HEAT KERNEL ESTIMATES FOR OPERATORS WITH BOUNDARY CONDITIONS

DANIEL DANERS

School of Mathematics and Statistics
University of Sydney, NSW 2006
Australia

April 16, 1997

ABSTRACT. We prove Gaussian upper bounds for kernels associated with non-symmetric, non-autonomous second order parabolic operators of divergence form subject to various boundary conditions. The growth of the kernel in time t is determined by the boundary conditions and geometric properties of the domain. The theory gives a unified treatment for Dirichlet, Neumann and Robin boundary conditions, and the existence of a Gaussian type bound is essentially reduced to verifying some properties of the Hilbert space in the weak formulation of the problem.

1. INTRODUCTION

In this paper we study kernels associated with non-autonomous second order parabolic operators (with lower order terms) in domains $\Omega \subset \mathbb{R}^N$ subject to boundary conditions of Dirichlet, Neumann or Robin type. We prove upper Gaussian bounds for operators with non-smooth, i.e. bounded and measurable (also in time) coefficient. The first to prove such estimates for the Dirichlet problem was Aronson, dealing with self adjoint operators in [6], and with the general case general in [7]. Some of the more recent results include those in [5, 9, 10, 18, 29, 31] for autonomous problems, and [19] for non-autonomous problems, to name just a few.

So far, no comprehensive treatment of the non-autonomous, non-selfadjoint case with boundary conditions seems to be available. The intention is to provide such a treatment in the present paper. For self adjoint operators with Dirichlet or Neumann boundary conditions bounds are proved in [16]. In [5] the case of non-symmetric autonomous operators is treated, some of which are allowed to have complex lower order coefficients. However, it is assumed that the drift terms (first order coefficients) have first order bounded derivatives. In the light of the parabolic L_p-regularity theory given e.g. in [8] this seems to be rather unnatural. We indeed remove these regularity conditions for all boundary conditions. We are also able to weaken the conditions on the domain in case of Neumann and Robin boundary conditions.

1991 *Mathematics Subject Classification.* 35K20.

Gaussian estimates are important for a number of reasons. They provide information about the spectrum of elliptic operators (e.g. [3, 16]), or ensure the existence of functional calculi for certain operators (e.g. [17]). The latter are rather important in the theory of evolution equations (e.g. [2, Chapter III]). Our motivation is to get global L_p-L_q estimates for solutions of parabolic equations with good control of the constants over the domain. Such estimates were the key in [12] to develop a general theory of domain perturbation for linear and nonlinear parabolic equations with Dirichlet boundary conditions. Using the results of the present paper one should be able to deal with other boundary conditions by similar methods.

The boundary value problems we deal with in this paper are of the form

$$\begin{aligned} \partial_t u + \mathcal{A}(t)u &= 0 && \text{in } \Omega \times (0,\infty) \\ \mathcal{B}(t)u &= 0 && \text{on } \partial\Omega \times (0,\infty) \\ u &= u_0 && \text{on } \Omega, \end{aligned} \tag{1.1}$$

with $\mathcal{A}$ given by

$$\mathcal{A}(t)u := -\partial_i\big(a_{ij}(x,t)\partial_j u + a_i(x,t)u\big) + b_i(x,t)\partial_i u + c_0(x,t)u, \tag{1.2}$$

with real valued coefficients $a_{ij}, a_i, b_i, c_0 \in L_\infty(\Omega \times (0,\infty))$ (we use the summation convention). Here, $\partial_i := \partial/\partial x_i$, and Ω is an open set in $\mathbb{R}^N$. The boundary operator $\mathcal{B}$ is of either Dirichlet, Neumann or Robin type. More precisely,

$$\mathcal{B}(t)u = \begin{cases} u & \text{(Dirichlet)} \\ \big(a_{ij}(x,t)\partial_j u + a_i(x,t)u\big)\nu_i & \text{(Neumann)} \\ \big(a_{ij}(x,t)\partial_j u + a_i(x,t)u\big)\nu_i + b_0(x,t)u & \text{(Robin).} \end{cases} \tag{1.3}$$

Here, as above we use the summation convention. Moreover we assume that $\mathcal{A}$ is uniformly strongly elliptic, that is, there exists $\alpha_0 > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0|\xi|^2 \tag{1.4}$$

for all $(x,t) \in \overline{\Omega} \times [0,\infty)$ and $\xi \in \mathbb{R}^N$. As usual $L_p(\Omega)$ ($p \in [1,\infty]$) are the Lebesgue spaces. We denote their norm by $\|\cdot\|_p$. The norm in the space of bounded linear operators from L_p to L_q, $\mathcal{L}(L_p, L_q)$, we denote by $\|\cdot\|_{p,q}$. Further, $W_p^k(\Omega)$ ($k \in \mathbb{N}$, $p \in [1,\infty]$) are the Sobolev spaces. If X is a subset of $\mathbb{R}^N$ then $\mathcal{D}(X)$ is the space of all smooth functions having compact support in X, and $\mathring{W}_2^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W_2^1(\Omega)$.

It is our aim to prove the existence of a fundamental solution $k(x,y;t,s)$ of (1.1), and to show that it satisfies a Gaussian estimate of the form

$$|k(x,y;t,s)| \leq C(t-s)^{-d/2}e^{\omega(t-s)}e^{|x-y|^2/c(t-s)} \tag{1.5}$$

for all $t > 0$ and $x, y \in \mathbb{R}^N$, where $d \in [N, \infty)$ depends on the boundary conditions and the regularity of the domain Ω (for the precise result see Section 6). We shall consider four model problems.

1. First we consider the Dirichlet problem on arbitrary domains (or even open sets) $\Omega \subset \mathbb{R}^N$. In this case it turns out that (1.5) holds with $d = N$, and C, c and ω just depend on the L_∞-norms of the coefficients of $\mathcal{A}$, the ellipticity constant α_0 and N. This reproves a theorem due to Aronson [7, Theorem 9 and 10]. An alternative, simpler proof of the result in the autonomous case but under more restrictive regularity conditions on the drift terms can be found in [5]. The regularity assumptions on the drift terms make sure that the corresponding semigroup $T(t)$ is a quasi contraction on L_1 and L_∞, that is, for some $\omega > 0$, $\|T(t)\|_{1,1} \leq e^{\omega t}$ and $\|T(t)\|_{\infty,\infty} \leq e^{\omega t}$, which is otherwise not necessarily the case as counterexamples show (e.g. [27, Remark 4.3]).

2. Second, we study the Neumann problem on bounded or unbounded domains $\Omega \subset \mathbb{R}^N$. Here, it turns out that d and C depend on the regularity properties of the domain. At least in the autonomous case it is known that (1.5) is true for $d = N$ if Ω has the extension property, that is, if there exists an operator $\mathcal{E} \in \mathcal{L}\big(W_2^1(\Omega), W_2^1(\mathbb{R}^N)\big)$ with $(\mathcal{E}u)|_\Omega = u$ for all $u \in W_2^1(\Omega)$. (see [16, Theorem 3.2.9] for the selfadjoint case, and [5, Theorem 4.4] for the autonomous non-selfadjoint case with regular first order terms). We do not need the regularity conditions on the first order term, and extend the result to a class of domains not having the extension property such as the domains shown in Fig. 1. We merely assume

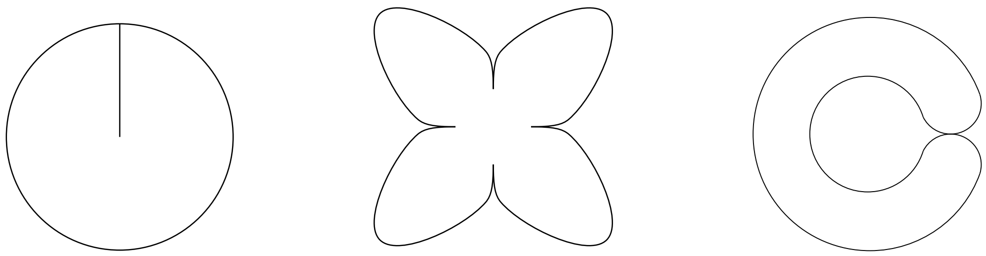

FIGURE 1. Some domains not having the extension property

that Ω satisfies an interior cone condition, which all domains in Fig. 1 clearly do. The extension property also fails for domains having an outward pointing cusp such as the ones in Fig. 2. If the cusps are polynomial we shall see that (1.5) still holds, with $d \in (N, \infty)$ depending on the sharpness of the cusp (see Example 4.6 and Theorem 6.1).

We finally consider Robin problems, where we look at the case of regular and irregular domains separately.

3. If $b_0 \geq 0$ and Ω is a bounded Lipschitz domain it turns out that (1.5) holds with $d = N$. This was shown in the autonomous case in [5, Theorem 4.8] in the autonomous case under some additional regularity assumptions on the first order terms. We remove all these additional assumptions.

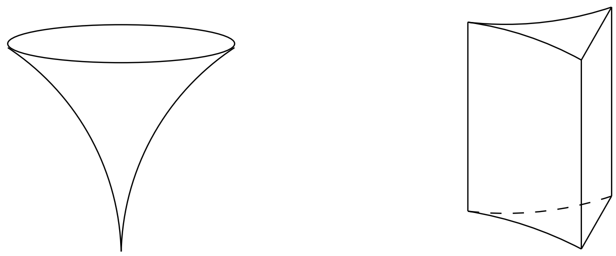

FIGURE 2. Domains with different types of cusps

4. Finally, if we assume that $b_0 \geq \beta_0$ for some constant $\beta_0 > 0$ we just need to assume that Ω has finite measure. Then, we show that (1.5) holds with $d = 2N$ and C depending only on N and upper bounds for the measure $|\Omega|$ of Ω and $\max\{\alpha_0^{-1}, \beta_0^{-1}\}$. This is the parabolic counterpart of the elliptic L_p-theory of Robin boundary value problems on arbitrary domains as presented in [13]. Depending on geometric properties of Ω one can achieve that $d \in [N, 2N)$. For instance for domains with polynomial outward pointing cusps we have $d \in (N, 2N)$ depending on the sharpness of the cusp. Finally, it turns out that $d = N$ for domains satisfying an interior cone condition.

We point out that the above are only model problems, and that various combinations and extensions are possible. The main aim is to give a unified treatment of all the cases under a simple set of assumptions. These mainly concern properties of the Hilbert space $V \hookrightarrow L_2(\Omega)$ used in the variational formulation of the boundary value problem. We need to assume that certain substitution operators are continuous on V (Lemma 3.2), and that bounded smooth functions are multipliers on V. Second, we need that for some $d > 0$ a Nash type inequality

$$\|u\|_2^{2+\frac{4}{d}} \leq C\|u\|_V^2\|u\|_1^{\frac{4}{d}}$$

holds for all $u \in V \cap L_1(\Omega)$. The existence of such a $d > 0$ is determined by the space V only, and in case of Neumann and Robin boundary conditions the geometric properties of Ω play an important rôle. The number d in the Nash inequality is the same as in (1.5). We refer to Section 4 for examples. To prove (1.5) we adapt the methods in [29] to get "diagonal bounds," and then use Davies perturbation method introduced in [15] (see [16]) to get (1.5).

An outline of the paper is as follows. In Section 2 we introduce our assumptions, and make precise what we mean by a solution of (1.1). We collect some facts from the theory of variational evolution equations which in particular imply the existence of an evolution system for (1.1). In Section 3 we prove some estimates related to the differential operator and the boundary conditions only. Section 4 deals with Nash inequalities for various domains and function spaces. In Section 5, they lead to "diagonal bounds," i.e. bounds of the evolution operator as a map from L_1 to L_∞. In Section 6 we employ Davies perturbation method to obtain (1.5) with explicit constants. In Section 7 we draw some consequences of the Gaussian estimates on the behaviour of the evolution system associated with (1.1) in $L_1(\Omega)$ and $L_\infty(\Omega)$. Our approach to prove kernel estimates does not rely on positivity or even the Harnack inequality. Nevertheless we can prove that the kernel is nonnegative, and that we have domination in case of the Dirichlet problem. This is done in Section 8.

