HIGHER-ORDER PAINLEVÉ EQUATIONS IN THE POLYNOMIAL CLASS I. BUREAU SYMBOL P2

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Abstract. In this paper, we construct all fourth- and fifth-order differential equations in the polynomial class having the Painlevé property and having the Bureau symbol $P2$. The fourth-order equations (including the Bureau barrier equation, $y^{(\mathrm{iv})} = 3yy'' - 4(y')^2$, which fails some Painlevé tests) are six in number and are denoted F-I, ..., F-VI; the fifth-order equations are four in number and are denoted Fif-I, ..., Fif-IV. The 12 remaining equations of the fourth order in the polynomial class (where the Bureau symbol is $P1$) are listed in the Appendix, their proof of uniqueness being postponed to a sequel (paper II). Earlier work on this problem by Bureau, Exton, and Martynov is incomplete, Martynov having found 13 of the 17 distinct reduced equations. Equations F-VI and Fif-IV are new equations defining new higher-order Painlevé transcendents. Other higher-order transcendents appearing here may be obtained by group-invariant reduction of the KdV5, Sawada-Kotera, and Kaup-Kuperschmidt equations, the latter two being related. Four sections are devoted to solutions, first integrals, properties, etc., and a section is devoted to the Bureau barrier equation, whose integrability status is unknown, and a new hierarchy built on this equation. Several of the equations are solved in terms of hyperelliptic functions of genus 2 by means of Jacobi's postmultiplier theory. Except for a classic solution of Drach, we believe that all of these hyperelliptic solutions are new. In an accompanying paper, the hyperelliptic solutions of F-V and F-VI are applied to the unsolved third-order Chazy classes IX and X.

Key words. Painlevé analysis, Painlevé transcendents, ordinary differential equations, hyperelliptic functions, nonlinear equations

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Running Head. Higher-order Painlevé equations I

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1. Introduction and Preliminary

In this paper, we carry out the first stage of the Painlevé classification of the class of differential equations of the fourth and fifth orders in the polynomial class. The subcases treated here have Bureau symbol $P2$ (see [1, 2, 3] for the notation) and may be identified by their reduced equations, which take the form,

$$y^{(\mathrm{iv})} = Ayy'' + B(y')^2 + Cy^4, \tag{1.1}$$

$$y^{(\mathrm{v})} = Ayy''' + By'y'' + Cy^2y'. \tag{1.2}$$

(All variables are complex and the prime denotes d/dx.)

The complete list of fourth-order equations of the type under investigation is found to be:

F-I: $$y^{(\mathrm{iv})} = 12yy'' + 12(y')^2 + (\alpha x+\beta)y' + 2\alpha y - \tfrac{1}{6}(\alpha x+\beta)^2; \tag{1.3}$$

F-II: $$y^{(\mathrm{iv})} = 3yy'' - 4(y')^2; \tag{1.4}$$

F-III: $$y^{(\mathrm{iv})} = 15yy'' + \tfrac{45}{4}(y')^2 - 15y^3 + \alpha y + \beta; \tag{1.5}$$

F-IV: $$y^{(\mathrm{iv})} = 30yy'' - 60y^3 + \alpha y + \beta; \tag{1.6}$$

F-V: $$y^{(\mathrm{iv})} = 20yy'' + 10(y')^2 - 40y^3 + \alpha y + \kappa x + \beta; \tag{1.7}$$

F-VI: $$y^{(\mathrm{iv})} = 18yy'' + 9(y')^2 - 24y^3 + \alpha y^2 + \tfrac{1}{9}\alpha^2 y + \kappa x + \beta. \tag{1.8}$$

The corresponding fifth-order equations Fif-I, ..., Fif-IV are listed in Section 2 below (see equations (2.69)–(2.75)). Except for F-II, we will prove that these are the unique equations, up to gauge transformations, of the fourth and fifth orders in the polynomial class that have the Painlevé property.

Of these, equation F-II, which we call the "Bureau barrier equation," does not have the Painlevé property and may not even be integrable. Nevertheless, it is a very interesting equation from the point of view of Painlevé analysis and has attracted recent attention, and so we have included it on our lists. Section 3 is devoted to a study of this equation and a new hierarchy built upon this equation.

