CHAZY CLASSES IX–XII
OF THIRD-ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study Classes IX–XII of the thirteen classes introduced by Chazy (1911) in his classification of third-order differential equations in the polynomial class having the Painlevé property. Classes IX and X are the only Chazy classes that have remained unsolved to this day. (There is an incorrect claim in the literature that these classes are unstable.) Here we construct their solutions in terms of hyperelliptic functions of genus 2, which are globally meromorphic. (We also add a parameter to Chazy Class X, overlooked in Chazy's original paper.) The method involves transforming to a more tractable class of fourth- and fifth-order differential equations, which is the subject of an accompanying paper (paper I). Most of the latter equations involve hyperelliptic functions and/or higher-order Painlevé transcendents. In the case of Chazy Class XI, the solution is elementary and well known, but there are interesting open problems associated with its coefficient functions, including the appearance of one of the aforementioned transcendents. Also, for completeness, we include a review of the well-known Classes III and XII, which have received much attention in the literature because of their movable natural barriers and beautiful symmetries. In an appendix, we present the full list of Chazy equations (in the third-order polynomial class) and the solutions of those that are not dealt with in the body of this paper.

Key words. Painlevé analysis, Painlevé transcendents, ordinary differential equations, hyperelliptic functions, nonlinear equations

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1. Introduction

In a celebrated paper, Chazy [1] attempted the complete classification of all third-order differential equations of the form,

$$y''' = F(x, y, y', y''), \tag{1.1}$$

the prime denoting d/dx, where F is a polynomial in y, y', and y'' and locally analytic in x, having the Painlevé property. Further details on this class can also be found in a major paper by Bureau [2]. A differential equation in the complex domain is said to have the Painlevé property if its general solution is free of movable branch points, i.e., branch points whose location depends on one or more integration constants [3, 4, 5]. Movable essential singularities or more complicated singular sets are permitted provided they are unbranched. (Painlevé-type equations that feature movable essential singularities were the subject of a recent study [6].)

Of the thirteen Chazy classes, denoted I–XIII, interesting open problems are associated with Classes IX, X, and XI. These equations are as follows:

IX:
$$y''' = 12(y')^2 + 72y^2y' + 54y^4 + \delta; \tag{1.2}$$

X:
$$\begin{aligned} y''' = {} & 6y^2y' + \tfrac{3}{11}(9+7\sqrt{3}\,)(y'+y^2)^2 - \tfrac{1}{22}(4-3\sqrt{3})\alpha y' \\ & + \tfrac{1}{44}(3-5\sqrt{3})\alpha y^2 - \tfrac{1}{352}(9+7\sqrt{3}\,)\alpha^2; \end{aligned} \tag{1.3}$$

XI:
$$\begin{aligned} y''' = {} & -2yy'' - 2(y')^2 + \frac{24}{N^2-1}\left\{y' + y^2 - g(x)\right\}^2 \\ & + g''(x) + \tfrac{1}{4}(N^2-1)(\alpha x + \beta) \end{aligned} \tag{1.4}$$

($N \geq 2$, N not a multiple of 6). The full version of Class X with $\alpha \neq 0$ is new here, the parameter α being overlooked in Chazy's original work. Also changing the sign of $\sqrt{3}$ throughout Class X yields an inequivalent equation, under the usual Möbius equivalence acting on Painlevé-type equations. The two versions, which we denote Chazy-X.a and Chazy-X.b, are related by a Bäcklund transformation, which is also new in this paper. In addition, we construct new first integrals for Classes IX and X of the second order and degrees 3 and 4, respectively.

The solutions of Chazy-IX and Chazy-X have not previously been found. In this paper, we solve these equations in terms of hyperelliptic functions of genus 2, the solution of Chazy-X being unexpectedly complicated. The method involves transforming to associated equations of fourth and fifth order, which lead to interesting Painlevé classification problems in their own right. The results of the latter work are the subject of the accompanying paper I [7] and a paper II in preparation [8]. The latter paper will complete the Painlevé classification of fourth-order equations in the polynomial class, earlier work on this problem having been carried out by Chazy [1, pp. 378–385], Bureau [2, pp. 74–116], Exton [9], and Martynov [10]. Our list contains 18 equations, denoted F-I, F-II, ..., F-XVIII, five of which are new (or at least new in the context of Painlevé analysis). Four involve higher-order Painlevé transcendents, of which two are well known, one is a reduction of a known soliton equation but has been overlooked in previous Painlevé classification attempts, and one (F-VI) is new. The hyperelliptic subcase of the latter plays an essential role in the solution of Chazy-X.

Of greatest relevance to the open problems in Chazy's classification are the class of fourth- and fifth-order Painlevé-type equations having the Bureau symbol $P2$. (See [11, 12, 13] for the explanation of Bureau symbols.) This restriction makes the Painlevé analysis substantially simpler and full details are presented in paper I. In the fourth-order case, the first six of the aforementioned 18 equations have Bureau symbol $P2$. They are as follows:

$$\text{F-I:} \qquad y^{(\mathrm{iv})} = 12yy'' + 12(y')^2 + (\alpha x+\beta)y' + 2\alpha y - \tfrac{1}{6}(\alpha x+\beta)^2; \tag{1.5}$$

$$\text{F-II:} \qquad y^{(\mathrm{iv})} = 3yy'' - 4(y')^2; \tag{1.6}$$

$$\text{F-III:} \qquad y^{(\mathrm{iv})} = 15yy'' + \tfrac{45}{4}(y')^2 - 15y^3 + \alpha y + \beta; \tag{1.7}$$

$$\text{F-IV:} \qquad y^{(\mathrm{iv})} = 30yy'' - 60y^3 + \alpha y + \beta; \tag{1.8}$$

$$\text{F-V:} \qquad y^{(\mathrm{iv})} = 20yy'' + 10(y')^2 - 40y^3 + \alpha y + \kappa x + \beta; \tag{1.9}$$

$$\text{F-VI:} \qquad y^{(\mathrm{iv})} = 18yy'' + 9(y')^2 - 24y^3 + \alpha y^2 + \tfrac{1}{9}\alpha^2 y + \kappa x + \beta. \tag{1.10}$$

The parameters α, β, and κ are arbitrary complex constants. The corresponding fifth-order equations Fif-I, ..., Fif-IV, which are natural extensions of F-III, ..., F-VI, respectively, are listed in [I, Section 2].

We have intentionally included the "Bureau barrier equation" F-II on the list, even though it is known to violate the Painlevé property [14, 15], because we anticipate that this equation will play an important role in studies of the relationship between integrability and natural barriers in the complex plane. A section in paper I is devoted to this equation as well as a new hierarchy with similar interesting structure.

When we transformed Chazy Class IX (by an elementary ansätz suggested in Chazy's original paper), we were fortunate enough to land on a fourth-order equation that we recognized, namely, a special case of F-V above. This equation, together with the corresponding fifth-order equation Fif-III [I, (6.2)], is well known, being a group-invariant reduction of the KdV5 equation [16]. Several authors have conjectured that the solution of (1.9) is an irreducible fourth-order Painlevé transcendent when $\kappa \neq 0$. The case relevant to Chazy-IX is the autonomous subcase $\kappa = 0$. The latter is an early member of a hierarchy of differential equations integrable in terms of hyperelliptic functions of every genus found by Drach [17] in 1919 using his own formulation of differential Galois theory. It is also only a short step away from the equations encountered by Kowalevski in her celebrated analysis of the integrable cases of the motion of a rigid body about a fixed point [18, 19] (see also Golubev [20]).

On the other hand, Chazy Class X (especially our augmented version with $\alpha \neq 0$) transformed into the most recalcitrant of the aforementioned fourth-order equations, namely, F-VI, which we believe is a new equation. The case $\kappa \neq 0$ defines a new fourth-order Painlevé transcendent and Fif-IV [I, (7.2)] is a further generalization. It is again the autonomous subcase $\kappa = 0$ that is needed for our present purposes.

We managed to solve the subcase $\kappa = 0$ of F-VI, and hence Chazy-X, in terms of hyperelliptic functions by a nontrivial application of Jacobi's postmultiplier theory. The results are given in paper I. Postmultipliers also played an important role in Kowalevski's work [18, 19, 20]. While attempting to get equation F-VI to reveal its secrets, we were able to use postmultiplier theory (or a simpler method) on all of the other associated fourth- and fifth-order equations that are integrable in terms of hyperelliptic functions. With the exception of Drach's example, we believe that all of these hyperelliptic solutions are new. Paper I includes a brief introduction to postmultiplier theory and uses it to solve F-III from first principles as well as the interesting equation (A.46) below.

Chazy speculated that Classes IX and X might be solved in terms of elliptic functions. This conjecture is correct in the case of the reduced Class X equation ($\alpha = 0$), but the solution is not doubly periodic. It involves two elliptic functions with different sets of periods. The hyperelliptic solutions of the full equations admit several other cases of degeneracy to simpler functions. Elementary Painlevé analysis shows, however, that no (nonconstant) solutions of Chazy-IX or Chazy-X can be doubly periodic because no finite collection of poles can have zero total residue and hence no period parallelogram can be formed.

The open problem associated with Chazy Class XI (1.4) is of a different character. Here the equation is very easily integrated in terms of a Riccati equation nested with a Painlevé-I equation, the coefficient function $g(x)$ being unrestricted. The interesting question concerns the constraint on $g(x)$ so that movable logarithms are absent at the resonance with resonance number N. (The equation with arbitrary $g(x)$ does not have the Painlevé property but is integrable anyway.) Chazy gave incomplete results about this constraint when $N \geq 8$, but the most interesting case is $N = 7$, where Chazy [1, p. 345] gave the fifth-order differential equation,

$$h^{(\mathrm{v})} = 15hh''' + \tfrac{75}{2}h'h'' - 45h^2h' - 24(\alpha x + \beta)h' - 48\alpha h, \tag{1.11}$$

for the function $h(x) := (4/15)g(x)$.

Except for a renaming of parameters, equation (1.11) is identical to equation Fif-I given by [I, (4.2)]. It is a group-invariant reduction of the Kaup-Kuperschmidt equation [21, 22, 23]. In [I, Section 4], we exhibit a first integral and then argue (but do not prove) that the generic solution must define a new fourth-order Painlevé transcendent when $\alpha \neq 0$. When $\alpha \neq 0$, we find a four-parameter particular solution, given by (5.60) below, expressed in terms of two distinct Painlevé-I functions. When $\alpha = 0$, the general solution is constructed from first principles using postmultiplier theory. These solutions involve hyperelliptic functions of genus 2. Since there do not appear to be enough free parameters to build the general solution of (1.9) out of hyperelliptic and Painlevé-I functions together, or known transcendents that generalize these functions, we conclude that the solution defines a new transcendent.

Chazy's original study of third- and higher-order equations was motivated by a search for new higher-order transcendents analogous to the classical Painlevé transcendents. Although he did encounter a number of differential equations that he was unable to solve, he speculated in each case that the solutions would be expressed in terms of known functions such as Painlevé transcendents or elliptic functions. When he encountered equation (1.11), he compared it to the many difficult differential equations for coefficient functions that had already appeared in earlier work on Painlevé classification, most notably Gambier's work [24] on the Painlevé-Gambier (PG) class (see Ince [3, Chapter 14] and Bureau [11]). In every case that Gambier encountered, he was able to express the coefficient functions in terms of Painlevé transcendents or simpler functions. In fact, the PG class exhibited a pattern that is not followed in other Painlevé classification problems: if the constants of integration appeared inside a particular Painlevé transcendent, then the coefficient functions were built out of Painlevé transcendents of the same species or else simpler functions. The most impressive example is Ince-XLVIII (Gambier's equation (50)). Chazy Class XI with $N = 7$ is the first known example of an equation of order lower than the fourth where a genuine higher-order transcendent appears in its coefficients.

While the case $N = 7$ of Chazy-XI has a new transcendent in its coefficients, and obviously the general solution of Chazy-XI depends on its coefficient functions, we want to argue against an interpretation that Chazy-XI itself defines a new higher-order Painlevé transcendent. If such a claim were accepted, then Riccati and linear equations and many of the linearizable equations in Ince [3, Chapter 14] would define new transcendents whenever new transcendents were substituted in place of arbitrary coefficient functions. According to Ince [3, p. 318], the transcendence of a solution should be judged on how the solution behaves as a function of its integration constants rather than its independent variable. While Chazy-XI does not satisfy Ince's definition of semi-transcendental, we would like to enlarge the definition of semi-transcendental to capture equations like Chazy-XI, especially since the idea of nesting a Riccati equation and a Painlevé equation can be generalized in a variety of ways. (For example, the fourth-order equations F-X and F-XI in [7, 8] involve the nesting of a Riccati equation and a Chazy-XIII equation.)

There is a long-standing conjecture, attributed to Chazy, that no Painlevé-type equations of the third order involve new transcendents in their general solutions. We do not have a counterexample, although we remark that Painlevé classification of rational third-order equations and higher-degree equations has hardly begun [1, 25, 26, 27, 28]. A number of researchers have communicated privately certain philosophical arguments in support of the conjecture, based on inverse scattering and isomonodromy theory. The lowest differential order at which the constants of integration are locked inside a higher-order Painlevé transcendent appears to be four. Nevertheless, it is not true, as has often been stated, that Chazy was unsuccessful in his search for new transcendents, because he did discover Fif-I (1.11) in its full generality. He also discovered equation F-IV [1, p. 380], given by (1.8) above, which, though hyperelliptic, can be lifted up to the equation for the transcendent Fif-II [I, (5.2)], which is related to Fif-I, by a routine calculation.

Bureau's paper includes a thorough treatment of Chazy Classes I–VIII (except III) and XIII, after which he directed his attention to fourth-order equations [2, pp. 74–116]. (His solution of Chazy-VI is postponed to a later paper [29]; see the Appendix below.) In [2, p. 39], Bureau stated that he intended to write a sequel covering Chazy Classes IX–XII. However, that sequel apparently never eventuated and Exton [9] attempted to fill in the missing details on Classes IX–XII as well as generalize the work to polynomial-type differential equations of higher order. Unfortunately there were major errors in Exton's analysis, which led to his conclusion that only trivial Painlevé equations exist at orders six and above in the polynomial class. This contradicts, for example, the aforementioned work of Drach [17] on the hyperelliptic hierarchy as well as the fact that numerous soliton hierarchies admit one-dimensional reductions to Painlevé transcendent hierarchies in the polynomial class. An early example of a hierarchy starting with Painlevé-II, together with a method of solution by the inverse scattering transform, was given by Ablowitz and Segur [30]. Exton also gave an incorrect proof [9, p. 398] that Chazy Class IX is not "stable" (which is Bureau's term describing a differential equation possessing the Painlevé property) and claimed that the same argument would also show that the reduced Class X equation is not stable. The solutions given below in Sections 3 and 4 are certainly stable because they are, in fact, globally meromorphic.

The present paper could be regarded as an alternative sequel to Bureau's work. To that end, we have included a review of the well known equations Chazy-III and XII, which admit movable natural barriers, in Section 6. We include our preferred method of determining recessive terms in these classes, the results being in agreement with Conte, Fordy, and Pickering [14]. We also include an invertible map from Chazy-XII to XI which appears to have gone unnoticed in the literature and examine one of the related equations introduced by Clarkson and Olver [31].

The Appendix contains a full list of Chazy equations in the third-order polynomial class together with the solutions of those that are not given detailed treatment in the body of this paper. Some of these results are scattered over several sources. We include a new direct solution of Chazy-VI and the unification of the various subcases of Chazy-I into a single master equation.

2. The Chazy polynomial class

Chazy [1, 32] proved that Painlevé-type differential equations of the third order in the polynomial class must take the form,

$$\begin{aligned} y''' \;=\; & Qyy'' + R(y')^2 + Sy^2y' + Ty^4 + A(x)y'' + B(x)yy' \\ & + C(x)y' + D(x)y^3 + E(x)y^2 + F(x)y + G(x), \end{aligned} \tag{2.1}$$

where, after a suitable normalization, Q, R, S, and T are certain rational numbers and the remaining coefficients are locally analytic functions of the complex variable x to be determined.

Most steps in that proof were quite routine, and follow methods established by Painlevé [4] for second-order equations, but the particular case,

$$y''' = y^ny'' - (n+1)y^{n-1}(y')^2 \qquad (n \geq 1), \tag{2.2}$$

presented special difficulties, and Chazy devoted a separate paper [32] to proving that this case always admits movable branch points. Cosgrove and Scoufis [33] gave a two-parameter formal series solution of (2.2) in descending powers of $\log(x-x_0)$, each coefficient being a polynomial in $\log\log(x-x_0)$, where x_0 is a constant of integration. We have not investigated the convergence of this series. (Note that for a differential equation of the first degree, where singular integrals cannot occur, any restricted solution is necessarily embedded in the general solution, however nongeneric it may be.)

An analogous two-parameter expansion can be given for the mth-order equation,

$$y^{(m)} \;=\; (m-2)y^ny'' - (n+m-2)y^{n-1}(y')^2 \tag{2.3}$$

($m \geq 3$, $n \geq 1$), the leading term of which is

$$y \sim \left((-1)^m \frac{\Gamma(m+(m-2)/n)}{\Gamma(1+(m-2)/n)}\right)^{1/n}$$

$$\times \left(\frac{\log(x-x_0)}{(x-x_0)^{m-2}}\right)^{1/n} \left\{1 + \mathrm{O}\left(\frac{\log\log(x-x_0)}{\log(x-x_0)}\right)\right\}. \tag{2.4}$$

In the same way, it can be shown that the equation,

$$y^{(\mathrm{iv})} = ayy''' + by'y'' + c(y^2y'' - 2y(y')^2), \tag{2.5}$$

never has the Painlevé property when $c \neq 0$ despite the fact that eight particular cases have admissible integer resonances, as observed by Martynov [10]. The special case $b = -3a$ was observed by Chazy [1, p. 378] to be problematic. Equations of this type were also considered by Valiron [34].

Having established the restricted form (2.1) for equations in the Chazy polynomial class, the next step in the Painlevé classification is to find all admissible reduced equations associated with the relevant Bureau symbols. Bureau introduced the notation, $P1$, $P2$, $Z1$, $Z2$, etc., in connection with his careful reworking of the Painlevé-Gambier classification problem in [11]. The notation has proved extremely useful in the Painlevé classification of rational second-order equations of degree two or more [6, 33, 35] and second-order partial differential equations [12, 13]. In the case of higher-order equations, the notation continues to be very useful provided no derivatives of the dependent variable appear in denominators, in which case a suitable generalization would need to be invented.

The differential equation (2.1) admits the Bureau symbol Pk, $k > 0$, if the α-test,

$$x = x_0 + \alpha X, \qquad y = \alpha^{-k}Y, \qquad \alpha \to 0, \tag{2.6}$$

yields a reduced equation containing the term Y''' and at least one other term. If no value of k yields such a reduced equation, as in the case of the linear equation, the Bureau symbol $P\infty$ is assigned. For equations of the first degree in the highest derivative, the index k must be a positive integer not exceeding the order of the highest derivative unless $k = \infty$. However, for equations of higher degree, fractional indices must be considered because singular integrals of Painlevé-type equations may admit movable branch points, as observed by Chazy [36]. In [6], we called such equations "exotic Painlevé equations" and constructed several examples.

The only indices compatible with (2.1) are $k = 1$, $3/2$, 2, 3, and ∞. The Bureau symbol is $P1$ when not all of the constants Q, R, S, and T are zero. The reduced equation associated with the Bureau symbol $P1$ is

$$Y''' = QYY'' + R(Y')^2 + SY^2Y' + TY^4. \tag{2.7}$$

Chazy [1] and Bureau [2] have determined all cases where (2.7) has the Painlevé property. Chazy listed twelve canonical reduced equations, denoted I–XII, and Bureau listed eleven because he chose to absorb Chazy-III into XII. (Note also that Bureau's Classes I, II, and III correspond to Chazy's Classes XIII, I, and II, respectively.) These reduced equations are listed below.