2. The Variational Setting

We shall show that (1.1) fits into the framework of variational evolution equations. The associated bilinear form and the Hilbert space used depend on the boundary conditions under consideration. As model problems we consider the Dirichlet, Neumann and Robin problem. The bilinear form used in the Dirichlet and Neumann problem is for $u, v \in W_2^1(\Omega)$ and $t \geq 0$ given by

$$a_0(t; u, v) := \int_\Omega \big(a_{ij}(x,t)\partial_j u + a_i(x,t)u\big)\partial_i v + b_i(x,t)\partial_i u + c_0(x,t)u\big)v\,dx. \tag{2.1}$$

The natural domain of this form in case of the Dirichlet problem is $\mathring{W}_2^1(\Omega)$, and for the Neumann problem $W_2^1(\Omega)$. For the Robin problem the associated bilinear form is

$$a_1(t; u, v) := a_0(t; u, v) + \int_{\partial\Omega} b_0 uv\,d\mathcal{H}_{N-1} \tag{2.2}$$

where $\mathcal{H}_{N-1}$ is the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$, which coincides with the surface measure if Ω is Lipschitz. The main problem is to make sure that the boundary integral is defined if b_0 is nontrivial. The natural choice of space is $W_{2,2}^1(\Omega, \partial\Omega)$ which includes the boundary integral. For an arbitrary open set of finite measure it is by definition the abstract completion of

$$V_0 := \big\{u \in W_2^1(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega) \colon \|u\|_{W_{2,2}^1(\Omega,\partial\Omega)} < \infty < \infty\big\} \tag{2.3}$$

with respect to the norm

$$\|u\|_{W_{2,2}^1(\Omega,\partial\Omega)} := \Big(\int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} |u|^2\,d\mathcal{H}_{N-1}\Big)^{\frac{1}{2}}, \tag{2.4}$$

This space was introduced by Maz'ja [24]. An extensive treatment of these spaces can be found in Maz'ja [25]. For our purposes, the most important features are the following. For all $u \in W_{2,2}^1(\Omega, \partial\Omega)$ and $N \geq 2$

$$\|u\|_{\frac{2N}{N-1}} \leq c(N, |\Omega|)\|u\|_{W_{2,2}^1(\Omega,\partial\Omega)}, \tag{2.5}$$

that is, $W_{2,2}^1(\Omega, \partial\Omega) \hookrightarrow L_{\frac{2N}{N-1}}(\Omega)$. The constant $c(N, |\Omega|) > 0$ only depends on the dimension N and the measure $|\Omega|$ of Ω. In particular, this this implies that $W_{2,2}^1(\Omega, \partial\Omega) \hookrightarrow W_2^1(\Omega)$. It turns out that for any bounded Lipschitz domain Ω

$$W_{2,2}^1(\Omega, \partial\Omega) = W_2^1(\Omega) \tag{2.6}$$

up to an equivalent norm. Indeed, this follows from (2.5), the fact that in this case the trace operator from $W_2^1(\Omega)$ to $L_2(\partial\Omega)$ is bounded (e.g. [26, Théorème 2.5.5], and the open mapping theorem. In general the norm (2.4) is stronger than the W_2^1-norm. This space was first applied to elliptic boundary value problems in Maz'ja [25]. An extensive treatment of the elliptic problem in an L_p-setting is contained in Daners [13].

We shall consider four sets of assumptions. However, as pointed out already these are model problems, and various combinations and extensions are possible.

(Dirichlet problem) $V := \mathring{W}_2^1(\Omega)$, $a(t;\cdot\,,\cdot) := a_0(t;\cdot\,,\cdot)$ and Ω is an arbitrary domain. (2.7)

(Neumann problem) $V := W_2^1(\Omega)$, $a(t;\cdot\,,\cdot) := a_0(t;\cdot\,,\cdot)$ and Ω is an arbitrary domain. (2.8)

(Robin problem, regular domains) $V := W_{2,2}^1(\Omega, \partial\Omega)$, $a(t;\cdot\,,\cdot) := a_1(t;\cdot\,,\cdot)$ and Ω is a bounded Lipschitz domain. (2.9)

(Robin problem, irregular domains) $V := W_{2,2}^1(\Omega, \partial\Omega)$, $a(t;\cdot\,,\cdot) := a_1(t;\cdot\,,\cdot)$, Ω is of finite measure and $b_0 \geq \beta_0$ for some constant $\beta_0 > 0$. (2.10)

2.1. Definition. A function u is called a weak solution of the Dirichlet, Neumann or Robin problem (1.1) on $\Omega \times [0, T]$, $T > 0$ arbitrary, if $u \in L_2((0, T), V)$ and

$$-\int_0^T \langle u(t), v\rangle \dot{\varphi}(t)\, dt + \int_0^T a(t; u(t), v)\varphi(t)\, dt - \langle u(0), v\rangle\varphi(0) = 0 \tag{2.11}$$

for all V and $\varphi \in \mathcal{D}([0, T))$, where V and $a(t;\cdot\,,\cdot)$ are as in (2.7)–(2.10).

2.2. Remark. As $\mathcal{D}(\Omega) \subset V$ in all cases it follows that a weak solution of the initial boundary value problem (1.1) is a weak solution of the equation $\partial_t u + \mathcal{A}(t)u = 0$ in $\Omega \times (0, T)$ in the usual sense. It also turns out that, if domain and coefficients are sufficiently regular, any weak solution is a pointwise solution of (1.1). On the other hand, any classical solution with $u_0 \in L_2(\Omega)$ is a weak solution provided it is not too fast growing.

In all the above cases we are given separable Hilbert spaces V and H such that $V \hookrightarrow H$, and V is dense in H. In our case, $H = L_2(\Omega)$ always. For all $t \geq 0$ a bilinear form

$$a(t;\cdot\,,\cdot)\colon V \times V \to \mathbb{R} \tag{2.12}$$

is defined. Moreover, for all $u, v \in V$ the function $a(\cdot\,; u, v)$ is measurable, and

$$|a(t; u, v)| \leq M\|u\|_V\|v\|_V. \tag{2.13}$$

Under these hypotheses there exist operators $A(t) \in \mathcal{L}(V, V')$ such that

$$\langle A(t)u, v\rangle = a(t; u, v) \tag{2.14}$$

for all $u, v \in V$. We are looking for solutions of the equation

$$\begin{aligned} \dot{u} + A(t)u &= 0 \qquad \text{in } (0, T] \\ u(0) &= u_0 \end{aligned} \tag{2.15}$$

for $u_0 \in H$. By a solution we mean a function u in

$$W := W(0,T;V,V') := \{v \in L_2((0,T),V) \colon \dot{v} \in L_2((0,T),V')\}, \tag{2.16}$$

where $\dot{u} := du/dt$ in the sense of distributions taking values in V'. Equipped with the norm

$$\|u\|_W := \Big(\int_0^T \|u(t)\|_V^2 \, dt + \int_0^T \|\dot{u}(t)\|_{V'}^2 \, dt \Big)^{\frac{1}{2}}$$

W is a Hilbert space. It is well known (e.g. [14, Section VXIII.1.2]) that

$$W(0,T;V,V') \hookrightarrow C([0,T],H) \tag{2.17}$$

so that it makes sense to write $u(t)$ for every $t \in [0,T]$. We also need to consider the final value problem

$$\begin{aligned} -\dot{v} + A(T-s)v &= 0 \qquad \text{in } [0,T) \\ v(T) &= v_T \end{aligned} \tag{2.18}$$

where $T > 0$ can be chosen arbitrarily.

2.3. Remark. A function $u \in W$ is a weak solution of (1.1) if and only if it is a solution of (2.15). This is the case if and only if for every $v \in V$ u satisfies the equation

$$\frac{d}{dt}\langle u(t), v\rangle + a(t; u(t), v) = 0$$

in the sense of distributions $\mathcal{D}'((0,T))$, and $u(0) = u_0$. Further, this is equivalent to u satisfying

$$-\int_0^T \langle u(t), \dot{v}(t)\rangle \, dt + \int_0^T a(t; u(t), v(t)) \, dt - \langle u(0), v(0)\rangle = 0$$

for all $v \in W$ satisfying $v(T) = 0$. (see [14, Section VXIII.3.2]).

We then have the following existence theorem.

2.4. Theorem. *Under the assumption* (2.7), (2.8), (2.9) *or* (2.10) *the problems* (2.15) *and* (2.18) *have a unique solution in* $W(0,T;V,V')$ *for all* $u_0, v_T \in H$.

Proof. The only point to prove the existence of a solution of (2.15) is that there exist $\lambda_0 \in \mathbb{R}$ and $\alpha > 0$ such that

$$\alpha\|u\|_V^2 \le a(t; u, u) + \lambda_0\|u\|_H^2 \tag{2.19}$$

for all $u \in V$ (e.g. [14, Section XVIII.3.3]). This however follows by a standard argument. (The inequality also follows from a more general result in Lemma 3.1 below by setting $p = 2$ and adding the appropriate additional terms.) Note that if u is the solution of the forward problem

$$\begin{aligned} \dot{u} + A(T-t)u &= 0 \qquad \text{in } (0,T] \\ u(0) &= v_T \end{aligned} \tag{2.20}$$

which exists by the previous result it follows that $v(s)=u(T-s)$ is a solution of (2.18). This shows that also (2.18) has a unique solution. $\square$

Obviously the initial and final times need not be 0 and T, respectively. Therefore we can make the following definition.

2.5. Definition. Let us denote the solution of (2.15) with initial time s by $U(\cdot,s)u_0$. Then, $U(\cdot\,,\cdot)$ defined on $\{(t,s)\colon 0\leq s\leq t<\infty\}$ is called the evolution system corresponding to the family $\bigl(A(t)\bigr)_{t\geq 0}$. Similarly, define $V(\cdot\,,t)v_t$ to be the solution of the final value problem (2.18) with final time t. We call $V(\cdot\,,\cdot)$ the backward evolution system corresponding to the family $\bigl(A(T-s)\bigr)_{s\in[0,T]}$. It is defined on $\{(t,s)\colon 0\leq s\leq t<T\}$

It turns out that $U(t,s),V(s,t)\in\mathcal{L}(H)$, which follows e.g. from [14, Section XVIII.3.4], (see also the following section, where we prove a more general result).

We finally consider the adjoint problem. We define the adjoint form $a^\sharp$ by

$$a^\sharp(t;u,v):=a(t;v,u) \tag{2.21}$$

for all $u,v\in V$ and $t\geq 0$. Clearly, this form has the same properties as $a(t;\cdot\,,\cdot)$. We then define the operator $A^\sharp(t)$ by (2.14) replacing a by $a^\sharp$. Hence, in all the theory above we can replace A by $A^\sharp$. We denote the corresponding evolution systems by $U^\sharp(\cdot\,,\cdot)$ and $V^\sharp(\cdot\,,\cdot)$. These evolution systems are related as follows.

2.6. Theorem. *Suppose that $U(\cdot\,,\cdot)$ and $U^\sharp(\cdot\,,\cdot)$ are the evolution systems corresponding to $\bigl(A(t)\bigr)_{t\in[0,T]}$ and $\bigl(A^\sharp(T-t)\bigr)_{t\in[0,T]}$, respectively. Further let $V(\cdot\,,\cdot)$ be the backward evolution system corresponding to $\bigl(A^\sharp(T-s)\bigr)_{s\in[0,T]}$. Then,*

$$V(s,t)=(U^\sharp(t,s))'=U(T-s,T-t) \tag{2.22}$$

for all $(t,s)\in\{(t,s)\colon 0\leq s\leq t\leq T\}$.