Equation F-VI is a new equation and defines a new fourth-order Painlevé transcendent when $\kappa \neq 0$. It has hyperelliptic asymptotics in sectors of angle $2\pi/7$ and appears to have no elementary particular solutions, as is known to be true for the Painlevé-I transcendent. Equation Fif-IV is a further generalization of F-VI, but has a first integral (of the third degree) and several particular solutions expressible in terms of Painlevé-II functions. The autonomous subcases ($\kappa = \lambda = 0$) are solved in Section 7 in terms of hyperelliptic functions of genus 2.

Equations F-III, F-IV, and F-V and their corresponding fifth-order extensions Fif-I, Fif-II, and Fif-III may be obtained by group-invariant reduction of the Kaup-Kuperschmidt, Sawada-Kotera, and KdV5 equations (see below). The members with nonconstant coefficients define higher-order Painlevé transcendents and the autonomous members (i.e., equations with constant coefficients) are all solvable in terms of hyperelliptic functions. Equations F-IV and Fif-I were first formed by Chazy [4]. Equation F-I can be obtained by nonclassical reduction of the KP and Boussinesq equations and admits the Chazy-XIII equation as a first integral.

A sequel [5] (paper II) will extend the analysis to the case of Bureau symbol $P1$, where the corresponding reduced equations are

$$
\begin{aligned}
y^{(\mathrm{iv})} &= Ayy''' + By'y'' + Cy^2y'' + Dy(y')^2 + Ey^3y' + Fy^5, && (1.9)\\
y^{(\mathrm{v})} &= Ayy^{(\mathrm{iv})} + By'y''' + Cy^2y''' + D(y'')^2 + Eyy'y'' \\
&\qquad + Fy^3y'' + G(y')^3 + Hy^2(y')^2 + Iy^4y' + Jy^6, && (1.10)
\end{aligned}
$$

and will bring the fourth-order problem to completion. We present the full list of fourth-order equations, denoted F-I, F-II, ... F-XVIII, in the Appendix. It is not difficult to show that there are no Painlevé-type equations of the fourth or fifth order in the polynomial class having Bureau symbols $P3$ or higher, except for the linear equations, whose symbol is $P\infty$. (These higher symbols are abundant among rational equations and higher-degree equations, there being examples in [6, 7, 8])

Partial results on the fourth-order polynomial class have been obtained by Chazy [4, pp. 378–385], Bureau [9, 10, pp. 74–116], Exton [11], Martynov [12], and Harada and Oishi [13]. Martynov attempted the complete Painlevé classification, but he missed four equations (F-III, F-VI, F-XV, and F-XVIII) as well as a non-Painlevé equation (A.3) below having integer resonances that his method should have produced. Exton also considered fifth and higher orders, but his work unfortunately contains a persistent error that caused him to reject most equations as unstable. He concluded at the end of his paper that only linearizable equations survive at orders six and above, which obviously contradicts the existence of one-dimensional reductions of numerous soliton equation hierarchies.

All differential equations obtained, except F-II, will be solved in terms of hyperelliptic functions, Painlevé transcendents, or simpler functions, or else evidence will be offered that the equations define higher-order Painlevé transcendents. In the latter case, we give a number of special solutions and transformation properties. We believe that all of the hyperelliptic solutions are new except for F-V with $\kappa = 0$ which appears in a hierarchy of equations found and solved by Drach [14] using his own formulation of differential Galois theory.

One of the most exciting results of the work of the French school on the analytic theory of differential equations at the turn of the century was the discovery of new special functions, namely, the six Painlevé transcendents [15, 16, 17, 18], here denoted P-I, ..., P-VI (or, more fully, Painlevé-I, etc.). They were revealed during the Painlevé classification of the class of differential equations,

$$y'' = F(x, y, y'), \tag{1.11}$$

where F is rational in y and y' (the prime denoting d/dx) and locally analytic in x, that was brought to completion by Gambier [17]. (This work is reviewed by Ince [18], Bureau [1], and the author [19].) At the time, it was not realised that these transcendents would have numerous applications throughout the physical sciences, e.g., fluid dynamics, plasma physics, fibre optics, quantum statistical mechanics, and general relativity, to name just a few.

