ for T sufficiently small and $q \in (1, \infty)$.*

Note that in the above theorem T may not be arbitrary because the solutions might exist only for finite time. If u_{0n} converges in L_q with q depending on the growth of f, T may be chosen arbitrarily close to the existence time of the solution on Ω as it turns out that the existence time is lower semicontinuous with respect to domain and initial values.

When studying the dynamic behavior of a periodic-parabolic equation one is interested in stationary points or periodic solutions and their stability properties. It is interesting to have methods to know how many T-periodic solutions there are, and to know their stability properties. If we assume that $f(\cdot\,, \xi)$ is T-periodic for all $\xi \in \mathbb{R}$ for some period $T > 0$ we shall prove that close to any nondegenerate stable (unstable) T-periodic solution of the limiting problem there exists a unique stable (unstable) T-periodic solution for (1.5) for n large. For details we refer to Section 7. In a similar manner as done in [4], [5] for elliptic equations our results can be used to construct domains, where a given periodic-parabolic equation has many T-periodic solutions. The idea is to take a domain where (1.5) has a unique nontrivial T-periodic solution. Taking Ω the two balls as above we can combine all the solutions and get three nontrivial periodic solutions. Then, our results tell us that we have at least three periodic solutions for the equation on the dumbbell.

In this paper we prove all the above results for general divergence form parabolic equations with measurable coefficients (also in time). The generalization is far from being straightforward. First of all we do not have an analogue for Proposition 1.1, so we have to work directly with the parabolic problem. Moreover, as we suppose the coefficients merely to be measurable in time it is not clear whether (1.4) holds in a space of continuous functions. The ideas used in this paper have there origin in several sources. One is the techniques used in Galerkin approximation of abstract parabolic equations. Another are techniques used when proving results for parabolic equations with measurable coefficients on nonsmooth domains. Our results contain these methods as a special case. The idea in that case is to approximate domains from the interior by smooth domains and measurable coefficients by smooth coefficients, using the results in the smooth case, and then trying to pass to the limit. The main difficulty there is to have estimates which do not depend on the smoothness of the coefficients and domains. This technique was extensively used for instance in Aronson [3]. Last but not least, many ideas have their roots in the corresponding theory for elliptic equations due to Dancer [4], [5]. However, some of our proofs considerably simplify those in [4], [5] when applied to the elliptic situation. In particular, we mention the proof of the continuity of the spectrum of the linearizations in stationary solutions.

We shall develop in Section 6 a theory for semilinear equations in L_p-spaces without making use of fractional power or interpolation spaces. For several reasons it is not possible to work with the theory of semilinear evolution equations such as contained for instance in [12], [1] or [7]. First of all they presuppose a certain smoothness of the coefficients and domain to be able to identify the domain of definition of $A(t)$ as a closed operator on L_p. When working with operators with measurable coefficients this is not possible. Even if we restricted ourselves to the case of smooth coefficients and domains it would not help much because for an initial value to belong to a 'higher' fractional power (or interpolation) space it necessarily has to satisfy certain boundary conditions. If we perturb the domain slightly these boundary conditions are no longer satisfied in general. The reason, we develop a pure L_p-theory is that L_p-spaces are stable under arbitrary restrictions and trivial extensions. Thus, we can consider all functions on the domain we need to without running into problems with boundary conditions. Apart from that we can not expect our approximation results to hold in a stronger norm such as L_∞ without posing further restrictions (see Remark 4.6). However, note that, given the growth of the nonlinearity, the exponent p we choose is exactly the same we would choose when working with fractional power or interpolation spaces. For instance there is no growth restriction if $p > N/2$.

The paper is organized as follows. Section 2 mainly introduces notation and provides convenient reference to well known facts on abstract variational evolution equations and their application to parabolic equations. Section 3 deals with the L_2-theory of domain perturbation. The results proved there are the basis to what follows in the subsequent sections. Section 4 provides an L_p-theory of domain perturbation. The key to 'lifting' the approximations from L_2 to L_p are good (i.e. domain independent) Gaussian estimates of the weak Green's function corresponding to parabolic operators due to Aronson [3], and interpolation. In Section 5, we establish analogous results for the periodic-parabolic problem. Moreover, we prove that the spectrum of the periodic-parabolic operator depends continuously on the domain (and the coefficients). Section 6 deals with the nonlinear problem. We conclude the paper by proving existence and uniqueness of periodic solutions for the perturbed problem in the neighborhood of a nondegenerate solution of the original problem.

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2. Preliminaries

In this section we introduce some concepts which are frequently used later. The intention is to introduce notation, and to provide convenient reference for well known facts which are not always explicitly stated in the literature. The only result which seems to be new is a compactness lemma which is suitable when dealing with varying domains. We divide the section into four parts. The first part introduces a class of function spaces proving in particular the above mentioned compactness lemma. In the second part we state some results on abstract variational evolution equations paying special attention to the constants appearing in the estimates. In the next part we summarize some results for the periodic problem, again keeping control on the constants. Finally, in the last part, we show the connection with parabolic evolution equations.

Before we start let us introduce some basic notation. If Ω is a set in the Euklidean space $\mathbb{R}^N$ we write $\overline{\Omega}$, $\mathring{\Omega}$, $\partial\Omega$ and $|\Omega|$ for its closure, interior, boundary and Lebesgue measure, respectively. If Ω' is a another set in $\mathbb{R}^N$ we write $\Omega' \subset\subset \Omega$ if $\overline{\Omega}' \subset \mathring{\Omega}$. By $L_p(\Omega, E)$ $(1 \leq p \leq \infty)$ we denote the usual Lebesgue spaces of functions taking values in the Banach space E. In case $E = \mathbb{R}$ we just write $L_p(\Omega)$ or L_p if there are no ambiguities. Further, $W_2^1(\Omega)$ is the Sobolev space of all functions in $L_2(\Omega)$ whose distributional derivatives up to the order one are in $L_2(\Omega)$. Moreover, $\mathring{W}_2^1(\Omega)$ is the closure of the space of the test function $\mathcal{D}(\Omega)$ in $W_2^1(\Omega)$. As usual $\mathcal{D}(\Omega)$ is the space of C^∞-functions having compact support in Ω. The support of a function u we denote by $\operatorname{supp} u$. If E, F are Banach spaces we write $\mathcal{L}(E, F)$ for the Banach space of bounded linear operators from E to F equipped with the uniform operator topology. We set $\mathcal{L}(E) := \mathcal{L}(E, E)$. If $E \subset F$ and the natural injection is continuous or compact we write $E \hookrightarrow F$ and $E \hookrightarrow\hookrightarrow F$, respectively.

A. Suppose that V and H are separable Hilbert spaces with $V \hookrightarrow H$ and V dense in H. By identifying H with its dual H' by means of the canonical isomorphism we can write

$$V \hookrightarrow H \hookrightarrow V' \tag{2.1}$$

If $J := (a, b)$ is a nontrivial interval in $\mathbb{R}$ we set

$$W(J; V, V') := \left\{u \in L_2(J, V) \colon \frac{du}{dt} \in L_2(J, V')\right\}, \tag{2.2}$$

where the derivative $\dot{u} := \frac{du}{dt}$ is taken in the sense of distributions taking values in V'. If equipped with the norm $\|\cdot\|_W$, given by

$$\|u\|_W := \left(\int_J \|u(\tau)\|_V^2 \, d\tau + \int_J \|\dot{u}(\tau)\|_{V'}^2 \, d\tau\right)^{\frac{1}{2}} \tag{2.3}$$

$W(J; V, V')$ is a Hilbert space. We just write W for $W(J; V, V')$ if there is no ambiguity. It is well known that (cf. [16], Theorem I.1.3.1)

$$W \hookrightarrow \mathrm{BC}(\bar{J}, H), \tag{2.4}$$

where BC denotes the space of bounded continuous functions equipped with the supremum norm. Therefore it makes sense to speak of $u(t)$ for each $t \in \bar{J}$. For functions $u, v \in W$ and $s, t \in \bar{J}$ with $s < t$ the following formula holds (integration by parts):

$$\big(u(t)\big|v(t)\big) - \big(u(s)\big|v(s)\big) = \int_s^t \langle \dot{u}(\tau), v(\tau)\rangle + \langle \dot{v}(\tau), u(\tau)\rangle \, d\tau \tag{2.5}$$

Here, as always, $(\cdot\,|\,\cdot)$ denotes the scalar product in H and $\langle\cdot\,,\cdot\rangle$ the duality pairing induced by $(\cdot\,|\,\cdot)$ on V. It is also well known (e.g. [15], Theorem 1.5.1) that if J is a bounded interval and $V \hookrightarrow H$ then

$$W \hookrightarrow L_2(J, H)\,. \tag{2.6}$$

As we will work with the spaces $V := \mathring{W}_2^1(\Omega)$ and $H := L_2(\Omega)$, with varying domains Ω in $\mathbb{R}^N$ this compactness result is not applicable in our case. We include here a version which is appropriate to our needs. If $(\Omega_n)_{n\in\mathbb{N}}$ is a sequence of domains we set

$$V_n := \mathring{W}_2^1(\Omega_n)\,, \qquad H_n := L_2(\Omega_n)\,, \qquad \text{and} \qquad W_n := W(J; V_n, V_n')\,. \tag{2.7}$$

By extending a function in W_n by zero outside Ω_n we may consider it as a function in $L_2\big(J, W_2^1(\mathbb{R}^N)\big)$. In that sense we can write

$$W_n \subset L_2\big(J, W_2^1(\mathbb{R}^N)\big) \subset L_2\big(J, L_q(\mathbb{R}^N)\big)\,. \tag{2.8}$$

for $q = 2N(N-2)^{-1}$ if $n \geq 3$ and $q > 1$ if $n = 2$, where for the second inclusion we used the Sobolev embedding theorem. Note, however, that $W_n \not\subset W\big(J; W_2^1(\mathbb{R}^N), W_2^{-1}(\mathbb{R}^N)\big)$ as $V_n' = W_2^{-1}(\Omega_n) \not\subset W_2^{-1}(\mathbb{R}^N)$. We prove now the following compactness lemma.

2.1 Lemma. *Suppose that B is a ball in $\mathbb{R}^N$ such that $\Omega_n \subset B$ for all $n \in \mathbb{N}$, and that J is a bounded interval. Assume that $(u_n)_{n\in\mathbb{N}}$ is a sequence of functions with $u_n \in W_n$, and that there exists $M > 0$ such that $\|u_n\|_{W_n} \leq M$ for all $n \in \mathbb{N}$. Then, the sequence $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $L_2\big(J, L_q(\mathbb{R}^N)\big)$ for all $q \in [1, 2N(N-2)^{-1})$.*

Proof. It is well known that there exists an extension operator $\mathcal{E} \in \mathcal{L}\big(W, W(\mathbb{R}; V, V')\big)$, that is an operator satisfying $\mathcal{E}u|_J = u$ for all $u \in W$. Moreover, an inspection of the proofs shows that, given any open neighborhood J_0 of $\bar{J}$, $\mathcal{E}$ can be chosen such that $\operatorname{supp} \mathcal{E}u \subset J_0$, and $\|\mathcal{E}\|$ depends only on the intervals J and J_0 (cf. [8], Section XVIII.§1.2 or [16], Theorem I.1.2.2). Hence, there exist functions $v_n \in W(\mathbb{R}; V_n, V_n')$ with $\operatorname{supp} v_n \subset J_0$, J_0 bounded, such that $v_n|_J = u_n$ and $\|v_n\|_{W(\mathbb{R};V_n,V_n')} \leq M_0$ for a constant M_0 independent of $n \in \mathbb{N}$. We first show that $(v_n)_{n\in\mathbb{N}}$ is relatively compact in $L_2\big(\mathbb{R}, L_2(\mathbb{R}^N)\big) = L_2(\mathbb{R}^{N+1})$. To do that we apply a well known compactness criterion in L_p-spaces (cf. [10], Theorem IV.8.21). The only nontrivial point is to show that the translation in any coordinate direction is equicontinuous with respect to the family $(v_n)_{n\in\mathbb{N}}$. In the following calculations $\|\cdot\|$ denotes the $L_2(\mathbb{R}^{N+1})$-norm. We start by showing equicontinuity of the translation in "time"-direction. If $h \in \mathbb{R}$ is arbitrary we can write

$$\|v_n(\cdot + h) - v_n(\cdot)\|^2 = \|v_n(\cdot + h)\|^2 - \|v_n(\cdot)\|^2 - 2\big(v_n(\cdot + h) - v_n(\cdot)\big|v_n(\cdot)\big)_{L_2(\mathbb{R}^{N+1})}.$$

Using (2.5), Fubini and the translation invariance of the integral we get that

$$\begin{aligned}\|v_n(\cdot + h)\|^2 - \|v_n(\cdot)\|^2 &= 2\int_{-\infty}^{\infty}\int_0^h \langle \dot v_n(t+\sigma h), v_n(t+\sigma h)\rangle \, d\sigma\, dt \\ &= 2|h|\int_{-\infty}^{\infty} \langle \dot v_n(t), v_n(t)\rangle\, dt \le |h| \|v_n\|^2_{W(\mathbb{R};V_n,V_n')} \le M_0^2|h|.\end{aligned}$$

for all $n \in \mathbb{N}$. Estimating the remaining term in a similar way we obtain

$$\|v_n(\cdot + h) - v_n(\cdot)\| \le M_0\sqrt{2|h|} \tag{2.9}$$

On the other hand, if $h \in \mathbb{R}^N$ is arbitrary, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{\mathbb{R}^N} |v_n(x+h,t) - v_n(x,t)|^2 dx\, dt &\le \int_{-\infty}^{\infty}\int_{\mathbb{R}^N}\Big(\int_0^1 |\nabla v_n(x+\sigma h, t)|\, d\sigma\Big)^2 dx\, dt|h|^2 \\ &\le |h|^2\int_{-\infty}^{\infty}\int_{\mathbb{R}^N}\int_0^1 |\nabla v_n(x+\sigma h,t)|^2\, d\sigma\, dx\, dt \\ &= |h|^2\|\nabla v_n\|^2 \le |h|^2 \|v_n\|^2_{W(\mathbb{R};V_n,V_n')} \le M_0^2|h|^2,\end{aligned}$$

where in the second step we made use of the Cauchy-Schwarz inequality. Here, as always, ∇ denotes the gradient with respect to $x \in \mathbb{R}^N$. This, together with (2.9) shows the relative compactness of (v_n) in $L_2\big((0,T), L_2\big)$. Suppose now that $2 < q < 2N(N-2)^{-1}$ is given and choose $q_1 \in (q, 2N(N-2)^{-1})$ arbitrary. As (v_n) is a bounded sequence in $L_2\big((0,T), W_2^1\big)$ it follows from the Sobolev embedding theorem that (v_n) is a bounded sequence in $L_2\big((0,T), L_{q_1}\big)$. Therefore, if (v_{n_k}) is a convergent subsequence of (v_n) in L_2, it follows by interpolation and Hölder's inequality that (v_{n_k}) is convergent in $L_2\big((0,T), L_q\big)$. This accomplishes the proof of the lemma. □