Proof. The first equality can be shown by using the fact that the assertion is true if H is finite dimensional and then by passing to the infinite dimensional case by means of Galerkin approximations for (2.15) and (2.18) (see [14, Section XVIII.3]). The last equality follows by a simple calculation. $\square$

3. Properties of the Form $a(t;\cdot\,,\cdot)$

In this section we prove some estimates of weak solutions of (1.1) which result from properties of the form $a(t;\cdot\,,\cdot)$ only, and which do not depend on the domain Ω. This provides one main ingredient needed to establish (1.5), namely an inequality given at the end of the section in Proposition 3.4. For $k\geq 1$ and $p\geq 2$ we introduce the functions $f_{k,p}\in C^1(\mathbb{R},\mathbb{R})$ given by

$$f_{k,p}(\xi):=\begin{cases}|\xi|^{\frac{p}{2}} & \text{if } |\xi|\leq k\\ k^{\frac{p}{2}}+\frac{p}{2}k^{\frac{p}{2}-1}\bigl(|\xi|-k\bigr) & \text{if } |\xi|\geq k\end{cases} \tag{3.1}$$

We then define $g_{k,p} \in C(\mathbb{R},\mathbb{R})$ by

$$g_{k,p}(\xi) := f'_{k,p}(\xi) f_{k,p}(\xi) = \begin{cases} \frac{p}{2} \operatorname{sign}\xi |\xi|^{p-1} & \text{if } |\xi| \le k \\ \frac{p}{2} \operatorname{sign}\xi \big(k^{p-1} + \frac{p}{2} k^{p-2}(|\xi| - k)\big) & \text{if } |\xi| \ge k \end{cases} \tag{3.2}$$

It is easily seen that $f'_{k,p}, g'_{k,p} \in L_\infty(\mathbb{R})$ are smooth on the real line except for $\xi = \pm k$. Therefore, $f_{k,p}(u), g_{k,p}(u) \in W_2^1(\Omega)$ for all $u \in W_2^1(\Omega)$ (e.g. [20, Theorem 7.8]). Using this fact we can prove the following lemma involving the form $a_0(t;\cdot,\cdot)$ defined in (2.1).

3.1. Lemma. *Let $u \in W_2^1(\Omega)$ and set $v_{k,p} := g_{k,p}(u)$ and $w_{k,p} := f_{k,p}(u)$. Then, for all $k \ge 1$, $p \ge 2$ and $t \ge 0$*

$$\alpha_0 \|\nabla w_{k,p}\|_2^2 \le 2a_0(t; u, v_{k,p}) + p(p-1)\delta_0 \|w_{k,p}\|_2^2, \tag{3.3}$$

where

$$\delta_0 := \alpha_0^{-1}\big(\|\mathbf{a}\|_\infty^2 + \|\mathbf{b}\|_\infty^2\big) + \|c_0^-\|_\infty. \tag{3.4}$$

Here, c_0^- is the negative part of c_0 and $\|\mathbf{a}\|_\infty = \big(\sum_{i=1}^N \|a_i\|_\infty^2\big)^{1/2}$. Likewise, define $\|\mathbf{b}\|_\infty$.

Proof. To simplify notation we set $v := v_{k,p}$ and $w := w_{k,p}$. Using [20, Theorem 7.8] an elementary calculation shows that if $|u| < k$

$$\partial_j w \partial_i w = \frac{p}{2(p-1)} \partial_j u \partial_i v \tag{3.5}$$

and

$$u\partial_i v = (p-1) w \partial_i w \qquad v\partial_i u = w\partial_i w \qquad uv = \frac{p}{2} w^2. \tag{3.6}$$

For $|u| > k$ we obtain

$$\partial_j w \partial_i w = \partial_j u \partial_i v \tag{3.7}$$

and

$$u\partial_i v = \frac{p}{2} k^{\frac{p}{2}-1} |u| \partial_i w \qquad v\partial_i u = w\partial_i w \qquad uv \le \frac{p}{2} w^2. \tag{3.8}$$

For the last inequality we used that for $p \ge 2$ and $|u| > k$

$$k^{\frac{p}{2}}|u| = k^{\frac{p}{2}} + k^{\frac{p}{2}-1}\big(|u| - k\big) \le k^{\frac{p}{2}} + \frac{p}{2} k^{\frac{p}{2}-1}\big(|u| - k\big) = w. \tag{3.9}$$

Now, (1.4), (3.5) and (3.6) imply

$$\begin{aligned} \alpha_0 |\nabla w|^2 \le a_{ij}\partial_j w \partial_i w &= \frac{p}{2(p-1)} a_{ij} \partial_j u \partial_i v \\ &= \frac{p}{2(p-1)}\big((a_{ij}\partial_j u + a_i u)\partial_i v + (b_i \partial_i u + c_0 u) v\big) \\ &\quad + \frac{p}{2}(\|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty)|\nabla w||w| + \frac{p}{2}\|c_0^-\|_\infty w^2. \end{aligned}$$

By an elementary inequality we have that $|\nabla w||w| \leq \varepsilon|\nabla w|^2 + (4\varepsilon)^{-1}|w|$ for all $\varepsilon > 0$ and thus, by choosing $\varepsilon > 0$ appropriately, we arrive at

$$\alpha_0|\nabla w|^2 \leq \frac{p}{p-1}\big((a_{ij}\partial_j u + a_i u)\partial_i v + (b_i\partial_i u + c_0 u)v + \frac{1}{2}p(p-1)\delta_0\|w\|_2^2\big). \tag{3.10}$$

For $|u| > k$ we proceed similarly using (3.7) and (3.8). In this case we have

$$\alpha|_0\nabla w|^2 \leq a_{ij}\partial_j w\partial_i w \leq \frac{p}{2(p-1)}a_{ij}\partial_j w\partial_i w = \frac{p}{2(p-1)}a_{ij}\partial_j u\partial_i v,$$

and to estimate $k^{\frac{p}{2}}|u||\nabla w|$ we use (3.9). This also yields (3.10). Integrating (3.10) over Ω, and taking into account that $p(p-1)^{-1} \leq 2$ for $p \geq 2$ the assertion of the lemma follows. □

Next we prove a lemma concerning substitution operators.

3.2. Lemma. *Suppose V is one of the spaces $\mathring{W}_2^1(\Omega)$, $W_2^1(\Omega)$ or $W_{2,2}^1(\Omega,\partial\Omega)$, and that f is a piecewise smooth function on $\mathbb{R}$ with $f' \in L_\infty(\mathbb{R})$ and $f(0) = 0$. Then, the substitution operator $u \mapsto f(u)$ acts continuously on V, $L_2((0,T),V)$ and $C([0,T],L_2(\Omega))$.*

Proof. We first prove continuity of the substitution operator on V. By [20, Theorem 7.8] we have that $f(u) \in W_2^1(\Omega)$ for all $u \in W_2^1(\Omega)$. It is easily seen that $u \mapsto f(u)$ is continuous on $W_2^1(\Omega)$ (see also [23]). As $\mathcal{D}(\Omega)$ is dense in $\mathring{W}_2^1(\Omega)$ and $f(0) = 0$ we have $f(u) \in \mathring{W}_2^1(\Omega)$ for all $u \in \mathring{W}_2^1(\Omega)$. Continuity of the substitution operator on $\mathring{W}_2^1(\Omega)$ is clear as this is a closed subspace of $W_2^1(\Omega)$. For $W_{2,2}^1(\Omega,\partial\Omega)$ continuity follows as it is true on $W_2^1(\Omega)$ and on $L_2(\partial\Omega)$. The assertion of the lemma for $L_2\big((0,T),V\big)$ follows easily from the above. Finally, by the mean value theorem we conclude that for any sequence u_n converging to u in $C([0,T],L_2(\Omega))$

$$\sup_{t\in[0,T]}\|f(u_n(t)) - f(u(t)\|_2 \leq \|f'\|_\infty \sup_{t\in[0,T]}\|u_n(t) - u(t)\|_2$$

holds. This shows continuity on $C([0,T],L_2(\Omega))$. □

3.3. Lemma. *Suppose that f is a piecewise smooth function on $\mathbb{R}$ with $f(0) = 0$, and that f' and $\big(f'(\cdot)f(\cdot)\big)'$ are bounded. Then,*

$$\frac{1}{2}\Big(\|f(u(t))\|_2^2 - \|f(u(0))\|_2^2\Big) = \int_0^t \big\langle \dot{u}(s), f(u(s))f'(u(s))\big\rangle\, ds \tag{3.11}$$

for all $u \in W(0,T;V,V')$ and $t \in [0,T]$, where V is one of the spaces $\mathring{W}_2^1(\Omega)$, $W_2^1(\Omega)$ or $W_{2,2}^1(\Omega,\partial\Omega)$, and $W(0,T;V,V')$ is defined by (2.16).

Proof. It is well known that $\mathcal{D}([0,T],V)$ is dense in $W := W(0,T;V,V')$ (see [14, Section XVIII.1.2]). Hence, given $u \in W$, there exists a sequence $u_n \in \mathcal{D}([0,T],V)$ approaching u in W. By integrating

$$\frac{1}{2}\frac{d}{dt}\|f(u_n(t))\|_2^2 = \big\langle f'(u_n(t))\dot{u}_n(t), f(u_n(t))\big\rangle = \big\langle \dot{u}_n(t), f'(u_n(t))f(u_n(t))\big\rangle$$

we get

$$\frac{1}{2}\Big(\|f(u_n(t))\|_2^2 - \|f(u_n(0))\|_2^2\Big) = \int_0^t \big\langle \dot u_n(s), f(u_n(s))f'(u_n(s))\big\rangle\,ds. \tag{3.12}$$

Further, note that $u_n \to u$ in $L_2((0,T),V)$, and $\dot u_n \to \dot u$ in $L_2((0,T),V')$ by definition of the norm in W. As $g(\xi) = f(\xi)f'(\xi)$ satisfies the assumption of Lemma 3.2 it follows that $g(u_n)$ converges to $g(u)$ in $L_2((0,T),V)$. Hence, the right hand side of (3.12) converges to the right hand side of (3.11). By (2.17) also $u_n \to u$ in $C([0,T],L_2(\Omega)$, and therefore, by Lemma 3.2, the left hand side converges to the corresponding expression as well. □

To formulate the main result of this section we define the following quantities:

$$\delta := \begin{cases} \delta_0 & \text{in case (2.7) and (2.10)} \\ \delta_0 + \alpha_0 & \text{in case (2.8) and (2.9)} \end{cases} \tag{3.13}$$

$$\alpha := \begin{cases} \alpha_0 & \text{in case (2.7), (2.8) and (2.9)} \\ \min\{\alpha_0, \beta_0\} & \text{in case (2.10)} \end{cases} \tag{3.14}$$

and for $w \in W_2^1(\Omega)$

$$q_V(w) := \begin{cases} \|\nabla w\|_2^2 & \text{in case (2.7)} \\ \|w\|_{W_2^1(\Omega)} & \text{in case (2.8) and (2.9)} \\ \|w\|_{W_{2,2}^1(\Omega,\partial\Omega)} & \text{in case (2.10)} \end{cases} \tag{3.15}$$

Finally, recall from Section 2 that (1.1) has a unique weak solution if one of the hypotheses (2.7)–(2.10) are satisfied. Suppose that $u \in W(0,T;V,V')$ is the weak solution and set

$$w_{k,p}(t) := f_{k,p}(u(t)) \qquad \text{and} \qquad v_{k,p}(t) := g_{k,p}(u(t)) \tag{3.16}$$

where $f_{k,p}$ and $g_{k,p}$ are as in (3.1) and (3.2). Then we have the following result.