Most of the transformation properties and elementary solutions of the Painlevé transcendents and their connections to soliton equations and integrable systems were discovered later. Painlevé himself knew the Airy function solution of P-II [20, p. 647] and Gambier knew the parabolic cylinder function solution of P-IV [17, p. 43]. Gambier also knew some of the transformation properties of P-II and P-IV, although he did not apparently attach much importance to them. For example, Gambier's construction [17, 21] of equation Ince-XLII [18, p. 341] involves three distinct P-IV functions that are related to each other by transformations usually attributed to Lukashevich [22]. The well-known scaling reduction of the sine-Gordon equation to a special case of P-III appears as an exercise (with an error) in Forsyth [23, p. 454], the result being attributed to Bianchi and Darboux.

Major assaults on the corresponding third-order problem,

$$y''' = F(x, y, y', y''), \tag{1.12}$$

have been carried out by Chazy [4], Garnier [24], and Bureau [10]. The Painlevé classification of rational third-order equations is a huge task, and only a very small part of the problem has been addressed in the literature. When restricted to the polynomial class, where F is a polynomial in y and its derivatives, fairly complete results have been obtained. Chazy divided the latter problem into 13 classes, here denoted Chazy-I, ..., Chazy-XIII, each having a reduced equation of the form,

$$y''' = Pyy'' + Q(y')^2 + Ry^2y' + Sy^4, \tag{1.13}$$

except for Chazy-XIII, whose reduced equation is

$$y''' = 12yy'. \tag{1.14}$$

Equations (1.13) and (1.14) have, respectively, the Bureau symbols $P1$ and $P2$, and so do the corresponding full equations built upon these reduced equations.

Chazy was able to attach complete recessive terms, or prove that none are allowed, in all classes except III, X, and XI (and it is not clear if he could prove the results he claimed for XII). Recessive terms for Chazy-X are easily determined but appear for the first time in [25]. The first published proof of the absence of recessive terms in III and XII, these being the celebrated "natural barrier equations," except for the parameter values $N = 2$ and $N = 3$, was given by Conte, Fordy, and Pickering [26] (see [25] for an alternative proof). Results for XI, conjectured to be complete, are given in [25].

Chazy was able to solve all of the third-order equations that he obtained, by reduction to classical Painlevé transcendents or simpler functions, except Chazy-IX and X. The latter equations were incorrectly claimed to be unstable (i.e., not have the Painlevé property) by Exton [11, p. 398]. The solutions of these equations have been obtained recently [25] and involve hyperelliptic functions of genus 2. (The solution of Chazy-X is remarkably complicated.)

It is sometimes stated in the literature (and often in conversation) that Chazy was looking for, but failed to find, higher-order transcendents in his study of third-order equations. A conjecture that no new transcendents appear at third order has been attributed to him. While this conjecture may be true if interpreted in a certain way, Chazy did actually find a new transcendent in the coefficient functions of Chazy-XI with parameter $N = 7$, namely, the transcendent defined by equation Fif-I.

In fact, it was the remaining unsolved problems associated with the Chazy polynomial class that led us to the fourth- and fifth-order Painlevé classification problems treated here. We solved Chazy-IX by transforming it a special case of the fourth-order equation F-V, which is one of the better-known equations on our list. Similarly, we solved Chazy-X by transforming it to a special case of our new equation F-VI. The hyperelliptic solution of F-VI was the most difficult to obtain by far, and we solved all of the other hyperelliptic cases during the assault on F-VI and Chazy-X.