The problem depends on a number of Diophantine equations that provide conditions for resonance numbers to be integers. The case where Q and T are both nonzero presents the greatest difficulties, and we refer the reader to Bureau's paper [2, pp. 25–36] for the proof that this case yields Chazy Classes XI and XII uniquely. The case where $Q = 0$ contains Chazy-IX and X, which have the interesting property that certain families of poles have irrational residues.

If Q, R, S, and T are all zero while $D(x) \neq 0$, the Bureau symbol is $P\frac{3}{2}$ and the corresponding reduced equation is

$$Y''' = DY^3, \tag{2.8}$$

where D is a nonzero constant. This equation is not a Painlevé-type equation on account of its fractional index $k = 3/2$ and also because its resonance numbers are not integers. So the vanishing of Q, R, S, and T forces the coefficient $D(x)$ to vanish also. If $B(x)$ is nonzero, then the Bureau symbol is $P2$ and the corresponding reduced equation is

$$Y''' = BYY', \tag{2.9}$$

where B is a nonzero constant. This equation is integrable with meromorphic (in fact, elliptic) functions and so has the Painlevé property. It defines the last of the Chazy classes, Class XIII (which is the first of the Bureau classes, denoted I in [2]). If $B(x)$ is also zero, but $E(x)$ is nonzero, the Bureau symbol is $P3$ and the corresponding reduced equation is

$$Y''' = EY^2, \tag{2.10}$$

where E is a nonzero constant. However, equation (2.10) admits the noninteger resonance numbers $(13 \pm \sqrt{-71}\,)/2$ and so does not have the Painlevé property. Finally, if $E(x)$ is also zero, equation (2.1) collapses to a linear equation of the third order. This is the case where the Bureau symbol is $P\infty$, the reduced equation being $Y''' = 0$. The third-order linear equation belongs on the list of Painlevé-type equations, and so we denote it "Chazy Class 0."

The canonical reduced equations defining each of the Chazy classes is as follows:

0: $$Y''' = 0; \tag{2.11}$$

I: $$Y''' = -6(Y')^2; \tag{2.12}$$

II: $$Y''' = -2YY'' - 2(Y')^2; \tag{2.13}$$

III:	$Y''' = 2YY'' - 3(Y')^2;$		(2.14)
IV:	$Y''' = -3YY'' - 3(Y')^2 - 3Y^2Y';$		(2.15)
V:	$Y''' = -2YY'' - 4(Y')^2 - 2Y^2Y';$		(2.16)
VI:	$Y''' = -YY'' - 5(Y')^2 - Y^2Y';$		(2.17)
VII:	$Y''' = -YY'' - 2(Y')^2 + 2Y^2Y';$		(2.18)
VIII:	$Y''' = 6Y^2Y';$		(2.19)
IX:	$Y''' = 12(Y')^2 + 72Y^2Y' + 54Y^4;$		(2.20)
X.a:	$Y''' = 6Y^2Y' + \frac{3}{11}(9 + 7\sqrt{3})\left(Y' + Y^2\right)^2;$		(2.21)
X.b:	$Y''' = 6Y^2Y' + \frac{3}{11}(9 - 7\sqrt{3})\left(Y' + Y^2\right)^2;$		(2.22)
XI:	$Y''' = -2YY'' - 2(Y')^2 + \dfrac{24}{N^2 - 1}\left(Y' + Y^2\right)^2;$		(2.23)
XII:	$Y''' = 2YY'' - 3(Y')^2 - \dfrac{4}{N^2 - 36}\left(6Y' - Y^2\right)^2;$		(2.24)
XIII:	$Y''' = 12YY'.$		(2.25)

In Chazy-XI, N is a positive integer not equal to 1 or a multiple of 6. In Chazy-XII, N is a positive integer not equal to 1 or 6. The full equations of the form (2.1) corresponding to these reduced equations are listed in the Appendix, together with the solutions of those that are not given special attention in the main part of this paper.

The splitting of Chazy Class X deserves a comment. Because its coefficients contain the irrationality $\sqrt{3}$ in an essential way, changing the sign of $\sqrt{3}$ yields a distinct equation, not equivalent under the usual Möbius equivalence that applies to Painlevé equations generally. There is, however, a nontrivial Bäcklund transformation relating Chazy Class X.a and Class X.b, which is new in this paper and given below by equation (4.19).

With the exception of Chazy Classes III, XI, and XII, the construction of the full equations is a straightforward exercise. The method involves determining conditions for the absence of logarithms at the positive-integer resonances. In every case except Chazy-III, which has no positive resonances, these conditions turn out to be both necessary and sufficient for the Painlevé property. In some Painlevé classification problems (in particular the original Painlevé-Gambier problem [37, 3, 24]), this analysis may run into difficult differential equations for coefficient functions, but the only member of the Chazy polynomial class that presents such difficulties is Chazy-XI.

Chazy's own analysis of Classes I, II, IV–VIII, and XIII is essentially complete, with all solutions found in terms of known functions, including Painlevé transcendents. An exception is Chazy-VI, where Chazy wrote down the linear equation into which he had transformed his equation and its second-degree integral, but he didn't reveal the transformation to his readers. This case was also set aside by Bureau [2], who eventually included the solution of the second-degree integral, without mentioning its relationship to Chazy-VI, in a series of steps in [29, pp. 277–281]. The direct solution of Chazy-VI is given in the Appendix below.

On the other hand, Chazy Class III, one of the celebrated natural barrier equations, has only negative-integer resonances and Classes XI and XII each have a positive-integer resonance with inconvenient value N. As a result, the full classification of Class III was not attempted by either Chazy or Bureau and the first successful treatment was given by Conte, Fordy, and Pickering (CFP) [14] in 1993. They introduced a powerful new method in Painlevé analysis, generalizing earlier work of Fordy and Pickering [38], in which positive- and negative-integer resonances were treated on an equal footing. Further development of the method in a direction towards Painlevé's original method of α-expansions was given by Musette and Conte [15] (see also [39]). The conclusion in [14] was that Chazy Class III does not admit any recessive terms except, of course, trivial gauge terms created by changing variables in the reduced equation. With gauge chosen appropriately, the reduced equation is the unique member of Class III. An alternative proof of this fact is included in Section 6 below.

Chazy himself [1, p. 345] gave the correct recessive terms in the case of Class XII, which admits natural barriers when $N \geq 7$, but he showed no details and we do not know if he was able to prove uniqueness. (He may have, for example, examined small values of N and conjectured the result for general N, although it is certainly possible to complete the proof by traditional methods.) This case was also first done properly by Conte et al. [14]. The result is that Class XII can take recessive terms only when $N = 2$ and $N = 3$. In Section 6, we include a proof of this fact by a classical α-expansion of the same sort that Painlevé himself used to analyse the class of equations, $y'' = 6y^2 + S(x)$, in [4, pp. 224–225] (reviewed by Ince in [3, pp. 329–330]). The same method is applied to the Bureau barrier equation in paper I. The method can be very powerful in many cases where the traditional Laurent series methods are slow or not applicable. However, it usually does not beat the traditional methods when there are enough positive resonances to test, and that is one possible reason why it does not work as well on Chazy-XI. In the two cases that accept recessive terms, Chazy-XII is reducible to a fourth-order linear equation. (Chazy's solution of the case $N = 3$ involves a fifth-order linear equation.)

The main purpose of this paper is to solve Chazy Classes IX and X for the first time in terms of hyperelliptic functions and examine the coefficient function $g(x)$ in Class XI. We also present new first integrals for Chazy-IX and Chazy-X, as well as an invertible map connecting X.a and X.b. We also look at some particular solutions expressible in terms of simpler functions, including the solution of reduced Chazy-X in terms of two elliptic functions. The solution of reduced Chazy-IX remains hyperelliptic.

In the case of Chazy-XI, we give special attention to six interesting cases, $N = 2$, 3, 4, 5, 7, and 11, the case $N = 7$ involving the higher-order transcendent Fif-I defined by (1.11). In many cases, the associated Riccati equation can be solved explicitly in terms of elliptic functions or the Painlevé-I transcendent, these exact solutions being mostly new. For higher values of N, we conclude on the basis of a computer-algebra analysis for $N \leq 25$ that there are no more interesting cases with nonconstant $g(x)$ when the logarithms corresponding to resonance number N are removed. (The proof for all N remains open.)

3. Solution of the Chazy-IX equation

In this section, we construct the general solution of the Chazy Class IX differential equation [1, pp. 336, 344],

$$y''' \;=\; 12(y')^2 + 72y^2y' + 54y^4 + \delta, \tag{3.1}$$

in terms of hyperelliptic functions by transforming it to a special case of equation F-V, given by (1.9). Chazy-IX and its companion equation Chazy-X have hitherto remained unsolved. Chazy speculated that these equations would not define new transcendents but would be solved in terms of known functions, for example, elliptic functions. The only exact solutions of each equation known until now are the trivial solutions, $y = a_0/(x - x_0)$, where a_0 takes the three values in (3.3)–(3.5) or (4.3)–(4.5) below. All solutions, first integrals, and transformations in this section are new.

The additive constant δ is, up to gauge, the only recessive term that Chazy-IX can admit. The general solution is also hyperelliptic in the case of the reduced equation $\delta = 0$.

While the solution of Chazy-IX here is self-contained, we will refer the reader to various equations in [I, Section 6] for details about the auxiliary equation F-V.

Substituting the formal Laurent series,

$$y \;=\; a_0(x - x_0)^{-1} + a_1 + a_2(x - x_0) + \;\ldots\; + a_n(x - x_0)^{n-1} + \;\ldots\;, \tag{3.2}$$

into Chazy-IX yields the following resonance data:

$$a_0 \;=\; \tfrac{1}{6}\left(1+\sqrt{5}\right): \qquad \text{resonances: } \{2,\,5\}; \tag{3.3}$$

$$a_0 \;=\; \tfrac{1}{6}\left(1-\sqrt{5}\right): \qquad \text{resonances: } \{2,\,5\}; \tag{3.4}$$

$$a_0 \;=\; 1: \qquad \text{resonances: } \{-3,\,10\}. \tag{3.5}$$

The general solution admits the first two species of simple poles, where the nontrivial resonance numbers are all positive. The third species of poles, where one nontrivial resonance is negative, occurs in a two-parameter family of particular solutions. The residues of different species have irrational ratio. Since no finite collection of poles can have zero total residue, it is not possible to draw a period parallelogram around any finite collection of poles. It follows that neither the general solution nor any (nonconstant) particular solution of Chazy IX is doubly periodic. Nevertheless, there are particular solutions expressible in terms of elliptic functions and quadratures or in terms of two elliptic functions with different periods.

It can be verified by direct differentiation that Chazy-IX admits the first integral,

$$\begin{aligned} y(y'')^3 \;&-\; \left\{\tfrac{1}{2}(y')^2 + \tfrac{1}{3}(54y^4+\delta)\right\}(y'')^2 \\ &- \left\{12y(y')^3 + 72y^3(y')^2 + 3y(54y^4+\delta)y'\right\}y'' \\ &+ \tfrac{24}{5}(y')^5 + 72y^2(y')^4 + 4(108y^4+\delta)(y')^3 \\ &+ 24y^2(54y^4+\delta)(y')^2 \;+\; \tfrac{2}{3}(54y^4+\delta)^2y' \\ &+ y^2(\tfrac{4374}{5}y^8 + 54\delta y^4 + \tfrac{3}{2}\delta^2) \;=\; K, \end{aligned} \tag{3.6}$$

which is of the second order and third degree. This is the lowest-degree differential equation of the second order having hyperelliptic solutions of which we are aware.

Because Chazy-IX admits the unit postmultiplier (see [I, Section 1]) and possesses one autonomous algebraic integral, we know at once that it is integrable by quadratures. In fact, by choosing y, y', and the constant K as new variables, an exact differential with coefficients built out of nested square roots and cube roots can be constructed explicitly. However, we want to present the solution in terms of standard hyperelliptic functions.

Following a suggestion of Chazy, we examine combinations of the form $a_0y' + y^2$ that cancel one of the above species of poles. Chazy explicitly mentioned the case $a_0 = 1$, which cancels the third species, but it is better to cancel either of the first two species. The first species is canceled by the variable,

$$w \;:=\; \tfrac{3}{10}\left\{(5+3\sqrt{5}\,)y' + 3(5+\sqrt{5}\,)y^2\right\}. \tag{3.7}$$

We have also included a scaling factor such that w inherits a double pole of strength 1 at the poles of y of the second species. Given that y satisfies (3.1), the inverse relation is found to be

$$y \;=\; \frac{1}{12w'}\left\{(1+\sqrt{5}\,)w'' - 2(5+\sqrt{5}\,)w^2 - \tfrac{6}{5}(5+2\sqrt{5}\,)\delta\right\}. \tag{3.8}$$

Elimination of y from (3.7) and (3.8) gives a third-order differential equation for w, namely,

$$w'w''' \;=\; \tfrac{1}{2}(w'')^2 + 10w(w')^2 - 10\left\{w^2 + \tfrac{3}{20}(3+\sqrt{5}\,)\delta\right\}^2, \tag{3.9}$$

which, after differentiating once, becomes

$$w^{(\mathrm{iv})} \;=\; 20ww'' + 10(w')^2 - 40w^3 - 6(3+\sqrt{5}\,)\delta\, w. \tag{3.10}$$

Equation (3.10) is a special case of equation F-V, given by (1.9) above, and (3.9) is a special case of the first integral [I, (6.3)] of F-V or, equivalently, [I, (6.7)]. The F-V parameters appearing in (1.9) and [I, Section 6] are given by

$$\alpha \;=\; -6(3+\sqrt{5}\,)\delta, \qquad \beta \;=\; 0, \qquad \kappa \;=\; 0, \tag{3.11}$$

$$\gamma \;=\; -\tfrac{1}{160}\alpha^2, \qquad K_1 \;=\; \tfrac{1}{40}\alpha^2. \tag{3.12}$$

A somewhat longer calculation identifies

$$K_2 \;=\; \tfrac{324}{25}(25+11\sqrt{5}\,)K, \qquad K \;=\; \tfrac{5}{1296}(25-11\sqrt{5}\,)K_2\,, \tag{3.13}$$

where K_2 and K are the constants of integration appearing in the first integrals [I, (6.8)] of F-V and (3.6) above, respectively.

Changing the sign of $\sqrt{5}$ can be used to map Chazy-IX into two F-V equations with different α parameters. Hence there is a transformation mapping a special case of F-V to itself with parameter $\bar{\alpha} \;=\; (7 \pm 3\sqrt{5}\,)\alpha/2$. Unfortunately, this type of transformation does not lift up to more general versions of F-V and, in particular, does not lift up to a Schlesinger transformation for the F-V transcendent. Applying the same idea in the reverse direction gives the following mapping of Chazy-IX to itself:

$$\begin{aligned}\bar{y} \;=\; &-\tfrac{1}{2}(3-\sqrt{5}\,)y \\ &\qquad - \frac{(1+\sqrt{5}\,)(y')^2 + 12y^2y' - (1-\sqrt{5}\,)(9y^4 + \frac{1}{6}\delta)}{y'' - 3(1-\sqrt{5}\,)yy'}\,,\end{aligned} \tag{3.14}$$

$$\bar{\delta} \;=\; (\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}\,)^4\delta, \qquad \bar{K} \;=\; (\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}\,)^{10}K. \tag{3.15}$$

The inverse transformation can be obtained from (3.15) by changing the sign of $\sqrt{5}$ wherever it appears. This transformation can be iterated.

The explicit solution of Chazy-IX can be deduced from the solution of F-V given in [I, Section 6]. The auxiliary variable w is given in terms of hyperelliptic functions by the right-hand side [I, (6.12)], with parameters given by (3.11)–(3.13). The latter solution can be substituted into (3.8) above and then simplified by canceling algebraic factors. We present the results of that calculation.

Let $u(x)$ and $v(x)$ be defined by inversion of the hyperelliptic integrals,

$$\int_{\infty}^{u(x)} \frac{dt}{\sqrt{P(t)}} + \int_{\infty}^{v(x)} \frac{dt}{\sqrt{P(t)}} \;=\; C_1\,, \tag{3.16}$$

$$\int_{\infty}^{u(x)} \frac{t\,dt}{\sqrt{P(t)}} + \int_{\infty}^{v(x)} \frac{t\,dt}{\sqrt{P(t)}} \;=\; x + C_2\,, \tag{3.17}$$

where $P(t)$ is the quintic polynomial,

$$P(t) \;:=\; t^5 - 2\alpha t^3 + \tfrac{4}{5}\alpha^2 t + 16K_2\,. \tag{3.18}$$

The coefficients α and K_2 are defined in terms of the Chazy-IX parameters δ and K by (3.11) and (3.13) above.

The general solution of Chazy-IX is

$$y \;=\; \frac{\frac{1}{12}\left(\sqrt{5}-1\right)\left\{\sqrt{P(u)}+\sqrt{P(v)}\right\}}{u^2+v^2+\frac{1}{2}\left(\sqrt{5}+1\right)uv+\frac{1}{5}\left(\sqrt{5}-5\right)\alpha}\,. \tag{3.19}$$

This solution is globally meromorphic. It can be verified by direct substitution into the differential equation (3.1) and its first integral (3.6). The first species of poles (3.3) occurs at the zeros of the denominator in (3.19). The second species (3.4) occurs where u or v (but not both) has a double pole, in which case w has a double pole of strength 1. The third species (3.5) occurs in the nongeneric situation where both u and v have a double pole, in which case w has a double pole of strength 3.

There are several cases where Chazy-IX can be solved in terms of elliptic or simpler functions. First, $P(t)$ has a double root whenever

$$K \;=\; \pm\,\tfrac{2}{45}\sqrt{6}\,(-\delta)^{5/2}. \tag{3.20}$$

In these cases $P(t)$ has, in fact, two double roots and the solution of Chazy-IX can be written in terms of trigonometric or exponential functions with two different periods. Also, Chazy-IX can be solved in terms of elliptic functions with two different periods whenever a linear transformation $t \to at + b$ exists such that the transformed $P(t)$ is divisible by t and $P(t)/t$ is a palindromic quartic. This occurs in the previous case and also when

$$K = 0, \qquad K = \pm \tfrac{11}{45}\sqrt{6}\,\delta^{5/2}. \tag{3.21}$$

There may be other cases of degeneracy to simpler functions. In the case $K = \delta = 0$, Chazy-IX admits the two-parameter rational solution,

$$y = \frac{(x - x_0)^2 \left\{(x - x_0)^3 - \frac{1}{2}(5 + 3\sqrt{5}\,)C\right\}}{\left\{(x - x_0)^3 - C\right\}\left\{(x - x_0)^3 + \frac{1}{2}(7 + 3\sqrt{5}\,)C\right\}}. \tag{3.22}$$

4. Solution of the Chazy-X equation

In this section, we construct the general solution of the full Chazy Class X.a and X.b differential equations,

$$\begin{aligned} y''' &= 6y^2y' + \tfrac{3}{11}(9 + 7\sqrt{3}\,)(y' + y^2)^2 - \tfrac{1}{22}(4 - 3\sqrt{3})\alpha y' \\ &\qquad + \tfrac{1}{44}(3 - 5\sqrt{3})\alpha y^2 - \tfrac{1}{352}(9 + 7\sqrt{3}\,)\alpha^2, \end{aligned} \tag{4.1}$$

$$\begin{aligned} y''' &= 6y^2y' + \tfrac{3}{11}(9 - 7\sqrt{3}\,)(y' + y^2)^2 - \tfrac{1}{22}(4 + 3\sqrt{3})\alpha y' \\ &\qquad + \tfrac{1}{44}(3 + 5\sqrt{3})\alpha y^2 - \tfrac{1}{352}(9 - 7\sqrt{3}\,)\alpha^2, \end{aligned} \tag{4.2}$$

which are new when $\alpha \neq 0$, in terms of hyperelliptic functions by transforming them to a special case of equation F-VI, given by (1.10). We also provide a direct transformation relating these two equations to each other. The reduced versions $\alpha = 0$ of Chazy-X.a and X.b, which were discovered by Chazy himself [1, pp. 336, 344], transform into the reduced F-VI equation and can be solved in terms of two elliptic functions with different periods. All solutions, first integrals, and transformations in this section are new.