3.4. Proposition. *For all $k \geq 1$ and $p \geq 2$ The function $t \mapsto \|w_{k,p}\|_2^2$ is locally absolutely continuous. Moreover,*

$$\frac{d}{dt}\|w_{k,p}(t)\|_2^2 \leq -\alpha_0\|\nabla w_{k,p}(t)\|_2 + p(p-1)\delta_0\|w_{k,p}(t)\|_2^2 \tag{3.17}$$

and

$$\frac{d}{dt}\|w_{k,p}(t)\|_2^2 \leq -\alpha\, q_V(w_{k,p}(t)) + p(p-1)\delta\|w_{k,p}(t)\|_2^2 \tag{3.18}$$

for almost all $t \geq 0$.

Proof. Using that $g_{k,p}(\xi) = f_{k,p}(\xi) f'_{k,p}(\xi)$ by definition, Lemma 3.3, and the fact that u is a solution of (2.15), we conclude that

$$\begin{aligned}\frac{1}{2}\Big(\|f_{k,p}(u(t))\|_2^2 - \|f_{k,p}(u(0))\|_2^2\Big) &= \int_0^t \big\langle \dot u(s), v_{k,p}(s)\big\rangle\, ds \\ &= -\int_0^t a(s; u(s), v_{k,p}(s))\, ds\end{aligned}$$

for all $k \geq 1$ and $p \geq 2$. As $s \mapsto a(s; u(s), v_{k,p}(s)) \in L_{1,\mathrm{loc}}((0,\infty))$ it follows by a well known characterisation of absolutely continuous functions (e.g. [21, Theorem 18.3]) that $\|f_{k,p}(u(\cdot))\|_2^2 = \|w_{k,p}(\cdot)\|_2^2$ is locally absolutely continuous. By differentiating the above equation we obtain

$$\frac{d}{dt}\|w_{k,p}\|_2^2 = -2a(t; u(t), v_{k,p}(t))$$

for almost all $t \geq 0$. Then, clearly,

$$\frac{d}{dt}\|w_{k,p}\|_2^2 = -\Big(2a(t; u(t), v_{k,p}(t)) + p(p-1)\delta_0\|w_{k,p}(t)\|_2^2\Big) + p(p-1)\delta_0\|w_{k,p}(t)\|_2^2.$$

On the other hand, by adding the appropriate terms in each case, Lemma 3.1 yields

$$\alpha_0\|\nabla w_{k,p}(t)\|_2^2 \leq 2a(t; u(t), v_{k,p}(t)) + p(p-1)\delta_0\|w_{k,p}(t)\|_2^2.$$

Now (3.17) follows by combining the two inequalities. The proof of (3.18) is an obvious modification of the one for (3.17). □

4. Nash Inequalities

Besides the inequality in Proposition 3.4 the second main ingredient to prove (1.5) are Nash inequalities. They provide estimates of L_p-norms of the solution u in terms of $q_V(u)$ allowing to do a bootstrapping argument. This is the part where the geometric properties of the domain Ω and the boundary condition play a crucial rôle. The classical two forms of the Nash inequality are the following. For a simple proof using the Fourier Transform we refer to [29, Section III.3].

4.1. Theorem. *For all $u \in W_2^1(\mathbb{R}^N) \cap L_1(\mathbb{R}^N)$ we have that*

$$\|u\|_2^2 \leq \varepsilon\|\nabla u\|_2^2 + \varepsilon^{-\frac{N}{2}} K_0\|u\|_1^2 \tag{4.1}$$

for all $\varepsilon > 0$, and

$$\|u\|_2^{2+\frac{4}{N}} \leq K_1\|\nabla u\|_2^2\|u\|_1^{\frac{4}{N}}, \tag{4.2}$$

where $K_0, K_1 > 0$ depend only on the dimension $N \geq 1$.

Note that the second form follows from the first by minimising the right hand side of (4.1) over $\varepsilon \in (0,\infty)$. Let us consider domains Ω having the L_1-W_2^1-extension property, that is, there exists an extension operator

$$\mathcal{E} \in \mathcal{L}(W_2^1(\Omega), W_2^1(\mathbb{R}^N)) \cap \mathcal{L}(L_1(\Omega), L_1(\mathbb{R}^N)). \tag{4.3}$$

By an extension operator we mean an operator such that $(\mathcal{E}u)|_\Omega = u$. For domains admitting such an extension operator it is known that the Nash inequalities hold. We include the simple proof of the well known fact for completeness.

4.2. Theorem. *Let Ω have the L_1-W_2^1-extension property. Then, for all $u \in W_2^1(\Omega) \cap L_1(\Omega)$ we have that*

$$\|u\|_2^2 \leq K_\Omega\Big(\varepsilon\|u\|^2_{W_2^1(\Omega)} + \varepsilon^{-\frac{N}{2}}K_0\|u\|_1^2\Big) \tag{4.4}$$

for all $\varepsilon > 0$, and

$$\|u\|_2^{2+\frac{4}{N}} \leq K_\Omega^{1+\frac{2}{N}}K_1\|u\|^2_{W_2^1(\Omega)}\|u\|_1^{\frac{4}{N}}, \tag{4.5}$$

where $K_\Omega = \max\{\|\mathcal{E}\|_{\mathcal{L}(W_2^1(\Omega),W_2^1(\mathbb{R}^N))}, \|\mathcal{E}\|_{\mathcal{L}(L_1(\Omega),L_1(\mathbb{R}^N))}\}$.

Proof. Using the properties of the extension operator and (4.1) we conclude that for all $u \in W_2^1(\Omega) \cap L_1(\Omega)$

$$\begin{aligned}\|u\|_2^2 \leq \|\mathcal{E}u\|_2^2 &\leq \varepsilon\|\mathcal{E}u\|^2_{W_2^1(\mathbb{R}^N)} + \varepsilon^{-\frac{N}{2}}K_0\|\mathcal{E}u\|_1^2\\ &\leq K_\Omega\Big(\varepsilon\|u\|^2_{W_2^1(\Omega)} + \varepsilon^{-\frac{N}{2}}K_0\|u\|^2_{1,\Omega}\Big)\end{aligned}$$

which proves (4.4). Inequality (4.5) follows by minimising the right hand side of (4.4) over $\varepsilon \in (0,\infty)$. □

4.3. Remark. An example of extension domains are bounded Lipschitz domains. In this case the norm of the extension operator just depends on the Lipschitz constants of the functions describing Ω and the dimension N (see [30, Chapter VI, Theorem 5 and 5'])

We shall prove that the Nash inequalities (4.4) and (4.5) not only hold on extension domains but on domains satisfying the cone condition. By this we mean that there exists a finite cone such that every point in Ω is the vertex of a cone in Ω congruent to the fixed given cone (see [1, Definition 4.3]). We then have the following result.

4.4. Theorem. *Suppose Ω is a bounded or unbounded domain satisfying the cone condition. Then the Nash inequalities* (4.4) *and* (4.5) *hold with a constant K_Ω just depending on the dimension N and the cone determining the cone condition.*

Proof. As (4.5) is a simple consequence of (4.4) we only prove (4.4). We shall use a result due to Gagliardo (see [1, Theorem 4.8]), which shows that any given domain satisfying the cone condition can be written as the union of a locally finite family of Lipschitz domains.

More precisely, in the first part of its proof given in [1, Theorem 4.8] it is shown that Ω is the union of finitely many domains $\tilde{\Omega}_i$ which are the union of parallelepipeds P_i

$$\tilde{\Omega}_i = \bigcup_{x\in\tilde{A}_i} (x+P_i)$$

for certain sets $\tilde{A}_i \subset \overline{\Omega}$. For any given $\rho > 0$ we subdivide $\mathbb{R}^N$ into cubes Q_j of side ρ and set

$$\Omega_{i,j} := \bigcup_{x\in\tilde{A}_i\cap Q_j} (x+P_i).$$

It turns out that, if ρ is small enough, the sets $\Omega_{i,j}$ are Lipschitz domains with uniformly bounded diameter, and the Lipschitz constants of the functions describing the domains just depend on N and the cone (see the second part of the proof of [1, Theorem 4.8]). Moreover, there exists an integer N_0 depending only on N and the cone such that no more than N_0 of the $\Omega_{i,j}$ have nonempty intersection. Suppose from now on that ρ is small enough such that $\Omega_{i,j}$ are Lipschitz. Then, by Remark 4.3 the domains $\Omega_{i,j}$ allow an extension operator $\mathcal{E}_{i,j}$, and there exists a constant $C > 0$ depending only on N and the cone such that $K_{\Omega_{i,j}} \leq C$ for all $i, j \in \mathbb{N}$, where $K_{\Omega_{i,j}} \leq C$ is defined as in Theorem 4.2. Hence, by Theorem 4.2 for all $W_2^1(\Omega) \cap L_1(\Omega)$ the Nash inequality (4.4) holds on Ω_{ij} with a constant $K_{\Omega_{ij}} \leq C$. Therefore, (4.4) yields

$$\begin{aligned}
\|u\|_{2,\Omega}^2 &\leq \sum_{i,j} \|u\|_{2,\Omega_{i,j}} \leq C \sum_{i,j} \Bigl(\varepsilon \|u\|^2_{W_2^1(\Omega_{i,j})} + \varepsilon^{-\frac{N}{2}} K_0 \|u\|^2_{L_1(\Omega_{i,j})}\Bigr) \\
&\leq C\Bigl(\varepsilon \sum_{i,j} \|u\|^2_{W_2^1(\Omega_{i,j})} + \varepsilon^{-\frac{N}{2}} K_0 \Bigl(\sum_{i,j} \|u\|_{L_1(\Omega_{i,j})}\Bigr)^2\Bigr) \\
&\leq C N_0^2 \bigl(\varepsilon \|u\|^2_{W_2^1(\Omega)} + \varepsilon^{-\frac{N}{2}} K_0 \|u\|^2_{L_1(\Omega)}\bigr),
\end{aligned}$$

where for the third inequality we used the fact that for any nonnegative sequence α_i

$$\sum_{i=1}^\infty \alpha_i^2 \leq \Bigl(\sum_{i=1}^\infty \alpha_i\Bigr)^2,$$

and for the last that at most N_0 of of the $\Omega_{i,j}$ have nonempty intersection. As C and N_0 just depend on the dimension and the cone C the assertion of the theorem follows. $\square$

The idea of the proof of the above theorem was similar to the one given in [1, Lemma 5.14] for the Sobolev inequality on domains satisfying the cone condition. It is easy to see that the Sobolev inequality implies the Nash inequality. The proof given below is taken from [11]. We remark that also the converse is true (see [11, Theorem 2.16]), but the proof is much more difficult.

4.5. Proposition. *Suppose that q_V is a quadratic form on $H = L_2(\Omega)$ with domain V, and that for some $d > 2$ and $C > 0$ the Sobolev inequality*

$$\|u\|^2_{\frac{2d}{d-2}} \leq C\, q_V(u) \tag{4.6}$$

holds for all $u \in V$. Then, for all $u \in V \cap L_1(\Omega)$ we have the Nash inequality

$$\|u\|_2^{2+\frac{4}{d}} \leq C\, q_V(u)\|u\|_1^{\frac{4}{d}}. \tag{4.7}$$

Proof. By a well known interpolation inequality (e.g. [20, Section 7.1]) we have that

$$\|u\|_2 \leq \|u\|^{\frac{d}{d+2}}_{\frac{2d}{d-2}} \|u\|_1^{\frac{2}{d+2}}$$

for all $u \in V \cap L_1(\Omega)$. Now the assertion follows by using the assumption (4.6) □

The reason why the above proposition is interesting for our purposes is that there are many results on Sobolev type inequalities available in the literature, where the exponent d depends on geometric properties of the domain. In the following examples q_V is the square of the norm of the Hilbert space V under consideration.