B. Let us now turn to variational evolution equations. Suppose again that V, H are Hilbert spaces with $V \hookrightarrow H$, V dense in H, and let $\mathrm{a}(t;\cdot,\cdot)$ be a continuous bilinear form on V for all $t \in J := [0,T]$. Suppose that for every $u, v \in V$ the map

$$[0,T] \longrightarrow \mathbb{R},\ t \longmapsto \mathrm{a}(t;u,v) \tag{2.10}$$

is measurable. Moreover, suppose that there exist constants $M, \alpha > 0$ and $\lambda_0 \in \mathbb{R}$ independent of $t \in [0,T]$ such that

$$\mathrm{a}(t;u,v) \le M\|u\|_V\|v\|_V \tag{2.11}$$

for all $u, v \in V$, and

$$\mathrm{a}(t; u, u) + \lambda_0 \|u\|_H^2 \geq \alpha \|u\|_V^2 \tag{2.12}$$

for all $u \in V$. For each $t \in [0, T]$ the bilinear form $\mathrm{a}(t; \cdot, \cdot)$ induces a linear operator $A(t) \in \mathcal{L}(V, V')$ such that

$$\mathrm{a}(t; u, v) = \langle A(t)u, v\rangle \tag{2.13}$$

holds for all $u, v \in V$. Moreover, due to (2.11) we have that

$$\sup_{t\in[0,T]} \|A(t)\|_{\mathcal{L}(V,V')} \leq M\,. \tag{2.14}$$

We then consider the equation

$$\begin{cases} \dot{u} + A(t)u = f(t) & \text{for } t \in (0, T] \\ \quad u(0) = u_0 \end{cases} \tag{2.15}$$

in V' with initial value $u_0 \in H$ and inhomogeneity $f \in L_2\big([0, T], V'\big)$. By a solution we mean a function $u \in W := W(0, T; V, V')$ satisfying (2.15). It is well known that u is a solution if and only if $u \in L_2\big((0, T), V\big)$ and

$$-\int_0^T \langle \dot{v}(\tau), u(\tau)\rangle\, d\tau + \int_0^T \mathrm{a}(\tau; u(\tau), v(\tau))\, d\tau = (u_0|v(0)) + \int_0^T \langle f(\tau), v(\tau)\rangle\, d\tau \tag{2.16}$$

holds for all $v \in W$ with $v(T) = 0$. Of course, the set $\{u \in W : v(T) = 0\}$ of test functions can be replaced by a dense subset, so for instance by $\mathcal{D}([0, T)) \otimes V_0$, where V_0 is a dense subspace of V. Here, $\mathcal{D}([0, T))$ is the space of C^∞-functions with support in $[0, T)$, and $\mathcal{D}([0, T)) \otimes V_0$ is the linear hull of all functions of the form $v\varphi(t)$, where $v \in V_0$ and $\varphi \in \mathcal{D}([0, T))$. Note that

$$v(t) := e^{-\lambda_0 t} u(t) \tag{2.17}$$

is a solution of (2.15) with $A(t)$ replaced by $A(t) + \lambda_0$. Therefore we can assume without loss of generality that $\lambda_0 = 0$ in (2.12). We shall do this henceforth. Concerning existence of solutions of (2.15) the following theorem is well known (cf. [16], Theorem III.4.4.1 or [8], Section XVIII.§3).

2.2 Theorem. *Suppose that (2.10)–(2.12) are satisfied (with $\lambda_0 = 0$). Then for any $u_0 \in H$ and $f \in L_2\big((0, T), V'\big)$ equation (2.15) has a unique solution. Moreover the solution satisfies the estimates*

$$\|u(t)\|_H^2 + \alpha \int_0^t \|u(\tau)\|_V^2\, d\tau \leq \|u_0\|_H^2 + \alpha^{-1} \int_0^t \|f(\tau)\|_{V'}^2\, d\tau \tag{2.18}$$

and

$$\int_0^t \|\dot u(\tau)\|_{V'}^2\,d\tau \le 2M\alpha^{-1}\|u_0\|_H^2 + (2M^2\alpha^{-2}+1)\int_0^t \|f(\tau)\|_{V'}^2\,d\tau \tag{2.19}$$

for all $t \in (0,T]$, where α and M are the constants from (2.11) *and* (2.12).

The above theorem tells us that the operator

$$\mathbf{L} := \Big(\frac{d}{dt} + A(\cdot), \gamma_0\Big) \in \mathcal{L}\big(W, L_2((0,T),V') \times H\big) \tag{2.20}$$

is a topological isomorphism. Here, $\gamma_0 \in \mathcal{L}(W,H)$ denotes the trace operator given by $\gamma_0 u := u(0)$ which is well defined by (2.4).

C. We finally state some results on the periodic problem, that is the equation

$$\begin{cases} \dot u + A(t)u = f(t) & \text{for } t \in (0,T) \\ \qquad u(0) = u(T)\,. \end{cases} \tag{2.21}$$

For simplicity, we assume in addition to the above section that the embedding constant of V in H is one. The following theorem is essentially known (cf. [16], Theorem III.6.2.1), but in order to have explicit estimates on the solution we include the easy proof.

2.3 Theorem. *For any $f \in L_2\big((0,T),V'\big)$ equation* (2.21) *has a unique solution $u \in W(0,T;V,V')$. Moreover, the initial value of the solution satisfies the estimate*

$$\|u(0)\|_H^2 \le \big(\alpha(1-e^{-\alpha T})\big)^{-1}\int_0^T \|f(\tau)\|_{V'}^2\,d\tau\,. \tag{2.22}$$

Proof. In a first step we prove an estimate on the time-T-operator of (2.15), that is the operator $U(T)$ mapping u_0 to the solution of (2.15) with $f \equiv 0$ at time T. To do so note that

$$\frac{d}{dt}\|u(t)\|_H^2 = 2\langle \dot u(t), u(t)\rangle = -2\,\mathrm{a}(t;u(t),u(t)) \le -2\alpha\|u(t)\|_H^2\,.$$

Therefore, $\|u(\cdot)\|_H^2$ is absolutely continuous, and by a comparison theorem for ordinary differential equations we conclude that $\|u(t)\|_H^2 \le e^{-2\alpha t}\|u(0)\|_H^2$. This proves that

$$\|U(T)\|_{\mathcal{L}(H)} \le e^{-\alpha T}\,. \tag{2.23}$$

Suppose now that w is the solution of (2.15) with $u_0 \equiv 0$. Note that by (2.23) $\|U(T)\|_{\mathcal{L}(H)} < 1$ and therefore $1 \in \varrho\big(U(T)\big)$. Thus, we can set $u_0 := \big(1-U(T)\big)^{-1}w(T)$. It is then easily verified that the solution u of (2.15) with initial value u_0 satisfies $u(0) = u(T)$, that is, it is a solution of (2.21). Moreover, it is clear that this solution is unique. The remaining assertion is now an easy consequence of (2.23) and (2.18). $\square$

If we set $W_T := \{u \in W \colon u(0) = u(T)\}$ the above theorem tells us that

$$L := \frac{d}{dt} + A(\cdot) \in \mathcal{L}\big(W_T, L_2((0,T),V')\big) \tag{2.24}$$

is a topological isomorphism.

D. The above results can be used to prove existence of weak solutions of the initial boundary value problem

$$\tag{2.25} \begin{cases} \partial_t u + \mathcal{A}(t)u = f & \text{in } \Omega \times (0,T] \\ u = 0 & \text{on } \partial\Omega \times (0,T] \\ u(\cdot,0) = u_0(\cdot) & \text{in } \Omega\,, \end{cases}$$

where Ω is an arbitrary domain in $\mathbb{R}^N$, and $\mathcal{A}(t)$ an elliptic operator of the form

$$\tag{2.26} \mathcal{A}(t)u := -\partial_i\big(a_{ij}(x,t)\partial_j u + a_i(x,t)u\big) + b_i(x,t)\partial_i u + c_0(x,t)u\,.$$

(We use summation convention, i,j running from 1 to N.) We assume that the coefficients of $\mathcal{A}(\cdot)$ satisfy the following conditions:

$$\tag{2.27} \begin{cases} \text{There exists } R_0, M_0 > 0 \text{ and } \nu > N \text{ such that} \\ \quad \text{(i) } a_{ij} \in L_\infty(\mathbb{R}^N \times (0,T)) \\ \quad \text{(ii) } a_i, b_i \in L_\infty((0,T), L_\nu(\mathbb{B}_{R_0})) \text{ and } |a_i(x,t)|, |b_i(x,t)| \leq M_0 \text{ if } \\ \qquad |x| \geq R_0 \\ \quad \text{(iii) } c_0 \in L_\infty((0,T), L_{\frac{\nu}{2}}(\mathbb{B}_{R_0})) \text{ and } c_0(x,t) \leq M_0 \text{ if } |x| \geq R_0 \\ \text{for all } i,j = 1,\dots,N, \end{cases}$$

where $\mathbb{B}_{R_0} := \{x \in \mathbb{R}^N\colon |x| < R_0\}$. Moreover, we suppose that there exists a constant $\alpha_0 > 0$ such that

$$\tag{2.28} a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0|\xi|^2$$

for all $(x,t) \in \mathbb{R}^N \times [0,T]$ and $\xi \in \mathbb{R}^N$. As usual, the Dirichlet form corresponding to $\mathcal{A}(t)$ is defined by

$$\tag{2.29} \mathrm{a}(t;u,v) := \int_\Omega \big(a_{ij}(x,t)\partial_j u + a_i(x,t)u\big)\partial_i v + \big(b_i(x,t)\partial_i u + c_0(x,t)u\big)v\,dx$$

for all $u \in W_2^1(\mathbb{R}^N)$. It is well known, and in fact easy to see, that there exist $M, \alpha > 0$ and $\lambda_0 \in \mathbb{R}$ only depending on α_0, N, M_0, p and an upper bound for the norms of the coefficients of $\mathcal{A}(\cdot)$ in L_∞, $L_\infty\big((0,T), L_p\big)$ and $L_\infty\big((0,T), L_{\frac{p}{2}}\big)$, respectively such that (2.10)–(2.12) hold for all $u, v \in W_2^1(\mathbb{R}^N)$. If we set $H := L_2(\Omega)$ and $V := \mathring{W}_2^1(\Omega)$ we can write (2.25) as an abstract equation of the form (2.15) in $W_2^{-1}(\Omega) = V'$. By the remarks in B. it is clear that u is a solution of the abstract equation iff it is a weak solution of (2.25), i.e $u \in L_2\big((0,T), V)\big)$ and (2.16) holds for all $v \in \mathcal{D}\big(\Omega \times [0,T)\big)$. Therefore, it follows from Theorem 2.2 that (2.25) has a unique weak solution for all $u_0 \in H$ and $f \in L_2\big((0,T), V'\big)$.

3. Domain Perturbation for Linear Equations: L_2-Theory

In this section we study how the (weak) solutions of the parabolic initial boundary value problems

$$
\left\{\begin{aligned} \partial_t u + \mathcal{A}_n(t)u &= f_n && \text{in } \Omega_n \times (0,T] \\ u &= 0 && \text{on } \partial\Omega_n \times (0,T] \\ u(\cdot,0) &= u_{0n}(\cdot) && \text{in } \Omega_n \end{aligned}\right. \tag{3.1}
$$

depend on data and domain. It is assumed that $\mathcal{A}_n(t)$ is an elliptic operator of the same structure as (2.26) and coefficients $a_{ij}^{(n)}, a_i^{(n)}, b_i^{(n)}, c_0^{(n)}$ satisfying a condition similar to (2.27) with R_0, M_0 and p independent of $n \in \mathbb{N}$, and norms uniformly bounded with respect to $n \in \mathbb{N}$ in L_∞, $L_\infty\big((0,T), L_\nu(\mathbb{B}_{R_0})\big)$ and $L_\infty\big((0,T), L_{\frac{\nu}{2}}(\mathbb{B}_{R_0}\big)$, respectively. Further, we suppose that the top order coefficients $a_{ij}^{(n)}$ satisfy an estimate similar to (2.28) uniformly with respect to $n \in \mathbb{N}$. Introducing the Dirichlet form $\mathrm{a}_n(t;\cdot\,,\cdot)$ corresponding to $\mathcal{A}_n(t)$, and taking into account the remarks in Section 2.D we find that there exist constants $M, \alpha > 0$ and $\lambda_0 \in \mathbb{R}$ such that for all $u, v \in W_2^1(\mathbb{R}^N)$

$$
\mathrm{a}_n(t;u,v) \le M\|u\|_{W_2^1}\|v\|_{W_2^1} \quad \text{and} \quad \mathrm{a}_n(t;u,u) + \lambda_0\|u\|_{L_2}^2 \ge \alpha\|u\|_{W_2^1}^2 \tag{3.2}
$$

for all $n \in \mathbb{N}$. As before, we assume without loss of generality that $\lambda_0 = 0$. We can therefore write equation (3.1) in the abstract form

$$
\left\{\begin{aligned} \dot u + A_n(t)u &= f_n(t) && \text{for } t \in (0,T] \\ u(0) &= u_{0n}\,, \end{aligned}\right. \tag{3.3}
$$

where $A_n(t)$ is the operator induced by $\mathrm{a}_n(t;\cdot\,,\cdot)$. Finally, we define H_n, V_n and W_n by (2.7). We assume throughout that

$$
\left\{\begin{array}{l} \text{For all } i,j = 1,\dots,N \text{ we have that } a_{ij}^{(n)}, a_i^{(n)}, b_i^{(n)} \text{ and } c_0^{(n)} \text{ con-} \\ \text{verge to the corresponding coefficients of } \mathcal{A}(\cdot) \text{ in } L_2((0,T), L_{1,\mathrm{loc}}). \end{array}\right. \tag{3.4}
$$

Following Dancer [4], [5] we say that a sequence Ω_n of domains in $\mathbb{R}^N$ ($N \ge 2$) converges to an open set Ω if

$$
\left\{\begin{array}{l} \text{There exists a compact set } K_0 \subset \Omega \text{ of capacity zero and a compact} \\ \text{set } K_1 \subset \mathbb{R}^N \text{ of measure zero such that} \\ \quad \text{(i) } \Omega' \subset\subset \Omega \setminus K_0 \text{ implies that } \Omega' \subset\subset \Omega_n \text{ for large } n \in \mathbb{N}, \\ \quad \text{(ii) For any open set } U \text{ with } \overline{\Omega} \cup K_1 \subset U \text{ we have } \Omega_n \subset U \text{ for} \\ \quad\quad \text{large } n \in \mathbb{N}. \end{array}\right. \tag{3.5}
$$

If this holds we simply write $\Omega_n \to \Omega$. Furthermore, we impose a very weak regularity assumption on the limiting domain Ω, namely that

$$
u \in W_2^1(\mathbb{R}^N) \text{ and } \operatorname{supp} u \subset \overline{\Omega} \text{ imply } u \in \mathring{W}_2^1(\Omega)\,. \tag{3.6}
$$

This property holds for all reasonably regular domains such as Lipschitz domains for instance. A discussion of this property may be found in [6].