While the solution of Chazy-X here is self-contained, we will refer the reader to various equations in [I, Section 7] for comparisons with equation F-VI and various identities and integrals associated with the latter equation.

Presumably, Chazy looked for recessive terms in his Class X but missed them on account of some minor calculation error. Bureau [2], having derived the 13 reduced Chazy classes from first principles, did not attempt to attach recessive terms to Classes III or IX–XII. Exton [9, p. 398], who incorrectly concluded that Chazy Classes IX and X were unstable equations, also apparently did not look for recessive terms. (Of course, recessive terms would have limited interest on an equation believed to be unstable.) Considering how well known the Chazy polynomial class of third-order equations is, it is quite surprising that the recessive terms in Chazy-X have escaped attention until now.

An attractive and exotic feature of Chazy-X is the appearance of the irrational number $\sqrt{3}$ in the coefficients. This irrational quantity is essential and cannot be scaled away or replaced by an arbitrary constant, for example. We have searched for equations of similar character among the Painlevé-type equations in the polynomial class of the fourth and fifth orders, but have not found a comparable equation. (The derivative of Chazy-IX appears among the fourth-order equations—see equation F-XIV in [7, 8].)

Substituting the formal Laurent series (3.2) into Chazy-X.a yields the following resonance data:

$$a_0 = \tfrac{1}{3}\sqrt{3} - 1 : \qquad \text{resonances:} \quad \{2,\ 5\}; \tag{4.3}$$

$$a_0 = 1 : \qquad \text{resonances:} \quad \{3,\ 4\}; \tag{4.4}$$

$$a_0 = 2\sqrt{3} - 1 : \qquad \text{resonances:} \quad \{-5,\ 12\}. \tag{4.5}$$

The corresponding resonance data for Chazy-X.b can be obtained by changing the sign of $\sqrt{3}$. As in the case of Chazy-IX, the general solution of Chazy-X.a admits the first two species of simple poles, where the nontrivial resonance numbers are all positive. The third species of poles, where one nontrivial resonance is negative, occurs in a two-parameter family of particular solutions. The residues of different species have irrational ratio and no finite collection of poles can have zero total residue. It follows that neither the general solution nor any (nonconstant) particular solution of Chazy-X.a or X.b is doubly periodic. Nevertheless, there are particular solutions expressible in terms of elliptic functions and quadratures or in terms of two elliptic functions with different periods.

It can be verified by direct differentiation that Chazy-X.a admits the (rather formidable) first integral,

$$(y'')^4 + P(y,y')(y'')^3 + Q(y,y')(y'')^2 + R(y,y')y'' + S(y,y') = K, \tag{4.6}$$

where the coefficients are given by

$$P = -\tfrac{8}{9}(12+5\sqrt{3})y^3 + \tfrac{2}{3}\alpha(3+\sqrt{3})y, \tag{4.7}$$

$$
\begin{aligned}
Q \;=\; & -\tfrac{4}{11}(9+7\sqrt{3}\,)(y')^3 + \Big\{ -\tfrac{32}{33}(6+\sqrt{3}\,)y^2 \\
& - \tfrac{1}{33}\alpha(21+20\sqrt{3}\,)\Big\}(y')^2 + \Big\{ -\tfrac{12}{11}(9+7\sqrt{3}\,)y^4 \\
& + \tfrac{1}{11}\alpha(-3+5\sqrt{3}\,)y^2 + \tfrac{1}{88}\alpha^2(9+7\sqrt{3}\,)\Big\}\, y' \\
& + \Big\{ \tfrac{24}{11}(15+8\sqrt{3}\,)y^6 - \tfrac{7}{33}\alpha(36+17\sqrt{3}\,)y^4 \\
& - \tfrac{1}{198}\alpha^2(21+31\sqrt{3}\,)y^2 + \tfrac{1}{3168}\alpha^3(216+113\sqrt{3}\,)\Big\}, \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
R \;=\; & \tfrac{16}{33}(6+\sqrt{3}\,)y(y')^4 + \Big\{ \tfrac{8}{11}(89+57\sqrt{3}\,)y^3 \\
& - \tfrac{2}{33}\alpha(141+95\sqrt{3}\,)y\Big\}(y')^3 + \Big\{ \tfrac{112}{11}(15+8\sqrt{3}\,)y^5 \\
& - \tfrac{4}{11}\alpha(80+39\sqrt{3}\,)y^3 + \tfrac{1}{99}\alpha^2(24-7\sqrt{3}\,)y\Big\}(y')^2 \\
& + \Big\{ \tfrac{24}{11}(71+43\sqrt{3}\,)y^7 - \tfrac{2}{11}\alpha(157+105\sqrt{3}\,)y^5 \\
& - \tfrac{1}{44}\alpha^2(59+19\sqrt{3}\,)y^3 + \tfrac{3}{88}\alpha^3(8+5\sqrt{3}\,)y\Big\}\, y', \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
S \;=\; & \tfrac{32}{1089}(240+139\sqrt{3}\,)(y')^6 + \Big\{ -\tfrac{24}{121}(25+17\sqrt{3}\,)y^2 \\
& + \tfrac{8}{363}\alpha(177+101\sqrt{3}\,)\Big\}(y')^5 + \Big\{ -\tfrac{4}{121}(3279+1828\sqrt{3}\,)y^4 \\
& + \tfrac{2}{363}\alpha(4926+2735\sqrt{3}\,)y^2 - \tfrac{1}{1089}\alpha^2(507+277\sqrt{3}\,)\Big\}(y')^4 \\
& + \Big\{ -\tfrac{8}{121}(5123+3019\sqrt{3}\,)y^6 + \tfrac{2}{121}\alpha(4637+2795\sqrt{3}\,)y^4 \\
& + \tfrac{1}{1452}\alpha^2(519-73\sqrt{3}\,)y^2 - \tfrac{1}{363}\alpha^3(139+80\sqrt{3}\,)\Big\}(y')^3 \\
& + \Big\{ -\tfrac{72}{121}(822+467\sqrt{3}\,)y^8 + \tfrac{2}{121}\alpha(7233+3994\sqrt{3}\,)y^6 \\
& + \tfrac{1}{726}\alpha^2(1305+1202\sqrt{3}\,)y^4 - \tfrac{1}{17424}\alpha^3(18597+11344\sqrt{3}\,)y^2 \\
& + \tfrac{5}{139392}\alpha^4(144+35\sqrt{3}\,)\Big\}(y')^2 + \Big\{ -\tfrac{432}{121}(101+59\sqrt{3}\,)y^{10} \\
& + \tfrac{6}{121}\alpha(1739+1047\sqrt{3}\,)y^8 + \tfrac{1}{121}\alpha^2(563+257\sqrt{3}\,)y^6 \\
& - \tfrac{5}{1452}\alpha^3(481+269\sqrt{3}\,)y^4 - \tfrac{1}{23232}\alpha^4(211+37\sqrt{3}\,)y^2 \\
& + \tfrac{1}{185856}\alpha^5(1439+843\sqrt{3}\,)\Big\}y' + \Big\{ -\tfrac{12}{121}(771+442\sqrt{3}\,)y^{12} \\
& + \tfrac{6}{121}\alpha(405+227\sqrt{3}\,)y^{10} + \tfrac{1}{484}\alpha^2(651+433\sqrt{3}\,)y^8 \\
& - \tfrac{1}{2904}\alpha^3(1728+1025\sqrt{3}\,)y^6 - \tfrac{1}{30976}\alpha^4(159+142\sqrt{3}\,)y^4 \\
& + \tfrac{3}{61952}\alpha^5(177+101\sqrt{3}\,)y^2\Big\}. \qquad (4.10)
\end{aligned}
$$

A corresponding first integral for Chazy-X.b can be obtained from (4.6)–(4.10) by changing the sign of $\sqrt{3}$ and renaming K to $\bar{K}$, say. These integrals are differential equations for y of the second order and fourth degree.

Because Chazy-X admits the unit postmultiplier (see [I, Section 1]) and possesses one autonomous algebraic integral, we know at once that it is integrable by quadratures, just as in the case of Chazy-IX. It is straightforward, in principle, to construct an exact differential with coefficients built out of nested square roots and cube roots. It would, however, be exceedingly complicated and would not give much insight into the true character of the solution. We want to present the solution in terms of standard hyperelliptic functions, and this requires a substantial amount of further work. All the big calculations in this section were done with the computer algebra package REDUCE.

We begin by following the procedure that transformed Chazy-IX into F-V. Choose a new auxiliary variable,

$$w \;:=\; -\tfrac{1}{11}\Big\{(5-\sqrt{3}\,)y' - (6+\sqrt{3}\,)y^2 + \tfrac{1}{18}(4-3\sqrt{3}\,)\alpha\Big\}. \tag{4.11}$$

This variable cancels the first species of poles (4.3). We have also included a scaling factor that gives w a double pole of strength 1 at the poles of y of the second species. The additive constant is included for later convenience. Given that y satisfies (4.1), the inverse relation is found to be

$$y \;=\; \frac{1-\sqrt{3}}{w'}\Big\{\tfrac{1}{6}\sqrt{3}\,w'' + w^2 - \tfrac{1}{36}\alpha w - \tfrac{13}{2592}\alpha^2\Big\}. \tag{4.12}$$

Elimination of y from (4.11) and (4.12) gives a third-order differential equation for w, namely,

$$w'w''' \;=\; \tfrac{1}{2}(w'')^2 + 9w(w')^2 - 6\Big\{w^2 - \tfrac{1}{36}\alpha w - \tfrac{13}{2592}\alpha^2\Big\}^2, \tag{4.13}$$

which, after differentiating once, becomes

$$w^{(\mathrm{iv})} \;=\; 18ww'' + 9(w')^2 - 24w^3 + \alpha w^2 + \tfrac{1}{9}\alpha^2 w - \tfrac{13}{7776}\alpha^3. \tag{4.14}$$

Equation (4.14) is a special case of equation F-VI, given by (1.10) above, and (4.13) is a special case of the first integral [I, (7.3)] of F-VI or, equivalently, [I, (7.7)]. The F-VI parameters appearing in (1.10) and [I, Section 7] are given by

$$\alpha \;=\; \alpha, \qquad \beta \;=\; -\tfrac{13}{7776}\alpha^3, \qquad \kappa \;=\; 0, \tag{4.15}$$

$$\gamma \;=\; -\tfrac{169}{1119744}\alpha^4, \qquad K_1 \;=\; -\tfrac{1}{1536}\alpha^4. \tag{4.16}$$

A considerably longer calculation identifies

$$K_2 \;=\; -\tfrac{9}{58564}(5-\sqrt{3}\,)^4 K + \frac{(5-\sqrt{3}\,)^6(225-19\sqrt{3}\,)}{2^{15}\cdot 3^3\cdot 11^6}\,\alpha^6, \tag{4.17}$$

$$K \;=\; -\tfrac{1}{36}(5+\sqrt{3}\,)^4 K_2 + \frac{3435-1391\sqrt{3}}{2^{12}\cdot 3^5\cdot 11^2}\,\alpha^6, \tag{4.18}$$

where K_2 and K are the constants of integration appearing in the first integrals [I, (7.8)] of F-VI and (4.6) above, respectively.

Note that the differential equations (4.13) and (4.14) do not contain any appearances of $\sqrt{3}$. It follows that these same equations would be obtained starting with Chazy-X.b if we changed the sign of $\sqrt{3}$ in the transformation formulas. Hence there is a direct transformation from Chazy-X.a to Chazy-X.b with the same α parameter. Let $y(x)$ satisfy Chazy-X.a. Then $\bar{y}(x)$ satisfies Chazy-X.b, where

$$\begin{aligned}\bar{y} \;=\; & (2+\sqrt{3}\,)y \;-\; \tfrac{4}{11}(1+2\sqrt{3}\,)\left\{y'' - (3+\sqrt{3}\,)yy'\right\}^{-1} \\ & \times\left\{\left[y' - \tfrac{1}{2}(3+\sqrt{3}\,)y^2 + \tfrac{1}{48}(3-\sqrt{3}\,)\alpha\right]^2 \right. \\ & \left. \qquad - \tfrac{1}{384}(14+5\sqrt{3}\,)\alpha^2\right\}. \end{aligned} \tag{4.19}$$

The constants of integration are related by

$$\bar{K} \;=\; \tfrac{1}{234256}(5-\sqrt{3}\,)^8 K + \frac{857861\sqrt{3}\,(5-\sqrt{3}\,)^4}{2^{13}\cdot 3^5\cdot 11^6}\,\alpha^6. \tag{4.20}$$

The inverse transformation can be constructed by changing the sign of $\sqrt{3}$.

The explicit solution of Chazy-X.a and Chazy-X.b can be deduced from the solution of F-VI given in [I, Section 7]. The cases $\alpha \neq 0$ and $\alpha = 0$ require separate treatment. When $\alpha \neq 0$, the auxiliary variable w is given in terms of hyperelliptic functions by the right-hand side [I, (7.17)] or, equivalently, [I, (7.26)], with parameters given by (4.15)–(4.18) above. When $\alpha = 0$, w is given in terms of elliptic functions by the right-hand side of [I, (7.37)]. The solution of F-VI can be substituted into (4.12) above and substantially simplified by canceling a polynomial factor (in the auxiliary variables u and v) common to the numerator and denominator. We present the results of these lengthy calculations.

In [I, Section 7], we gave two alternative presentations of the hyperelliptic functions appearing in the general solution of F-VI, one employing an octic (8th-degree) polynomial under the square root, the other employing a sextic (6th-degree) polynomial. They are related by a simple quadratic transformation. While the latter is more standard, and can be further reduced to a quintic polynomial by taking a suitable Möbius transformation in the integration variable t, we prefer the octic formulation because there are fewer square roots appearing explicitly. This is especially true when applied to Chazy-X.

First, we deal with the case $\alpha \neq 0$. Rename the coefficient parameters and integration constants in Chazy-X.a, Chazy-X.b, and F-VI, as well as the auxiliary parameters appearing in [I, (7.9)–(7.12)], according to

$$\alpha \;=\; 24\mu^2 B, \qquad \beta \;=\; -\tfrac{208}{9}\mu^6 B^3, \qquad \gamma \;=\; -\tfrac{1352}{27}\mu^8 B^4, \tag{4.21}$$

$$C \;=\; 3B^2 - 6B + 2, \qquad E \;=\; 48B^2, \qquad F \;=\; 72B^2(1-B), \tag{4.22}$$

$$K_1 \;=\; -216\mu^8 B^4, \qquad K_2 \;=\; -7776\mu^{12}B^7(B-1)(B-2), \tag{4.23}$$

$$K = 216(5+\sqrt{3}\,)^4\mu^{12}B^7(B-1)(B-2) + \tfrac{192}{121}(3435-1391\sqrt{3}\,)\mu^{12}B^6, \qquad (4.24)$$

$$\bar{K} = 216(5-\sqrt{3}\,)^4\mu^{12}B^7(B-1)(B-2) + \tfrac{192}{121}(3435+1391\sqrt{3}\,)\mu^{12}B^6. \qquad (4.25)$$

All parameters are expressed in terms of the nonzero constants μ and B, which can be constructed from the original Chazy-X.a parameters α and K by solving a cubic equation. Note that this construction elevates the status of μ from a trivial scaling parameter (as it is in [I, Section 7]) to an essential parameter depending on α and K. (Of course, one can still normalize μ by the scaling $x \to \mu x$, $y \to y/\mu$, $\alpha \to \alpha/\mu^2$, $K \to K/\mu^{12}$.)

Define the octic polynomial $P(t)$ by

$$\begin{aligned} P(t) &= (t^2-C)^4 - 96Bt^2(t^2-C)^2 + 8Et^3(t^2+C) + 16Ft^4 \\ &= \Big\{\Big(t^2+2\sqrt{3}\,Bt+C\Big)^2 - 8\Big(1+(3+2\sqrt{3}\,)B\Big)\,t^2\Big\} \\ &\qquad \times \Big\{\Big(t^2-2\sqrt{3}\,Bt+C\Big)^2 - 8\Big(1+(3-2\sqrt{3}\,)B\Big)\,t^2\Big\}. \end{aligned} \qquad (4.26)$$

Next, let $u(x)$ and $v(x)$ be defined by inversion of the hyperelliptic integrals,

$$\int_\infty^{u(x)} \frac{t\,dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{t\,dt}{\sqrt{P(t)}} = \tfrac{1}{4}C_1\,, \qquad (4.27)$$

$$\int_\infty^{u(x)} \frac{(t^2+C)\,dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{(t^2+C)\,dt}{\sqrt{P(t)}} = \tfrac{1}{2}\mu(x+C_2). \qquad (4.28)$$

Before proceeding, we need to define the following auxiliary polynomials in the variables u and v:

$$\begin{aligned} D_1(u,v) := {} & (6B+1)(u+v)^2(u^2-C)(v^2-C) \\ & - (u+v)(uv+C)\Big\{6B(B-2)(u+v)^2 \\ & + 24Buv\Big\} + 6BC(u+v)^4 + 24B(B-2)uv(u+v)^2 \\ & - 192B^2u^2v^2, \end{aligned} \qquad (4.29)$$

$$
\begin{aligned}
D_2(u,v) :=\ & -(B+1)(u+v)^2(u^2-C)(v^2-C) \\
& + (u+v)(uv+C)\Big\{(u^2-C)(v^2-C) \\
& + 2B(3B-4)(u+v)^2 + 40Buv\Big\} - 2BC(u+v)^4 \\
& - 8B(3B-4)uv(u+v)^2 - 192B^2u^2v^2, \qquad (4.30) \\
N_1(u,v) :=\ & (u+v)^2(uv-C)(v^2-C)^2 \\
& - v^4(u+v)^2\Big\{(12B+1)u - v\Big\} \\
& + 12Bv^3(u+v)\Big\{(B-4)u^2 + 2(B-1)uv + (B-2)v^2\Big\} \\
& + 2(18B^3 - 21B^2 - 18B + 2)v^2(u-v)(u+v)^2 \\
& - 192B^2uv^3(u-v) - 12BCv(u+v)\Big\{(B-2)u^2 \\
& + 2(B-1)uv + (B-4)v^2\Big\} \\
& - C^2(u+v)^2\Big\{u - (12B+1)v\Big\}, \qquad (4.31) \\
N_2(u,v) :=\ & -2v(u+v)(u^2-C)(v^2-C)^2 \\
& + v^4(u+v)^2\Big\{(3B+1)u + (B-1)v\Big\} \\
& - 4Bv^3(u+v)\Big\{(3B-10)u^2 + 6(B+1)uv \\
& + (3B-4)v^2\Big\} - 2(3B^3 + 9B^2 \\
& - 20B + 2)v^2(u-v)(u+v)^2 \\
& - 192B^2uv^3(u-v) + 4BCv(u+v)\Big\{(3B-4)u^2 \\
& + 6(B+1)uv + (3B-10)v^2\Big\} \\
& - C^2(u+v)^2\Big\{(B-1)u + (3B+1)v\Big\}. \qquad (4.32)
\end{aligned}
$$

The general solution of Chazy-X.a when $\alpha \neq 0$ is

$$
y = \frac{\Big\{N_1(u,v) + \sqrt{3}\,N_2(u,v)\Big\}uu' + \Big\{N_1(v,u) + \sqrt{3}\,N_2(v,u)\Big\}vv'}{2uv(u+v)\Big\{D_1(u,v) + \sqrt{3}\,D_2(u,v)\Big\}}, \qquad (4.33)
$$

and the general solution of Chazy-X.b when $\alpha \neq 0$ is

$$y \;=\; \frac{\left\{N_1(u,v) - \sqrt{3}\,N_2(u,v)\right\} uu' + \left\{N_1(v,u) - \sqrt{3}\,N_2(v,u)\right\} vv'}{2uv(u+v)\left\{D_1(u,v) - \sqrt{3}\,D_2(u,v)\right\}}\,. \tag{4.34}$$

These solutions are globally meromorphic. The complexity of these solutions is quite remarkable for such a simple and elegant equation as Chazy-X.