Bureau also encountered fourth-order equations having the symbol $P2$ in a major study of second-order second-degree (SD) equations. In [27, pp. 258–260], he obtained, but did not solve, the equations,

$$R^{(\mathrm{iv})} = 8RR'' + 4(R')^2 - \tfrac{32}{5}R^3 + c_1 R + c_2, \tag{1.15}$$

$$R^{(\mathrm{iv})} = \tfrac{13}{4}RR'' + \tfrac{33}{16}(R')^2 - \tfrac{3}{4}R^3 + c_1. \tag{1.16}$$

These equations define coefficient functions $R(x)$ appearing in two SD equations. Equation (1.15) becomes the subcase $\kappa = 0$ of F-V under the scaling $R = (5/2)y$. It is solved in terms of hyperelliptic functions in Section 6 below. Equation (1.16) does not have the Painlevé property, the general solution admitting movable quadratic branch points. Nevertheless, it is integrable and has the elementary solution,

$$x = \int \{P(t)\}^{-1/3}\, dt, \qquad R(x) = -\tfrac{4}{9} P^{-4/3}(6P\ddot{P} - 5\dot{P}^2), \tag{1.17}$$

where $P(t)$ is the sextic polynomial,

$$P(t) = at^6 + bt^5 + ct^4 + dt^3 + et^2 + ft + g, \tag{1.18}$$

the dot denotes d/dt, and $c_1 = -16(120ag - 20bf + 8ce - 3d^2)$. The polynomial class of SD equations will be the subject of a forthcoming paper. Other special classes of SD equations are studied in [6, 7, 8].

The hyperelliptic functions that arise in our work are meromorphic functions of a single complex variable x. (In the full theory, a hyperelliptic function of genus g is a meromorphic function of g complex variables.) They are certain rational or algebraic combinations of two functions $u(x)$ and $v(x)$ that are defined by inversion of hyperelliptic integrals of the form,

$$\int_\infty^{u(x)} \frac{dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{dt}{\sqrt{P(t)}} = c_1\,, \tag{1.19}$$

$$\int_\infty^{u(x)} \frac{t\,dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{t\,dt}{\sqrt{P(t)}} = x + c_2\,, \tag{1.20}$$

where $P(t)$ is a polynomial of the fifth or sixth degree. The square root of $P(t)$ is single-valued on a compact Riemann surface of genus 2.

Although $u(x)$ and $v(x)$ are not meromorphic separately, each having movable quadratic branch points, the combinations that satisfy the aforementioned differential equations are all globally meromorphic. All symmetric rational functions of u and v are meromorphic. So also are rational expressions involving the derivatives that obey the symmetry, $R(v, u, v', u') = R(u, v, u', v')$. If λ is a root of $P(t)$, then the combination $(u - \lambda)(v - \lambda)$ is actually the square of a meromorphic function. The above construction gives a hyperelliptic function of two complex variables if we rename $z_1 := c_1$ and $z_2 := x + c_2$ and use these independent variables as arguments for u and v.

For F-VI, whose solution is quite complicated in these variables, we give an alternative presentation where the polynomial under the square root in the hyperelliptic integrals is of degree 8. (The genus remains 2, of course.)

All of the equations in this paper having hyperelliptic solutions also admit several cases of degeneracy to elliptic or simpler functions. Some cases can be recognized immediately. If $P(t)$ has a double root, then $u(x)$ and $v(x)$ can be expressed in terms of an elliptic function and a quadrature. When $P(t)$ has two double roots, the hyperelliptic functions are expressible in terms of trigonometric or exponential functions. When $P(t)$ is an even sextic polynomial, the hyperelliptic functions again degenerate to elliptic functions. A more general case occurs when $P(t)$ is transformable to an even sextic by a Möbius transformation in t. Then, the hyperelliptic functions are expressible, in general, in terms of two elliptic functions with different periods. A quintic $P(t)$ is in this category if, after applying a suitable linear transformation, $P(t)$ is divisible by t and $P(t)/t$ is a palindromic quartic. There are also a number of doubly periodic solutions where the degeneracy is not obvious by inspection of the roots of $P(t)$, one such example being given in Section 6. Solutions involving elliptic functions are often not doubly periodic because of additional quadratures or because two elliptic functions with different periods are superposed.

The hyperelliptic solutions in this paper were obtained by a nontrivial application of Jacobi's postmultiplier theory (see [28]), or else by techniques suggested by the knowledge of a postmultiplier. Postmultipliers also played an important role in Kowalevski's work [28, 29, 30] on the motion of a rigid body about a fixed point. In fact, the autonomous subcase of F-V is only a short step away from equations appearing in Kowalevski's work.