4.6. Example. Consider domains with outward pointing polynomial cusps. It is shown e.g. in [1, Theorem 5.35] that (4.6) holds for all $u \in V = W_2^1(\Omega)$, where $d = N + \nu > N$, and $\nu > 0$ is determined by the type and sharpness of the cusp.

4.7. Example. In Maz'ja [25, Section 4.9] geometric criteria on domains are given which imply

$$\|u\|_q \leq C\|u\|_{W_2^1(\Omega)} \tag{4.8}$$

for all $u \in V := W_2^1(\Omega)$, for some $q > 2$. By setting $d := 2q(q-2)^{-1}$ we immediately obtain an inequality of the form (4.6) and thus a Nash type inequality.

4.8. Example. Suppose that Ω is any domain of finite measure in $\mathbb{R}^N$ ($N \geq 2$) and $V := W_{2,2}^1(\Omega, \partial\Omega)$. Then, by (2.5) a Sobolev type inequality holds with $d := 2N$. We can also consider classes of domains for which $d \in [N, 2N)$. We refer to [25, Section 4.11.1] for detailed results. Finally, if Ω satisfies an interior cone condition then $d = N$ ([25, Example 3.6.1/3 and Theorem 3.6.3]).

5. Estimates for the Evolution System

Let $U(\cdot\,,\cdot)$ be the evolution system introduced in Definition 2.5. We prove estimates of the norm of $U(\cdot\,,\cdot)$ as a map between various L_p-spaces. We start by showing that it acts on L_p. This theorem is true under no additional regularity assumptions on the domains Ω.

5.1. Theorem. *Suppose that one of the assumptions* (2.7)–(2.10) *is satisfied. Then, for all $p \in (1, \infty)$ There exists and operator $U_p(t,s) \in \mathcal{L}(L_p(\Omega))$ such that $U_p(t,s)u_0 =$*

$U(t,s)u_0$ for all $u_0 \in L_2(\Omega) \cap L_p(\Omega)$. Moreover, for $s \geq 0$ the map $U_p(\cdot\,,s)$ is strongly continuous from $[s,\infty)$ to $\mathcal{L}(L_p(\Omega))$, and for all $t \geq s$

$$\|U_p(t,s)\|_{p,p} \leq \begin{cases} e^{(p-1)\delta_0(t-s)} & \text{if } p \in [2,\infty) \\ e^{(p'-1)\delta_0(t-s)} & \text{if } p \in (1,2]. \end{cases} \tag{5.1}$$

Here, δ_0 is defined in (3.4) and p' the dual exponent to p.

Proof. We first consider the case $p \geq 2$. Suppose that $u_0 \in L_p(\Omega) \cap L_2(\Omega)$ for some $p \in [2,\infty)$, and let u be the solution of (2.15). Then, from Proposition 3.4

$$\frac{d}{dt}\|w_{k,p}(t)\|_2^2 \leq p(p-1)\delta_0\|w_{k,p}(t)\|_2^2$$

for almost all $t \geq 0$. As $\|w_{k,p}(\cdot)\|_2^2$ is locally absolutely continuous this implies that

$$\|w_{k,p}(t)\|_2^2 \leq e^{p(p-1)\delta_0 t}\|w_{k,p}(0)\|_2^2$$

for all $t \geq 0$. Now, as $w_{k,p} \nearrow |u|^{\frac{p}{2}}$ as k goes to infinity, the monotone convergence theorem implies that

$$\|U_p(t,0)u_0\|_{p,p} \leq e^{(p-1)\delta_0 t}\|u_0\|_p.$$

As $L_2 \cap L_p$ is dense in L_p we can continue $U(t,0)|_{L_2\cap L_p}$ in a unique way to a bounded linear operator $U_p(t,0)$ on L_p. It clearly satisfies (5.1) for $s=0$. The inequality holds also for $s \neq 0$ as there is nothing special about $s=0$. It remains to show the continuity of the evolution system. It is sufficient to show continuity for $p=2$. In the other cases it follows from (5.1) and the Riesz-Thorin interpolation theorem. For fixed $s \geq 0$ and $u_0 \in L_2(\Omega)$ continuity of $U(\cdot\,,s)u_0$ follows from (2.17) and Theorem 2.4. We next define the evolution system $U(\cdot\,,\cdot)$ on $L_p(\Omega)$ for $p \in (1,2)$. As in the semigroup case we do this by duality. The above arguments clearly remain true if we replace U by the dual system $U^\sharp$ associated with $\big(A^\sharp(T-t)\big)_{t\in[0,T]}$. For $0 \leq s \leq t \leq T$ ($T>0$ arbitrary) we define

$$U_p(t,s) := \big(U^\sharp_{p'}(T-s,T-t)\big)'$$

for $p \in (1,2)$. By Theorem 2.6 $U_p(t,s)$ and $U_2(t,s) = U(t,s)$ coincide on $L_2 \cap L_p$, so that $U_p(t,s)$ is an extension of $U(t,s)$. Moreover, as the norm of the dual operator is the same as for the original one (5.1) follows. Continuity follows from the Riesz-Thorin interpolation theorem as before. This completes the proof of the theorem. $\square$

Next we prove that $U(t,s)$ maps L_2 into L_∞. Apart from Proposition 3.4 Nash inequalities come into action. In case (2.7), (2.9) and (2.10) there is always such an inequality. Indeed, if we set q_V as in (3.15) then by Theorem 4.1, Remark 4.3 and Example 4.8 inequality (4.7) holds with

$$d := \begin{cases} N & \text{in case (2.7) and (2.9)} \\ 2N\ (N \geq 2) & \text{in case (2.10)} \end{cases} \tag{5.2}$$

In case (2.8) we need to assume that Ω has some additional regularity properties. We suppose that there exists $d > 0$ such that (4.7) is satisfied. Geometric properties ensuring this are given in Example 4.6 and 4.7. By Theorem 4.4 it holds with $d = N$ if Ω satisfies an interior cone condition.

5.2. Theorem. *Suppose that one of the above assumptions is satisfied. Moreover, suppose that*

$$\|U(t,s)\|_{2,2} \leq g_2(t-s) \tag{5.3}$$

for some nondecreasing function $g_2\colon [0,\infty) \to [0,\infty)$. *Then, there exists a constant* $c_d > 0$ *depending only on* d *such that*

$$\|U(t,s)\|_{2,\infty} \leq c_d(\varepsilon\alpha)^{-\frac{d}{4}} C^{\frac{d}{4}} (t-s)^{-\frac{d}{4}} e^{\varepsilon\delta(t-s)} g_2(t-s) \tag{5.4}$$

where α *and* δ *are as in* (3.14) *and* (3.13), *and* C *from the Nash inequality* (4.7).

The proof will be given in a series of lemmas. The method we use is a variation of the arguments given in [29, pp. 261–264] in case of diffusion semigroups on Lie Groups. For the readers convenience we include the full proof. Before we do so we discuss some consequences. First of all we conclude that $U(t,s) \in \mathcal{L}(L_2, L_\infty)$ for all $t > s$. The same is true if we replace $U(t,s)$ by the adjoint $U^\sharp(t,s)$ associated with $\big(A^\sharp(T-t)\big)_{t\in[0,T]}$. Then, for $0 \leq s < t \leq T$ we define

$$U_1(t,s) := \big(U^\sharp(T-s,T-t)\big)'|_{L_1(\Omega)} \in \mathcal{L}(L_1, L_2).$$

As $U(t,s) = \big(U^\sharp(T-s,T-t)\big)'$ as operators on L_2 we find that $U_1(t,s)u_0 = U(t,s)u_0$ for all $u_0 \in L_1 \cap L_2$. Moreover, as $T > 0$ is arbitrary we can define $U_1(t,s)$ for all $0 \leq s < t < \infty$. We then get the following estimate.

5.3. Corollary. *Suppose that the assumptions of Theorem 5.2 are satisfied, and that* $\|U^\sharp(t,s)\|_{2,2} \leq g_2(t-s)$ *as well. Then there exists a constant* C_d *depending only on* $d > 0$ *such that*

$$\|U_1(t,s)\|_{1,\infty} \leq C_d(\varepsilon\alpha)^{-\frac{d}{2}} C^{\frac{d}{2}} (t-s)^{-\frac{d}{2}} e^{\varepsilon\delta(t-s)} \big(g_2((t-s)/2)\big)^2 \tag{5.5}$$

for all $t > s \geq 0$.

Proof. It is sufficient to prove the estimate for $0 \leq s < t \leq T$ for every $T > 0$. As the dual operator has the same norm as the original one we have that

$$\|U_1(t,s)\|_{1,2} = \big\|\big(U^\sharp(T-s,T-t)\big)'\big\|_{1,2} = \|U^\sharp(T-s,T-t)\|_{2,\infty}$$

Obviously, Theorem 5.2 also applies to $U^\sharp(\cdot\,,\cdot)$, and as $(T-s)-(T-t) = t-s$ we have

$$\|U_1(t,s)\|_{1,2} \leq c_d(\varepsilon\alpha)^{-\frac{d}{4}} C^{\frac{d}{4}} (t-s)^{-\frac{d}{4}} e^{\varepsilon\delta(t-s)} g_2(t-s). \tag{5.6}$$

Finally, as $t-(t+s)/2 = (t+s)/2 - s = (t-s)/2$ it follows from (5.4) and (5.6) that

$$\begin{aligned}\|U_1(t,s)\|_{1,\infty} &\leq \|U(t,(t+s)/2)\|_{2,\infty}\|U((t+s)/2,s)\|_{1,2}\\ &\leq 2^{\frac{d}{2}} c_d^2(\varepsilon\alpha)^{-\frac{d}{2}} C^{\frac{d}{2}} (t-s)^{-\frac{d}{2}} e^{\varepsilon\delta(t-s)} \big(g_2((t-s)/2)\big)^2,\end{aligned}$$

which finishes the proof of the corollary. □

5.4. Remark. According to Theorem 5.1 we can always choose $g_2(t) := e^{\delta_0 t}$ in the above estimates. In that case, (5.5) becomes

$$\|U_1(t,s)\|_{1,\infty} \leq C_d(\varepsilon\alpha)^{-\frac{d}{2}} C^{\frac{d}{2}} (t-s)^{-\frac{d}{2}} e^{\varepsilon\delta(t-s)} e^{\delta_0(t-s)} \tag{5.7}$$

The remainder of the section is devoted to the proof of Theorem 5.2. For the calculations we fix $k \geq 1$, and set for arbitrary $p \geq 2$

$$\varphi_p(t) := \|w_{k,p}(t)\|_2^{\frac{2}{p}}, \tag{5.8}$$

where $w_{k,p}(t) = f_{k,p}(u(t))$ with $f_{k,p}$ defined by (3.1), and $u(\cdot)$ is the solution of (2.15) with $0 \neq u_0 \in L_2(\Omega)$. By Proposition 3.4 $\varphi_p(\cdot)$ is locally absolutely continuous and thus differentiable almost everywhere. As (2.15) is uniquely solvable it follows that $u(t) = 0$ for all $t \geq t_0$ if $u(t_0) = 0$ for some $t_0 > 0$. Hence, there exists an interval $J := [0, t_0)$ such that $\varphi_p(t) \neq 0$ for all $t \in [0, t_0)$ and $\varphi_p(t) = 0$ for $t \geq t_0$. (Note that, using the Harnack inequality for parabolic equations, one could show that $t_0 = \infty$ always. However, we want a proof of (1.5) which does not rely on the Harnack inequality.)