It is quite difficult to verify these solutions by direct substitution into the differential equations and first integrals satisfied by y. It is much easier to verify the identities (4.11)–(4.14) above involving the auxiliary variable w, and deduce that y does indeed satisfy Chazy-X.a or X.b and its first integral. The variable w is given by the simpler formula,

$$w \;=\; \tfrac{5}{36}\alpha \;+\; \tfrac{1}{8}\mu^2\,\frac{P(u,v) + \sqrt{P(u)}\,\sqrt{P(v)}}{(u+v)^2(uv-C)^2}\,, \tag{4.35}$$

where

$$\begin{aligned} P(u,v) := \;& (u^2-C)^2(v^2-C)^2 - 48B(u^2+v^2)(u^2v^2+C^2) \\ & + 192BCu^2v^2 + 4Euv(u+v)(uv+C) + 16Fu^2v^2. \end{aligned} \tag{4.36}$$

The first species of poles (4.3) occurs at the zeros of the denominator in (4.33). The second species (4.4) occurs where w has a double pole of strength 1. The third species (4.5) occurs in the nongeneric situation where w has a double pole of strength 5.

The alternative presentation using the variables $\bar{u}$ and $\bar{v}$ in [I, Section 7] is less convenient, having more square roots appearing explicitly. The solution of F-VI in these variables is given by [I, (7.26)]. However, it is useful to have the hyperelliptic integrals in this form when discussing degenerate cases. Apply the quadratic transformation,

$$\bar{u} \;=\; \frac{u^2+C}{2u}\,, \qquad \sqrt{\bar{u}^2 - C} \;=\; \frac{u^2-C}{2u}\,, \tag{4.37}$$

where C is given by (4.22), and similarly for the variables v and $\bar{v}$. Then $\bar{u}(x)$ and $\bar{v}(x)$ are defined by inversion of the hyperelliptic integrals,

$$\int_\infty^{\bar{u}(x)} \frac{dt}{\sqrt{\bar{P}(t)}} \;+\; \int_\infty^{\bar{v}(x)} \frac{dt}{\sqrt{\bar{P}(t)}} \;=\; K_3\,, \tag{4.38}$$

$$\int_\infty^{\bar{u}(x)} \frac{t\,dt}{\sqrt{\bar{P}(t)}} \;+\; \int_\infty^{\bar{v}(x)} \frac{t\,dt}{\sqrt{\bar{P}(t)}} \;=\; \mu(x+K_4)\,, \tag{4.39}$$

where $\bar{P}(t)$ is the sextic polynomial,

$$\bar{P}(t) = (t^2 - C)\{t^2 + C - 4 + 2\sqrt{3}\,(t-2)\}\{t^2 + C - 4 - 2\sqrt{3}\,(t-2)\}. \qquad (4.40)$$

There are a number of cases where the above solution can be expressed in terms of elliptic or simpler functions. We set aside the degenerate case $B = 0$ till the next paragraph. The sextic polynomial $\bar{P}(t)$ has double roots in the cases,

$$B = 1 \pm \tfrac{1}{3}\sqrt{3}, \qquad B = 1 \pm \tfrac{2}{3}\sqrt{3}. \qquad (4.41)$$

In fact, there are two double roots in these four cases and Chazy-X.a and X.b can therefore be solved in terms of trigonometric or exponential functions with two different periods. Also Chazy-X.a and X.b can be solved in terms of elliptic functions with two different periods when there exists a Möbius transformation in t that makes $\bar{P}(t)$ an even sextic. In addition to the cases given by (4.41) and $B = 0$, this type of degeneracy occurs for the following 20 values of B:

$$B = 1, \qquad B = 2, \qquad (4.42)$$

$$B = 1 \pm \tfrac{1}{6}\sqrt{3} \pm \tfrac{1}{2}i, \qquad (4.43)$$

$$B = 1 \pm \tfrac{1}{6}\sqrt{3} \pm \tfrac{1}{6}\sqrt{39}, \qquad (4.44)$$

the $\pm$ signs being independent, and the roots of the tenth-degree equation,

$$\begin{aligned} &19683B^{10} - 139968B^9 + 454977B^8 - 868158B^7 \\ &+ 1150254B^6 - 1209168B^5 + 917352B^4 \\ &- 374112B^3 + 58272B^2 + 6144B - 2432 = 0. \end{aligned} \qquad (4.45)$$

We now turn to the reduced versions of Chazy-X.a and Chazy-X.b, namely,

$$y''' = 6y^2y' + \tfrac{3}{11}(9 + 7\sqrt{3}\,)(y' + y^2)^2, \qquad (4.46)$$

$$y''' = 6y^2y' + \tfrac{3}{11}(9 - 7\sqrt{3}\,)(y' + y^2)^2, \qquad (4.47)$$

where $\alpha = 0$. These cases transform into equation F-VI with

$$\alpha = \beta = \gamma = K_1 = 0,$$

which is one of the degenerate cases discussed in [I, Section 7] (see [I, (7.37)–(7.42)]). The solutions can be given in terms of two elliptic functions with different periods. The first integral for (4.46) is given by (4.6)–(4.10) with $\alpha = 0$, and the first integral for (4.47) can be given by changing the sign of $\sqrt{3}$.

The remaining parameters are K, $\bar{K}$, and K_2. They are related by (4.17), (4.18), and (4.20) with $\alpha = 0$. Rename these parameters according to

$$K_2 \;=\; 2304 D^3, \qquad K \;=\; -\,64(5+\sqrt{3}\,)^4 D^3, \tag{4.48}$$

and, similarly, $\bar{K} = -\,64(5-\sqrt{3}\,)^4 D^3$. When $D \neq 0$, define the elliptic functions $u(x)$ and $v(x)$ by

$$(u')^2 \;=\; u^4 + (3+2\sqrt{3}\,)D, \tag{4.49}$$

$$(v')^2 \;=\; v^4 + (3-2\sqrt{3}\,)D. \tag{4.50}$$

The phases of these elliptic functions will be the second and third constants of integration in the solution.

The solutions of the reduced versions of Chazy-X.a and Chazy-X.b are, respectively,

$$\begin{aligned} y \;=&\; \frac{u'-v'}{u-v} \\ &+\; \frac{u(u-v)\left\{(1-\sqrt{3}\,)uu' - 2vu' - (1+\sqrt{3}\,)uv'\right\}}{u^2(u-v)\left\{(\sqrt{3}\,-1)u + (\sqrt{3}\,+1)v\right\} + 2(3+\sqrt{3}\,)D}, \end{aligned} \tag{4.51}$$

$$\begin{aligned} y \;=&\; \frac{u'-v'}{u-v} \\ &+\; \frac{v(u-v)\left\{(1-\sqrt{3}\,)vu' + 2uv' - (1+\sqrt{3}\,)vv'\right\}}{v^2(u-v)\left\{(\sqrt{3}\,-1)u + (\sqrt{3}\,+1)v\right\} + 2(3-\sqrt{3}\,)D}. \end{aligned} \tag{4.52}$$

When $D = K = 0$, Chazy-X.a and Chazy-X.b admit the respective rational solutions:

$$y \;=\; \frac{(2\sqrt{3}\,-1)x^{10} + (37-19\sqrt{3}\,)Cx^5 + (14-8\sqrt{3}\,)C^2}{x(x^5-C)\left\{x^5 + (9-5\sqrt{3}\,)C\right\}}, \tag{4.53}$$

$$y \;=\; -\,\frac{(2\sqrt{3}\,+1)x^{10} - (37+19\sqrt{3}\,)Cx^5 - (14+8\sqrt{3}\,)C^2}{x(x^5-C)\left\{x^5 + (9+5\sqrt{3}\,)C\right\}}. \tag{4.54}$$

A second parameter can be included by allowing an arbitrary translation in x.

5. Coefficient functions of the Chazy-XI equation

In this section, we examine the coefficient function $g(x)$ in the Chazy Class XI differential equation,

$$\begin{aligned} y''' \;=\; & -2yy'' - 2(y')^2 + \frac{24}{N^2-1}\left\{y' + y^2 - g(x)\right\}^2 \\ & + g''(x) + \tfrac{1}{4}(N^2-1)(\alpha x + \beta) \end{aligned} \tag{5.1}$$

($N \geq 2$, N not a multiple of 6). This equation has been somewhat neglected in the literature, probably because it is well known to be integrable via a Riccati equation and a Painlevé-I (P-I) equation, with N and $g(x)$ unrestricted. However, there are interesting open problems associated with this equation. So far the most complete (correct) results appear in Chazy's original paper [1, pp. 344–345], at least for $N \leq 7$. Exton [9, pp. 390–392] examined the case $N = 2$ in detail but made incorrect statements about the cases $N = 4$ and $N = 5$. Additional brief remarks appear in [2, 14]. (We omit discussion of the limiting case $N = \infty$, which belongs to Chazy-II, and the limiting case $N = \pm 1$, which belongs to Chazy-I.)

While Chazy-XI is integrable for every complex N and for every locally analytic function $g(x)$, it has the Painlevé property only when N is an integer that is not a multiple of 6 and when $g(x)$ satisfies certain differential constraints. The equation becomes interesting when we impose these differential constraints. The most distinguished cases are

$$N \;=\; 2,\ 3,\ 4,\ 5,\ 7,\ \text{ and }\ 11. \tag{5.2}$$

The case $N = 7$ is especially interesting because $g(x)$ involves the Fif-I transcendent defined by (1.11) and studied in [I, Section 4]. The fact that $N = 11$ is distinguished has been overlooked in the literature.

The solution of Chazy-XI in the general case is given by the Riccati equation,

$$y' \;=\; -y^2 + \tfrac{1}{4}(N^2-1)w(x) + g(x), \tag{5.3}$$

where $w(x)$ satisfies the P-I equation,

$$w'' \;=\; 6w^2 + \alpha x + \beta. \tag{5.4}$$

When $\alpha = 0$, $w(x)$ reduces to a Weierstrass elliptic function and satisfies

$$(w')^2 \;=\; 4w^3 + 2\beta w + K, \tag{5.5}$$

where K is a constant of integration. The Riccati equation (5.3) is linearized by the substitution,

$$y \;=\; u'/u, \tag{5.6}$$

where $u(x)$ satisfies the linear Schrödinger-like equation,

$$u'' = \left\{ \tfrac{1}{4}(N^2-1)w(x) + g(x) \right\} u. \tag{5.7}$$

This section contains a number of examples where a well-behaved $u(x)$ (such that u'/u is meromorphic) satisfies a Schrödinger equation (5.7) having two distinct transcendents for a potential. This is quite unusual. There are a number of generalizations of Chazy-XI to higher orders where the role of the P-I equation (5.4) is replaced by some other Painlevé-type differential equation having the Bureau symbol $P2$. However, $g(x)$ is severely restricted by the condition for the absence of logarithms and is not often allowed to be a genuine transcendent. One case occurs when $w(x)$ satisfies Fif-I and $N = 4$. This is a natural dual of Chazy-XI when $N = 7$ where the roles of $w(x)$ and $g(x)$ have been interchanged. Another case occurs when $N = 5$ in the fourth-order equation F-XI [7, 8]. Here, both $w(x)$ and $g(x)$ satisfy the special case of Chazy-XIII that is solvable with Painlevé-II functions. There are many opportunities for further generalization. One place to start is to replace the Schrödinger equation (5.7) with the type of third-order equation used by Kaup [21] in the inverse scattering for the Sawada-Kotera and Kaup-Kuperschmidt equations.

When $\alpha = 0$ and $g(x) = g_0$ (constant), Chazy-XI has constant coefficients and equation (5.7) is a Lamé equation. It can be put in rational Fuchsian form by choosing w as the independent variable instead of x. The equation then has four regular singular points, the exponent differences being $1/2$, $1/2$, $1/2$, and $N/2$. (In the case of the reduced Chazy-XI equation, the Lamé equation can be put in hypergeometric form by choosing w^3 as the independent variable.) When N is odd, these Lamé functions are meromorphic in x and so there are no further conditions for the Painlevé property. In this case, the Lamé equation can be solved explicitly in terms of Weierstrass sigma functions [3, pp. 375–381]. When N is even, there is a condition for the absence of logarithmic branch points at $w = \infty$. (There is also a family of trivial quadratic branch points that cancel out of u^2 and $y = u'/u$.) When that condition is satisfied, the solution can also be written explicitly in terms of Weierstrass sigma functions. When $N = 2$ and $N = 4$, the freedom-from-logarithms condition forces $g_0 = 0$ and $(g_0)^2 = -(3/2)\beta$, respectively, β being unrestricted in both cases. In the latter case $g(x)$ can also be nonconstant (see (5.41) below).

When $N \geq 6$ and even, the condition for the Lamé function to be free of logarithms depends on K and must be satisfied identically in K if Chazy-XI (with constant coefficients) is to have the Painlevé property. When N is a multiple of 6, this condition can never be satisfied, not even in the case of the reduced equation. In this case, Chazy-XI still admits meromorphic particular solutions expressible explicitly in terms of w. For other even $N \geq 8$, the condition can be satisfied only if $g_0 = 0$ and $\beta = 0$, in other words, only for the corresponding reduced equation. For the reduced equation, let $u_N(x)$ be the solution of (5.7) with $g(x) = 0$. Then the u_N obey the recurrence relation,

$$u_{N\pm6} = w(u_N)' + \tfrac{1}{4}(1 \pm N)w'u_N\,. \tag{5.8}$$

This formula shows directly that u_N and y have movable logarithms whenever N is a multiple of 6 because they do so when $N = 6$.

The interesting open questions concerning Chazy-XI are (i) the determination of the most general coefficient function $g(x)$ for each N such that Chazy-XI possesses the Painlevé property, and (ii) determining those cases for which the linear equation (5.7) has explicit solutions. We have an answer to the first question, but we cannot prove it for every N. In answer to the second question, we present a number of exact solutions but do not claim to have them all. Chazy himself found the constraints on $g(x)$ for $N \le 7$. He did not attempt to analyse the multiple constraints on $g(x)$ for $N \ge 8$.

Substituting the formal Laurent series (3.2) into Chazy-XI yields the following resonance data:

$$a_0 = 1: \qquad \text{resonances: } \{2,\,3\}; \tag{5.9}$$

$$a_0 = \tfrac{1}{2}(1-N): \qquad \text{resonances: } \{6,\,N\}; \tag{5.10}$$

$$a_0 = \tfrac{1}{2}(1+N): \qquad \text{resonances: } \{6,\,-N\}. \tag{5.11}$$

We see at once that N must be an integer (nonnegative, without loss of generality) and cannot be 0 or 6. The general form of Chazy-XI obeys the conditions for freedom from logarithms at all the positive resonances except N in (5.10).

The resonance N in (5.10) forces $g(x)$ to satisfy a differential constraint of order $N-2$. When $N \ge 8$, this constraint contains powers of the coefficient a_6 in (3.2), this coefficient being arbitrary because 6 is also a resonance number. Since the differential constraint must be satisfied identically in a_6, $g(x)$ satisfies more than one differential equation when $N \ge 8$. When $N = 6m + k$, $k = 1,\ 2,\ 3,\ 4,\ 5$, the lowest-order differential equation satisfied by $g(x)$ has order 5, 0, 1, 2, 3, respectively (provided the highest derivative does not have zero coefficient). When $N = 2,\ 3,\ 4,\ 5$, and 7, this equation for $g(x)$ is the only equation that $g(x)$ is required to satisfy. All of these cases are interesting and details are presented below. When $N \ge 8$, there is a high degree of incompatibility between the lowest-order equation and the next equation, whose order is 6 higher, except for the single case $N = 11$.

By examining the cases $8 \le N \le 25$, we have arrived at the following conjecture:

$$\text{Odd } N \ge 9 \text{ (except 11):} \qquad g(x) = g_0 \ \text{(constant)}, \quad \alpha = 0; \tag{5.12}$$

$$\text{Even } N \ge 8 \text{ (except } 6m\text{):} \qquad g(x) = 0, \quad \alpha = \beta = 0. \tag{5.13}$$

We can prove (5.13) for the case $N = 6m+2$ because there is a simple limiting contraction to the Lamé function case discussed above. Here, the coefficient of $(a_6)^m$ in the constraint for $g(x)$ is just a nonzero rational number times $g(x)$. This coefficient is given by a series of ${}_3F_2(1)$ type and can be evaluated as a product of gamma functions. Similarly, when $N = 6m+3$, the coefficient of $(a_6)^m$ is a nonzero rational number times $g'(x)$, which forces $g(x) = g_0$. In this case, we have not evaluated the coefficient explicitly but we can show that it is nonzero. The corresponding calculations for $N = 6m+4$, $6m+5$, and $6m+1$ are considerably more difficult and we have not made much progress. The companion equation Chazy-XII has many structural similarities to XI, including a resonance number N, but it can be analysed quite readily by a natural α-test (given in the next section) and by the Conte-Fordy-Pickering (CFP) test. These more advanced techniques are much less effective when applied to Chazy-XI.

On the basis of (5.12) and (5.13), we conclude Chazy-XI has constant coefficients and can be solved in terms of Lamé functions except in the six interesting cases (5.2). In the latter cases, a straightforward calculation gives the following constraints for $g(x)$:

$$N=2: \qquad g(x) = 0; \qquad \alpha,\ \beta \text{ unrestricted}; \tag{5.14}$$

$$N=3: \qquad g(x) = g_0; \qquad \alpha,\ \beta \text{ unrestricted}; \tag{5.15}$$

$$N=4: \qquad g'' = \tfrac{1}{2}g^2 + \tfrac{3}{4}(\alpha x + \beta); \tag{5.16}$$

$$N=5: \qquad g'' = g^2 + 6(\alpha x + \gamma); \tag{5.17}$$

$$N=7: \qquad g(x) = \tfrac{15}{4}h(x) \text{ where } h(x) \text{ satisfies}$$

$$h^{(\mathrm{v})} = 15hh''' + \tfrac{75}{2}h'h'' - 45h^2h' - 24(\alpha x + \beta)h' - 48\alpha h; \tag{5.18}$$

$$N=11: \qquad \textit{Case 1}: \quad \alpha = 0 \quad \text{and} \quad g(x) = g_0,$$

$$\textit{Case 2}: \quad \alpha = 0 \quad \text{and} \quad g'' = \tfrac{1}{2}g^2 + 132\beta. \tag{5.19}$$

We now turn to a discussion of these six cases. Some of the exact solutions given below are new. In all cases, the solution $y(x)$ is a globally meromorphic function, having movable poles at the poles of $w(x)$ and fixed poles at the poles, if any, of $g(x)$. (Note that there are various nongeneric parameter-values where the solutions shown are degenerate, in which case some simple limiting process must be invoked to recover maximal generality.)