In abuse of notation we shall often write $u \in L_p(\mathbb{R}^N)$ for the trivial extension (i.e. u=0 outside Ω) of a function $u \in L_p(\Omega)$. On the other hand, if $u \in L_p(\mathbb{R}^N)$ we write $u \in L_p(\Omega)$ and mean its restriction to Ω. For instance, if we write $u_{0n} \to u_0$ in $L_2(\mathbb{R}^N)$ for $u_{0n} \in L_2(\Omega_n)$ we always mean that the trivial extensions converge in $L_2(\mathbb{R}^N)$. If we write $u_{0n}|_\Omega$ we mean that the trivial extension is restricted to Ω. We shall use this convention throughout without further comment.

We are now in a position to prove one of our main results.

3.1 Theorem. *Suppose that Ω is a bounded open set satisfying (3.6), that Ω_n is a sequence of domains with $\Omega_n \to \Omega$ as n tends to infinity, and that (3.4) holds. Moreover, assume that $u_{0n} \in L_2(\Omega_n)$ and $f_n \in L_2\big((0,T), L_{\hat{2}}(\Omega_n)\big)$ such that*

$$u_{0n}|\Omega \rightharpoonup u_0 \quad \text{in } L_2(\Omega) \qquad \text{and} \qquad f_n|\Omega \rightharpoonup f \quad \text{in } L_2\big((0,T), L_{\hat{2}}(\Omega)\big) \text{ weakly,}$$

where $\hat{2} = 2N(N+2)^{-1}$ if $N \geq 3$ and $\hat{2} > 1$ arbitrary if $N = 2$. If u_n is the solution of (3.1) and u the solution of (2.25), then

$$\lim_{n\to\infty} u_n = u$$

in $L_2\big((0,T), L_q(\mathbb{R}^N)\big)$ for all $q \in [1, 2N(N-2)^{-1})$, and weakly in $L_2\big((0,T), W_2^1(\mathbb{R}^N)\big)$.

Proof. Note first that by Sobolev's embedding theorem and duality $L_{\hat{2}}(\Omega_n) \hookrightarrow V_n' = W_2^{-1}(\Omega_n)$ with an embedding constant independent of $n \in \mathbb{N}$. On the other hand, weakly convergent sequences are bounded. Hence, $\|f_n\|_{L_2((0,T),V_n')}$ is uniformly bounded and therefore we conclude from (3.2) and Theorem 2.2 that there exists $M_0 > 0$ such that

$$\|u_n(t)\|_{H_n}^2 + \|u_n\|_{W_n}^2 \leq M_0 \tag{3.7}$$

for all $n \in \mathbb{N}$ and $t \in [0,T]$. In particular, it follows that (u_n) is uniformly bounded in $L_2\big((0,T), W_2^1(\mathbb{R}^N)\big)$ (if we extend u_n outside Ω_n by zero). Thus we can extract a subsequence, which we denote again by (u_n) converging to some u in $L_2\big((0,T), W_2^1(\mathbb{R}^N)\big)$ weakly. By Lemma 2.1 this sequence converges strongly in $L_2\big((0,T), L_q(\mathbb{R}^N)\big)$ for all $q \in [1, 2N(N-2)^{-1})$. It remains to show that u is the solution of (2.25). As the solution is unique it follows that the whole sequence converges.

Let $v \in \mathcal{D}(\Omega \setminus K_0)$ and $\varphi \in \mathcal{D}\big([0,T)\big)$ arbitrary. Then as u_n is a weak solution of (3.1) and $\operatorname{supp} v \subset \Omega_n$ for large $n \in \mathbb{N}$ we have that

$$\begin{aligned}
-\int_0^T (u_n(\tau)|v)\dot{\varphi}(\tau)\,d\tau + \int_0^T \mathrm{a}_n(\tau; u_n(\tau), v)\varphi(\tau)\,d\tau& \\
= (u_{n0}|v)\varphi(0) + \int_0^T \langle f_n(\tau), v\rangle\varphi(\tau)\,d\tau\,.&
\end{aligned} \tag{3.8}$$

By (3.4), the uniform boundedness of the coefficients of $\mathcal{A}_n(\cdot)$ in L_∞, $L_\infty\big((0,T), L_\nu\big)$ and $L_\infty\big((0,T), L_{\frac{\nu}{2}}\big)$, respectively and interpolation it follows that the coefficients converge in L_2, $L_2\big((0,T),L_N\big)$ and $L_2\big((0,T),L_{\frac{N}{2}}\big)$, respectively. Thus, by the weak convergence of (u_n) in $L_2\big((0,T),W_2^1(\mathbb{R}^N)\big)$ and the weak convergence of u_{0n} and f_n it follows from (3.8) that

$$-\int_0^T (u(\tau)|v)\dot{\varphi}(\tau)\,d\tau+\int_0^T \mathrm{a}(\tau;u(\tau),v)\varphi(\tau)\,d\tau = (u_0|v)\varphi(0)+\int_0^T \langle f(\tau),v\rangle\varphi(\tau)\,d\tau$$

as n tends to infinity. As K_0 has zero capacity it follows that $\mathcal{D}(\Omega\setminus K_0)$ is dense in $\mathring{W}_2^1(\Omega)$. Thus, by our earlier remarks in Section 2.B $\mathcal{D}\big([0,T)\big)\otimes\mathcal{D}(\Omega\setminus K_0)$ is a dense set of test functions. Therefore, u is a weak solution if we show that $u(t)\in\mathring{W}_2^1(\Omega)$ for almost all $t\in(0,T)$. To do so let B_0 be any closed ball not intersecting $\overline{\Omega}\cup K_1$. Then it follows from (3.3) that $u_n=0$ almost everywhere in $B_0\times(0,T)$ whenever n is large enough. As u_n converges in $L_2\big(\mathbb{R}^N\times(0,T)\big)$ it follows that the same is true for u. As B_0 was chosen arbitrarily, and K_1 is of measure zero, it follows that $\operatorname{supp}u\subset\overline{\Omega}\times[0,T]$. By Fubini's Theorem it follows that $\operatorname{supp}u(t)\subset\overline{\Omega}$ for almost all $t\in(0,T)$. Hence, (3.6) shows that $u(t)\in\mathring{W}_2^1(\Omega)$ for almost all t. This completes the proof of the theorem. □

3.2 Remarks. (a) If $\Omega_n\subset\Omega$ for all $n\in\mathbb{N}$ the assumption that Ω satisfies (3.6) is not necessary. This is clear from the proof because in that case $u_n\in L_2\big((0,T),\mathring{W}_2^1(\Omega)\big)$ for all $n\in\mathbb{N}$, and hence, we can work with $L_2\big((0,T),\mathring{W}_2^1(\Omega)\big)$ rather than $L_2\big((0,T),W_2^1(\mathbb{R}^N)\big)$. This was in fact used in [3] to approximate weak solutions by sequences of classical solutions of problems with smooth domains and coefficients. More recently it was observed in [6] in the case of elliptic problems. As all the subsequent approximation results rely on Theorem 3.1 the above remark remains valid for all approximation results.

(b) One could slightly generalize the notion of convergence of domains by replacing (3.5ii) by assuming that

$$\text{(3.9)}\qquad\left\{\begin{array}{l}\text{There exists a bounded set } K_1\subset\mathbb{R}^N\setminus\overline{\Omega} \text{ of measure zero which}\\ \text{is closed in } \mathbb{R}^N\setminus\overline{\Omega} \text{ such that if } U \text{ is an open set with } \overline{\Omega}\cup K_1\subset U\\ \text{then } \Omega_n\subset U \text{ for large } n\in\mathbb{N}.\end{array}\right.$$

This condition was introduced in [6] for the elliptic problem. An inspection of the above proof shows that we only make use of this condition. For this reason, the above theorem remains true if we replace (3.5ii) by (3.9). As all the subsequent approximation results are based on the above theorem the same is true for all approximation results in this paper. If $\overline{\Omega}\subset\Omega_n$ for all $n\in\mathbb{N}$ (3.9) is equivalent to $|\Omega_n\setminus\overline{\Omega}|$ tending to zero as n goes to infinity. A situation which is covered by (3.9) but not by (3.5) is the following example due to Dancer (personal communication): Take a domain whose boundary has positive measure and which satisfies (3.6). For the existence of such a domain see Dancer [6]. Then take a set K_1 of measure zero such that every point of $\partial\Omega$ is a limit point of this set, and $\overline{\Omega}\cup K_1$ is closed. Further, choose an arbitrary decreasing sequence of open sets Ω_n with $\overline{\Omega}\cup K_1\subset\Omega_n$ for all $n\in\mathbb{N}$ and $\bigcap_{n=0}^\infty\Omega_n=\overline{\Omega}\cap K_1$. Obviously, in this situation the above condition is satisfied but not (3.5ii). Note also that this is an example where the natural projections form $L_2(\mathbb{R}^N)$ onto $L_2(\Omega_n)$ do not converge strongly to the corresponding projection onto $L_2(\Omega)$. To have an example such that (3.5) is satisfied we can choose $K_1 = \emptyset$.

(c) Define $\mathbf{L}_n$ and $\mathbf{L}$ similarly to (2.20). As

$$L_2\big((0,T), L_{\hat{2}}(\Omega_n)\big) \hookrightarrow L_2\big((0,T), V_n'\big)$$

with an embedding constant independent of $n \in \mathbb{N}$ we can consider $\mathbf{L}_n^{-1}$ as a bounded linear operator from $L_2\big((0,T), L_{\hat{2}}(\mathbb{R}^N)\big) \times L_2(\mathbb{R}^N)$ into $L_2\big((0,T), L_q\big)$ (q as in the above theorem) by restricting functions first to Ω_n, using the above embedding, applying $\mathbf{L}_n^{-1}$ and finally extending the function trivially outside Ω_n. If we write again $\mathbf{L}_n^{-1}$ for this operator, the above theorem shows that

$$\lim_{n\to\infty} \mathbf{L}_n^{-1}(f, u_0) = \mathbf{L}^{-1}(f, u_0)$$

in $L_2\big((0,T), L_q(\mathbb{R}^N)\big)$ for all $(f, u_0) \in L_2\big((0,T), L_{\hat{2}}(\mathbb{R}^N)\big) \times L_2(\mathbb{R}^N)$, i.e. the operators converge in the strong operator topology. We shall in fact show more, namely convergence in the uniform operator topology. □

3.3 Corollary. *Suppose that the assumptions of the above theorem hold and denote by $\mathbf{L}_n^{-1}$ and $\mathbf{L}^{-1}$ the operators defined in the preceding remark. Then,*

$$\lim_{n\to\infty} \mathbf{L}_n^{-1} = \mathbf{L}^{-1} \tag{3.10}$$

in $\mathcal{L}\big(L_2((0,T), L_{\hat{2}}) \times L_2, L_2((0,T), L_q(\mathbb{R}^N))\big)$ for all $q \in [1, 2N(N-2)^{-1})$.

Proof. Suppose the assertion is not true. Then there exist $\varepsilon_0 > 0$ and a sequence $(f_n, u_{0n}) \in L_2\big((0,T), L_{\hat{2}}\big) \times L_2$ with $\|u_{0n}\|_2^2 + \|f_n\|^2_{L_2((0,T),L_{\hat{2}})} = 1$ such that

$$\|(\mathbf{L}_n^{-1} - \mathbf{L}^{-1})(f_n, u_{0n})\|_{L_2((0,T),L_q)} \geq \varepsilon_0 \tag{3.11}$$

for all $n \in \mathbb{N}$. Because (f_n, u_{0n}) is a uniformly bounded sequence it has a subsequence, which we denote again by (f_n, u_{0n}), converging weakly in $L_2((0,T), L_{\hat{2}}) \times L_2$ to some (f, u_0). Then it follows from Theorem 3.1 that

$$\lim_{n\to\infty} \mathbf{L}_n^{-1}(f_n, u_{0n}) = \mathbf{L}^{-1}(f, u_0) \quad \text{and} \quad \lim_{n\to\infty} \mathbf{L}^{-1}(f_n, u_{0n}) = \mathbf{L}^{-1}(f, u_0)\,.$$

in $L_2\big((0,T), L_q(\mathbb{R}^N)\big)$. (The latter follows by taking $\Omega_n = \Omega$ and $\mathcal{A}_n = \mathcal{A}$ for all $n \in \mathbb{N}$.) As this contradicts (3.11) this concludes the proof of the corollary. □

In a first step towards showing convergence in stronger norms we prove pointwise weak convergence of $u_n(t)$ in $L_2(\Omega)$. Because of (2.4) it makes sense to speak about pointwise convergence.

3.4 Lemma. *Suppose that the hypotheses of Theorem 3.1 hold. Then, $u_n(t) \rightharpoonup u(t)$ weakly in $L_2(\Omega)$ for all $t \in [0,T]$.*

Proof. By (3.7) $u_n(t)$ is uniformly bounded with respect to $n \in \mathbb{N}$ in L_2, and therefore has a subsequence (which we denote again by $u_n(t)$) with weak limit $\bar{u}$. Take $v \in \mathcal{D}(\Omega \setminus K_0)$ and $\varphi \in \mathcal{D}\big((0,t]\big)$ arbitrary. As u_n is the weak solution of (3.1) it follows that an equation similar to (3.8) holds with the integral taken over $(0,t)$, and $(u_{0n}|v)\varphi(0)$ replaced by $-(u_n(t)|v)\varphi(t)$, whenever $n \in \mathbb{N}$ is large enough. Passing to the limit we obtain

$$-\int_0^t (u(\tau)|v)\dot{\varphi}(\tau)\,d\tau + \int_0^t \mathrm{a}(\tau; u(\tau), v)\varphi(\tau)\,d\tau = -(\bar{u}|v)\varphi(t) + \int_0^t \langle f(\tau), v\rangle \varphi(\tau)\,d\tau\,.$$

As u is the weak solution of (2.25) a similar equation holds with $(\bar{u}|v)$ replaced by $(u(t)|v)$. Therefore, $(\bar{u}|v) = (u(t)|v)$ holds for all $v \in \mathcal{D}(\Omega \setminus K_0)$. Hence the density of $\mathcal{D}(\Omega \setminus K_0)$ in $L_2(\Omega)$ implies that $\bar{u} = u(t)$. By uniqueness the whole sequence converges and the proof of the lemma is complete. $\square$

We are now ready to prove convergence of u_n in stronger topologies.