5.5. Lemma. *For all $t \in J$ and $p \geq 2$ we have*

$$\Big(\varphi_{2p}(t)e^{-(2p-1)\delta t}\Big)^{-\frac{4p}{d}} \geq \frac{2\alpha}{Cd}\int_0^t \Big(\varphi_p(t)e^{-(2p-1)\delta s}\Big)^{-\frac{4p}{d}}\,ds. \tag{5.9}$$

Proof. From Proposition 3.4 and the Nash inequality (4.7) we conclude that

$$\frac{d}{dt}\|w_{k,2p}(t)\|_2^2 \leq -\alpha C^{-1}\|w_{k,2p}(t)\|_2^{2+\frac{4}{d}}\|w_{k,2p}(t)\|_1^{-\frac{4}{d}} + 2p(2p-1)\delta\|w_{k,2p}(t)\|_2^2.$$

Next observe that $\|w_{k,2p}(t)\|_1^{\frac{1}{p}} \leq \varphi_p(t)$. Indeed, by definition of $w_{k,2p}$ we have that $|w_{k,p}|^2 = |w_{k,2p}|$ if $|u| \leq k$, and

$$|w_{k,p}|^2 = \Big(k^{\frac{p}{2}} + \frac{p}{2}k^{\frac{p}{2}-1}(|u|-k)\Big)^2 \geq k^p + pk^{p-1}(|u|-k) = |w_{k,2p}|$$

if $|u| \geq k$ which implies the inequality. Therefore, by definition of φ_p we obtain

$$\frac{d}{dt}\big(\varphi_{2p}(t)\big)^{2p} \leq -\alpha C^{-1}\big(\varphi_{2p}(t)\big)^{2p+\frac{4p}{d}}\big(\varphi_p(t)\big)^{-\frac{4p}{d}} + 2p(2p-1)\delta\big(\varphi_{2p}(t)\big)^{2p}$$

almost everywhere. This yields

$$\frac{d}{dt}\big(\varphi_{2p}(t)\big)^{-\frac{4p}{d}} + \frac{4p}{d}(2p-1)\delta\big(\varphi_{2p}(t)\big)^{-\frac{4p}{d}} \geq \frac{2\alpha}{Cd}\big(\varphi_{2p}(t)\big)^{-\frac{4p}{d}},$$

and finally,

$$\frac{d}{dt}\Big(\varphi_{2p}(t)e^{-(2p-1)\delta t}\Big)^{-\frac{4p}{d}} \geq \frac{2\alpha}{Cd}\Big(\varphi_p(t)e^{-(2p-1)\delta t}\Big)^{-\frac{4p}{d}}$$

for almost all $t \in J$. Now the assertion of the lemma follows by integration since $\varphi_p(\cdot)$ is locally absolutely continuous for all $p \geq 2$. □

For p of the form 2^n $(n \geq 1)$ and $\varepsilon \in (0,1]$ we inductively define the functions

$$f_2(t) := g_2(t)\|u_0\|_2 \qquad \text{and} \qquad f_{2p}(t) := (\varepsilon M)^{-\frac{d}{4p}} e^{\varepsilon\delta\frac{t}{p}}(p-1)^{\frac{d}{2p}} f_p(t),$$

where we set $M := \alpha(3Cd)^{-1}$. We then have the following result.

5.6. Lemma. *For all p of the form 2^n $(n \geq 1)$ we have that*

$$\varphi_p(t) \leq t^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})} f_p(t). \tag{5.10}$$

Proof. To prove the inequality we need some properties of the function

$$\sigma_\varepsilon(p) := 6\varepsilon^{-1}(p-1)\Big(1-\Big(1-\frac{2\varepsilon}{2p(2p-1)}\Big)^{p-1}\Big)$$

which is defined for all $\varepsilon \in (0,1]$ and $p \geq 2$. For every fixed $\varepsilon \in (0,1]$ this turns out to be an increasing function of $p \geq 2$. To see this rewrite the function in the form

$$\sigma_\varepsilon(p) = 6\varepsilon^{-1}(p-1)\gamma_\varepsilon(p)\frac{e^{-\gamma_\varepsilon(p)}-1}{-\gamma_\varepsilon(p)},$$

where $\gamma_\varepsilon(p) := -(p-1)\log(1-2\varepsilon/2p(2p-1))$. An elementary calculation then shows that $\gamma_\varepsilon(p)$ is decreasing, and $(p-1)\gamma_\varepsilon(p)$ increasing in $p \geq 2$. Hence, as a product of increasing two increasing functions σ_ε is increasing. This shows that

$$\inf_{p\geq 2}\sigma_\varepsilon(p) = \sigma_\varepsilon(2) = 1. \tag{5.11}$$

We now use induction to prove (5.10). By assumption, it holds for $p=2$. Suppose now it is true for p. Then, by Lemma 5.5 and the fact that f_p is increasing we get that for $t \in J$

$$\begin{aligned}
\Big(\varphi_{2p}(t)e^{-(2p-1)\delta t}\Big)^{-\frac{4p}{d}} &\geq 6M\int_0^t \Big(\varphi_p(t)e^{-(2p-1)\delta s}\Big)^{-\frac{4p}{d}}\,ds\\
&\geq 6M\big(f_p(t)\big)^{-\frac{4p}{d}}\int_0^t s^{p-2}e^{-\frac{4p}{d}(2p-1)\delta s}\,ds\\
&\geq 6M\big(f_p(t)\big)^{-\frac{4p}{d}}e^{-\frac{4p}{d}(2p-1)\delta(1-\frac{2\varepsilon}{2p(2p-1)})t}\int_{(1-\frac{2\varepsilon}{2p(2p-1)})t}^t s^{p-2}\,ds.
\end{aligned}$$

By cancelling the common exponential factor, computing the integral, and taking into account the definition of f_{2p} and σ_ε this implies that

$$\begin{aligned}
\big(\varphi_{2p}(t)\big)^{-\frac{4p}{d}} &\geq 6M\big(f_p(t)\big)^{-\frac{4p}{d}}e^{-\frac{4}{d}\varepsilon\delta t}\int_{(1-\frac{2\varepsilon}{2p(2p-1)})t}^t s^{p-2}\,ds\\
&= \sigma_\varepsilon(p)\big(f_{2p}(t)\big)^{-\frac{4p}{d}}t^{p-1} \geq \big(t^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{2p})}f_{2p}(t)\big)^{-\frac{4p}{d}},
\end{aligned}$$

where we used (5.11) in the last inequality. Hence, (5.10) holds for all $t \in J$. It trivially holds for larger t, which completes the proof of the lemma. $\square$

5.7. Lemma. *For all $t > 0$ and $\varepsilon \in (0, 1]$ we have that*

$$\|u(t)\|_\infty \le c_d(\varepsilon\alpha)^{-\frac{d}{4}} C^{\frac{d}{4}} t^{-\frac{d}{4}} e^{\varepsilon\delta t} f_2(t),$$

where c_d is a constant just depending on $d > 0$.

Proof. Note that $|w_{k,p}(t)| \nearrow |u(t)|^{\frac{p}{2}}$ as k tends to infinity. Hence, by definition (5.8) of φ_p we can replace φ_p by $\|u(t)\|_p$ in (5.10). Further, by induction we easily prove that

$$f_{2^{n+1}}(t) \le (M\varepsilon)^{-\frac{d}{4}\sum_{l=1}^n 2^{-l}} e^{\varepsilon\delta t \sum_{l=1}^n 2^{-l}} 2^{\frac{d}{2}\sum_{l=1}^n l2^{-l}} f_2(t).$$

Combining this, letting n to infinity, and recalling the definition of M we obtain

$$\|u(t)\|_\infty = \limsup_{n\to\infty} \|u(t)\|_{2^n} \le c_d(\varepsilon\alpha)^{-\frac{d}{4}} C^{\frac{d}{4}} t^{-\frac{d}{4}} f_2(t),$$

where $c_d := (3d)^{\frac{d}{4}} 2^{\frac{d}{2}\sum_{l=1}^\infty l2^{-l}} < \infty$. This finishes the proof. □

Theorem 5.2 is a simple consequence of the above result. It follows as $f_2(t) = g_2(t)\|u_0\|_2$, and because it is not relevant which initial time we choose.

6. Gaussian Estimates

Throughout this section we suppose that the assumptions of Theorem 5.2 are satisfied. Then, as shown in Theorem 5.1 and Corollary 5.3, $U(t, s) \in \mathcal{L}(L_2) \cap \mathcal{L}(L_1, L_\infty)$. Hence, the Dunford-Pettis Theorem makes sure that it is an integral operator with kernel $k(x, y; t, s) \in L_\infty(\Omega \times \Omega)$ for all $0 \le s < t \le \infty$ (see e.g. [4, Theorem 1.3] for a short proof). We use Davies perturbation method to prove estimates for $k(x, y; t, s)$ from the estimates in Corollary 5.3. Throughout, the constants involved are the same as in Corollary 5.3, unless otherwise stated.

6.1. Theorem. *Suppose the above assumptions hold. Then, there exist $\omega_1 \ge 0$ and $\omega_2 > 0$ such that for all $\varepsilon \in (0, 1]$, $0 \le s < t < \infty$ and $(x, y) \in \Omega$*

$$|k(x, y; t, s)| \le C_d(\alpha\varepsilon)^{-d/2} C^{d/2} (t - s)^{-d/2} \exp\Big(\omega_1(1 + \varepsilon)t + \frac{|x - y|^2}{4\omega_2(1 + \varepsilon)(t - s)}\Big) \qquad (6.1)$$

Moreover, ω_1 depends only on upper bounds for α_0^{-1}, $\|c_0^-\|_\infty$, $\|\mathbf{a}\|_\infty$ and $\|\mathbf{b}\|_\infty$, whereas ω_2 depends on upper bounds for $\|a_{ij}\|_\infty$ and α_0^{-1}. If $\|c_0^-\|_\infty = \|\mathbf{a}\|_\infty = \|\mathbf{b}\|_\infty = 0$ we can set $\omega_1 = 0$ in case (2.7) and (2.10), and $\omega_1 = \alpha_0$ in the other two cases.

We prove this theorem in a series of lemmas. We need to consider functions in the set

$$\Psi := \Big\{\psi \in C^\infty(\mathbb{R}^N) \text{ bounded: } \sum_{i=1}^N \|\partial_i\psi(x)\|_\infty^2 \le 1\Big\}$$

For every $\rho \in \mathbb{R}$ and $\psi \in \Psi$ we define

$$U^\rho(t, s)u_0 := e^{\rho\psi} U(t, s)(e^{-\rho\psi} u_0)$$

This definition makes sense because $e^{-\rho\psi} u_0 \in L_p$ whenever $u_0 \in L_p$ for all $p \in [1, \infty]$.

6.2. Lemma. *$U^\rho(\cdot\,,\cdot)$ is the evolution system associated with the form*

$$a^\rho(t;u,v) := \int_\Omega \big(a_{ij}(x,t)\partial_j u + a_i^\rho(x,t)u\big)\partial_i v + b_i^\rho(x,t)\partial_i u + c_0^\rho(x,t)u\big)v\,dx$$

in case of the Dirichlet and Neumann problem, and

$$a^\rho(t;u,v) := \int_\Omega \big(a_{ij}\partial_j u + a_i^\rho u\big)\partial_i v + b_i^\rho \partial_i u + c_0^\rho u\big)v\,dx + \int_{\partial\Omega} b_0 uv\,d\mathcal{H}_{N-1}$$

in case of the Robin problem, where

$$a_i^\rho = a_i - \rho\; a_{ij}\partial_j\psi \tag{6.2}$$

$$b_i^\rho = b_i + \rho\; a_{ji}\partial_j\psi \tag{6.3}$$

$$c_0^\rho = c_0 - \rho^2\; a_{ji}\partial_j\psi\partial_i\psi + \rho\; a_i\partial_i\psi - \rho\; b_i\partial_i\psi. \tag{6.4}$$

Here, as always, we use the summation convention.