5.1. *Chazy-XI when $N = 2$*

When $N = 2$, the freedom-from-logarithms condition is

$$g(x) = 0. \tag{5.20}$$

The parameters α and β are unrestricted. In this case, Chazy-XI is solved in terms of P-I functions and a linear differential equation by

$$y = u'/u, \qquad w'' = 6w^2 + \alpha x + \beta, \tag{5.21}$$

$$u'' = \tfrac{3}{4} w(x) u. \tag{5.22}$$

When $\alpha = 0$ the linear equation (5.22) can be solved explicitly. Let $v(x)$ be the elliptic function defined by the differential equation,

$$(v')^2 = v^4 + 6Cv^2 + 2Dv - 3C^2 - \tfrac{1}{8}\beta, \tag{5.23}$$

C and D being constants of integration. Then the w-equation (5.21) is solved by

$$w(x) = (v'')^2/(v')^2 - 4(v^2 + C). \tag{5.24}$$

The constant of integration K appearing in (5.5) is given by

$$K = 8(32C^3 + 2D^2 + \beta C). \tag{5.25}$$

In terms of $v(x)$, the u-equation (5.22) with $\alpha = 0$ has the general solution,

$$u(x) = (v')^{-1/2}(K_1 + K_2 v). \tag{5.26}$$

We appear to have picked up four integration constants in the solution of Chazy-XI, namely, C, D, K_2/K_1, and the phase of the elliptic function $v(x)$. The first three are not independent and we can fix one of them without loss of generality. There is a two-parameter family of Möbius transformations in v that preserve the form of equation (5.23) while varying the parameters C and D. This transformation leaves $w(x)$ and K invariant. It also preserves the form of (5.26) while varying the ratio K_2/K_1. We could, for example, set $C = 0$ or $K_2 = 0$ without losing generality (except, perhaps, for the loss of some isolated particular solutions).

5.2. *Chazy-XI when $N = 3$*

When $N = 3$, the freedom-from-logarithms condition is

$$g'(x) = 0, \qquad g(x) = g_0. \tag{5.27}$$

The parameters α and β are unrestricted. In this case, Chazy-XI is solved in terms of P-I functions and a linear differential equation by

$$y = u'/u, \qquad w'' = 6w^2 + \alpha x + \beta, \tag{5.28}$$

$$u'' = \{2w(x) + g_0\}u. \tag{5.29}$$

When $N = 3$, the solution of Chazy-XI has an interesting interpretation. It is related to the isomonodromy problem for the first Painlevé transcendent and to the solution of the Modified KdV equation that corresponds to the Galilean-invariant solution of the KdV equation.

For the isomonodromy problem, we may refer to the original papers Garnier [25, 40, 41] or the more recent paper by Jimbo and Miwa [42] that has a very useful appendix listing the associated linear problems for the six Painlevé transcendents. Let the variables y and u be functions of two variables x and t. Also replace the constant function $g(x)$ with

$$g(t) := g_0 + 2\alpha t. \tag{5.30}$$

Take the 21- and 22-components of the matrix $Y(x,t)$ appearing in [42, pp. 436–437] and swap x and t. After suitable rescaling, a linear combination $u(x,t)$ of those matrix elements satisfies the compatible system of partial differential equations,

$$u_{xx} = \{2w(x) + g(t)\}u, \tag{5.31}$$

$$u_t = -4\{w(x) - g(t)\}u_x + 2w'(x)u, \tag{5.32}$$

the subscripts denoting partial differentiation. Then

$$y(x,t) = u_x/u \tag{5.33}$$

satisfies Chazy-XI with $N = 3$ and $g = g(t)$.

Introduce the similarity variable,

$$\eta := x - 6\alpha t^2 - 6g_0 t. \tag{5.34}$$

Then the function,

$$U(x,t) := 2w(\eta) + g(t), \tag{5.35}$$

satisfies the KdV equation,

$$U_{xxx} = 6UU_x + U_t\,, \tag{5.36}$$

and the function,

$$V(x,t) := y(\eta,t), \tag{5.37}$$

satisfies both the Chazy-XI equation and the Modified KdV equation,

$$V_{xxx} \;=\; 6V^2V_x + V_t\,. \tag{5.38}$$

These two solutions are related to each other by the usual Miura transformation. The solution (5.35) of the KdV equation is well known, being one of its three group-invariant reductions. The corresponding solution (5.37) of the Modified KdV equation does not appear among its group-invariant reductions.

When $\alpha = 0$, the solution of the u-equation can be given explicitly in terms of elliptic functions. Now $g(x) = g_0$, and $w(x)$ satisfies

$$(w')^2 \;=\; 4w^3 + 2\beta w + K. \tag{5.39}$$

Define the constant,

$$g_1 \;:=\; \sqrt{4(g_0)^3 + 2\beta g_0 + K}\,.$$

Then

$$\begin{aligned} u(x) \;&=\; K_1 \exp\left\{ \int \frac{w'(x) + g_1}{2(w(x) - g_0)}\, dx \right\} \\ &\quad +\; K_2 \exp\left\{ \int \frac{w'(x) - g_1}{2(w(x) - g_0)}\, dx \right\}. \end{aligned} \tag{5.40}$$

A simple limit is needed in the degenerate case $g_1 = 0$.

5.3. Chazy-XI when $N = 4$

When $N = 4$, the freedom from logarithms condition is

$$g'' \;=\; \tfrac{1}{2}g^2 + \tfrac{3}{4}(\alpha x + \beta). \tag{5.41}$$

Hence both $g(x)$ and $w(x)$ are P-I functions if $\alpha \neq 0$, elliptic functions if $\alpha = 0$. Rescale $g(x)$ according to

$$g(x) \;=\; 12h(x), \qquad h'' \;=\; 6h^2 + \tfrac{1}{16}(\alpha x + \beta). \tag{5.42}$$

In the general case, the solution of Chazy-XI is given by

$$y \;=\; u'/u, \qquad w'' \;=\; 6w^2 + \alpha x + \beta, \tag{5.43}$$

$$u'' \;=\; \{\tfrac{15}{4}w(x) + 12h(x)\}u. \tag{5.44}$$

The u-equation (5.44) can be solved explicitly when $\alpha = 0$, all other parameters being unrestricted. First, let $v(x)$ be the elliptic function defined by (5.23) above.

Then the solution of the w-equation (5.43) and the constant appearing in (5.5) are given by (5.24) and (5.25). The general solution of the u-equation (5.44) is

$$u(x) = K_1 u_1(x) + K_2 u_2(x), \tag{5.45}$$

where

$$u_1(x) = (v')^{-3/2} \left\{ 2(w-4h)v'' + (w'+8h')v' \right\}, \tag{5.46}$$

$$u_2(x) = (v')^{-3/2} \left\{ 2(w-4h)\left[vv'' - 2(v')^2 \right] + (w'+8h')vv' \right\}. \tag{5.47}$$

As in the $N = 2$ case above, we have four apparent integration constants, of which only three are essential, and the remarks at the end of Subsection 5.1 apply.

5.4. Chazy-XI when $N = 5$

When $N = 5$, the freedom-from-logarithms condition is

$$g''' = 2gg' + 6\alpha, \qquad g'' = g^2 + 6(\alpha x + \gamma). \tag{5.48}$$

The parameters α, β, and γ are unrestricted. Rescale $g(x)$ according to

$$g(x) = 6h(x), \qquad h'' = 6h^2 + \alpha x + \gamma. \tag{5.49}$$

In the general case, the solution of Chazy-XI is given by

$$y = u'/u, \qquad w'' = 6w^2 + \alpha x + \beta, \tag{5.50}$$

$$u'' = 6\{w(x) + h(x)\}u. \tag{5.51}$$

When $\gamma = \beta$, the u-equation possesses the simple solution, $u(x) = w(x) - h(x)$. The general solution is

$$u(x) = (w-h)\left\{ K_1 + K_2 \int (w-h)^{-2}\, dx \right\}. \tag{5.52}$$

All other parameters, including α, are unrestricted.

When $\alpha = 0$, the quadrature appearing in (5.52) can be evaluated in terms of Weierstrass zeta functions (i.e., antiderivatives of $w(x)$ and $h(x)$). The result is

$$\int (w-h)^{-2}\, dx = \frac{1}{K-\delta}\left\{ 2\int w(x)\, dx - 2\int h(x)\, dx - \frac{w'+h'}{w-h} \right\}, \tag{5.53}$$

where K and δ are the constants appearing in the integrals,

$$(w')^2 = 4w^3 + 2\beta w + K, \qquad (h')^2 = 4h^3 + 2\gamma h + \delta. \tag{5.54}$$

A simple limit is needed in the degenerate case $K = \delta$.

5.5. Chazy-XI when $N = 7$

When $N = 7$, the freedom-from-logarithms condition is a fifth-order differential equation for $g(x)$. Rescale $g(x)$ according to

$$g(x) = \tfrac{15}{4}h(x).$$

Then $h(x)$ satisfies equation Fif-I, namely,

$$h^{(\mathrm{v})} = 15hh''' + \tfrac{75}{2}h'h'' - 45h^2h' - 24(\alpha x + \beta)h' - 48\alpha h, \tag{5.55}$$

that we studied in [I, Section 4]. (The parameters λ and α in the latter reference can be identified with -24α and -24β here, respectively.) We argued in paper I that Fif-I defines a new higher-order Painlevé transcendent, in general, when $\alpha \neq 0$ and $K_1 \neq 576\alpha^2$, K_1 being the constant appearing in the first integral [I, (4.46)] of Fif-I and α being the parameter appearing in Chazy-XI. When $\alpha = 0$, we gave the general solution in terms of hyperelliptic functions of genus 2. When $K_1 = 576\alpha^2$, we gave a 4-parameter solution in terms of two P-I transcendents, that solution being given by (5.60) below. There are also numerous simpler solutions.

In the general case, the solution of Chazy-XI with $N = 7$ is

$$y = u'/u, \qquad w'' = 6w^2 + \alpha x + \beta, \tag{5.56}$$

$$u'' = \{12w(x) + \tfrac{15}{4}h(x)\}u. \tag{5.57}$$

As mentioned earlier in this section, it is quite unusual for a Schrödinger equation to free of logarithms when it takes a sum (or some other superposition) of two distinct transcendents for a potential. In fact the potential in (5.57) is a superposition of three P-I functions when $h(x)$ is the particular solution [I, (4.48)] of Fif-I. It is instructive to examine an equation like (5.57) when the roles of $w(x)$ and $h(x)$ are interchanged and $h(x)$ is given different rational multipliers. The example,

$$u'' = \tfrac{3}{4}h(x)u, \tag{5.58}$$

is free of logarithms, but it does not have a second transcendent on the right-hand side. (In this case, the variable $y := -2u'/u$ satisfies the equation F-XVIII in [7, 8].) It is easy to construct examples having a single transcendent for a potential. An example having two potentials that is outside the Chazy-XI family is the equation,

$$u'' = \{w(x) + h(x)\}u,$$

where $w(x)$ and $h(x)$ both satisfy the special Chazy-XIII equation,

$$w''' = 2ww' + \alpha w - \alpha^2 x + \beta, \tag{5.59}$$

which is solvable in terms of Painlevé-II functions.

The u-equation (5.57) has an explicit solution expressible in terms of three P-I transcendents when $h(x)$ is the particular solution [I, (4.48)] of the Fif-I equation. The latter solution is given by

$$h(x) = \frac{(p' - q')^2}{(p-q)^2} - 4(p+q), \tag{5.60}$$

where $p(x)$ and $q(x)$ are P-I functions (elliptic if $\alpha = 0$) satisfying

$$p'' = 6p^2 + \alpha x + \beta, \qquad q'' = 6q^2 + \alpha x + \beta. \tag{5.61}$$

The general solution of the u-equation (5.57) in this case is

$$u(x) = K_1 u_1(x) + K_2 u_2(x), \tag{5.62}$$

where

$$u_1(x) = (p-q)^{-3/2}\left\{(p-q)w' - (p'-q')w + qp' - pq'\right\}, \tag{5.63}$$

$$u_2(x) = u_1(x)\int \{u_1(x)\}^{-2}\, dx. \tag{5.64}$$

When $\alpha = 0$, the quadrature appearing in (5.64) can be evaluated explicitly in terms of Weierstrass zeta functions. First, introduce the constants K, δ_1, and δ_2 by means of the first integrals,

$$(w')^2 = 4w^3 + 2\beta w + K, \tag{5.65}$$

$$(p')^2 = 4p^3 + 2\beta p + \delta_1, \qquad (q')^2 = 4q^3 + 2\beta q + \delta_2. \tag{5.66}$$

Define also

$$c_1 := \tfrac{1}{6}(K - \delta_1)^{-1}, \qquad c_2 := \tfrac{1}{6}(\delta_2 - K)^{-1}. \tag{5.67}$$

Then

$$\int \{u_1(x)\}^{-2}\,dx$$

$$\begin{aligned}
&= 12c_1c_2(\delta_1-\delta_2)\int w(x)\,dx \;+\; 2c_1\int p(x)\,dx \;+\; 2c_2\int q(x)\,dx \\
&\quad + (p-q)^{-3/2}\{u_1(x)\}^{-1}\Big\{6c_1c_2(\delta_1-\delta_2) \\
&\quad \times \Big[(p'-q')w' - 4(p-q)w^2\Big] \\
&\quad + c_1\Big[4p(p-q)w + p'q' - 4p^2q - 2\beta q - \delta_1\Big] \\
&\quad + c_2\Big[4q(p-q)w - p'q' + 4pq^2 + 2\beta p + \delta_2\Big]\Big\}. \qquad (5.68)
\end{aligned}$$

Limits are needed in the three degenerate cases, $\delta_2 = \delta_1$, $K = \delta_1$, and $K = \delta_2$.

5.6. *Chazy-XI when $N = 11$*

This is the only case with $N \geq 8$ that admits a nonconstant $g(x)$. When $N = 11$, the condition for the absence of logarithms consists of two differential equations for $g(x)$, one of order 3, the other of order 9:

$$g''' \;=\; gg' + 27\alpha, \qquad (5.69)$$

$$\begin{aligned}
g^{(\mathrm{ix})} \;=\; & 8gg^{(\mathrm{vii})} + 42g'g^{(\mathrm{vi})} + \Big\{108g'' - \tfrac{64}{3}g^2 - 648(\alpha x+\beta)\Big\}g^{(\mathrm{v})} \\
& - 504\alpha g^{(\mathrm{iv})} - \Big\{240gg'' + 210(g')^2 - \tfrac{1600}{81}g^3 - 1440(\alpha x+\beta)g\Big\}g''' \\
& - \tfrac{2700}{7}g'(g'')^2 + \Big\{\tfrac{2080}{21}g^2g' + \tfrac{32400}{7}(\alpha x+\beta)g' + 7392\alpha g\Big\}g'' \\
& + 35g(g')^3 + 6489\alpha(g')^2 - \Big\{\tfrac{2560}{567}g^4 + \tfrac{4160}{7}(\alpha x+\beta)g^2 \\
& + \tfrac{76032}{7}(\alpha x+\beta)^2\Big\}g' - \tfrac{1792}{3}\alpha g^3 - 36288\alpha(\alpha x+\beta)g. \qquad (5.70)
\end{aligned}$$

The first equation (5.69) integrates easily to

$$g'' \;=\; \tfrac{1}{2}g^2 + 27\alpha x + \gamma. \qquad (5.71)$$

When g'' and all higher-order derivatives are replaced using (5.71), we find that (5.70) reduces to a first-order equation (of the second degree). But it is well known that the P-I transcendent does not satisfy a first-order equation of this type. Hence we can immediately set $\alpha = 0$. The first-order equation now simplifies to

$$(245g^2 - 1728\beta + 414\gamma)(\gamma - 132\beta)g' \;=\; 0. \qquad (5.72)$$

The vanishing of the last factor gives *Case 1*:

$$g(x) = g_0, \qquad \alpha = 0, \qquad \beta \text{ unrestricted.} \tag{5.73}$$

This is in agreement with the conjectured result (5.12) for all other odd $N \geq 9$. The solution of the u-equation in this case is a linear combination of Lamé functions. The vanishing of the first factor gives a restricted form of *Case 1*. We will not discuss this case further.

The vanishing of the middle factor in (5.72) gives a distinct case (*Case 2*) where $g(x)$ is an elliptic function. The final form of the constraint is

$$\alpha = 0, \qquad g(x) = 12h(x), \tag{5.74}$$

where $h(x)$ is the elliptic function defined by

$$h'' = 6h^2 + 11\beta, \qquad (h')^2 = 4h^3 + 22\beta h + \delta. \tag{5.75}$$

The solution of Chazy-XI with $N = 11$ in the general case is

$$y = u'/u, \qquad (w')^2 = 4w^3 + 2\beta w + K, \tag{5.76}$$

$$u'' = 6\{5w(x) + 2h(x)\}u. \tag{5.77}$$

We can give an explicit solution of the u-equation (5.77) when $\beta = 0$. In this case, Chazy-XI admits a first integral of the second order and second degree. The general solution of the u-equation is

$$u(x) = K_1 u_1(x) + K_2 u_2(x), \tag{5.78}$$

where

$$u_1(x) = 3ww' - 2hw' - h'w, \tag{5.79}$$

$$\begin{aligned} u_2(x) = {} & u_1(x)\left\{(15K - \delta)\int w(x)\,dx - 14K\int h(x)\,dx\right\} \\ & - 6(15K - \delta)w^3 - 4(6K + \delta)hw^2 \\ & - \tfrac{1}{2}(27K + \delta)(h'w' + K) - 28Kh^2 w. \end{aligned} \tag{5.80}$$

If K coincides with 0, δ, or $-(1/27)\delta$, the solutions $u_1(x)$ and $u_2(x)$ are not linearly independent and so a different expression for $u_2(x)$ must be obtained by taking a suitable limit.

6. Chazy Classes III and XII

For completeness, we include a review of Chazy Classes III and XII [1, 43, 44], which are probably the best known and most celebrated of the 13 Chazy equations. We call these equations the Chazy "natural barrier equations," on account of the movable circular natural barrier in the general solution for integers $N \geq 7$ and $N = \infty$. With this section included, we offer this paper as an alternative sequel to Bureau's paper [2] covering Chazy Classes I, II, IV–VIII and XIII, the other candidate for a sequel being by Exton [9].

Being such thoroughly trampled equations in the literature, we may be a little ambitious in attempting to say something new about them. But we have not seen in the literature the mapping between the reduced Chazy-XI and XII equations given by (6.15)–(6.18) below. We also believe that our method of solution below in the case $N = 3$ is new, having lowered the order of the auxiliary linear equation from five to four. Finally, we remark that a recent paper by Clarkson and Olver [31] opens up the possibility of higher-order integrable equations generalizing the Chazy natural barrier equations and suggests new lines of research in the wider theory of integrable differential equations. But the main discussion in this section is a confirmation of a known result reported by Chazy (for XII) and proved by Conte, Fordy, and Pickering that Chazy-III cannot take recessive terms and XII can only take them when $N = 2$ and $N = 3$. We would like to draw the reader's attention to the power and simplicity of Painlevé's original method for handling such problems, negative resonances notwithstanding.

The reduced equations in Classes III and XII are given, respectively, by

III: $$y''' = 2yy'' - 3(y')^2; \tag{6.1}$$

XII: $$y''' = 2yy'' - 3(y')^2 - \frac{4}{N^2 - 36}\left(6y' - y^2\right)^2. \tag{6.2}$$

It is convenient to treat III as the special case $N = \infty$ of XII. The reduced Chazy-XII equation is linearizable via a hypergeometric equation for every complex N, including $N = \infty$, and so is integrable for N unrestricted. (The limiting case $N = \pm 6$ belongs to Chazy Class I.) The cases of XII that take recessive terms are written down and solved in the paragraphs containing (6.60)–(6.67) below.