3.5 Theorem. *Suppose that the hypotheses of Theorem 3.1 hold and that in addition f_n converges weakly in $L_2\big((0,T), L_p\big)$ for some $p > 2N(N+2)^{-1}$. Then, for any $\delta \in (0,T]$ the solution u_n of (3.1) converges to the solution u of (2.25) in $C\big([\delta,T], L_2\big)$ and in $L_2\big((\delta,T), W_2^1\big)$. Moreover, if u_{0n} converges strongly, the above assertion holds for $\delta = 0$.*

Proof. In a first step we show that $u_n(t)$ converges in L_2 for all $t \in (0,T]$. As u_n converges in $L_2\big((0,T), L_2\big)$ we may extract a subsequence (which we denote again by u_n) converging pointwise almost everywhere in L_2. Choose now δ arbitrarily close to zero in $(0,T]$ such that $u_n(\delta)$ converges in L_2 to $u(\delta)$. Then, for $t \in (\delta, T]$ we set

$$d_n(t) := \frac{1}{2}\|u_n(t) - u(t)\|_{L_2}^2 + \alpha \int_\delta^t \|u_n(\tau) - u(\tau)\|_{W_2^1}^2\,d\tau\,. \tag{3.12}$$

Using (3.2) (with $\lambda_0 = 0$) we obtain

$$\begin{aligned} d_n(t) \le{} & \frac{1}{2}\|u(t)\|_H^2 + \int_\delta^t \mathrm{a}_n(\tau; u(\tau), u(\tau))\,d\tau \\ & + \frac{1}{2}\|u_n(t)\|_{H_n}^2 + \int_\delta^t \mathrm{a}_n(\tau; u_n(\tau), u_n(\tau))\,d\tau \\ & - (u_n(t)|u(t)) - \int_\delta^t \mathrm{a}_n(\tau; u(\tau), u_n(\tau))\,d\tau - \int_\delta^t \mathrm{a}_n(\tau; u_n(\tau), u(\tau))\,d\tau \end{aligned} \tag{3.13}$$

for all $n \in \mathbb{N}$. By (2.5), and the fact that u_n is a weak solution we have that

$$\frac{1}{2}\|u_n(t)\|_{H_n}^2 + \int_\delta^t \mathrm{a}_n(\tau; u_n(\tau), u_n(\tau))\,d\tau = \frac{1}{2}\|u_n(\delta)\|_{H_n}^2 + \int_\delta^t \langle f_n(\tau), u_n(\tau)\rangle\,d\tau\,.$$

By our assumption on p we have that the dual exponent p' given by $1/p+1/p'=1$ satisfies $p'<2N(N-2)^{-1}$. Hence, by Theorem 3.1 u_n converges strongly in $L_2\big((0,T),L_{p'}\big)$. As f_n converges weakly in $L_2\big((0,T),L_p\big)$ by assumption, $\lim_{n\to\infty}u_n(\delta)=u(\delta)$ in L_2 and u is a weak solution of (2.25) we conclude that

$$\begin{aligned}\lim_{n\to\infty}\Big(\frac{1}{2}\|u_n(t)\|_{H_n}^2+\int_\delta^t \mathrm{a}_n(\tau;u_n(\tau),u_n(\tau))\,d\tau\Big)&\\ =\frac{1}{2}\|u(\delta)\|_H^2+\int_\delta^t\langle f(\tau),u(\tau)\rangle\,d\tau=\frac{1}{2}\|u(t)\|_H^2+\int_\delta^t \mathrm{a}(\tau;u(\tau),u(\tau))\,d\tau\,.&\end{aligned}\tag{3.14}$$

By the weak convergence of u_n in $L_2\big((\delta,t),W_2^1\big)$ we get that

$$\begin{aligned}\lim_{n\to\infty}\int_\delta^t \mathrm{a}_n(\tau;u_n(\tau),u(\tau))\,d\tau+\int_\delta^t \mathrm{a}_n(\tau;u(\tau),u_n(\tau))\,d\tau&\\ =2\int_\delta^t \mathrm{a}(\tau;u(\tau),u(\tau))\,d\tau\,.&\end{aligned}\tag{3.15}$$

Further, it is easy to see from the assumptions on the coefficients of $\mathcal{A}_n(\cdot)$ that

$$\lim_{n\to\infty}\int_\delta^t \mathrm{a}_n(\tau;u(\tau),u(\tau))\,d\tau=\int_\delta^t \mathrm{a}(\tau;u(\tau),u(\tau))\,d\tau\,.\tag{3.16}$$

Finally we have by Lemma 3.4 that

$$\lim_{n\to\infty}(u_n(t)|u(t))=\|u(t)\|_H^2\,.\tag{3.17}$$

Hence, (3.13)–(3.17) imply that $d_n(t)$ tends to zero as n goes to infinity. This proves pointwise convergence, and convergence in $L_2\big((\delta,T),W_2^1\big)$ for all $\delta\in(0,T)$.

To prove uniform convergence note first that

$$\lim_{n\to\infty}\int_\delta^t\|u_n(\tau)-u(\tau)\|_{W_2^1}^2\,d\tau=0\tag{3.18}$$

uniformly with respect to $t\in[\delta,T]$. Indeed, this follows from what we proved before since by (3.12)

$$\int_\delta^t\|u_n(\tau)-u(\tau)\|_{W_2^1}^2\,d\tau\le\alpha^{-1}d_n(T)$$

for all $n\in\mathbb{N}$ and $t\in[\delta,T]$. From this it is clear that the limits (3.14) and (3.15) are uniform with respect to $t\in[\delta,T]$. It is also clear that (3.16) holds uniformly with respect to $t\in[\delta,T]$. Hence it remains to show that (3.17) holds uniformly. As $[\delta,T]$ is a compact interval it is enough to show local uniform convergence. We divide the proof into two parts. The first shows uniform convergence in any compact subset of $\Omega\setminus K_0$

and the second near $\partial\Omega$. Let $\psi \in \mathcal{D}(\Omega \setminus K_0)$ be given. Using (2.5) and the fact that u_n and u are weak solutions of (2.25) and (3.1), respectively, we can write

$$\begin{aligned}(u_n(t)|\psi u(t)) = (u_n(\delta)|\psi u(\delta)) - \int_\delta^t \mathrm{a}(\tau; u(\tau), \psi u_n(\tau))\,d\tau + \int_\delta^t \langle f(\tau), \psi u_n(\tau)\rangle\,d\tau \\ - \int_\delta^t \mathrm{a}_n(\tau; u_n(\tau), \psi u(\tau))\,d\tau + \int_\delta^t \langle f_n(\tau), \psi u(\tau)\rangle\,d\tau\end{aligned}$$

for n large. Taking into account that $u_n(\delta)$ converges in $L_2(\mathbb{R}^N)$, we conclude from (3.18) and the assumption on f_n that

$$\lim_{n\to\infty}(u_n(t)|\psi u(t)) = (u(t)|\psi u(t)) \tag{3.19}$$

uniformly with respect to $t \in [\delta, T]$ for all $\psi \in \mathcal{D}(\Omega \setminus K_0)$. To estimate the part near the boundary fix $s \in [\delta, T]$ and an open set $\Omega' \subset\subset \Omega \setminus K_0$ arbitrary. We show that $\|u(t)\|_{L_2(\Omega\setminus\Omega')}$ is uniformly small in a neighborhood of s in $[\delta, T]$. To see this recall that $u \in C\big([\delta, T], L_2(\Omega)\big)$ and therefore for any $\varepsilon > 0$ there exists $\eta > 0$ only depending on ε such that $\|u(t) - u(\tau)\|_{L_2} \le \varepsilon/2$ for all $t, \tau \in [\delta, T]$ with $|t - \tau| \le 2\eta$. Thus, the triangle inequality implies that

$$\|u(t)\|_{L_2(\Omega\setminus\Omega')} \le \|u(t) - u(s)\|_{L_2} + \|u(s)\|_{L_2(\Omega\setminus\Omega')} \le \frac{\varepsilon}{2} + \|u(s)\|_{L_2(\Omega\setminus\Omega')} \tag{3.20}$$

for all $t \in (s - \eta, s - \eta) \cap [\delta, T]$. Therefore, given $\varepsilon > 0$, choose $\Omega' \subset\subset \Omega$ such that $\|u(s)\|_{L_2(\Omega\setminus\Omega')} \le \varepsilon/2$. Then, by (3.20) there exists $\eta > 0$ such that $\|u(t)\|_{L_2(\Omega\setminus\Omega')} \le \varepsilon$ for all $t \in (s - \eta, s + \eta) \cap [\delta, T]$ and $n \in \mathbb{N}$. Finally choose a cut-off function for Ω', i.e. $\psi \in \mathcal{D}(\Omega \setminus K_0)$ nonnegative with $\psi = 1$ on Ω'. Then, by writing $u = \psi u + (1 - \psi)u$ and using the Cauchy-Schwarz inequality to estimate the term $|(u_n(t) - u(t)|(1 - \psi)u(t))|$ we obtain

$$|(u_n(t) - u(t)|u(t))| \le |(u_n(t) - u(t)|\psi u(t))| + 2M_0\varepsilon$$

for all $t \in (s - \eta, s + \eta) \cap [\delta, T]$ and $n \in \mathbb{N}$, where M_0 is the constant from (3.7). By (3.19) the first term is uniformly small for large $n \in \mathbb{N}$. As η depends only on ε this shows local uniform, and hence uniform convergence of (3.17).

Finally, by choosing $\delta = 0$ in case u_{0n} converges in L_2 it is obvious, that all the arguments above remain true. This proves the last assertion of the theorem. $\square$

3.6 Corollary. *Suppose that $\Omega_n \to \Omega$ with Ω satisfying* (3.6), *and that* (3.4) *holds. Moreover, let $\mathbf{L}_n^{-1}$ and $\mathbf{L}^{-1}$ be as in Remark 3.2(c). Then, assertion* (3.10) *holds in $\mathcal{L}\big(L_2((0,T), L_p) \times L_2, C((0,T], L_2)\big)$ for all $p > 2N(N+2)^{-1}$.*

Proof. The proof is analogous to that of Corollary 3.3. We find a bounded sequence (f_n, u_{0n}) in $L_2\big((0,T), L_p\big) \times L_2$ and $\varepsilon_0 > 0$ such that (3.11) holds in the norm of $C\big([\delta, T], L_2\big)$, where $\delta \in (0, T)$ is arbitrary. Then, using the fact that (f_n, u_{0n}) has a weakly convergent subsequence and Theorem 3.5, we obtain a contradiction. $\square$

3.7 Remarks. (a) We can not expect the assertion of Corollary 3.6 to hold if we replace $C\big((0,T],L_2\big)$ by $C\big([0,T],L_2\big)$. Indeed, $\mathbf{L}_n^{-1}(0,\cdot)$ at time 0 is the projection p_n given by $u \mapsto u|\Omega_n$. Similarly, $\mathbf{L}^{-1}(0,\cdot)$ at time 0 is the projection $u \mapsto u|\Omega$. In general, these projections do not converge in the uniform operator topology on $\mathcal{L}(L_2)$ but only in the strong operator topology. For instance if $\Omega_n \subset\subset \Omega$ for all $n \in \mathbb{N}$ we choose a sequence of $u_n \in L_2$ supported in $\Omega \setminus \Omega_n$ with $\|u_n\|_2 = 1$. Then, $\|\,\mathrm{p}_n\, u_n - \mathrm{p}\, u_n\|_2 = 1$ for all $n \in \mathbb{N}$. Similarly the same assertion follows in the general case provided Ω and Ω_n differ by a set of positive measure for all $n \in \mathbb{N}$. For this reason we can not expect (3.10) to hold in L_2 uniformly with respect to $t \in [s,T]$.

(b) On the other hand, we have that

$$\lim_{n\to\infty} \mathbf{L}_n^{-1}(\cdot\,,0) = \mathbf{L}^{-1}(\cdot\,,0) \tag{3.21}$$

in $\mathcal{L}\big(L_2((0,T),L_p),C([0,T],L_2)\big)$ for p as in the above corollary. The proof is an obvious modification of the proof of Corollary 3.6. $\square$

4. Domain Perturbation for Linear Equations: L_p-Theory

In this section we obtain approximations in higher norms. The main tool which enables us to do that are good estimates on the weak Green's function due to Aronson [3], and interpolation. We first establish a representation formula for weak solutions of (2.25). The proof makes use of the results in the previous section. We start by showing some properties of the evolution operator $U(t,s)$ corresponding to the family of operators $\big(A(t)\big)_{t\in[0,T]}$. By definition $U(\cdot,s)u_0$ is the solution of the equation

$$\begin{cases} \dot u + A(t)u = 0 & \text{for } t \in (s,T] \\ \qquad u(s) = u_0\,. \end{cases} \tag{4.1}$$

where $s \in [0,T)$ is arbitrary. It is defined for all $(t,s) \in J_\Delta := \{(t,s)\colon 0 \le s \le t \le T\}$. Moreover, it follows from Theorem 2.2 and (2.4) that

$$U(\cdot\,,s) \in C\big([s,T],\mathcal{L}(H)\big)\,. \tag{4.2}$$

Further, by uniqueness of the solution of (4.1) it follows that

$$U(t,t) = \mathrm{I} \qquad \text{and} \qquad U(t,s) = U(t,\tau)U(\tau,s) \tag{4.3}$$

for all $0 \le s \le \tau \le t \le T$. Among other things, it is shown in [3] Theorem 9(vi) that the evolution operator has a representation of the form

$$U(t,s)u_0 = \int_\Omega \gamma(\cdot\,,t;\xi,\tau)u_0(\xi)\,d\xi \tag{4.4}$$

for all $u_0 \in L_2(\Omega)$, where $\gamma(x,t;\xi,\tau)$ is the weak Green's function corresponding to $\partial_t + \mathcal{A}(\cdot)$ and Ω. From this representation we obtain important estimates for the evolution operator in L_p-spaces for $p \ne 2$. In the sequel we denote by $\|\cdot\|_{p,q}$ the norm in $\mathcal{L}(L_p,L_q)$.

4.1 Lemma. *For all $1 \leq p \leq q \leq \infty$ we have that $U(t,s) \in \mathcal{L}(L_p, L_q)$. Moreover,*

$$\|U(t,s)\|_{p,q} \leq C(t-s)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}, \tag{4.5}$$

where C is a constant depending only on p, q, N, α_0 and upper bounds for the norms of the coefficients of $\mathcal{A}$. Moreover, for all $s \in [0,T)$ $U(\cdot\,,s) \in C\big([s,T],\mathcal{L}(L_p)\big) \cap C\big((s,T],\mathcal{L}(L_p,L_q)\big)$ for $1 < p < q < \infty$, and $U(\cdot\,,s) \in L_\infty\big((s,T),\mathcal{L}(L_\infty)\cap\mathcal{L}(L_1)\big)$.