Proof. First of all note that $e^{\rho\psi}v \in V$ for all $v \in V$. By definition of $U(\cdot\,,\cdot)$ and $U^\rho(\cdot\,,\cdot)$ for all $u_0 \in L_2(\Omega)$ and $v \in V$

$$\begin{aligned}\frac{\partial}{\partial t}\big\langle U^\rho(t,s)u_0, v\big\rangle &= \frac{\partial}{\partial t}\big\langle U(t,s)e^{-\rho\psi}u_0, e^{\rho\psi}v\big\rangle = \big\langle A(t)U(t,s)e^{-\rho\psi}u_0, e^{\rho\psi}v\big\rangle \\ &= a(t; U(t,s)e^{-\rho\psi}u_0, e^{-\rho\psi}v) = a(t; e^{-\rho\psi}U^\rho(t,s)u_0, e^{\rho\psi}v)\end{aligned}$$

in the sense of distributions (see Remark 2.3). Moreover, by an elementary calculation $a(t; e^{-\rho\psi}u, e^{\rho\psi}v) = a^\rho(t;u,v)$ for all $u, v \in V$, which completes the proof of the lemma. $\square$

6.3. Lemma. *There exist $\omega_1 \geq 0$ and $\omega_2 \geq 0$ such that for all $\psi \in \Psi$ and $\rho \in \mathbb{R}$*

$$\|U^\rho(t,s)\|_{1,\infty} \leq C_d(\alpha\varepsilon)^{-\frac{d}{2}} C^{\frac{d}{2}} (t-s)^{-\frac{d}{2}} e^{\omega_1(1+\varepsilon)(t-s)} e^{\rho^2\omega_2(1+\varepsilon)(t-s)}. \tag{6.5}$$

The constant ω_1 just depends on upper bounds for α_0^{-1}, $\|c_0^-\|_\infty$, $\|\mathbf{a}\|_\infty$ and $\|\mathbf{b}\|_\infty$, whereas ω_2 depends on upper bounds for α_0^{-1} and

$$\|[a_{ij}]\|_\infty := \Big(\sum_{i,j=1}^N \|a_{ij}\|_\infty^2\Big)^{1/2}.$$

Moreover, if $\|c_0^-\|_\infty = \|\mathbf{a}\|_\infty = \|\mathbf{b}\|_\infty = 0$ then we can set $\omega_1 = 0$ in case (2.7) *and* (2.10)*, and $\omega_1 = \alpha_0^{-1}$ in the other two cases.*

Proof. We apply Corollary 5.3 to $U^\rho(t,s)$ and estimate the constant δ_0 using (6.2)–(6.4). As $\|\nabla\psi\|_\infty \leq 1$ we get that

$$\begin{aligned}\|(c_0^\rho)^-\|_\infty &\leq \|c_0^-\|_\infty + \rho^2\|[a_{ij}]\|_\infty + \rho(\|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty) \\ &\leq \|c_0^-\|_\infty + \|\mathbf{a}\|_\infty^2 + \|\mathbf{b}\|_\infty^2 + \rho^2\big(1 + \|[a_{ij}]\|_\infty\big)\end{aligned}$$

and

$$\|\mathbf{a}^\rho\|_\infty^2 + \|\mathbf{b}^\rho\|_\infty^2 \leq 2\big(\|\mathbf{a}\|_\infty^2 + \|\mathbf{b}\|_\infty^2\big) + 4\rho^2\|[a_{ij}]\|_\infty^2.$$

Define now

$$\omega_1 := \begin{cases} \tilde{\omega}_1 & \text{in case (2.7) and (2.10)} \\ \tilde{\omega}_1 + \alpha_0 & \text{in case (2.8) and (2.9),} \end{cases}$$

where

$$\tilde{\omega}_1 := \|c_0^-\|_\infty + (1 + 2\alpha_0^{-1})\big(\|\mathbf{a}\|_\infty^2 + \|\mathbf{b}\|_\infty^2\big),$$

and

$$\omega_2 := 1 + \|[a_{ij}]\|_\infty + 4\alpha_0^{-1}\|[a_{ij}]\|_\infty^2.$$

The assertion of the lemma follows now from Corollary 5.3, Remark 5.4 and the definition (3.4) and (3.13) of δ_0 and δ, respectively. □

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. If k_ρ is the kernel of $U^\rho(t,s)$ we clearly have that

$$k(x,y;t,s) = e^{-\rho(\psi(x)-\psi(y))}k_\rho(x,y;t,s).$$

For the following we fix $0 \leq s < t < \infty$. Further, all inequalities are meant to be true for almost all $(x,y) \in \Omega \times \Omega$. The above lemma implies that

$$|k(x,y;t,s)| = C_0(t-s)^{-\frac{d}{2}}e^{\omega_1(1+\varepsilon)(t-s)}e^{\rho^2\omega_2(1+\varepsilon)(t-s)}e^{-\rho(\psi(x)-\psi(y))}$$

for all $\rho \in \mathbb{R}$ and $\psi \in \Psi$, where $C_0 := C_d(\alpha\varepsilon)^{-\frac{d}{2}}C^{\frac{d}{2}}$. Replacing ρ by $-\rho$ we obtain

$$|k(x,y;t,s)| = C_0(t-s)^{-\frac{d}{2}}e^{\omega_1(1+\varepsilon)(t-s)}e^{\rho^2\omega_2(1+\varepsilon)(t-s)}e^{-\rho|\psi(x)-\psi(y)|}.$$

Note that $|x-y| = \sup_{\psi\in\Psi}|\psi(x)-\psi(y)|$. Minimising the right hand side of the above inequality over $\psi \in \Psi$ we therefore obtain

$$|k(x,y;t,s)| = C_0(t-s)^{-\frac{d}{2}}e^{\omega_1(1+\varepsilon)(t-s)}e^{\rho^2\omega_2(1+\varepsilon)(t-s)}e^{-\rho|x-y|}$$

for all $\rho \in \mathbb{R}$. Here we used a technical lemma on infima of measurable functions (see [5, Lemma 3.4]). The right hand side of this inequality takes its minimum over $\rho \in \mathbb{R}$ at $\rho = \big(2\omega_2(1+\varepsilon)(t-s)\big)^{-1}|x-y|$. Hence, (6.1) follows by minimising the right hand side of the above inequality over $\rho \in \mathbb{R}$, using again [5, Lemma 4.3]. The dependence of ω_1, ω_2 on the data follows from the last part of Lemma 6.3. This concludes the proof of Theorem 6.1. □

7. Exponential boundedness

In this section we discuss the behaviour of $U(t,s)$ on L_1 and L_∞. From Theorem 5.1 it is not clear whether $U(t,s)$ is exponentially bounded on these spaces, that is, whether there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that for $p = 1, \infty$ and $0 \leq s \leq t$

$$\|U(t,s)\|_{p,p} \leq Me^{\omega(t-s)}. \tag{7.1}$$

However, if $d = N$ in the Gaussian estimate this is true as the following proposition shows.

7.1. Proposition. *If $d = N$ in (6.1) then, there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that (7.1) holds for $p = 1, \infty$ and $0 \leq s \leq t < \infty$. The constant ω only depends on upper bounds for the L_∞-norm of the coefficients of $\mathcal{A}$ and α^{-1}, whereas M also depends on the constant from the Nash inequality (4.7)*

Proof. From (6.1) we conclude that for some $M \geq 1$ and $\omega \in \mathbb{R}$

$$\int_\Omega |k(x, y; t, s)|\, dy \leq M(t-s)^{\frac{N}{2}-\frac{d}{2}} e^{\omega(t-s)} \tag{7.2}$$

for all $0 \leq s < t < \infty$. The assertion of the proposition easily follows from this. □

Recall, that d is the number appearing in the Nash inequality (4.7). Therefore, $d = N$ for Dirichlet boundary condition no matter how Ω looks like (Theorem 4.1), and for Neumann or Robin boundary conditions if Ω satisfies an interior cone condition (Theorem 4.4 and Example 4.8).

From the Riesz-Thorin interpolation theorem and Corollary 5.3 we therefore get the following result.

7.2. Corollary. *If $d = N$ in (6.1) then, there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|U(t, s)\|_{p,q} \leq M(t-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} e^{\omega(t-s)} \tag{7.3}$$

for all $1 \leq p \leq q \leq \infty$ and $t - s > 0$, where M, ω are as in the above Proposition.

7.3. Remark. For $d > N$ the above argument to get exponential bounds on L_1 and L_∞ does not work as in this case a singularity appears on the right hand side of (7.2) in $(t-s) = 0$. However, the estimate (7.2) was very crude. We replaced the integral over Ω by an integral over $\mathbb{R}^N$. The singularity $t^{-\frac{d}{2}}$ in (6.1) can only occur near a "bad point" of the boundary of Ω. Indeed, this follows since for any weak solution of (1.1) we always have a local lower Gaussian bound (e.g [7, Theorem 8]. Hence, it is still possible that $k(x, \cdot\,; t, s)$ and $k(\cdot\,, y; t, s)$ are bounded in $L_1(\Omega)$ uniformly with respect to $0 < t - s \leq 1$ and $x, y \in \Omega$ but it seems not to be obvious.

We that this is true in some cases. Due to Theorem 5.1 the estimate (7.1) holds with $M = 1$ if the drift terms are zero. Here, we do not need the Gaussian bounds but only one of the assumptions (2.7)–(2.10). (We can assume without loss of generality that $c_0 \geq 0$ by replacing c_0 by $c_0 + \lambda$ for $\lambda \geq \|c_0\|_\infty$. If the solution of the new problem is $v(t)$ then the solution of the original is $e^{\lambda(t-s)}u(t)$.)

As in the semigroup case we can get exponential bounds if either a_i or b_i vanish, but only for p larger or smaller than two.

7.4. Proposition. *Suppose that one of the assumptions (2.7)–(2.10) are satisfied. Then, if $a_i = 0$ for all $i = 1, \ldots, N$ then, (7.1) holds for $p \in [2, \infty]$, and if $b_i = 0$ for all $i = 1, \ldots, N$ for $p \in [1, 2]$. In both cases we can choose $M = 1$ and $\omega = \delta_0$.*

Proof. We suppose that $a_i = 0$ for all $i = 1, \dots, N$. Then, an analysis of the proof of Lemma 3.1 shows that (3.3) holds with $p(p-1)\delta_0$ replaced by $p\delta_0$ on the right hand side. (Dealing with the coefficients b_i we used a crude estimate to make (3.3) symmetric in a_i and b_i.) Then, using the same arguments as in the proof of Theorem 5.1 we get the assertion for all $p \in [2, \infty)$, whence also for $p = \infty$. The assertion for $b_i = 0$ follows by the same duality argument we used to get (5.1) for $p \in (1, 2]$. □

8. Positivity and Comparison

The arguments leading to the upper Gaussian bound did not use the positivity of solutions for positive initial conditions. We nevertheless have the following result.