These equations can be deduced from the Halphen system [45, 46, 47],

$$\frac{d\omega_i}{dx} = \frac{N-2}{N-6}\left\{\omega_j\omega_k - \omega_i(\omega_j + \omega_k)\right\} + \frac{4}{N-6}(\omega_i)^2, \tag{6.3}$$

where the indices (i, j, k) run over the three cyclic permutations of $(1, 2, 3)$. Then the variable $y = -2(\omega_1 + \omega_2 + \omega_3)$ satisfies XII (including III when $N = \infty$). They can also be deduced from the differential equations governing certain elliptic modular functions and Schwarz triangle functions [47]. They are also the simplest members of a hierarchy of differential equations admitting a certain transformation group locally isomorphic to $\mathrm{SL}(2, C)$ [31].

Chazy-III appears in several physical contexts, for example, self-dual Yang-Mills equations [48] and the Mixmaster universe in general relativity [49, 50, 51]. It is actually the Halphen system (with $N = \infty$) that arises in these applications. Clarkson and Olver [31] obtained III among the group-invariant reductions of the partial differential equation,

$$u_{xxx} = u_y u_{xx} - u_x u_{xy} , \tag{6.4}$$

which has applications in boundary layer theory. Ablowitz and Clarkson [52, pp. 335–343] have given a detailed discussion of III, including asymptotics near the circular barrier. Detailed discussions of XII may be found in [1, 31, 47].

Substituting the formal Laurent series (3.2) into XII yields the following resonance data:

$$a_0 = -6 : \qquad \text{resonances: } \{-2, -3\}; \tag{6.5}$$

$$a_0 = \tfrac{1}{2}(N-6) : \qquad \text{resonances: } \{1, N\}; \tag{6.6}$$

$$a_0 = -\tfrac{1}{2}(N+6) : \qquad \text{resonances: } \{1, -N\}. \tag{6.7}$$

We see at once that, for the Painlevé property, N must be an integer (nonnegative, without loss of generality) and cannot be 0 or 1. In the case of III, when $N = \infty$, only the first species of poles (6.5) is applicable.

The problem of determining recessive terms in these classes remained essentially open until the early 1990's, undoubtedly because researchers were uncertain how to deal with negative resonances, the standard methods based on Laurent series being unable to handle them. Chazy himself [1, p. 345] claimed correctly, but without proof, that XII could support nontrivial recessive terms only in the cases $N = 2$ and $N = 3$. As far as we know, he made no corresponding claim about III. It is certainly possible that Chazy was able to prove his claim because there are enough positive resonances to completely analyse XII by the traditional methods, the resonance number N in (6.6) presenting the greatest difficulties. (In this regard, Chazy-XII is much easier than XI.) The required terms (or sufficient parts thereof) in the Laurent series can be calculated explicitly by working backwards from the α-expansion below, and therefore can be calculated directly by skilful handling of the recurrence relations for the coefficients. There are many occasions in Chazy's paper [1] where Chazy presents the results of a calculation without proof and where the reader cannot doubt that Chazy knew the proof. Exton [9] confirmed Chazy's result for small values of N. The first published proof was by Conte, Fordy, and Pickering [14] using a method (the

CFP test) specifically designed to handle negative-integer resonances. They discovered that III also could not support any nontrivial recessive terms. We include below an alternative proof based on a fairly routine Painlevé α-expansion.

The general solution of (6.2) is expressed parametrically in terms of hypergeometric functions by

$$x = \frac{z_1(t)}{z(t)}, \tag{6.8}$$

$$y(x) = \frac{6z(t)\dot{z}(t)}{z(t)\dot{z}_1(t) - z_1(t)\dot{z}(t)}, \tag{6.9}$$

where $z(t)$ and $z_1(t)$ are any two linearly independent solutions of the hypergeometric equation,

$$t(1-t)\ddot{z} + (\tfrac{1}{2} - \tfrac{7}{6}t)\dot{z} + (\tfrac{1}{4}N^{-2} - \tfrac{1}{144})z = 0, \tag{6.10}$$

the dot denoting differentiation with respect to t. This solution is also applicable to (6.1) by setting $N = \infty$. The solution is rational for $2 \leq N \leq 5$, the results appearing below. Note (6.8) and (6.9) remain invariant under a change of variable $t \to f(t)$.

The case $N = 0$ is not captured by the above construction. Although this case does not have the Painlevé property, it was pointed out by Clarkson and Olver [31] that it can be solved by inversion of an Airy function. In (6.10), change variable,

$$t \to 1 + \tfrac{4}{9}N^2 t^3,$$

and take the limit as $N \to 0$. Then (6.10) simplifies to the Airy equation,

$$\ddot{z} = tz. \tag{6.11}$$

If $y_0(x)$ is a generically chosen particular solution, then the general solution takes the form,

$$y(x) = \frac{\alpha\delta - \beta\gamma}{(\gamma x + \delta)^2}\, y_0(\bar{x}) - \frac{6\gamma}{\gamma x + \delta}, \tag{6.12}$$

where

$$\bar{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \tag{6.13}$$

and $\alpha\delta - \beta\gamma = 1$. This transformation group, which is locally isomorphic to $\mathrm{SL}(2, C)$, admits infinitely many differential invariants, the simplest three (relative) invariants being

$$J_1 := y' - \tfrac{1}{6}y^2, \tag{6.14}$$

$$J_2 := y'' - yy' + \tfrac{1}{9}y^3, \tag{6.15}$$

$$J_3 := y''' - 2yy'' + 3(y')^2. \tag{6.16}$$

The ratios $J_2/(J_1)^{3/2}$ and $J_3/(J_1)^2$ are absolute invariants, the latter being a constant or zero if $y(x)$ satisfies Chazy-XII or Chazy-III, respectively.

The inverse of the relations (6.8)–(6.10) is given by

$$z(t) \;=\; K\left\{(J_2)^2 + \frac{8N^2}{3(N^2-36)}\,(J_1)^3\right\}^{1/12}, \tag{6.17}$$

$$t \;=\; \{K/z(t)\}^{12}(J_2)^2, \tag{6.18}$$

where K is a constant.

There is a direct mapping between the reduced Chazy-XI and XII equations that appears to have been overlooked in the literature. It arises because both equations are reducible to the same hypergeometric equation. Let N be any complex number except ± 6 and let $M := 6/N$. Let $y(x)$ satisfy XII and define $v(x)$ to be the right-hand side of (6.17), i.e., $v(x) = z(t(x))$. Hence

$$y \;=\; 6v'/v. \tag{6.19}$$

Define variables X and $Y(X)$ according to

$$X \;=\; \tfrac{1}{6}\int v^2\,dx, \qquad Y \;=\; y/v^2. \tag{6.20}$$

The inverse relations are

$$x \;=\; 6\int (1/V^2)\,dX, \qquad y \;=\; V^2 Y, \qquad Y \;=\; \frac{1}{V}\frac{dV}{dX}, \tag{6.21}$$

where $V(X) = v(x)$. Then $Y(X)$ satisfies the reduced Chazy-XI equation,

$$Y''' \;=\; -\,2YY'' - 2(Y')^2 + \frac{24}{M^2-1}\left(Y' + Y^2\right)^2, \tag{6.22}$$

where the prime denotes d/dX.

Included in the above is the case $N = 0$ and $M = \infty$. Then Chazy-XI simplifies to the reduced Chazy-II equation, which is easily solved in terms of Airy functions. Hence we arrive at the solution of XII when $N = 0$ given above by (6.8), (6.9), and (6.11).

Equation (6.8) may be regarded as a conformal mapping from the Riemann surface parametrized by the complex variable t, being the domain of $z(t)$ and $z_1(t)$, to a region of the complex x-plane. In fact, the inverse function $t(x)$ is the Schwarz triangle function $S(x) = S(x;\pi/2,\pi/3,\pi/N)$ [1, 47, 53, 54, 55]. The general Schwarz triangle function $S(x;\rho\pi,\sigma\pi,\tau\pi)$ satisfies the third-order equation,

$$S''' \;=\; \frac{3(S'')^2}{2S'} + \frac{1}{2}\left\{\frac{\rho^2-1}{S^2} + \frac{\sigma^2-1}{(S-1)^2} + \frac{1-\rho^2-\sigma^2+\tau^2}{S(S-1)}\right\}(S')^3. \tag{6.23}$$

It has the Painlevé property when each of ρ, σ, and τ are zero or reciprocals of integers ≥ 2. It is solved by inverting $x = z_1(t)/z(t)$ for $S(x) = t(x)$, where $z(t)$ and $z_1(t)$ are solutions of the hypergeometric equation having exponent differences ρ, σ, and τ at the regular singular points $t = 0$, 1, and ∞, respectively. $S(x)$ has a natural barrier if $\rho + \sigma + \tau < 1$.

In terms of $S(x)$, the solution of Chazy-XII (including III when $N = \infty$) is given by

$$y = \frac{3S''}{S'} - \left\{\frac{3}{2S} + \frac{2}{S-1}\right\} S', \tag{6.24}$$

where $\rho = 1/2$, $\sigma = 1/3$, and $\tau = 1/N$.

On any given sheet of the Riemann surface for $z(t)$ and $z_1(t)$, the upper half plane together with the real line segments joining the three branch points $t = 0$, $t = 1$, and $t = \infty$, map to a closed circular-arc triangle in the complex x-plane having interior angles $\rho\pi$, $\sigma\pi$, and $\tau\pi$. (In this context, where Möbius transformations are involved, straight lines and line segments are to be counted as special cases of circles and circular arcs.) Three distinct lower half-planes on different Riemann sheets join to this upper half-plane. They map to three circular-arc triangles having a common edge with the first and having the same three interior angles. When this process is continued, the triangles tesselate all or part of the x-plane when the aforementioned condition for the Painlevé property is satisfied. When $\rho + \sigma + \tau > 1$, the process terminates at a finite (necessarily integer) number $4/(\rho + \sigma + \tau - 1)$ of triangles, including a triangle on the Riemann sphere containing the point $x = \infty$. If $\rho + \sigma + \tau = 1$, there are infinitely many triangles tiling the plane.

If $\rho+\sigma+\tau < 1$, there are infinitely many triangles accumulating against a circular barrier from either the inside or outside, the barrier being a natural barrier to analytic continuation of $S(x)$. If any interior angles are zero, as in the case of Chazy-III, the corresponding vertices must lie on the barrier itself, as in the illustration in [52, p. 338]. In this case, the asymptotics near the barrier are relatively easy to determine from the hypergeometric solution. There are regular patterns in the arrangement of the triangles. For example, circular arcs running from the barrier to another part of the barrier form the edges of infinitely many triangles, in general, and these arcs always strike the barrier at right angles. The domain of the particular case $S(x; \pi/2, \pi/4, \pi/6)$ having a pole at the centre was used by the artist M. C. Escher in his woodcuts titled *Circle Limit I* and *Circle Limit IV* [56].

The Chazy-XII solution $y(x)$ inherits the properties of $S(x)$ having $\rho = 1/2$, $\sigma = 1/3$, and $\tau = 1/N$. When $2 \leq N \leq 5$, the Riemann sphere is covered by $24N/(6-N)$ triangles. Taken 6 at a time appropriately, these triangles form the faces of a dihedron, tetrahedron, octohedron, and icosahedron, respectively. Taken $2N$ at a time, they form the faces of a hosohedron, tetrahedron, hexahedron (cube), and dodecahedron, respectively [57, p. 10]. When $N = 6$, the triangles tile the plane and, taken 24 at a time, they form the period parallelograms of the elliptic function appearing in (6.37) below. When $N \geq 7$, including $N = \infty$, a circular natural barrier is formed. It is a movable barrier because the Möbius transformation (6.13) maps circles to (in general, different) circles. Also Möbius transformations can be employed to swap the domain of $y(x)$ from the outside to the inside of the circle and to place a pole, for example, at the center of the circle.

When $7 \leq N < \infty$, every point on the barrier is a limit point of poles of $y(x)$. Then $y(x)$ is meromorphic on one side or the other (but not both) of the barrier. These poles correspond to the points $t = \infty$ on the Riemann surface for the hypergeometric function. When $N = \infty$, $y(x)$ has no poles, but every point on the barrier is a limit point of zeros of the invariants $J_1(x)$ and $J_2(x)$, which correspond, respectively, to the points $t = 0$ and $t = 1$ on the Riemann surface. The latter is equally true for all finite $N \geq 7$.

Joshi and Kruskal [58] have studied the possibility of identifying the circular barrier of Chazy-III by a local Painlevé test. They examined solutions in the neighborhood of the rational solution (6.41) below where the pole would open up into a small circular barrier with the domain of $y(x)$ on the outside. By picking suitable dominant terms in each term of an expansion similar to (6.39) below, they were able to show that the expansion broke down as it approached the circle. We are not sure if this method confirms that the boundary of the domain of validity of the expansion is necessarily a singular curve.

The radius of the barrier can be determined from the connection formulas for the hypergeometric function. Consider the general Schwarz triangle function $S(x; \pi/m, \pi/n, \pi/p)$ with $1/m + 1/n + 1/p < 1$. If $m \neq \infty$, choose the integration constants so that $z(t)$ is the hypergeometric function that is analytic at $t = 0$ and takes the value $z(0) = 1$ and $z_1(t)$ is the linearly independent solution of the hypergeometric equation such that $t^{-1/m} z_1(t)$ is analytic and takes the value λ at $t = 0$. This puts the center of the circle at $x = 0$, and this point is a vertex common to $2m$ triangles, each subtending an angle π/m at $x = 0$ and each having two straight edges meeting at $x = 0$. The square of the radius R of the barrier is

$$R^2 = |\lambda|^2 \frac{\Gamma^2(1 + 1/m)}{\Gamma^2(1 - 1/m)} \prod_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\Gamma\left(\frac{1}{2}(1 - 1/m + \epsilon_1/n + \epsilon_2/p)\right)}{\Gamma\left(\frac{1}{2}(1 + 1/m + \epsilon_1/n + \epsilon_2/p)\right)}. \tag{6.25}$$

For Chazy-XII, there are three families of distinguished points in the complex x-plane at which it is possible to give a closed form expression for the center and radius of the barrier in terms of initial data. The distinguished points are zeros of $J_1(x)$, zeros of $J_2(x)$, and poles of $y(x)$, corresponding, respectively, to $t = 0$, 1, and ∞. The first two types are also available for Chazy-III. These points are all equivalent to each other under special Möbius transformations, and the exact connection formulas can be deduced from the theory of the hypergeometric function. The most symmetric results occur when we use the above transformation group to place any one of these distinguished points at the center of the circle and the latter at $x = 0$, and we assume that this has been done in the next paragraph.

The origin is a zero of $J_1(x)$ when $y(x)$ is odd and admits a convergent power series expansion of the form,

$$y(x) \;=\; b_0 x + b_1 x^3 + b_2 x^5 + \dots . \tag{6.26}$$

The radius of the barrier is given by (6.25) with $m = 2$, $n = 3$, $p = N$, and

$$\lambda^2 \;=\; \frac{N^2 - 36}{6N^2 b_0}\,.$$

The origin is a zero of $J_2(x)$ when $y(x)$ admits a convergent power series of the form,

$$y(x) \;=\; c_0 x^2 + c_1 x^5 + c_2 x^8 + \dots . \tag{6.27}$$

The radius of the barrier is given by (6.25) with $m = 3$, $n = 2$, $p = N$, and

$$\lambda^3 \;=\; \frac{3(N^2 - 36)}{16N^2 c_0}\,.$$

Both of these results are valid for the case $N = \infty$, in which case the formulas simplify a little.

The origin is a pole of $y(x)$ when $y(x)$ admits a convergent Laurent expansion of the form,

$$y(x) \;=\; \tfrac{1}{2}(N-6)/x + a_N x^{N-1} + a_{2N} x^{2N-1} + \dots , \tag{6.28}$$

which can be compared with (6.6). The radius of the barrier is given by (6.25) with $m = N$, $n = 2$, $p = 3$, and

$$\lambda^N \;=\; \frac{N(N^2 - 36)}{144(N^2 - 1)a_N}\,.$$

This formula for the radius was obtained by Halphen [45, 46] and Chazy [1, p. 350]. (It can be simplified with the help of the gamma function duplication formula.)

For Chazy-XII (but not III), a useful auxiliary function $u(x)$ is defined by

$$y \;=\; \tfrac{1}{2}(N - 6)\, u'/u. \tag{6.29}$$

Then $u(x)$ satisfies the fourth-order equation,

$$u u^{(\mathrm{iv})} - (N - 2)u'u''' + \frac{3N(N-2)}{2(N+6)}(u'')^2 \;=\; 0. \tag{6.30}$$

This equation is also integrable for all complex N (the cases $N=-6$ and ∞ requiring a limit). In terms of the above hypergeometric functions, the solution is given by

$$x = \frac{z_1(t)}{z(t)}, \qquad u(x) = K_4\{z(t)\}^{12/(N-6)}. \tag{6.31}$$

The symmetry corresponding to (6.12) and (6.13) is given by

$$u(x) = (\gamma x+\delta)^{12/(6-N)} u_0(\bar{x}). \tag{6.32}$$

The u-equation has the Painlevé property for integers $2 \leq N \leq 10$ as well as $N=12$ and $N=18$, the case $N=6$ being no longer exceptional. For $N \geq 7$, $u(x)$ has the same movable natural barrier as $y(x)$. If $u(x)$ is defined inside the circular barrier, then it is analytic and single-valued inside that circle. If the barrier is a straight line, then $u(x)$ is analytic and single-valued on the side on which it is defined. If $u(x)$ is defined outside the circular barrier, then $x=\infty$ is a point of analyticity or an algebraic branch point, according to whether the particular integral $u = x^{12/(6-N)}$ is analytic or branched at $x=\infty$, implying that the barrier is also unbranched or algebraically branched in the same fashion. (A branch cut can be drawn from any point of the barrier to $x=\infty$.)

For $N = 2,\ 3,\ 4$, and 5, $u(x)$ is a polynomial of generic degree $3,\ 4,\ 6$, and 12, respectively. For $N=6$, $u(x)$ is expressible in terms of the Weierstrass elliptic function $\wp(x; g_2, g_3)$. Specifically, the general integrals of (6.30) in these cases are

$$N=2: \qquad u = Ax^3+Bx^2+Cx+D; \tag{6.33}$$

$$N=3: \qquad u = Ax^4+Bx^3+Cx^2+Dx+E, \tag{6.34}$$

where $12AE - 3BD + C^2 = 0$;

$$N=4: \qquad u = (\alpha x+\beta)^5(\gamma x+\delta) + (\alpha x+\beta)(\gamma x+\delta)^5; \tag{6.35}$$

$$\begin{aligned} N=5: \qquad u = {}& (\alpha x+\beta)^{11}(\gamma x+\delta) + 11(\alpha x+\beta)^6(\gamma x+\delta)^6 \\ & - (\alpha x+\beta)(\gamma x+\delta)^{11}; \end{aligned} \tag{6.36}$$

$$N=6: \qquad u = Ae^{Bx}\exp\left(\iint \wp(x-C; 0, D)\, dx dx\right), \tag{6.37}$$

all coefficients being arbitrary constants.

So far, we have restricted attention to the reduced equations in Chazy Classes III and XII. We now turn to the problem of determining the full equations in these classes. In other words, we wish to construct all Painlevé-type equations of the form,

$$\begin{aligned} y''' = {}& 2yy'' - 3(y')^2 + \frac{4}{36-N^2}(6y'-y^2)^2 + A(x)y'' + B(x)yy' \\ & + C(x)y' + D(x)y^3 + E(x)y^2 + F(x)y + G(x), \end{aligned} \tag{6.38}$$

where the coefficients $A(x), \ldots, G(x)$ are locally analytic functions to be determined. (See [14] for an alternative derivation.) The method shown here is essentially an application of the method used originally by Painlevé himself to derive his Painlevé-I equation [4, pp. 224–225] (reviewed by Ince [3, pp. 329–330]). It is also the method we used in [I, Section 3] to analyze the Bureau barrier equation (1.6).