Proof. It is shown in [3], Theorem 9(iii) that the weak Green's function has a Gaussian upper bound whose constants depend only on the quantities listed in the lemma (except p,q). This, (4.4) and Young's inequality for convolutions yield (4.5). In particular we get from (4.5) that $U(\cdot\,,s) \in L_\infty\big((s,T),\mathcal{L}(L_\infty)\cap\mathcal{L}(L_1)\big)$ holds. By interpolation it follows that $U(\cdot\,,s) \in C\big([s,T],\mathcal{L}(L_p)\big)$ for $p \in (1,\infty)$ since we already know this for $p=2$. The remaining assertion follows by writing $U(t,s) = U(t,\delta)U(\delta,s)$ for arbitrary $\delta \in (s,T)$ and $t \in [\delta,T]$, and using the previous results. □

We now prove that the weak solution of (2.25) can be represented by means of the variation of constants formula. The proof makes use of the results in Section 3 and is based on an idea in [3], Theorem 9(vi). In [3], the assertion is proved under conditions on f which always imply that the range of f is in L_p for p at least $N/2$. We shall prove the formula under much weaker assumptions on f.

4.2 Proposition. *Let $f \in L_2\big((0,T),L_p(\Omega)\big)$ for some $p > 2N(N+2)^{-1}$, and $u_0 \in L_2(\Omega)$. Then, the weak solution of (2.25) can be represented by the variation of constants formula*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)f(\tau)\,d\tau\,. \tag{4.6}$$

Proof. In view of (4.4) it is sufficient to prove the result in case $u_0 = 0$. As in the proof of Theorem 9(vi) in [3] we approximate the coefficients of $\mathcal{A}_n$ in $L_{2,\mathrm{loc}}$ by smooth coefficients. This can be achieved by convolution with appropriate smoothing kernels. Moreover, chose a sequence of smooth domains Ω_n such that $\Omega_n \subset \Omega_{n+1} \subset \Omega$ holds for all $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}}\Omega_n = \Omega$. Denote the Green's functions to $\partial_t + \mathcal{A}_n(\cdot)$ on Ω_n by γ_n. Then, the evolution operator $U_n(\cdot\,,\cdot)$ corresponding to the approximating problems has a representation similar to (4.4) with γ replaced by γ_n. We also approximate f in $L_2\big((0,T),L_p(\Omega)\big)$ by smooth functions f_n. From the classical theory it is well known that the solution u_n of the approximating smooth problems is given by

$$u_n(\cdot\,,t) = \int_0^t U_n(t,\tau)f_n(\tau)\,d\tau\,.$$

As by Theorem 3.5 and Remark 3.2(a) $u_n(t)$ is converging in $L_2(\Omega)$ to the solution u of the unperturbed problem it is sufficient to show that the right hand side converges in $L_2(\Omega)$ to the corresponding integral for any $t \in (0,T]$. We have to show that

$$\lim_{n\to\infty}\int_0^t \|U_n(t,\tau)f_n(\tau) - U(t,\tau)f(\tau)\|_2\,d\tau = 0 \tag{4.7}$$

for all $t \in (0,T]$. By our assumption on p we have that

$$\frac{N}{2}\Big(\frac{1}{p}-\frac{1}{2}\Big)+\frac{1}{2}<1 \tag{4.8}$$

Hence, it follows from Lemma 4.1, Hölder's inequality and the fact that f_n converges to f in $L_2\big((0,T),L_p(\Omega)\big)$ that the integrand in (4.7) has an integrable majorant independent of $n \in \mathbb{N}$. Therefore by Lebesgue's Theorem on dominated convergence it is sufficient to show that the integrand tends to zero for almost all $\tau \in (0,t)$. Fix $\tau \in (0,t)$ arbitrary. As f_n converges in $L_2\big((0,T),L_p(\Omega)\big)$ it has a subsequence which converges pointwise almost everywhere in $L_p(\Omega)$. Thus we can assume that $f_n(\tau)$ converges to $f(\tau)$ for almost all $\tau \in (0,t)$ in $L_p(\Omega)$. Now we can apply Theorem 3.5 to conclude that $\|U_n(t,\tau)f_n(\tau)-U(t,\tau)f(\tau)\|_2$ tends to zero for almost all $\tau \in (0,t)$. As it is enough to show (4.7) for a subsequence this accomplishes the proof of the proposition. $\square$

4.3 Remark. Suppose $\mathbf{L}$ is defined as in (2.20). Then, (2.29) is a representation formula for $\mathbf{L}^{-1}(f,u_0)$. Moreover, combining this with Lemma 4.1 we obtain higher integrability for the solutions. More precisely, if $r \geq 2$ and $p > 2N(N+2)^{-1}$ we obtain that

$$\mathbf{L}^{-1} \in \mathcal{L}\big(L_r((0,T),L_p)\times L_q, C([0,T],L_q)\big) \tag{4.9}$$

whenever $q \in [2,\infty)$ is such that

$$\frac{N}{2}\Big(\frac{1}{p}-\frac{1}{q}\Big)+\frac{1}{r}<1\,, \tag{4.10}$$

and the norm depends on the same quantities as C in (4.5). Similarly, if (4.10) holds with $q=\infty$ we have that

$$\mathbf{L}^{-1} \in \mathcal{L}\big(L_r((0,T),L_p)\times L_\infty, L_\infty(\mathbb{R}^N\times(0,T))\big)\,, \tag{4.11}$$

and the same assertion on the norm as above is true. Note that (4.9) and (4.11) follow from embedding theorems for Sobolev spaces provided the domain and coefficients are sufficiently smooth. $\square$

In the following we suppose that $\mathbf{L}_n^{-1},\mathbf{L}^{-1}$ are as in Remark 3.2(c). We shall also consider $U(t,s)$ as an operator in $\mathcal{L}\big(L_p(\mathbb{R}^N)\big)$ by restricting a function u_0 first to Ω, applying $U(t,s)$ and extending the result trivially to $\mathbb{R}^N$. Moreover, we assume that $\Omega_n \to \Omega$ and that (3.6) and (3.4) are satisfied. After these preparations we prove the main result of this section.

4.4 Theorem. *Suppose that $r \geq 2$, $p > 2N(N+2)^{-1}$ and $q \in [2,\infty)$ are such that* (4.10) *holds. Then, the following assertions hold.*

(a) *If $u_{0n} \rightharpoonup u_0$ weakly in $L_2(\Omega)$ and $f_n \rightharpoonup f$ in $L_r\big((0,T),L_p\big)$ weakly, then the solution $u_n := \mathbf{L}_n^{-1}(f_n,u_{0n})$ of* (3.1) *tends to the solution $u := \mathbf{L}^{-1}(f,u_0)$ of* (2.25) *in $C\big((0,T],L_q\big)$*

(b) *If in* (a) *$u_{0n} \to u_0$ in $L_q(\mathbb{R}^N)$ we have that u_n converges to u in $C\big([0,T],L_q\big)$.*

(c) $\mathbf{L}_n^{-1}$ *converges to* $\mathbf{L}^{-1}$ *in* $\mathcal{L}\big(L_r((0,T),L_p)\times L_2, C((0,T],L_q)\big)$

Proof. (a) Suppose that $q_1>q$ is such that (4.10) is satisfied with q replaced by q_1. Then, by (4.6) and (4.5) we get that for any $\delta>0$ and $t\in[\delta,T]$

$$\|\mathbf{L}_n^{-1}(f_n,u_{0n})(t)\|_{q_1}\leq c_0\big(\|u_{0n}\|_2+\|f_n\|_{L_r((0,T),L_p)}\big) \tag{4.12}$$

holds with a constant $c_0>0$ depending on $\delta>0,q_1,r,p$ but independent of $n\in\mathbb{N}$. A similar estimate holds if we replace $\mathbf{L}_n^{-1}$ by $\mathbf{L}^{-1}$. Thus, by interpolation we get that for all $t\in[\delta,T]$

$$\begin{aligned}&\|\mathbf{L}_n^{-1}(f_n,u_{0n})(t)-\mathbf{L}^{-1}(f,u_0)(t)\|_q\\&\quad\leq\|\mathbf{L}_n^{-1}(f_n,u_{0n})(t)-\mathbf{L}^{-1}(f,u_0)(t)\|_2^{1-\theta}\|\mathbf{L}_n^{-1}(f_n,u_{0n})(t)-\mathbf{L}^{-1}(f,u_0)(t)\|_{q_1}^{\theta}\end{aligned} \tag{4.13}$$

where $\theta\in(0,1)$ is given by $1/q=(1-\theta)/2+\theta/q_1$. By the fact that weakly convergent sequences are bounded it follows from (4.12) that the second factor on the right hand side of the above inequality is uniformly bounded with respect to $n\in\mathbb{N}$ and $t\in[\delta,T]$. It is also clear that $f_n\rightharpoonup f$ in $L_2\big((0,T),L_p\big)$. Hence, the assertion of (a) follows from (4.13) and Theorem 3.5.

(c) Similarly to the proof of (a) we get that for $\delta>0$ and $t\in[\delta,T]$

$$\begin{aligned}&\|\mathbf{L}_n^{-1}(f,u_0)(t)-\mathbf{L}^{-1}(f,u_0)(t)\|_q\\&\quad\leq\|\mathbf{L}_n^{-1}(f,u_0)(t)-\mathbf{L}^{-1}(f,u_0)(t)\|_2^{1-\theta}\|\mathbf{L}_n^{-1}(f,u_0)(t)-\mathbf{L}^{-1}(f,u_0)(t)\|_{q_1}^{\theta}\\&\quad\leq c_0\|\mathbf{L}_n^{-1}-\mathbf{L}^{-1}\|^{1-\theta}_{\mathcal{L}(L_2((0,T),L_p)\times L_2,C([\delta,T],L_2))}\big(\|u_0\|_2+\|f\|_{L_r((0,T),L_p)}\big)\end{aligned}$$

with a constant c_0 independent of $n\in\mathbb{N}$. Now, (c) is obvious from Corollary 3.6.

(b) It is obviously enough to show that $\mathbf{L}_n^{-1}(0,u_{0n})$ and $\mathbf{L}_n^{-1}(f_n,0)$ converge to $\mathbf{L}^{-1}(0,u_0)$ and $\mathbf{L}^{-1}(f,0)$, respectively. Convergence of the latter is an easy consequence of Remark 3.7(b) and an argument similar to that used in the proof of (c). Note that $\mathbf{L}_n^{-1}(0,u_{0n})=U_n(\,\cdot\,,0)u_{0n}$. Therefore, we only have to show that

$$\lim_{n\to\infty}U_n(\,\cdot\,,0)u_{0n}=U(\,\cdot\,,0)u_0 \tag{4.14}$$

holds in $C(\big([0,T],L_q\big)$ provided u_{0n} converges in L_q. Here, we are not able to apply interpolation directly since u_{0n} is not necessarily a bounded sequence in L_{q_1} for some $q_1>q$. For this reason we use a truncation argument and replace u_{0n} by

$$u_{0n,k}:=\begin{cases}k & \text{if } |u_{0n}|\geq k\\ u_{0n} & \text{if } |u_{0n}|\leq k\,,\end{cases}$$

where $k>0$ is arbitrary. Similarly we define $u_{0,k}$. Then we can write

$$\begin{aligned}\|U_n(t,0)u_{0n}-U(t,0)u_0\|_q&\leq\|U_n(t,0)\|_{q,q}\|u_{0n}-u_{0n,k}\|_q\\&\quad+\|U_n(t,0)u_{0n,k}-U(t,0)u_{0,k}\|_q+\|U(t,0)\|_{q,q}\|u_{0,k}-u_0\|_q\,.\end{aligned}$$

By Lemma 4.1 and the fact that a convergent sequence in L_q is equi-integrable the first and the last terms are small uniformly with respect to $n \in \mathbb{N}$ and $t \in [0,T]$. Therefore, given $\varepsilon > 0$ we find k such that the first and the last term are smaller than ε for all $n \in \mathbb{N}$ and $t \in [0,T]$. It is also easy to see that $u_{0n,k}$ converges to $u_{0,k}$ in L_q and hence in L_2 for all $k > 0$. By definition, the family $u_{0n,k}$ is uniformly bounded in L_∞ by k. Thus, by Lemma 4.1, Theorem 3.5 and interpolation the middle term converges to zero uniformly with respect to $t \in [0,T]$ as n tends to infinity. This concludes the proof of the theorem. $\square$

4.5 Remark. As an easy consequence of the above theorem we see that

$$\lim_{n\to\infty} U_n(\,\cdot\,,s) = U(\,\cdot\,,s)$$

in $C\big((0,T],\mathcal{L}(L_p,L_q)\big)$ for all $p,q \in (1,\infty)$. Indeed, observe that $U_n(\,\cdot\,,0) = \mathbf{L}_n^{-1}(0,\cdot)$. Further note that which initial time we start with is irrelevant. This proves our claim. $\square$

4.6 Remarks. So far we always excluded the case $q = \infty$. In fact, the above results do not hold in this case. Take for instance a domain Ω where the solutions of (2.25) are not continuous up to the boundary and approximate this domain by smooth domains from inside. If a result as the one above was true the solution of (2.25) would be continuous up to the boundary which contradicts our assumption. Concerning approximation in the L_∞-norm we include the following two results:

(a) Suppose the hypotheses of Theorem 4.4(a) are satisfied with p, r such that (4.10) holds with $q = \infty$ (note that this implies $p > N/2$). Then, u_n converges to u in $C\big(\Omega' \times [\delta,T]\big)$ for every $\Omega' \subset\subset \Omega\setminus K_0$ and $\delta \in (0,T)$, that is we have uniform convergence away from the parabolic boundary.

Proof. It follows from our assumptions on f_n, the variation of constants formula (4.6) and Lemma 4.1 that $\|u_n\|_{L_\infty(\mathbb{R}^N \times [\delta/2,T])}$ is uniformly bounded with respect to $n \in \mathbb{N}$. By the interior Hölder continuity of weak solutions of a parabolic equation (e.g. [3], Theorem C) u_n is Hölder continuous in $\Omega' \times [\delta,T]$ with a Hölder constant not depending on $n \in \mathbb{N}$. Moreover, the same result tells us that u_n is bounded in a Hölder norm on $\Omega' \times [\delta,T]$. Hence there is a convergent subsequence in $C\big(\Omega' \times [\delta,T]\big)$. As we know already that u_n is convergent in a weaker topology it follows that the whole sequence converges in that space.