8.1. Proposition. *The evolution operator $U(t, s)$ is positive on L_p for all $p \in (1, \infty)$.*

Proof. Let u be the solution of (1.1) with nonnegative $u_0 \in L_2(\Omega)$, and set $v(t) := e^{-t\delta_0}u(t)$. Further, define $f(\xi) = 0$ for $\xi \geq 0$ and $f(\xi) = -\xi$ if $\xi \leq 0$. Note that $f(v(t)) = v^-(t)$ is the negative part of $v(t)$. By Lemma 3.3, Lemma 3.1 with $p = 2$, and the fact that u is the solution of (1.1) it follows that

$$\begin{aligned}\|v^-(t)\|_2^2 &= \|v^-(s)\|_2^2 - 2\int_s^t a(\tau; v(\tau), v^-(\tau)) + \delta_0 v(\tau)v^-(\tau)\, d\tau \\ &= \|v^-(s)\|_2^2 - 2\int_s^t a(\tau; v^-(\tau), v^-(\tau))\, d\tau + \delta_0\|v^-(\tau)\|_2^2 \leq \|v^-(s)\|_2^2.\end{aligned}$$

As $v^-(s) = u_0^- = 0$ by assumption it follows that $v^-(t) = 0$ for all $t \geq s$ and therefore $u(t)$ is nonnegative for all $t \geq s$. Hence, $U(t, s)$ is positive on $L_2(\Omega)$. On L_p for $p \neq 2$ this follows by a density argument. □

8.2. Corollary. *Suppose that $U(t, s)$ has a kernel $k(x, y; t, s)$. Then, $k(x, y; t, s) \geq 0$ almost everywhere.*

Proof. For all $u_0 \in L_2(\Omega)$, $u_0 \geq 0$ we have that $\int_\Omega k(x, y; t, s)u_0(y)\, dy = U(t, s)u_0(x) \geq 0$ by the above proposition. Hence, the assertion of the corollary follows. □

Next we prove a prove that the kernel of the Dirichlet problem on a sub-domain of Ω is always dominated by the kernel on Ω. To prove this result the idea is to regularize our problem and then use theorems for classical solutions and pass to the limit afterwards, a technique employed e.g. in [7] to prove the existence of weak solutions and a kernel to the Dirichlet problem (1.1) on any bounded domain. We point out that such a domination result is not true for other boundary conditions.

8.3. Theorem. *Suppose that $k_i(\cdot\,, \cdot\,; \cdot\,, \cdot)$ is the kernel associated with the Dirichlet problem on Ω_i, $i = 1, 2$. If $\Omega_1 \subset \Omega_2$ then $k_1(x, y; t, s) \leq k_2(x, y; t, s)$ almost everywhere. Here, we extend $k_1(x, y; t, s)$ by zero outside Ω.*

Proof. We show that for any $u_0 \in \mathcal{D}(\Omega_1)$ the corresponding solutions u_1 and u_2 of (1.1) on Ω_1 and Ω_2, respectively satisfy

$$u_1 \leq u_2 \tag{8.1}$$

almost everywhere. This implies that

$$u_1(x) = \int_{\Omega_1} k_1(x, y; t, s) u_0(y)\, dy \leq \int_{\Omega_1} k_2(x, y; t, s) u_0(y)\, dy = u_2(x)$$

for all $u_0 \in \mathcal{D}(\Omega_1)$. This proves the assertion of the theorem. It remains to prove (8.1) for $u_0 \in \mathcal{D}(\Omega_1)$. We use an approximation argument to prove (8.1). Suppose that Ω_{1n}, Ω_{2n} are smooth domains with $\Omega_{1n} \subset \Omega_{2n}$. We can choose them such that $\Omega_{in} \subset \Omega_i$, and $\bigcup_{n\in\mathbb{N}} \Omega_{in} = \Omega_i$ for $i = 1, 2$. Further we regularize the coefficients of $\mathcal{A}$ by setting $a_i = b_i = c_0 = 0$ and $a_{ij} = \delta_{ij}$ (the Kronecker symbol) in the complement of Ω_2, and convolving them with an appropriate smoothing kernel. This way we get operators $\mathcal{A}_n$ with uniformly bounded smooth coefficients converging to those of $\mathcal{A}$ pointwise almost everywhere. We then look at the regularized problem, where we replace Ω_i by Ω_{in} and $\mathcal{A}$ by $\mathcal{A}_n$. As for n large enough $u_0 \in \mathcal{D}(\Omega_{1n})$ it follows that the solutions u_{1n} and u_{2n} of the regularized problem on Ω_{1n} and Ω_{2n} are smooth (see [22, Theorem IV.5.1]). Hence, by the maximum principle (see [28, Theorem 3.7 and subsequent Remark (ii)]) we have that $u_{1n} \leq u_{2n}$ for all $n \in \mathbb{N}$. By Lemma 8.4 below $u_{in} \rightharpoonup u_i$ weakly in $\mathcal{L}_2(\Omega \times (0, T))$ for all $T > 0$ and $i = 1, 2$. As $u_{2n} - u_{1n} \geq 0$ this implies that

$$\int_0^T \langle u_2(t) - u_1(t), v(t)\rangle\, dt \geq 0$$

for all $v \in L_2(\Omega \times (0, T)$ with $v \geq 0$. This clearly implies (8.1) and thus finishes the proof of the theorem. □

We made use of the following approximation theorem. For bounded domains, much more general domain perturbation results can be found in [12]. Their proof is based on the estimates in Corollary 7.2.

8.4. Lemma. *Suppose that $\Omega_n \subset \Omega$ is a sequence of domains with $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. Further suppose that $\mathcal{A}_n$ is of the form (1.2) with coefficients $a_{ij}^{(n)}, a_i^{(n)}, b_i^{(n)}, c_i^{(n)}$ bounded in $L_\infty(\Omega \times (0, \infty))$ uniformly with respect to $n \in \mathbb{N}$, and converging pointwise almost everywhere to the corresponding coefficients of $\mathcal{A}$. Suppose that u_n is the solution of*

$$\begin{aligned} \partial_t u + \mathcal{A}_n(t) u &= 0 && \text{in } \Omega_n \times (0, \infty)\\ u &= 0 && \text{on } \partial\Omega_n \times (0, \infty)\\ u &= u_{0n} && \text{on } \Omega_n, \end{aligned} \tag{8.2}$$

with u_{0n} converging to u_0 weakly in $L_2(\Omega)$. Then, u_n converges to the solution u of (1.1) weakly in $L_2((0, T), \mathring{W}_2^1(\Omega))$ for all $T > 0$ as n tends to infinity.

Proof. We first show that u_n is bounded in $L_2((0,T),\mathring{W}_2^1(\Omega_n))$. To do so first note that by Lemma 3.1 (with $p=2$) we have that

$$\int_0^T \|u_n(t)\|^2_{W_2^1(\Omega_n)}\,dt \le 2\int_0^T a_n(t;u_n(t),u_n(t))\,dt + (\delta_0+\alpha_0)\int_0^T \|u_n(t)\|_2^2\,dt,$$

where $a_n(t;\cdot\,,\cdot)$ is the form corresponding to $\mathcal{A}_n$. Using that u_n is a weak solution of (8.2) and Lemma 3.3 (with $f=$ identity) we obtain

$$2\Big|\int_0^T a_n(t;u_n(t),u_n(t))\,dt\Big| = \big|\|u_n(0)\|_2^2 - \|u_n(T)\|_2^2\big| \le (1+e^{2\delta_0 T})\|u_{0n}\|_2^2,$$

where we used Theorem 5.1 for the last inequality. Also from Theorem 5.1 we conclude that

$$\int_0^T \|u_n(t)\|_2^2\,dt \le \frac{e^{2\delta_0 T}-1}{2\delta_0}\|u_{0n}\|_2^2$$

Putting everything together we see that

$$\int_0^T \|u_n(t)\|^2_{W_2^1(\Omega_n)}\,dt \le \Big(1+e^{2\delta_0 T}+\frac{e^{2\delta_0 T}-1}{2\delta_0}\Big)\|u_{0n}\|_2^2.$$

Hence, as δ_0 is bounded by a constant independent of $n\in\mathbb{N}$, and $\|u_{0n}\|_2$ bounded by the weak convergence, the solutions u_n are bounded in $L_2((0,T),\mathring{W}_2^1(\Omega))$. Here we continued the functions by zero outside Ω_n. This makes sure that u_n has a weakly convergent subsequence in $L_2((0,T),\mathring{W}_2^1(\Omega_n))$. We show that the limiting function u is the unique solution of (1.1), and so the whole sequence converges. if $v\in\mathcal{D}(\Omega)$ then $v\in\mathcal{D}(\Omega_n)$ for large $n\in\mathbb{N}$. Hence, for all $\varphi\in\mathcal{D}([0,T))$ and n large enough we have

$$-\int_0^T\langle u_n(t),v\rangle\dot\varphi(t)\,dt + \int_0^T a_n(t;u_n(t),v)\varphi(t)\,dt - \langle u_{0n},v\rangle\varphi(0)=0.$$

Passing to the limit, and using the fact that u_n and u_{0n} converge weakly we conclude that u satisfies (2.11) for all $\varphi\in\mathcal{D}([0,T))$ and $v\in\mathcal{D}(\Omega)$, showing that u is the weak solution of (1.1). □

References

1. Robert A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.
2. Herbert Amann, *Linear and quasilinear parabolic problems*, vol. I. Abstract Linear Theory, Birkhäuser, Basel, 1995.
3. W. Arendt, *Gaussian estimates and interpolation of the spectrum in L_p*, Differential and Integral Equations **7** (1994), 1153–1168.
4. W. Arendt and A.V. Bukhvalov, *Integral representations of resolvents and semigroups*, Forum Math. (1994), 111–135.
5. W. Arendt and A.F.M. ter Elst, *Gaussian estimates for second order elliptic operators with boundary conditions*, Preprint, 1995.
6. D.G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc. **73** (1967), 890–896.

7. ———, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa **22** (1968), 607–694.
8. D.G. Aronson and J. Serrin, *Local behavior of solutions of quasilinear parabolic equations*, Arch. Rational Mech. Anal. **25** (1967), 81–122.
9. Pascal Auscher, *Regularity theorems and hear kernel for elliptic operators*, J. London Math. Soc. **54** (1996), 284–296.
10. Pascal Auscher, Alan McIntosh, and Philippe Tchamitchian, *Heat kernels of second order complex elliptic operators and applications*, Preprint, 1994.
11. E.A. Carlen, S. Kusuoka, and D.W. Stroock, *Upper bounds for symmetric Markov transition functions*, Ann. Inst. H. Poincaré. Probab. Statist. **23** (1987), no. 2 (sup), 245–287.
12. Daniel Daners, *Domain perturbation for linear and nonlinear parabolic equations*, J. Differential Equations **129** (1996), 358–402.
13. ———, *Robin boundary value problems on arbitrary domains*, Tech. report, University of Sydney, 1996.
14. R. Dautray and J.L. Lions, *Mathematical analysis and numerical methods for science and technology*, vol. 5, Springer, Berlin, 1992.
15. E.B. Davies, *Explicit constants for Gaussian upper bounds on heat kernels*, Amer. J. Math. **109** (1987), 319–334.
16. ———, *Heat kernels and spectral theory*, Cambridge University Press, Cambridge, 1989.
17. Xuan T. Duong and Derek W. Robinson, *Semigroup kernels, poisson bounds, and holomorphic fuctional calculus*, J. Funct. Anal. **142** (1996), 89–128.
18. A.F.M. ter Elst and Derek W. Robinson, *Second-order subelliptic operators on Lie groups II: real measurable principal coefficients*, Tech. report, Australian National University, 1996.
19. E.B. Fabes and D.W. Stroock, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Rational Mech. Anal. **96** (1986), 327–338.
20. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, Berlin, 1983.
21. Edwin Hewitt and Karl Stromberg, *Real and abstract analysis*, Springer, Berlin, 1965.
22. O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc., Providence, Rhode Island, 1968.
23. M. Marcus and V.J. Mizel, *Every superposition operator mapping one Sobolev space into another is continuous*, J. Funct. Anal. **33** (1979), 217–229.
24. V.G. Maz'ja, *Classes of regions and embedding theorems for function spaces*, Soviet Math. Dokl. **1** (1960), 882–885.
25. ———, *Sobolev spaces*, Springer, Berlin, 1985.
26. Jindřich Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967.
27. El-Maati Ouhabaz, *L_∞-contractivity of semigroups generated by sectorial forms*, J. London Math. Soc. **46** (1991), 529–542.
28. M.H. Protter and H.F. Weinberger, *Maximum principles in differential equations*, Prentice Hall, Englewood Cliffs, NJ, 1967.
29. Derek W. Robinson, *Elliptic operators on Lie groups*, Oxford University Press, Oxford, 1991.
30. Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970.
31. N.Th. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and geometry on groups*, Cambridge University Press, Cambridge, 1992.