First, a change of gauge, $\bar{x} = \phi(x)$, $\bar{y} = \lambda(x)y + \mu(x)$, can be invoked to set $A(x) \equiv 0$ and $C(x) \equiv 0$ for every N (except the inadmissible values $N = \pm 6i/\sqrt{11}$). Apply the Painlevé α-expansion,

$$\begin{aligned} x &= x_0 + \alpha X, \\ y &= \alpha^{-1} Y_0(X) + Y_1(X) + \alpha Y_2(X) + \alpha^2 Y_3(X) + \alpha^3 Y_4(X) + \ldots, \end{aligned} \tag{6.39}$$

$\alpha \to 0$, where x_0 is arbitrary, to equation (6.38). We need to go out as far as $Y_4(X)$ to get information about the coefficient $G(x)$. However, we never need to consider more than one term at a time beyond the leading term, so the problem is essentially an easy first-order perturbation problem.

The leading coefficient $Y_0(X)$ satisfies the reduced Chazy equation (6.2). It is sufficient for our purposes to let $Y_0(X)$ be the following two-parameter particular solutions:

$$N \neq \infty: \qquad Y_0(X) = -\frac{1}{2}\left(\frac{6-N}{X-X_0} + \frac{6+N}{X-X_1}\right), \tag{6.40}$$

$$N = \infty: \qquad Y_0(X) = -\frac{6}{X-X_0} + \frac{C_0}{(X-X_0)^2}, \tag{6.41}$$

where X_0, X_1, and C_0 are integration constants. A necessary condition for (6.38) to have the Painlevé property is that each of the $Y_n(X)$ be single-valued in the neighborhood of $X = X_0$ and $X = X_1$. We may exclude $N = 0$ and $N = 1$ from the outset because the reduced equation has movable logarithms in these cases. The limiting case $N = 6$ belongs to a different class of equations from the point of view of Painlevé classification, namely, Chazy Class I, whose full version is given in the Appendix.

For each index $n \geq 1$, $Y_n(X)$ satisfies a third-order linear equation of the form,

$$\mathcal{L}[Y_n] = \Phi_n[Y_0, Y_1, \ldots, Y_{n-1}], \tag{6.42}$$

where $\mathcal{L}$ denotes the linear differential operator,

$$\begin{aligned} \mathcal{L}[Y] := Y''' - 2Y_0 Y'' + 6\left(Y_0' + \frac{8(6Y_0' - Y_0^2)}{N^2-36}\right) Y' \\ - 2\left(Y_0'' + \frac{8Y_0(6Y_0' - Y_0^2)}{N^2-36}\right) Y, \end{aligned} \tag{6.43}$$

and Φ_n denotes some function of the earlier Y_i, $i < n$, and their derivatives.

When $N \neq \infty$, the homogeneous equation $\mathcal{L}[Y] = 0$ has linearly independent solutions,

$$Y = (X - X_0)^{-2}, \quad (X - X_1)^{-2}, \quad (X - X_0)^{N-1}(X - X_1)^{-N-1}. \tag{6.44}$$

When $N = \infty$, the linearly independent solutions are,

$$Y = (X - X_0)^{-2}, \quad (X - X_0)^{-3}, \quad (X - X_0)^{-2} e^{-2C_0/(X - X_0)}. \tag{6.45}$$

From these, we deduce the following integrating factors for equation (6.42). When $N \neq \infty$, three independent integrating factors are

$$I_1(X) = (X - X_0)^{3-N}(X - X_1)^{3+N}, \tag{6.46}$$

$$I_2(X) = (X - X_0)^2 (X - X_1)^4, \tag{6.47}$$

$$I_3(X) = (X - X_0)^4 (X - X_1)^2. \tag{6.48}$$

When $N = \infty$, three independent integrating factors are

$$I_1(X) = (X - X_0)^6 e^{2C_0/(X - X_0)}, \tag{6.49}$$

$$I_2(X) = (X - X_0)^6, \tag{6.50}$$

$$I_3(X) = (X - X_0)^5. \tag{6.51}$$

It is easily verified in each case that $\int I_\mu(X)\mathcal{L}[Y_n]\, dX$, $\mu = 1, 2, 3$, is a linear combination of Y_n and its first and second derivatives with single-valued coefficients. Hence a necessary condition for (6.38) to have the Painlevé property is that all of the products $I_\mu(X)\Phi_n[Y_0, \ldots, Y_{n-1}]$, $\mu = 1, 2, 3$, $n \geq 1$, have zero residue at the singularities $X = X_0$ and $X = X_1$. The problem has now been reduced to the calculation of residues.

Begin with the series (6.39) truncated at the $Y_1(X)$ term. Substitution into (6.38) gives

$$\mathcal{L}[Y_1] = \Phi_1[Y_0] = B(x_0) Y_0 Y_0' + D(x_0) Y_0^3. \tag{6.52}$$

When $N \neq \infty$, multiplication of Φ_1 by $I_2(X)$ produces a simple pole at $X = X_0$ with residue a nonzero multiple of $2B(x_0) + (6-N)D(x_0)$. Similarly, multiplication by $I_3(X)$ produces a simple pole at $X = X_1$ with residue a nonzero multiple of $2B(x_0) + (6+N)D(x_0)$. Since x_0 is arbitrary, the vanishing of both residues forces

$$B(x) \equiv 0, \qquad D(x) \equiv 0. \tag{6.53}$$

This is in agreement with the freedom-from-logarithms conditions implied by the resonance number 1 appearing in (6.6) and (6.7).

When $N = \infty$, the latter approach is not available because there are no positive resonance numbers, but the α-expansion works well. Multiplication of Φ_1 by $I_3(X)$ produces a simple pole at $X = X_0$ with residue $C_0^3 D(x_0)$. Multiplication by $I_1(X)$ produces an isolated essential singularity at $X = X_0$ with residue $-C_0^4(4B(x_0) + 34D(x_0))$. The vanishing of both residues again leads immediately to (6.53).

Since (6.53) holds in all cases, we may choose the particular solution $Y_1(X) \equiv 0$. The equation for $Y_2(X)$ now takes the simple form,

$$\mathcal{L}[Y_2] \;=\; E(x_0){Y_0}^2. \tag{6.54}$$

We use the integrating factor $I_1(X)$ exclusively for the rest of this calculation. When $N \neq \infty$, the product $I_1(X)E(x_0){Y_0}^2$ has a (nonsimple) pole at $X = X_0$ with residue,

$$\tfrac{1}{120} N(N^2-1)(11N^2+36)(X_0-X_1)^5 E(x_0).$$

The only legal way to make this residue vanish is to set

$$E(x) \;\equiv\; 0. \tag{6.55}$$

(Note that the factor $11N^2 + 36$ is an artefact of our choice of gauge.) When $N = \infty$, the corresponding residue is $(44/15)C_0^5 E(x_0)$. Again, we arrive at (6.55).

With (6.55) now shown to hold in all cases, we may choose $Y_2(X) \equiv 0$. The equation for $Y_3(X)$ becomes

$$\mathcal{L}[Y_3] \;=\; F(x_0)Y_0. \tag{6.56}$$

When $N \neq \infty$, $I_1(X)F(x_0)Y_0$ has a pole at $X = X_0$ with residue,

$$-\tfrac{1}{240} N^2(N^2-1)(N^2-4)(X_0-X_1)^6 F(x_0).$$

Setting this residue to zero gives two possibilities, namely,

$$F(x) \;\equiv\; 0 \qquad \text{or} \qquad N = 2. \tag{6.57}$$

When $N = 2$, we will see below that both $F(x)$ and $G(x)$ can remain arbitrary. When $N = \infty$, the corresponding residue is $-(4/15)C_0^6 F(x_0)$, which can only vanish when $F(x)$ is identically zero.

Finally, we postpone consideration of the case $N = 2$ to the next paragraph and put $F(x) \equiv 0$ and $Y_3(X) \equiv 0$. The equation for $Y_4(X)$ becomes

$$\mathcal{L}[Y_4] \;=\; G(x_0). \tag{6.58}$$

When $N \neq \infty$, $I_1(X)G(x_0)$ has a pole at $X = X_0$ with residue,

$$\tfrac{1}{5040}N(N^2-1)(N^2-4)(N^2-9)(X_0-X_1)^7 G(x_0).$$

Setting this residue to zero also gives two possibilities, namely,

$$G(x) \equiv 0 \qquad \text{or} \qquad N = 3. \tag{6.59}$$

When $N = 3$, $G(x)$ can remain arbitrary. When $N = \infty$, the corresponding residue is $(8/315)C_0^{\,7}G(x_0)$, which can only vanish when $G(x)$ is identically zero. This completes the Painlevé analysis of the family of equations (6.38).

We conclude from the above calculation that no nontrivial recessive terms can be added to the reduced equations in Chazy Classes III or XII except for the two cases $N = 2$ and $N = 3$. In particular, none of the cases involving natural barriers can admit recessive terms.

These results agree with the results announced by Chazy (at least for XII) and proved by Conte, Fordy, and Pickering [14]. With the CFP test, we can weaken the hypotheses. By examining only the first species of poles (6.5), we can relax the requirement that N be an integer and let it take arbitrary complex values (including ∞, excluding ± 6). Some irrational and complex values of N require special handling, as well as $N = 0$, but the conclusion reached at the second perturbation stage in the CFP test is again that only the cases $N = 2$ and $N = 3$ are able to support recessive terms.

When $N = 2$, the full equation reads

$$y''' = 2yy'' - 3(y')^2 + \tfrac{1}{8}(6y' - y^2)^2 + F(x)y + G(x). \tag{6.60}$$

The change of variable $y = -2u'/u$ yields the fourth-order linear differential equation,

$$u^{(\text{iv})} = F(x)u' - \tfrac{1}{2}G(x)u. \tag{6.61}$$

When $N = 3$, the full equation reads

$$y''' = 2yy'' - 3(y')^2 + \tfrac{4}{27}(6y' - y^2)^2 + G(x). \tag{6.62}$$

The change of variable $y = -(3/2)u'/u$ yields the fourth-order equation,

$$uu^{(\text{iv})} - u'u''' + \tfrac{1}{2}(u'')^2 = -\tfrac{2}{3}G(x)u^2. \tag{6.63}$$

Differentiation gives the fifth-order linear equation,

$$u^{(\text{v})} = -\tfrac{2}{3}\{2G(x)u' + G'(x)u\}. \tag{6.64}$$

The linearizations (6.61) and (6.64) were obtained by Chazy [1, p. 346].

It is possible to solve (6.63) in terms of a fourth-order linear differential equation, which would be best-possible in general. Let $v_1(x)$ and $v_2(x)$ be any two linearly independent solutions of

$$v^{(\mathrm{iv})} = \tfrac{1}{3}G(x)v, \tag{6.65}$$

subject to the constraint,

$$v_1 v_2''' - v_1' v_2'' + v_1'' v_2' - v_1''' v_2 = 0. \tag{6.66}$$

If $v_1(x)$ and $v_2(x)$ were unrestricted, the left-hand side of (6.66) would be a constant on account of (6.65). Thus (6.66) is a single algebraic constraint acting on the eight constants of integration appearing in $v_1(x)$ and $v_2(x)$. The general solution of (6.63) is

$$u(x) = v_1 v_2' - v_1' v_2 \,. \tag{6.67}$$

We believe that this presentation of the solution of XII when $N = 3$ is new.

Clarkson and Olver [31] have made a study of differential equations of any order that admit an $\mathrm{SL}(2, C)$ transformation group similar to (6.12)–(6.16) above. They found that such differential equations having order $n \geq 3$ can be expressed in terms of differential invariants and the resulting equation has order $n - 3$. Furthermore, they were able to recover the original variables from the solution of the latter equation by a series of quadratures and a single Riccati equation. In this theory, the Chazy-XII equation is the simplest case and expresses the simple fact that the absolute invariant $J_3/(J_1)^2$ is constant. (Since the Painlevé property is not essential in their work, they allowed the parameter $\alpha := 4/(36 - N^2)$ to take unrestricted complex values.) When the corresponding Riccati equation is linearized in the usual way, the hypergeometric equation (6.10) or an equivalent Lamé equation is obtained. (The authors expressed surprise at this equivalence, but it comes from a routine manipulation of the Riemann P-function for either equation.)

Clarkson [64] pointed out that the next simplest equation admitting the same type of transformation group as Chazy-XII is the fourth-order equation,

$$U' - \tfrac{4}{3}yU = 0, \tag{6.68}$$

where U is the relative invariant,

$$U := J_3 - 36\alpha(J_1)^2. \tag{6.69}$$

Equation (6.68) does not have the Painlevé property except when $\alpha = 4/27$, in which case it takes recessive terms and can be identified with equation F-XVI. In general, we can expect a very complicated singularity structure in view of the fact that setting

$U = 0$ provides a 3-parameter solution involving, in general, a natural barrier that may or may not have branch points accumulating against it. Nevertheless, equation (6.68) can be integrated without difficulty using the theory in [31]. When the equation is written in terms of absolute invariants, it becomes a first-order equation of Abel type. In general, Abel equations are presumed to be nonintegrable, some test cases having failed a poly-Painlevé test in [65, pp. 142–153], but the Abel equation obtained here is gauge-equivalent to one with constant coefficients and is therefore trivially integrable.

The solution of the aforementioned Abel equation translates into a first integral for (6.68) of the third order and third degree. It is

$$U^3 = K\left\{6J_1U - 3(J_2)^2 + 8(9\alpha - 1)(J_1)^3\right\}^2, \tag{6.70}$$

where K is a constant of integration. Equation (6.70) is an algebraic relation connecting the absolute invariants $J_3/(J_1)^2$ and $J_2/(J_1)^{3/2}$. This is precisely the type of equation that can be reduced to quadratures, functional inversions, and a Riccati equation by following the procedure in [31]. Clarkson [64] has speculated that higher-order equations constructed in a similar fashion might also be integrable and perhaps there is a Chazy-XII hierarchy. We leave this interesting question for another occasion.

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Appendix. The 13 Chazy equations and their solutions

We present here the full versions of the 13 Chazy equations in the third-order polynomial class and the solutions of those that are not already the subject of Sections 3–6 above. Most of this information appears in the classic papers of Chazy [1] and Bureau [2], but there are some gaps in their work making it desirable to have a handy compilation of the main results.

A few new results appear here. The solution of the full Chazy-VI equation, for example, does not appear in the aforementioned references but is well hidden (with no mention of Chazy-VI) in Bureau's paper on second-order second-degree (SD) equations [29, pp. 277–281]. The direct solution provided below is new. Also, in the literature (for example, [1, 2, 29, 66]), Chazy-I always appears as a list of separate equations. The single master equation (A.3) below is new. (It is still necessary to split the equation up to express its solutions in terms of all of the classical Painlevé transcendents.) We use a generalization of Chazy-VII to illustrate the concept of partial integrability. As already mentioned, the recessive terms attached to Chazy-X are new.

Each Chazy class is actually an equivalence class of differential equations, two equations being regarded as equivalent if they are related by a gauge transformation of the form,

$$\bar{y} = \lambda(x)y + \mu(x), \qquad \bar{x} = \phi(x). \tag{A.1}$$

This group of gauge transformations applies to equations of any order in the polynomial class. When equations that are rational in y and/or its derivatives are under consideration, the linear transformation in y should be enlarged to a Möbius transformation in y with coefficients locally analytic in x. The list below selects a canonical representative from each Chazy class. In the case of some of the linearizable equations (II, IV, V), we have included additional coefficient functions that are not essential and could be transformed away by a stricter choice of gauge. Usually it is inconvenient to force linearizable equations into the form with minimal gauge. (The same principle has been applied to a revision of the Painlevé-Gambier class of equations in [37].) Except for Chazy-I and Chazy-II, we have selected a gauge where the full equation consists of the reduced equation plus recessive terms (if any are allowed).

The list of reduced Chazy equations (in the third-order polynomial class) is given above in Section 2. Bureau [2] used the same numbering as Chazy for Classes IV–XII, he absorbed Chazy-III into XII, and his classes I, II, and III are Chazy's Classes XIII, I, and II, respectively. The linear equation is included for completeness; we remark that Gambier [24] and Ince [3] included the second-order linear equation on their lists.

The 13 full Chazy equations, plus the linear equation, are as follows:

0:
$$y''' = A(x)y'' + B(x)y' + C(x)y + D(x); \tag{A.2}$$

I:
$$\begin{aligned} y''' = &-\frac{2}{f^2(x)}\big\{3c_1x(xy'-y)^2 + c_2(xy'-y)(3xy'-y) \\ &+ c_3y'(3xy'-2y) + 3c_4(y')^2 + 2c_5x(xy'-y) \\ &+ c_6(2xy'-y) + 2c_7y' + c_8x + c_9\big\} - \frac{f'(x)}{f(x)}\,y'', \end{aligned} \tag{A.3}$$

where $f(x) := c_1x^3 + c_2x^2 + c_3x + c_4$; an equivalent version of Chazy-I is

I′: $$y''' = 6\Big\{ -(y')^2 + A(x)(y' + y^2) + B(x)y + C(x)\Big\}, \tag{A.4}$$

where the coefficient functions $A(x)$, $B(x)$, and $C(x)$ satisfy

$$A'' = 6A^2, \qquad B'' = 6AB, \qquad C'' = B^2 + 2AC;$$

II: $$\begin{aligned} y''' = {} & \frac{d^2}{dx^2}\Big\{P(x)y^2 + Q(x)y + R(x)\Big\} \\ & + A(x)\frac{d}{dx}\Big\{y' - P(x)y^2 - Q(x)y - R(x)\Big\} \\ & + B(x)\Big\{y' - P(x)y^2 - Q(x)y - R(x)\Big\} + C(x); \end{aligned} \tag{A.5}$$

III: $$y''' = 2yy'' - 3(y')^2; \tag{A.6}$$

IV: $$\begin{aligned} y''' = {} & -3yy'' - 3(y')^2 - 3y^2y' + R(x)(y'' + 2yy') \\ & + R'(x)(y' + y^2) + S(x)y' + S'(x)y + T(x); \end{aligned} \tag{A.7}$$

V: $$\begin{aligned} y''' = {} & -2yy'' - 4(y')^2 - 2y^2y' \\ & + R(x)(3y'' + 7yy' + y^3) + (R' - 2R^2)(y' + y^2) \\ & + 2S(x)y' + (S' - 2RS)y; \end{aligned} \tag{A.8}$$

VI: $$y''' = -yy'' - 5(y')^2 - y^2y' + 3R(x)y' + R'(x)y - R''(x), \tag{A.9}$$

where $R(x)$ is an elliptic function satisfying $R'' = (1/2)R^2 + \beta$;

VII: $$y''' = -yy'' - 2(y')^2 + 2y^2y'; \tag{A.10}$$

VIII: $$\begin{aligned} y''' = {} & 6y^2y' + (-2\alpha^2x^2 + \beta x + \gamma)(y' + \alpha) \\ & + 2\alpha y^2 + (-4\alpha^2x + \beta)y; \end{aligned} \tag{A.11}$$

IX: $$y''' = 12(y')^2 + 72y^2y' + 54y^4 + \delta; \tag{A.12}$$

X.a: $$\begin{aligned} y''' = {} & 6y^2y' + \tfrac{3}{11}(9 + 7\sqrt{3})(y' + y^2)^2 - \tfrac{1}{22}(4 - 3\sqrt{3})\alpha y' \\ & + \tfrac{1}{44}(3 - 5\sqrt{3})\alpha y^2 - \tfrac{1}{352}(9 + 7\sqrt{3})\alpha^2; \end{aligned} \tag{A.13}$$

X.b: $$\begin{aligned} y''' = {} & 6y^2y' + \tfrac{3}{11}(9 - 7\sqrt{3})(y' + y^2)^2 - \tfrac{1}{22}(4 + 3\sqrt{3})\alpha y' \\ & + \tfrac{1}{44}(3 + 5\sqrt{3})\alpha y^2 - \tfrac{1}{352}(9 - 7\sqrt{3})\alpha^2; \end{aligned} \tag{A.14}$$

$$\textbf{XI}: \qquad y''' = -2yy'' - 2(y')^2 + \frac{24}{N^2-1}\left\{y' + y^2 - g(x)\right\}^2 + g''(x) + \tfrac{1}{4}(N^2-1)(\alpha x + \beta); \tag{A.15}$$

$$\textbf{XII}: \qquad y''' = 2yy'' - 3(y')^2 - \frac{4}{N^2-36}\left(6y' - y^2\right)^2; \tag{A.16}$$

$$\textbf{XII}_{(N=2)}: \qquad y''' = 2yy'' - 3(y')^2 - \tfrac{1}{8}(6y' - y^2)^2 + A(x)y + B(x); \tag{A.17}$$

$$\textbf{XII}_{(N=3)}: \qquad y''' = 2yy'' - 3(y')^2 - \tfrac{4}{27}(6y' - y^2)^2 + B(x); \tag{A.18}$$

$$\textbf{XIII}.\text{a}: \qquad y''' = 12yy' + (1/x)(y'' - 6y^2) + \alpha xy - \tfrac{1}{24}\alpha^2 x^3 + \mu/x; \tag{A.19}$$

$$\textbf{XIII}.\text{b}: \qquad y''' = 12yy' + \beta y - \tfrac{1}{6}\beta^2 x + \mu. \tag{A.20}$$

We now examine the solutions of each of these equations in turn. For Classes III and IX–XII, we refer the reader to Sections 3–6 above for a detailed treatment. For Class I, we refer the reader to [33] for details. The remaining equations are solved below by reduction to linear equations or known transcendents.