(b) If we assume in addition that $\partial\Omega$ and $\partial\Omega_n$ satisfy a regularity condition uniformly with respect to $n \in \mathbb{N}$ then u_n converges in $C\big([\delta,T] \times \mathbb{R}^N\big)$, where as usual we extend u_n trivially outside Ω_n. The regularity condition is the following (e.g. [3], p. 616/17): There exists $0 < r_0$ and $\delta_0 \in (0,1)$ such that for all $x \in \partial\Omega$

$$|\mathbb{B}(x,\rho) \cap \Omega)| \leq (1-\delta_0)|\mathbb{B}(x,\rho)|$$

holds whenever $\rho \leq r_0$. If the constants r_0 and δ_0 are independent of $n \in \mathbb{N}$ solutions are Hölder continuous up to the boundary of Ω for positive time with a Hölder constant indepent of n (e.g. [3], Theorem D). Hence the arguments of the proof of Remark (a) apply and we get the assertion. $\square$

5. Periodic Problems

In this section we prove results analogous to those in Section 3 and 4 for the periodic-parabolic problem. Moreover, we show continuity properties of the spectrum of the periodic-parabolic eigenvalue problem. Before we start let us remark that in case of the periodic problem it is no longer possible to use the transformation (2.17) because this means shifting the periodic-parabolic operator by λ_0. However, in the applications to nonlinear problems in Section 7 below we are able to reduce the general case to the case $\lambda_0 = 0$ by modifying the nonlinearity slightly. For this reason we assume throughout that $\lambda_0 = 0$. We start by showing that the solution of the periodic problem

$$\begin{cases} \partial_t u + \mathcal{A}_n(t)u = f_n & \text{in } \Omega_n \times (0,T] \\ u = 0 & \text{on } \partial\Omega_n \times (0,T] \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega_n \end{cases} \tag{5.1}$$

converge to the solution of the limiting problem

$$\begin{cases} \partial_t u + \mathcal{A}(t)u = f & \text{in } \Omega \times (0,T] \\ u = 0 & \text{on } \partial\Omega \times (0,T] \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega \end{cases} \tag{5.2}$$

as n tends to infinity. On the operators $\mathcal{A}_n$ and $\mathcal{A}$ we impose the same hypotheses as in Section 3. As (3.1) we can write (5.1) in the abstract form

$$\begin{cases} \dot{u} + A_n(t)u = f_n(t) & \text{for } t \in (0,T] \\ u(0) = u(T)\,. \end{cases} \tag{5.3}$$

By Theorem 2.3 this equation has a unique solution for all $n \in \mathbb{N}$, which is the weak solution of (5.1). A similar result is true for (5.3). Let L and L_n be defined as in (2.24). In the same way as in Remark 3.2(c) we can see them as operators in

$$\mathcal{L}\big(L_2((0,T),L_{\hat{2}}),L_2((0,T),L_q)\big)$$

for all $q \in [1, 2N(N-2)^{-1})$. Assume now that $\Omega_n \to \Omega$, that Ω satisfies (3.6) and that (3.4) holds. Then, we have the following approximation theorem.

5.1 Theorem. *Suppose that $r \geq 2$, $p > 2N(N+2)^{-1}$ and $q \in [2,\infty)$ are such that (4.10) holds. Then, the following assertions hold.*

(a) *If $f_n \rightharpoonup f$ in $L_r\big((0,T),L_p\big)$ weakly, then the solution $u_n := L_n^{-1}(f_n, u_{0n})$ of (5.1) tends to the solution $u := L^{-1}(f)$ of (5.2) in $C\big([0,T],L_q\big)$ as n goes to infinity.*

(b) *L_n^{-1} converges to L^{-1} in $\mathcal{L}\big(L_r((0,T),L_p),C([0,T],L_q)\big)$ as n tends to infinity.*

Proof. It follows from Lemma 4.1 and e.g. [1], Lemma V.1.1.1 that the spectrum of $U_n(T,0)$ and $U(T,0)$ considered as an operator on L_p does not depend on $p \in [1,\infty]$.

Therefore, $\big(\mathrm{I} - U_n(T,0)\big)^{-1} \in \mathcal{L}(L_q)$ is well defined. Recall that the initial value of (5.3) is given by

$$u_n(0) = \big(\mathrm{I} - U_n(T,0)\big)^{-1}\mathbf{L}_n^{-1}(f,0)(T)\,. \tag{5.4}$$

As f_n converges weakly it follows from Theorem 4.4(a), the continuity of the resolvent with respect to the operator and (5.4) that $u_n(0)$ converges to $u(0)$ in L_q. Thus, assertion (a) follows from Theorem 4.4(b). Assertion (b) follows from (a) by an argument similar to that in the proof of Corollary 3.3 or 3.6. The details are left to the reader. $\square$

As an easy consequence we obtain the following useful compactness result.

5.2 Corollary. *Suppose the assumptions of Theorem 5.1 hold. Then, the family of maps* (L_n^{-1}, L^{-1}) *is collectively compact in* $\mathcal{L}\big(L_r((0,T),L_p), C([0,T],L_q)\big)$ *whenever* $r \geq 2$, $p \geq 2N(N+2)^{-1}$ *and* $s,q \in [2,\infty)$ *satisfy* (4.10), *that is, the union of the images of a bounded set in* $L_r\big((0,T),L_p\big)$ *under* L_n^{-1} *and* L^{-1} *is relatively compact in* $C\big([0,T],L_q\big)$. *In particular, any of the maps* L_n^{-1}, L^{-1} *is compact.*

Before we turn to eigenvalue problems we remark that the assertions of Remark 4.6 hold. In fact, due to periodicity we can replace the time intervals $[\delta,T]$ by $[0,T]$.

We consider now the periodic-parabolic eigenvalue problems

$$\begin{cases} \partial_t\varphi + \mathcal{A}_n(t)\varphi = \mu_n\varphi & \text{in } \Omega_n \times [0,T] \\ \varphi = 0 & \text{on } \partial\Omega_n \times [0,T] \\ \varphi(\cdot,0) = \varphi(\cdot,T) & \text{in } \Omega_n\,, \end{cases} \tag{5.5}$$

and show that the eigenvalues μ_n converge to the eigenvalues of the problem

$$\begin{cases} \partial_t\varphi + \mathcal{A}(t)\varphi = \mu\varphi & \text{in } \Omega \times [0,T] \\ \varphi = 0 & \text{on } \partial\Omega \times [0,T] \\ \varphi(\cdot,0) = \varphi(\cdot,T) & \text{in } \Omega\,. \end{cases} \tag{5.6}$$

as n tends to infinity. Note that μ is an eigenvalue of (5.5) iff μ^{-1} is an eigenvalue of L_n^{-1}. A similar statement holds for the eigenvalues of (5.6). Hence, it is sufficient to consider the behavior of the eigenvalues of L_n^{-1} as n tends to infinity. By Corollary 5.2 the operators L_n^{-1}, L^{-1} are compact and therefore have got a discrete spectrum. We shall prove the following result on the dependence of the eigenvalues on n.

5.3 Theorem. *Let* Σ *be a finite system of eigenvalues of* L^{-1} *and* U *an open neighborhood of* Σ *in* $\mathbb{C}$ *such that no other eigenvalue of* L^{-1} *is contained in* U. *Then, counting multiplicity,* L_n^{-1} *has exactly the same number of eigenvalues in* U *as* L^{-1}.

Proof. As already remarked the operators are compact and therefore have discrete spectrum. By combining Theorem 5.1(b) and the perturbation result in [13], Section IV.3.5 we obtain the continuity properties of any finite system of eigenvalues. $\square$

6. Nonlinear Problems

In this section we prove continuous domain dependence for solutions of the nonlinear initial boundary value problem

$$\begin{cases} \partial_t u + \mathcal{A}(t)u = f(x,t,u) & \text{in } \Omega \times (0,T] \\ u = 0 & \text{on } \partial\Omega \times (0,T] \\ u(\cdot,0) = u_0(\cdot) & \text{in } \Omega\,, \end{cases} \tag{6.1}$$

where $f \in C\big(\mathbb{R}^N \times [0,T] \times \mathbb{R}\big)$, $\mathcal{A}(\cdot)$ and Ω as in Section 2.D. We may write this problem as an abstract parabolic equation of the form

$$\begin{cases} \dot{u} + A(t)u = F(t,u) & \text{in } (0,T] \\ u(0) = u_0 \end{cases} \tag{6.2}$$

where $F(t,u)(x) := f(x,t,u(x))$ is the substitution operator induced by f. Recall that $F(t,\cdot) \in C(L_q, L_p)$ if and only if there exists a $c > 0$ and $g \in L_p$ such that

$$f(t,x,\xi) \leq g(x) + c|\xi|^{\frac{q}{p}} \tag{6.3}$$

for all $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}$ (cf. [14], Section I.2 or [2], Chapter 3). No growth condition is needed if $q = \infty$. To prove existence of solutions of (6.3) we also need that F is (locally) Lipschitz in u uniformly with respect to $t \in [0,T]$. A sufficient condition implying this is that there exists a function g such that

$$|f(x,t,\xi_1) - f(x,t,\xi_2)| \leq g(x,\zeta)|\xi_1 - \xi_2| \tag{6.4}$$

holds for all $|\xi_1|, |\xi_2| \leq \zeta$, and the substitution operator induced by g is bounded from L_q into $L_{\frac{pq}{q-p}}$. For necessary and sufficient conditions see [2], Theorem 4.4. If $q = \infty$ it is sufficient that f is locally Lipschitz in ξ uniformly with respect to $(x,t) \in \mathbb{R}^{N+1}$

We shall first prove existence of solutions of the integral equation

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)F(\tau,u(\tau))\,d\tau \tag{6.5}$$

under suitable assumptions on f. A solution of (6.5) is called a mild solution of (6.2). Note also that by Proposition 4.2 a mild solution of (6.2) is a weak solution of (6.1) provided $u_0 \in L_2$ and $F(\cdot\,,u(\cdot)) \in L_2\big((0,T),L_p\big)$ for some $p > 2N(N+2)^{-1}$. Existence of a solution of (6.5) is proved in a standard way similar to Picard-Lindelöf iteration in ordinary differential equations. We shall use here a weaker metric which allows us to prove continuous dependence of the solutions on the domains even if the initial values are not supposed to converge strongly.

6.1 Proposition. *Suppose that $F \in C\big([0,T] \times L_q, L_p\big)$ is uniformly Lipschitz continuous with respect to $t \in [0,T]$ on bounded sets of L_q, and that p, q satisfy $2N(N+2)^{-1} < p \leq q \leq \infty$ with*

$$\gamma := \frac{N}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big) < 1\,. \tag{6.6}$$

Then, for any $\rho > 0$ there exists $t_1 > 0$ such that for $u_0 \in L_q$ with $\|u_0\|_q < \rho$ (6.3) has a unique mild solution. If $q < \infty$ we have that

$$u \in C\big([0,t_1], L_q\big)\,. \tag{6.7}$$

If $p < q = \infty$ we have that

$$u \in L_\infty\big((0,t_1), L_\infty\big) \cap C\big([0,t_1], L_p\big)\,. \tag{6.8}$$

Proof. For any $u \in L_\infty\big((0,T), L_q\big)$ and $u_0 \in L_q$ we define

$$G(u)(t) := U(t,0)u_0 + \int_0^t U(t,\tau)F(\tau, u(\tau))\,d\tau\,. \tag{6.9}$$

It is clear that u is a mild solution of (6.2) iff u is a fixed point of G. It follows from Lemma 4.1 and our assumption on F that G maps $L_\infty\big((0,T), L_q\big)$ into itself. We shall show that G is a contraction in a suitable subset. Fix $u_0 \in L_q$ such that $\|u\|_q \leq \rho$ and choose $c_1 > 0$ such that

$$\|U(t,0)u_0 - u_0\|_q \leq c_1\|u_0\|_q \tag{6.10}$$

holds for all $t \in [0,T]$. Then, from (4.5) and the assumption on F we conclude that

$$\|\int_0^t U(t,\tau)F(\tau,u(\tau))\,d\tau\|_q \leq \frac{C}{1-\gamma}\,t^{1-\gamma}\|F(\cdot,u(\cdot))\|_{L_\infty((0,T),L_p)} = c_2 t^{1-\gamma} \tag{6.11}$$

for all $t \in (0,T]$. Choose $t_0 \in (0,T]$ such that $c_2 t_0^{1-\gamma} \leq 1$ and set $\varepsilon := c_1\rho + 1$. It is then easy to check that the set

$$M_{t_1} := L_\infty\big((0,t_1), \mathbb{B}(u_0,\varepsilon)\big) \tag{6.12}$$

is invariant under G for any $t_1 \in (0, t_0]$, where $\mathbb{B}(u_0,\varepsilon)$ is the ball with center u_0 and radius ε in L_q. In the classical proof it is shown that G is a contraction on M_{t_1} for t_1 small enough in the metric induced by the norm of $L_\infty\big((0,T), L_q\big)$. We choose now a weaker metric, namely that induced by the Fréchet space $L_{\infty,\mathrm{loc}}\big((0,t_1], L_q\big)$. The topology in this space is generated by any family of seminorms given by

$$q_i(w) := \sup_{t\in(\varepsilon_i t_1, t_1)} \|w(t)\|_q\,, \tag{6.13}$$

where $(\varepsilon_i)_{i\in\mathbb{N}}$ is a strictly decreasing sequence in $(0,1)$ with limit zero. It is well known that for any sequence $(\beta_i)_{i\in\mathbb{N}}$ in $(0,\infty)$ with $\sum_{i=1}^\infty \beta_i < \infty$

$$d(u,v) := \sum_{i=1}^\infty \beta_i \frac{q_i(u-v)}{1+q_i(u-v)} \tag{6.14}$$

is a metric in $L_{\infty,\mathrm{loc}}\big((0,t_1],L_q\big)$. By definition, for all $u,v\in M_{t_1}$ $\|u(t)-v(t)\|\leq \|u(t)-u_0\|+\|u_0-v(t)\|\leq 2\varepsilon$ and therefore

$$q_i(u-v)\leq 2\varepsilon \tag{6.14}$$

for all $u,v\in M_{t_1}$ and $i\in\mathbb{N}$. Set now $\varepsilon_i:=2^{-i}$ and $\beta_i:=2^{-i(1-\gamma)}$ for all $i\in\mathbb{N}$. Then, for all $t\geq 2^{-i}t_1$ and $u,v\in M_{t_1}$ we have that

$$\begin{aligned}\|G(u)(t)-G(v)(t)\|_q &\leq c_3t^{1-\gamma}q_i(u-v)+c\sum_{j=i+1}^\infty\int_{2^{-j}t_1}^{2^{-j+1}t_1}(t-\tau)^{-\gamma}\,d\tau\, q_j(u-v)\\ &\leq ct_1^{1-\gamma}\Big(q_i(u-v)+\sum_{j=i+1}^\infty 2^{-j(1-\gamma)}q_j(u-v)\Big)\end{aligned}$$

where c depends only on C in (4.5), and the Lipschitz constant of F (which depends on ε). To estimate the integrals we used the elementary inequality

$$(t-a)^{1-\gamma}-(t-b)^{1-\gamma}\leq b^{1-\gamma}-a^{1-\gamma}\leq b^{1-\gamma}$$

which holds whenever $0\leq a\leq b\leq t$. Further note that the series converges because of (6.14). Therefore, we have that

$$q_i\big(G(u)-G(v)\big)\leq ct_1^{1-\gamma}\Big(q_i(u-v)+\sum_{j=i+1}^\infty\beta_jq_j(u-v)\Big)$$

for all $i\in\mathbb{N}$. This, together with (6.14), shows that

$$\frac{q_i\big(G(u)-G(v)\big)}{1+q_i\big(G(u)-G(v)\big)}\leq c(2\varepsilon+1)\Big(\frac{q_i(u-v)}{1+q_i(u-v)}+\sum_{j=i+1}^\infty\beta_i\frac{q_j(u-v)}{1+q_j(u-v)}\Big)$$

for all $i\in\mathbb{N}$. Therefore, by definition of the metric we find that

$$\begin{aligned}d\big(G(u),G(v)\big)&\leq c(2\varepsilon+1)t_1^{1-\gamma}\Big(d(u,v)+\sum_{i=1}^\infty\beta_i\sum_{j=i+1}^\infty\beta_j\frac{q_j(u-v)}{1+q_j(u-v)}\Big)\\ &\leq c(2\varepsilon+1)t_1^{1-\gamma}\Big(d(u,v)+d(u,v)\sum_{i=1}^\infty\beta_i\Big)\\ &= c(2\varepsilon+1)\Big(1+\frac{1}{2^{\gamma-1}-1}\Big)t_1^{1-\gamma}d(u,v)=c_3t_1^{1-\gamma}d(u,v)\end{aligned} \tag{6.16}$$

for all $u, v \in M_{t_1}$. As c_3 only depends on C from (4.5), the Lipschitz constant of F, γ and ε we can choose $t_1 > 0$ such that G is a contraction on M_{t_1} in the metric (6.14). Therefore, by the contraction mapping principle G has a unique fixed point in M_{t_1}, which means that (6.2) has a unique mild solution u with initial value u_0 in $L_\infty\big((0,t_1), L_q\big)$. From the continuity properties of the evolution operator it follows easily that (6.7) and (6.8) hold. $\square$

6.2 Remarks. (a) In the standard proof of the above Lemma it is shown that G is a contraction in M_{t_1} using the metric induced by the norm of $L_\infty\big((0,t_1), L_q\big)$. This is clear from the above proof. Indeed, in that case we easily see that

$$\|G(u) - G(v)\|_{L_\infty((0,t_1),L_q)} \leq c t_1^{1-\gamma} \|u - v\|_{L_\infty((0,t_1),L_q)}$$

for a constant depending only on ε, γ, the Lipschitz constant of F and the constant C in (4.5). Hence, G is a contraction in that metric if t_1 is chosen suitably. But as emphasized already we will need the more complicated proof in order to show domain perturbation results for the nonlinear equation.

(b) In a standard way (cf. [7],[12]) it is easy to prove from the above proposition that (6.2) has a unique maximal solution. $\square$

We turn now to the problem of approximation of a solution of problem (6.1) by the solution of

$$\begin{cases} \partial_t u + \mathcal{A}_n(t) u = f(x,t,u) & \text{in } \Omega_n \times (0,T] \\ u = 0 & \text{on } \partial\Omega_n \times (0,T] \\ u(\cdot,0) = u_{0n}(\cdot) & \text{in } \Omega_n\,, \end{cases} \tag{6.17}$$

6.3 Theorem. *Suppose $\Omega_n \to \Omega$ with Ω satisfying (3.6), that $\mathcal{A}_n$ is as in Section 3, and that (3.5) holds. Further assume that f satisfies the conditions of Proposition 6.1.*

Then, for any $\rho > 0$ there exists $t_1 \in (0,T]$ such that for all $u_0, u_{0n} \in L_q$ with $\|u_0\|_q, \|u_{0n}\|_q \leq \rho$ there exist unique solutions u and u_n of (6.1) and (6.17) satisfying (6.7) and (6.8), respectively for all $n \in \mathbb{N}$. If $u_{0n} \rightharpoonup u_0$ weakly in L_2 then,

$$\lim_{n\to\infty} u_n = u \tag{6.18}$$

exists in $C\big((0,t_1], L_r\big)$, where $r \leq q$ if $q < \infty$, and $r < \infty$ if $q = \infty$. Moreover, if u_{0n} converges strongly in L_r, then, (3.19) holds in $C\big([0,t_1], L_r\big)$, where r is as before.

Proof. Define G_n similar as G by replacing $U(\cdot\,,\cdot)$ by $U_n(\cdot\,,\cdot)$ and u_0 by u_{0n} in (6.9). As the constants c_1, c_2, c_3 in (6.10), (6.11) and (6.16), respectively can be chosen independently of $n \in \mathbb{N}$ we find that G_n is a contraction on M_{t_1} for a $t_1 \in (0,T]$ independent of $n \in \mathbb{N}$. Moreover, the Lipschitz constant of G_n does not depend on $n \in \mathbb{N}$, i.e. the G_n are uniform contractions in the metric (6.14). Hence, the first assertion follows from Proposition 6.1.

To prove the approximation result we start with the case $q < \infty$. It is an easy consequence of Theorem 4.4(a) that $G_n(u)$ tends to $G(u)$ in M_{t_1} for all $u \in M_{t_1}$. As G_n

are uniform contractions it follows that their fixed points u_n converge to a fixed point of G as n tends to infinity (cf. [12], Section 1.3.6). This proves (3.19) in case $q < \infty$. Assume now that $q = \infty$. Then, as the solutions of (6.1) and (6.17) are in M_{t_1}, they are bounded in $L_\infty\big(\mathbb{R}^N \times (0,t_1)\big)$, uniformly with respect to $n \in \mathbb{N}$. Hence, we may replace f by a truncation of f without changing the solutions. Then, the hypotheses of our theorem are fulfilled for any $q < \infty$ and we can apply what we proved before.

Finally, it remains to prove the assertion on uniform convergence on $[0,t_1]$. In this case we proceed exactly as above but replacing the metric (6.14) by the norm in $L_\infty\big((0,t_1),L_q\big)$ which is possible by Remark 6.2, and use Theorem 4.4(b) to see that $G_n(u)$ converges to $G(u)$ as n tends to infinity. This completes the proof of the theorem. □

It follows from Remark 6.2 that (6.17) has a unique maximal solution for all initial values in the appropriate space for all $n \in \mathbb{N}$. Denote by $t^+(u_0)$ the maximal existence time of the solution of (6.2) with initial value u_0. Similarly, we define $t_n^+(u_{0n})$ the maximal existence time of (6.17). Next we prove lower semicontinuity of the existence time with respect to the domain and the initial values.

6.4 Theorem. *Suppose the same assumptions as in Theorem 6.3 are satisfied. Fix $u_0 \in L_q$ arbitrary. Then, for all $t_0 < t^+(u_0)$ there exists a neighborhood W of u_0 and $n_0 \in \mathbb{N}$ such that $t_0 < t_n^+(v)$ for all $v \in W$.*

Proof. Denote the solution of (6.2) and (6.17) for the initial value v by $u(\cdot\,;v)$ and $u_n(\cdot\,;v)$, respectively. Fix $u_0 \in L_q$ arbitrary and let $t_0 < t^+(u_0)$ be given. Since the solution is bounded in L_q on any compact subinterval of $[0,t^+(u_0))$ we find for any $R > 0$ a constant $\rho > 0$ such that $\|v\|_q \le \rho$ for all $v \in B_R := \big\{w \in L_q \colon \|u(t;u_0) - w\|_q \le R \text{ for all } t \in [0,t_0]\big\}$. We are done if we show that the solution of (6.17) stays in B_R for all $t \in [0,t_0]$ for initial values close to v and n large. By Theorem 6.1 there exists $t_1 > 0$ such that $t_n^+(v), t^+(v) > t_1$ for all $n \in \mathbb{N}$.

We first assume that $q < \infty$ and suppose that there exist sequences $n_k \in \mathbb{N}$ tending to infinity and u_{0k} converging to u_0 in L_q such that for all $k \in \mathbb{N}$ $u_{n_k}(t;u_{0k})$ does not stay in B_R for all $t \in [0,t_1]$. Therefore, there exist $t_k \in [0,t_1]$ such that

$$\|u_{n_k}(t_k;u_{0k}) - u(t_k;u_0)\|_q = R \tag{6.19}$$

for all $k \in \mathbb{N}$. Selecting a suitable subsequence we may assume without loss of generality that t_k converges to $\bar{t} \in [0,t_1]$. But by the second part of the above theorem this is not possible. Hence, we find $n_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that $u_n(t;v)$ remains in B_R for all $v \in \mathbb{B}_{\varepsilon_1}(u_0) := \{w \in L_q \colon \|w - u_0\|_q \le \varepsilon_1\}$ and $n \ge n_1$. The first part of the above theorem then implies that $t_n^+(v) > 2t_1$ for all $v \in \mathbb{B}_{\varepsilon_1}(u_0)$. Repeating the process above we see that there exists $n_2 \in \mathbb{N}$ and $\varepsilon_2 > 0$ such that $u_n(t;v) \in B_R$ for all $n \ge n_2$ and $v \in \mathbb{B}_{\varepsilon_2}(u_0)$. Since in finitely many steps we reach t_0 this shows the assertion of the theorem in case $q < \infty$.

If $q = \infty$ we have to modify the above proof because then u_n does not necessarily converge to u in L_∞, and therefore we do not get a contradiction from (6.19). On the other hand, it is sufficient to show that for all $r < \infty$ large enough

$$\|u_n(t;v) - u(t;u_0)\|_r < R$$

holds for all v close to u_0 in L_∞ and n large enough. Assuming the contrary we obtain a contradiction by similar arguments as in the case $q < \infty$ using the second part of Theorem 6.3. This concludes the proof of the theorem. $\square$

Combining Theorems 6.3 and 6.4 we easily deduce the following corollary.

6.5 Corollary. *Suppose that the assumptions of Theorem 6.3 hold. If u_{0n} converges to u_0 in L_q then (6.18) holds in $C([0,t_0],L_r)$, where $t_0 < t^+(u_0)$ is arbitrary and r is as in Theorem 6.3.*

7. Nonlinear Periodic Problems

In this section we obtain results about the existence and uniqueness of T-periodic solutions of (6.17) which are close to a known (nondegenerate) T-periodic solution of (6.1). We always suppose that $\Omega_n \to \Omega$ in the sense of (3.5) with Ω satisfying (3.6), and that (3.4) holds. Before we state the main result of this section we introduce the assumptions on the nonlinearity. Let us suppose throughout that

$$\begin{cases} f \in C^1(\mathbb{R}^N \times [0,T] \times \mathbb{R}) \text{ and there exist } c, \beta > 0 \text{ such that} \\ \qquad\qquad |\partial_3 f(x,t,\xi)| \le c\big(1 + |\xi|^{\beta-1}\big) \\ \text{for all } (x,t,\xi) \in \mathbb{R}^N \times [0,T] \times \mathbb{R}. \end{cases} \tag{7.1}$$

We say a periodic solution of (6.1) is nondegenerate if

$$\begin{cases} \partial_t v + \mathcal{A}(t)v = \partial_3 f(x,t,u)v & \text{in } \Omega \times [0,T] \\ v = 0 & \text{on } \partial\Omega \times (0,T] \\ v(\cdot,0) = v(\cdot,T) & \text{in } \Omega\,, \end{cases} \tag{7.2}$$

has only the trivial solution. To be able to use the results of Section 4 we replace $\mathcal{A}_n(t)$ by $\mathcal{A}_n(t) + \lambda_0$ and $f(x,t,\xi)$ by $\tilde f(x,t,\xi) := f(x,t,\xi) + \lambda_0\xi$. Then, it is clear that $\tilde f$ satisfies (7.1) with the same constants as f. So without loss of generality we may assume that $\lambda_0 = 0$. The main result is the following.

7.1 Theorem. *Suppose f satisfies (7.1) and that u is a T-periodic nondegenerate solution of (6.1) in $L_\infty(\mathbb{R}^N \times (0,T))$. Then the following assertions hold.*

(a) *For n large, there exists a T-periodic solution of (6.17) which is close to u in $C([0,T],L_q)$ for all $q \in [1,\infty)$. Further, this solution is unique in a suitable neighborhood of u in $C([0,T],L_q)$ if $\max\big\{\frac{2N\beta}{N+2}, \frac{N}{2}(\beta-1)\big\} < q < \infty$ if n is large.*

(b) *Suppose that Σ is a finite system of eigenvalues of the linearization of (6.1) in u, and U is an open neighborhood of Σ in $\mathbb{C}$. Then, counting multiplicity, the linearization of (6.17) in u_n has exactly the same number of eigenvalues in U provided n is large enough.*

(c) *Given $K > 0$, every T-periodic solution of* (6.17) *with* $\|u_n\|_\infty \leq K$ *must be close to a T-periodic solution of* (6.1) *in* $C\big([0,T], L_q\big)$ *for all* $q \in [1, \infty)$.

Proof. We first prove uniqueness in (a). To do so let us assume the contrary. Then for all $n \in \mathbb{N}$ there exist T-periodic solutions $u_n \neq v_n$ of (6.17) converging to u in $L_\infty\big((0,T), L_q\big)$. For all $n \in \mathbb{N}$ set $w_n := (u_n - v_n)\|u_n - v_n\|^{-1}_{L_2(\mathbb{R}^N \times (0,T))}$. Obviously, w_n satisfies the (linear) equation

$$\left\{\begin{aligned} \partial_t w_n + \mathcal{A}_n(t) w_n &= d_n(x,t) w_n && \text{in } \Omega_n \times (0,T] \\ w_n &= 0 && \text{on } \partial\Omega_n \times (0,T] \\ w_n(\cdot\,,0) &= w_n(\cdot\,,T) && \text{in } \Omega_n\,, \end{aligned}\right. \tag{7.3}$$

where

$$d_n(x,t) := \int_0^1 \partial_3 f(x,t,u_n + \tau(u_n - v_n))\, d\tau\,.$$

By assumption u_n, v_n converge to u in $L_\infty\big((0,T), L_q\big)$ and therefore, by our conditions on f it is not hard to show that

$$\lim_{n\to\infty} d_n(x,t) = \partial_3 f(x,t,u(x,t)) \tag{7.4}$$

in $L_2\big((0,T), L_1\big)$. Moreover, $d_n(x,t)$ is a bounded sequence in $L_\infty\big((0,T), L_{\frac{\nu}{2}}\big)$ for some $\nu > N$. Thus, from the remarks in Section 2.D and Theorem 2.2 it follows that $\|w_n\|_{W_n}$ is a bounded sequence. In view of Lemma 2.1 we can choose a subsequence (which we denote again by w_n) converging to some w in $L_2\big(\mathbb{R}^N \times (0,T)\big)$. It follows from (7.3), (7.4) and Theorem 5.1 that w is a solution of (7.2). By our assumption on nondegeneracy w must be zero. But as $\|w_n\|_{L_2} = 1$ for all $n \in \mathbb{N}$, and w_n is convergent in L_2 this is a contradiction.