Chazy Class I

This is the only case where we ask the reader to refer to another paper [33] for full details. Here we set up the canonical forms of the equation and its first integral to permit easy use of the results in the latter reference.

We gave above two qualitatively different canonical forms for the Chazy-I equation. They are equivalent under a gauge transformation of the form (A.1). The version (A.3), which is new here, is better suited to the direct solution of Chazy-I in terms of Painlevé transcendents, all six being needed for various subcases. The version (A.4) was formed by Bureau [2, (27.1)] and must have been formed by Chazy before he split the class into six subcases [1, 67].

Equation (A.3) admits the first integral,

$$(y'')^2 = -\frac{4}{f^2(x)}\Big\{c_1(xy'-y)^3 + c_2y'(xy'-y)^2 + c_3(y')^2(xy'-y) + c_4(y')^3 + c_5(xy'-y)^2 + c_6y'(xy'-y) + c_7(y')^2 + c_8(xy'-y) + c_9y' + c_{10}\Big\}, \tag{A.21}$$

where c_{10} is the constant of integration. This is equation SD-I, introduced in [33, pp. 57,65–73]. It was called there the "master Painlevé equation" because it unified all of the six Painlevé transcendents into a single equation. The same name could be applied to Chazy-I and a number of other associated equations.

Only four of the 10 parameters in SD-I and only three of the 9 parameters in Chazy-I are essential, because the equation retains its shape under gauge transformations of the form,

$$\bar{x} = \frac{ax+b}{cx+d}, \qquad \bar{y} = \frac{hy+kx+m}{cx+d}. \tag{A.22}$$

By using this gauge freedom appropriately, equations Chazy-I and SD-I can, without loss of generality, be split into six canonical subcases. With the ordered 10-tuples denoting $\{c_1, c_2, \ldots, c_{10}\}$, the six subcases are identified by the parameters:

Chazy-I.a and SD-I.a: $\{0, 1, -1, 0, 0, 0, c_7, c_8, c_9, c_{10}\}$,

Chazy-I.b and SD-I.b: $\{0, 0, 1, 0, c_5, 0, 0, c_8, c_9, c_{10}\}$,

Chazy-I.c and SD-I.c: $\{0, 0, 0, 1, -1, 0, 0, 0, c_9, c_{10}\}$,

Chazy-I.d and SD-I.d: $\{0, 0, 0, 1, 0, \frac{1}{2}, 0, 0, 0, c_{10}\}$,

Chazy-I.e and SD-I.e: $\{0, 0, 0, 1, 0, 0, 0, \frac{1}{2}, 0, c_{10}\}$,

Chazy-I.f and SD-I.f: $\{0, 0, 0, 1, 0, 0, 0, 0, c_9, c_{10}\}$.

In SD-I.e, the parameter c_{10} can be absorbed by a translation in y, but we keep it here because it is an integration constant for Chazy-I.e. Apart from this translation, the above choices of parameters agree with those in [33, p. 66] apart from an obvious scaling and renaming of parameters.

These six canonical subcases can be solved, respectively, in terms of the Painlevé transcendents VI, III/V, IV, II, I, and the Weierstrass zeta function. Full details are provided in [33, pp. 65–73] (see also [42, Appendix C]). A special case of Chazy-I.a and SD-I.a appearing in [68] is perhaps the first application of Painlevé-VI to a physical problem, namely, an exact solution of Einstein's equations representing the gravitational field of a rotating spheroid.

The corresponding first integral of (A.4) is

$$\begin{aligned}(y'' - 2Ay - B)^2 = & -4(y')^3 + 12Ay^2y' - 4A'y^3 + 4A(y')^2 \\ & + (12B - 4A')yy' + (4A^2 - 6B')y^2 \\ & + (12C - 2B')y' + (4AB - 12C')y \\ & + B^2 - 12\int BC\,dx + K,\end{aligned} \tag{A.23}$$

which is simply SD-I in an alternative gauge. Lists of second-degree equations of the form (A.23) are given by Chazy [1, 66] and Bureau [2, 29]. Chazy did not provide the explicit solutions but obviously knew them because he deduced these second-degree equations and another set from the six Painlevé equations. (The latter set contains the Fokas-Yortsos equation [69] in an alternative gauge.) In the generic case, $A(x)$ is a Weierstrass elliptic function and $B(x)$ and $C(x)$ can be expressed in terms of $A(x)$ by quadratures. In this case, (A.4) is Chazy-I.a and (A.23) is SD-I.a. The special case $A(x) = 1/x^2$ is Chazy-I.b and SD-I.b. The transformation of these cases into our preferred canonical form is given explicitly in [33, p. 68]. These cases are also given special attention in [70]. The remaining four subcases correspond to $A(x) = 0$, the transformation to our canonical form being relatively straightforward.

Chazy Class II

The version of Chazy-II presented above is in a general gauge. By being more strict with gauge, we can set $P(x) = -1$ and $Q(x) = R(x) = A(x) = 0$, for example. Only two of the arbitrary coefficient functions are essential. The general solution of Chazy-II is given by solving the Riccati equation,

$$y' = P(x)y^2 + Q(x)y + R(x) + w(x), \tag{A.24}$$

where $w(x)$ satisfies the second-order linear equation,

$$w'' = A(x)w' + B(x)w + C(x). \tag{A.25}$$

Chazy Class III

This may be regarded as the special case $N = \infty$ of Chazy-XII. It is the simplest Painlevé-type equation admitting a movable natural barrier and has been much studied. Chazy's own solution by inversion of a hypergeometric function is given by setting $N = \infty$ in (6.8)–(6.10). The solution in terms of the Schwarz triangle function $S(x; \pi/2, \pi/3, 0)$ is given by (6.24). A proof that III takes no recessive terms (except for gauge-induced terms) is also provided in Section 6.

Chazy Class IV

In the version of Chazy-IV presented, the coefficient function $R(x)$ can be set to zero by using gauge freedom. Chazy-IV admits the first integral,

$$y'' = -3yy' - y^3 + R(x)(y' + y^2) + S(x)y + \int T(x)\,dx + K, \tag{A.26}$$

K being the integration constant. This is equation Ince-VI [3, p. 331]. The substitution $y = w'/w$ yields the third-order linear equation,

$$w''' = R(x)w'' + S(x)w' + \left\{\int T(x)\,dx + K\right\} w. \tag{A.27}$$

Chazy Class V

In the version of Chazy-V presented, the function $R(x)$ can be set to zero by using gauge freedom. Chazy-V admits the first integral,

$$y'' = \frac{1}{2y}(y')^2 - 2yy' - \tfrac{1}{2}y^3 + R(x)(y' + y^2) + S(x)y - \frac{K^2q^2(x)}{2y}, \tag{A.28}$$

where $q(x) := \exp\{\int R(x)\,dx\}$ and K is an integration constant. This is a special case of equation Ince-XXVII [3, p. 338], the general case of which has been derived and solved by Gambier [24, 71] and Bureau [11].

When $K \neq 0$, the general solution of Chazy-V and (A.28) is

$$y = \frac{Kq(x)}{u(x) - v(x)}, \tag{A.29}$$

where $u(x)$ and $v(x)$ satisfy the Riccati equations,

$$u' = -u^2 + R(x)u + \tfrac{1}{2}\{S(x) + Kq(x)\}, \tag{A.30}$$

$$v' = -v^2 + R(x)v + \tfrac{1}{2}\{S(x) - Kq(x)\}. \tag{A.31}$$

When $K = 0$, the general solution is

$$y = \frac{w^2}{\int w^2\,dx + K_2}, \tag{A.32}$$

where K_2 is another constant of integration and $w(x)$ satisfies the linear equation,

$$w'' = R(x)w' + \tfrac{1}{2}S(x)w. \tag{A.33}$$

Chazy Class VI

For Chazy-VI, Chazy [1, p. 343] gave the first integral,

$$\begin{aligned}\Big\{y'' + yy' - y^3 + Ry - R'\Big\}^2 &\\ = -\tfrac{8}{3}\Big\{&(y' + y^2 - R)^2(y' - \tfrac{1}{2}y^2 + \tfrac{1}{2}R)\\ &- \tfrac{3}{2}(R'y - R'')(y' + y^2 - R)\Big\} + K,\end{aligned} \tag{A.34}$$

where K is the integration constant. In a proposed paper on second-order second-degree equations, we will give this equation the name SD-LVI, it being the 56th in a complete list of 60 equations of this type having the Painlevé property. (Approximately 20 of those equations were obtained by Bureau [29].) It is clear that Chazy knew how to solve this equation, because he mentioned that he could reduce it to (A.37) below, but he did not reveal the method. Bureau, in [2], solved the special case $R(x) = 0$, and eight years later, in [29, pp. 277–281], he solved the general case by a series of transformations. The latter solution is well hidden and no mention is made of its connection to Chazy-VI.

The direct solution of Chazy-VI and (A.34) is as follows:

$$y = U'/U, \tag{A.35}$$

where

$$U = 2ww'w'' - (w')^3 - R(x)w^2w' + \tfrac{1}{3}\{R'(x) - \sqrt{K}\,\}w^3, \tag{A.36}$$

and where $w(x)$ satisfies the third-order linear equation,

$$w''' = R(x)w' + \tfrac{1}{3}\sqrt{K}\,w. \tag{A.37}$$

Chazy Class VII

Chazy-VII does not take any recessive terms. It has the first integral,

$$y'' = \frac{1}{2y}(y')^2 - yy' + \tfrac{1}{2}y^3 + \frac{K}{y}, \tag{A.38}$$

K being the integration constant, which is a special case of equation Ince-XXVIII [3, p. 339], first found and solved by Gambier [24, 72]. The solution of Chazy-VII and (A.38) is

$$y = w''/w', \tag{A.39}$$

where $w(x)$ is the general solution of the equation,

$$w'' = 6w^2 + \tfrac{1}{12}K, \tag{A.40}$$

solvable in terms of elliptic functions.

Equation VII with certain recessive terms attached is a good candidate for a discussion of the concept of partial integrability, a term for which we will not attempt a precise definition. A poorly understood problem in Painlevé analysis is determining when equations that pass some Painlevé tests and fail others can admit well-behaved (e.g., meromorphic) solutions containing fewer than the maximal number of integration constants. (This is a natural companion problem to the problem posed by Kruskal and Clarkson [65] of determining when a non-Painlevé equation is integrable.)

Chazy-VII admits two species of simple poles. The first species has residue 1 and resonances 2 and 4. The second species has residue -3 and resonances 4 and 6. If the condition for freedom from logarithms at the last resonance is not imposed, then, with a certain choice of gauge, the full equation takes the form,

$$\begin{aligned} y''' \;=\; & -yy'' - 2(y')^2 + 2y^2y' \\ & + q(x)(3y'' - 4yy' - y^3) + (7q' + q^2)y' \\ & - 2(q' - 2q^2)y^2 + 3(q'' - qq' - q^3 + k_0r^2)y, \end{aligned} \tag{A.41}$$

where $q(x)$ is arbitrary and $r(x) = \exp\{\int q(x)\,dx\}$. Set $k_0 = 0$. Then (A.41) admits the first integral,

$$y'' = \frac{1}{2y}(y')^2 - yy' + \tfrac{1}{2}y^3 + q(x)(y' - 2y^2) + \tfrac{3}{2}(2q' + q^2)y - \frac{72Kr^2}{y}. \tag{A.42}$$

Equation (A.42) has the Painlevé property under circumstances more general than VII provided we allow the integration constant K to be restricted. In fact, (A.42) becomes the full version of Ince-XXVIII if we set

$$K = 1, \qquad q = r'/r, \qquad r = \tfrac{1}{2}(V_2 - V_1), \tag{A.43}$$

where $V_1(x)$ and $V_2(x)$ are any two solutions of the Painlevé-I equation,

$$w'' = 6w^2 + \alpha x + \beta. \tag{A.44}$$

The general solution of (A.42) is now

$$y = \frac{12(w - V_1)(w - V_2)}{2w' - V_1' - V_2' - q(x)(2w - V_1 - V_2)}, \tag{A.45}$$

where $w(x)$ is the general solution of (A.44). This is a globally meromorphic solution of the generalized Chazy-VII equation (A.41) with $k_0 = 0$ and $q(x)$ given by (A.43). It contains two of the three constants of integration.

We remark here that the equation,

$$y''' = yy'' - 2(y')^2 + 6y^2y'. \tag{A.46}$$

has the same basic shape as the reduced equations in Chazy Classes IV–VII but it has the complex resonance numbers $3 \pm i\sqrt{3}$, implying a relatively complicated singularity structure. Nevertheless, it is a fully integrable equation and its solution is given in paper I as an illustration of the method of postmultipliers.

Chazy Class VIII

We distinguish two canonical subcases. If $\alpha \neq 0$, we may set $\beta = 0$ by a translation in x. This gives equation VIII.a, whose solution is

$$y = w(x) - \alpha x, \tag{A.47}$$

where $w(x)$ satisfies the Painlevé-IV equation,

$$w'' = \frac{1}{2w}(w')^2 + \tfrac{3}{2}w^3 - 4\alpha x w^2 + \tfrac{1}{2}(4\alpha^2 x^2 + \gamma)w + \frac{K}{w}, \tag{A.48}$$

K being a constant of integration.

If $\alpha = 0$, we call the equation Chazy-VIII.b. A first integral of VIII.b is the Painlevé-II equation,

$$y'' = 2y^3 + (\beta x + \gamma)y + K. \tag{A.49}$$

The case $\alpha = \beta = 0$ is solvable with elliptic functions.

Chazy Classes IX–XII

These classes are given detailed treatment in Sections 3–6 above. The solution of Chazy-IX in terms of hyperelliptic functions, which is new here, is given by (3.16)–(3.19). A first integral of the second order and third degree is given by (3.6).

The solution of Chazy-X in terms of hyperelliptic functions, which is also new here, is given by (4.21)–(4.36). This solution is remarkably intricate. It was deduced from a study of the new fourth-order Painlevé equation F-VI, given by (1.10). The solution of the reduced equation in terms of two elliptic functions is given by (4.48)–(4.52). A first integral of the second order and fourth degree is given by (4.6)–(4.10). The transformation connecting X.a and X.b is given by (4.19) and (4.20). The solutions of IX and X are globally meromorphic, contradicting a claim in the literature that these two equations are unstable.

Chazy-XI is solved in terms of a Painlevé-I equation and a Riccati or Schrödinger equation according to (5.3)–(5.7). Imposing the Painlevé property strictly on XI led to a number of interesting special cases having meromorphic solutions, including a case where the higher-order transcendent Fif-I appears as a coefficient function.

Chazy-XII is the well-known Chazy natural barrier equation. It is solved by inversion of a hypergeometric function according to (6.8)–(6.10) and is related to the Schwarz triangle function $S(x; \pi/2, \pi/3, \pi/N)$ by (6.24). The cases $N = 2$ and $N = 3$ that take recessive terms can both be reduced to fourth-order linear equations according to (6.60)–(6.67). A proof that XII does not take recessive terms for any $N \geq 4$ (including $N = \infty$, which is Chazy-III) is given in Section 6.

Chazy Class XIII

This is the only Chazy equation having Bureau symbol $P2$. It splits into two canonical subcases XIII.a and XIII.b. In the generic case, the solution of XIII.a can be deduced from VIII.a and I.c and the solution of XIII.b can be deduced from VIII.b and I.d. In the corresponding Painlevé classification problem for partial differential equations of the third order, we find that the KdV equation belongs to Chazy Class XIII, the modified KdV equation belongs to Class VIII, and the potential KdV equation (as well as the soliton equation, $u_{xxt} = -6u_x u_t + \alpha$ [73, 74]) belongs to Class I.

In the generic case $\alpha \neq 0$, XIII.a can be solved in terms of Painlevé-IV functions. Introduce two parameters γ and δ, either one of which may be regarded as a constant of integration, while the other is determined in terms of the parameters α and μ in XIII.a according to

$$\mu \;=\; \alpha\delta + \tfrac{1}{6}\gamma(\gamma - 2\alpha_1), \qquad \alpha_1 \;:=\; \pm\sqrt{-\alpha}\,.$$

Then the solution of XIII.a is

$$y \;=\; \tfrac{1}{2}\alpha_1 w' - \tfrac{1}{2}\alpha(w^2 + xw + \tfrac{1}{6}x^2) - \tfrac{1}{6}(\gamma - \alpha_1), \tag{A.50}$$

where $w(x)$ satisfies the Painlevé-IV equation,

$$w'' \;=\; \frac{1}{2w}\,(w')^2 - \tfrac{3}{2}\alpha w^3 - 2\alpha x w^2 - (\tfrac{1}{2}\alpha x^2 + \gamma)w + \frac{\delta}{w}\,. \tag{A.51}$$

When $\alpha = 0$, XIII.a has the first integral,

$$y'' \;=\; 6y^2 + Kx - \mu, \tag{A.52}$$

which is a Painlevé-I equation.

Chazy-XIII.b is the equation obtained by each of the three group-invariant reductions of the KdV equation. In the generic case $\beta \neq 0$, XIII.b can be solved in terms of Painlevé-II functions. Let δ be a constant of integration and let $\gamma := -6\mu/\beta$. Then the solution of XIII.b is

$$y \;=\; \tfrac{1}{2}(\pm w' + w^2) + \tfrac{1}{12}(\beta x + \gamma), \tag{A.53}$$

where $w(x)$ satisfies the Painlevé-II equation,

$$w'' \;=\; 2w^3 + (\beta x + \gamma)w + \delta. \tag{A.54}$$

When $\beta = 0$, XIII.b has the first integral,

$$y'' \;=\; 6y^2 + \mu x + K, \tag{A.55}$$

which is, once again, the Painlevé-I equation.

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