We now prove existence of a solution for the perturbed problem. We define operators Q and Q_n by setting

$$Q(u) := L^{-1}F(\cdot\,,u) \qquad \text{and} \qquad Q_n(u) := L_n^{-1}F(\cdot\,,u)\,,$$

respectively, where L^{-1}, L_n^{-1} are as in Section 5, and F is the substitution operator to f. It is clear that u is a T-periodic solution of (6.1) iff u is a fixed point of u. A similar statement is true for Q_n. It follows from our assumptions on f that

$$F \in C^1\big(L_q(\mathbb{R}^N \times (0,T)), L_p(\mathbb{R}^N \times (0,T)\big)$$

is bounded whenever p, q are large enough (see e.g. [2], Theorem 3.12). Moreover, we can make sure that (4.10) holds with r replaced by p. Therefore, it follows from Corollary 5.2 that $Q, Q_n \in C^1\big(L_q(\mathbb{R}^N \times (0,T)\big)$ are completely continuous. Thus, the Leray-Schauder Fixed Point Index ([9]) is defined. As u is a nondegenerate fixed point of Q it follows that $\mathrm{I} - Q'(u)$ has trivial kernel and therefore the fixed point index of u is ± 1. Let $\mathbb{B}_\varepsilon(u)$ be a ball in $L_q\big(\mathbb{R}^N \times (0,T)\big)$ with center u and radius $\varepsilon > 0$ such that Q has no further fixed point in $\mathbb{B}_\varepsilon(u)$. We prove now that if n is large $v \neq \theta Q(v)+(1-\theta)Q_n(v)$ whenever $v \in \partial\mathbb{B}_\varepsilon(u)$ and $\theta \in [0,1]$. If not there exist $\theta_n \in [0,1]$ and $u_n \in \partial\mathbb{B}_\varepsilon(u)$ such that

$$u_n = \theta_n Q(u_n) + (1-\theta_n)Q_n(u_n) \tag{7.5}$$

It follows from Theorem 2.3 and 2.2 that $\|u_n\|_{W_n}$ is a bounded sequence. Therefore, by Lemma 2.1, u_n has a convergent subsequence in $L_2(\mathbb{R}^N \times (0,T))$, which we denote again by u_n. Its limit we denote by v. By our assumptions on f we have that $g_n(t) := F(t, u_n(t))$ is a bounded sequence in $L_p((0,T), L_p)$. Hence, by Theorem 5.1 $Q_n(u_n) = L_n^{-1}g_n$ and $Q(u_n) = L^{-1}g_n$ are bounded sequences in $L_{q_1}(\mathbb{R}^N \times (0,T))$ for some $q_1 > q$. Thus, by interpolation, u_n converges to v in $L_q(\mathbb{R}^N \times (0,T))$. By continuity of F this implies that g_n converges to $g := F(\cdot, v(\cdot))$ in $L_p(\mathbb{R}^N \times (0,T))$. Now, Theorem 5.1 implies that

$$\lim_{n\to\infty} Q_n(u_n) = \lim_{n\to\infty} Q(u_n) = Q(v)$$

in $L_q(\mathbb{R}^N \times (0,T))$. Selecting a further subsequence we can also assume that θ_n converges to some θ in $[0,1]$. But then, (7.5) implies that $v = Q(v)$ which is a contradiction to the choice of ε. By the homotopy invariance of the Leray-Schauder degree (e.g. [9]) this implies that $\deg(\mathrm{I}-Q_n, 0, \mathbb{B}_\varepsilon(u)) = \deg(\mathrm{I}-Q, 0, \mathbb{B}_\varepsilon(u))$ for large n. As already mentioned $\deg(\mathrm{I}-Q, 0, \mathbb{B}_\varepsilon(u)) = \pm 1$. Hence, Q_n has a fixed point u_n in $\mathbb{B}_\varepsilon(u)$. This shows the existence of periodic solutions to (6.17) for all n large enough. By construction it is clear that u_n is close to u in $L_q(\mathbb{R}^N \times (0,T))$. So by part one of the proof it is unique. Set g_n, g similar as above. But as g_n is close to g in $L_p(\mathbb{R}^N \times (0,T))$ it follows from Theorem 5.1 that u_n is close to u in $C((0,T), L_q)$. As q large was chosen arbitrary, the assertion follows.

Assertion (b) follows from Theorem 5.3 by replacing $\mathcal{A}_n(t)$ by $\mathcal{A}_n(t) - \partial_3 f(x,t,u_n)$ and taking into account the continuity properties of $\partial_3 f$.

We now prove (c). Suppose that u_n is a sequence of T-periodic solutions of (6.17) with $\|u_n\|_\infty \leq K$. It is clear that F is bounded from $L_\infty(\mathbb{R}^N \times (0,T))$ to $L_p(\mathbb{R}^N \times (0,T))$ for all $p \in [1,\infty]$. Therefore, it follows from Theorem 2.2 that there exists a constant c such that$\|u_n\|_{W_n} \leq c$ for all $n \in \mathbb{N}$. By Lemma 2.1 we can choose a subsequence u_{n_k} converging to some v in $L_2(\mathbb{R}^N \times (0,T))$. By interpolation convergence takes place in $L_q(\mathbb{R}^N \times (0,T))$ for all $q \in [2,\infty)$. Choosing q large enough it follows from Theorem 5.2 and the continuity properties of F that $\lim_{k\to\infty} Q_{n_k}(u_{n_k}) = Q(v) = v$ is a T-periodic solution of (6.1). As we can do this for every convergent subsequence this proves our claim. $\square$

The idea of the proof of existence and uniqueness is essentially the same as that in [4] in case of the elliptic problem but we make consequent use of the approximation theorems for linear equations. The idea of the proof of the continuity of the spectrum of the linearizations is new and provides a much shorter proof also in case of elliptic equations.

7.2 Remark. (a) Continuity of the spectrum of the linearization together with the principle of linearized stability implies that stability or instability of the T-periodic solution u_n is preserved for $n \in \mathbb{N}$ large.

(b) One could go further and prove results on the existence of positive solutions in a neighborhood of a positive solution, or to prove results on bifurcation as is was done in [4] and [5] for the elliptic problem. □

In the next proposition we shall give conditions under which a periodic solution lies in $L_\infty\big(\mathbb{R}^N \times (0,T)\big)$. This result also enables us to establish existence results similar to those in the above theorem under no growth conditions on the nonlinearity.

7.3 Proposition. *Suppose that f satisfies (7.1) and u is a T-periodic solution of (6.1) in $L_s\big((0,T),L_q\big)$ with*

$$s > 2\beta, \qquad q > \frac{2N\beta}{N+2} \qquad \text{and} \qquad \Big(\frac{N}{2q}+\frac{1}{s}\Big)(\beta-1) < 1\,. \tag{7.6}$$

Then, $u \in L_\infty\big(\mathbb{R}^N \times (0,T)\big)$. Moreover, the bound depends only on N,q,s,T, the ellipticity constant of $\mathcal{A}(t)$, the norms of the coefficients of $\mathcal{A}$, the constants c and β in (7.1) as well as an upper bound of $|\Omega|$ and the norm of u in $L_s\big((0,T),L_q\big)$

Proof. Suppose that F is the substitution operator induced by f. Then, from (7.1) we easily see that for all u in $L_q(\Omega)$ and $t \in [0,T]$

$$\|F(t,u)\|_{\frac{q}{\beta}} \le c\big(|\Omega| + \|u\|_q^\beta\big)\,.$$

Therefore, using Lemma 4.1 we see that for $q_1 > q$ satisfying

$$\frac{N}{2}\Big(\frac{\beta}{q}-\frac{1}{q_1}\Big)+\frac{\beta}{s} < 1 \tag{7.7}$$

we have that

$$\Big\|\int_0^t U(t,\tau)F(\tau,u(\tau))\,d\tau\Big\|_{q_1} \le c_0 t^{-\frac{N}{2}(\frac{\beta}{q}-\frac{1}{q_1})\frac{s}{s-\beta}+1} \tag{7.8}$$

for all $u \in L_s\big((0,T),L_q\big)$ with c_0 depending only on f,s,q,β, the constant C in (4.5) and an upper bound for $|\Omega|$ and $\|u\|_{L_s((0,T),L_q)}$. Further, as we know that the spectrum of $U(T,0)$ as an operator on L_p is independent of $p \in [0,\infty]$ (see proof of Theorem 5.1) $\big(\mathrm{I}-U(T,0)\big)^{-1} \in \mathcal{L}(L_p)$ is well defined for all $p \in [1,\infty]$. By (4.5) and (2.23) we conclude that

$$\begin{aligned}\big\|\big(\mathrm{I}-U(T,0)\big)^{-1}\big\|_{p,p} &= \big\|\,\mathrm{I}+U(T,0)\big(\mathrm{I}-U(T,0)\big)^{-1}\big\|_{p,p}\\ &\le 1 + Ce^{-\alpha T}T^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{p})}\end{aligned} \tag{7.9}$$

for all $p \in [2,\infty]$. Thus, from (7.8) and (7.9) we get that for any $u \in L_s\big((0,T),L_q\big)$

$$\Big\|\big(\mathrm{I}-U(T,0)\big)^{-1}\int_0^T U(t,\tau)F(\tau,u(\tau))\,d\tau\Big\|_{q_1} \le c_1 T^{-\frac{N}{2}(\frac{\beta}{q}-\frac{1}{q_1})\frac{s}{s-\beta}+1} \tag{7.10}$$

holds whenever $q_1 > q$ is such that (7.7) is satisfied, and c_1 depends on the same quantities as c_0. Recall that if $u \in L_s\big((0,T),L_q\big)$ is a periodic solution of (6.1) then

$$u(0) = \big(\mathrm{I} - U(T,0)\big)^{-1} \int_0^T u(t,\tau)F(\tau,u(\tau))\,d\tau\,.$$

Therefore, using (6.5), (7.8), (7.10) and Lemma 4.1 we see that there exists a constant $c_3 > 0$ depending on the same quantities as c_0 as well as $s_1 > s$, $q_1 > q$ and T such that

$$\|u\|_{L_{s_1}((0,T),L_{q_1})} \leq c_3$$

whenever

$$\frac{N}{2}\Big(\frac{\beta}{q} - \frac{1}{q_1}\Big) + \frac{\beta}{s} - \frac{1}{s_1} < 1 \tag{7.11}$$

holds. The assertion of the proposition follows if we can choose $s_1 = q_1 = \infty$. If this is not the case take $\eta > 1$ such that (7.11) is satisfied with $s_1 = \eta s$ and $q_1 = \eta q$. This choice of η is possible because of (7.6). This proves that if $u \in L_s\big((0,T),L_q\big)$ is a periodic solution of (6.1) then $u \in L_{\eta s}\big((0,T),L_{\eta q}\big)$. We repeat now the whole procedure with s,q replaced by $\eta s, \eta q$, respectively. By iteration we see that $u \in L_{\eta^k s}\big((0,T),L_{\eta^k q}\big)$ for all $k \in \mathbb{N}$. For some finite k we have that (7.11) holds for $s_1 = q_1 = \infty$ and s,q replaced by $\eta^k s$ and $\eta^k q$, respectively. This completes the proof of the proposition. □

7.4 Remarks. (a) It is a consequence of the above proposition that the T-periodic solutions u_n in Theorem 7.1 form a bounded sequence in $L_\infty\big(\mathbb{R}^N \times (0,T)\big)$.

(b) In Theorem 7.1(c) we can replace the condition that $\|u_n\|_\infty \leq K$ by the condition that $\|u_n\|_{L_s((0,T),L_q)} \leq K$ with s,q satisfying (7.6). Indeed, in this case the above proposition implies that u_n is a bounded sequence in L_∞.

(c) By (2.4) every weak solution of (6.1) lies in $L_\infty\big((0,T),L_2\big)$. Hence, it follows from the above proposition that any T-periodic solution of (6.1) lies in $L_\infty\big(\mathbb{R}^N \times (0,T)\big)$ whenever $\beta < 1 + 4/N$. However, it does not seem to be known whether this is true for a less restrictive growth condition on f. □

We prove now a result similar to that of Theorem 7.1 but without any growth condition on f.

7.5 Theorem. *Suppose that $f \in C^1\big(\mathbb{R}^N \times [0,T] \times \mathbb{R}\big)$ and that $u \in L_\infty\big(\mathbb{R}^N \times (0,T)\big)$ is a nondegenerate T-periodic solution of (6.1). Then, apart from the uniqueness all the assertions of Theorem 7.1 hold.*

Proof. To prove the assertion we use a truncation argument. Set

$$f_M(x,t,\xi) := \begin{cases} f(x,t,\xi) & \text{if } |\xi| \leq M \\ M & \text{if } |\xi| \geq M, \end{cases}$$

where $M > \|u\|_\infty$ is determined later. Then, u obviously is a nondegenerate T-periodic solution of (6.1) with f replaced by f_M. According to Theorem 7.1 and Remark 7.4(a) there exists a solution $u_n \in L_\infty\big(\mathbb{R}^N \times (0,T)\big)$ of (6.17) with f replaced by f_M if n is large enough. Our claim follows if we can show that $\|u_n\|_\infty \le M$. As a consequence of Proposition 7.3 there exists an increasing function $c\colon \mathbb{R} \to \mathbb{R}$ independent of $n \in \mathbb{N}$ and $M > \|u\|_\infty$ such that

$$\|u_n\|_\infty \le c\big(\|u_n\|_{C([0,T],L_q)}\big) \tag{7.12}$$

if q is large enough. Let us choose $M := c(\|u\|_{C([0,T],L_q)} + 1)$. Theorem 7.1 tells us that u_n is close to u in $C\big([0,T],L_q\big)$ which implies that

$$\|u_n\|_{C([0,T],L_q)} \le \|u\|_{C([0,T],L_q)} + 1$$

holds for all n large enough. Thus it follows from (7.12) that $\|u_n\|_\infty \le M$ for n large enough. This accomplishes the proof of the theorem. $\square$

7.6 Remarks. (a) According to Remark 7.4(b) the T-periodic solutions u_n whose existence was shown in Theorem 7.1 and 7.5 form a bounded sequence in L_∞. Therefore, a similar argument as in Remark 4.6(a) together with periodicity shows that the u_n converge uniformly on sets of the form $\Omega' \times [0,T]$ with $\Omega' \subset\subset \Omega \setminus K_0$. Moreover, if Ω, Ω_n satisfy the regularity condition of Remark 4.6(b) uniformly with respect to $n \in \mathbb{N}$ the convergence of u_n is uniform on $\mathbb{R}^N \times [0,T]$.

(b) The u_n of Theorem 7.5 are unique among those T-periodic solutions close to u which are uniformly bounded with respect to $n \in \mathbb{N}$ and uniformly (or merely pointwise) convergent on sets of the form $\Omega' \times [0,T]$ with Ω' as above. Indeed, using the uniform boundedness and a truncation argument as in the proof of Theorem 7.6 we can apply the uniqueness result from Theorem 7.1(a). $\square$

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