Because postmultipliers are not yet a widely used tool in Painlevé analysis, we conclude this section with a brief discussion of this powerful theory. Given an autonomous system of first-order differential equations,

$$y_i' \;=\; f_i(y_1, y_2, \ldots, y_n), \qquad i = 1, 2, \ldots, n, \tag{1.21}$$

a postmultiplier for this system is any nonzero function $I(y_1, y_2, \ldots, y_n)$ that satisfies the partial differential equation,

$$\sum_{j=1}^{n} \frac{\partial}{\partial y_j}\Big(f_j I\Big) \;=\; 0. \tag{1.22}$$

In this paper, we will often speak of a postmultiplier for a single autonomous differential equation of order n, say,

$$y^{(n)} \;=\; F(y, y', y'', \ldots, y^{(n-1)}). \tag{1.23}$$

By that we mean the postmultiplier for the following default equivalent system,

$$y_i' \;=\; y_{i+1}, \qquad i = 1, 2, \ldots, n-1, \qquad y_n' \;=\; F(y_1, y_2, \ldots, y_n), \tag{1.24}$$

where $y_1 = y$ and $y_i = d^{i-1}y/dx^{i-1}$ for $i > 1$. A nonautonomous equation of order n can always be rewritten in a standard way as an autonomous first-order system of order $n+1$.

If the system of equations (1.24) has $n-2$ known first integrals (also autonomous) and any known postmultiplier, it can be integrated by quadratures. (Also, if two distinct postmultipliers are known, their ratio is a constant of integration.) Consider, first, the effect of a locally invertible transformation,

$$\bar{y}_i \;=\; \bar{y}_i(y_1, y_2, \ldots, y_n), \tag{1.25}$$

yielding a transformed system of differential equations,

$$\bar{y}_i' \;=\; \bar{f}_i(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n). \tag{1.26}$$

The postmultiplier for the system (1.26) is given by the simple formula,

$$\bar{I}(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \;=\; \frac{I(y_1, y_2, \ldots, y_n)}{\det|\partial\bar{y}_i/\partial y_j|}, \tag{1.27}$$

or any constant multiple thereof. Now if (1.21) has $n-2$ known integrals, we may choose new variables $\bar{y}_3$, $\bar{y}_4$, $\ldots$, $\bar{y}_n$ to coincide with the corresponding integration constants. In that case, $\bar{f}_3 = \bar{f}_4 = \ldots = 0$ and the partial differential equation for the postmultiplier $\bar{I}$ has just two terms and is easily seen to be the necessary and sufficient condition for the left-hand side of

$$\bar{I}\{\bar{f}_2 d\bar{y}_1 - \bar{f}_1 d\bar{y}_2\} \;=\; 0, \tag{1.28}$$

to be the exact differential of a locally analytic function. Hence (1.28) is integrable by quadratures and can be used to express $\bar{y}_2$ in terms of $\bar{y}_1$ and constants of integration. When that is done, the equation $\bar{y}_1' = \bar{y}_2$ is obviously integrable by quadratures and the system (1.21) is now completely integrated.

In practice the method is difficult to apply because of the need to carry out functional inversions explicitly. This usually involves solving polynomial equations of high degree if the variables are not chosen carefully. To achieve a neat hyperelliptic presentation as in equations (1.19) and (1.20) above, many changes of variable are often needed, the aim being to try to reduce the functional inversions to a nested sequence of quadratic equations.

Now it so happens that the autonomous members of F-I–F-VI each have the unit postmultiplier $I(y, y', y'') = 1$. Any autonomous equation in which the penultimate derivative is absent admits the unit postmultiplier. (A unit postmultiplier may be called "trivial" but it has exactly the same power as any nonconstant postmultiplier.) Given that we have found two autonomous first integrals for each of F-III–F-VI (see Sections 4–7 below), we know at once that F-III–F-VI are integrable by quadratures. But the first integrals, while not being especially complicated, certainly do not reveal any obvious clues how to present the solutions in terms of standard hyperelliptic functions, except in the case of F-V. While it is true that some of the equations in this paper could be solved by experimenting with various hyperelliptic ansätzen with undetermined coefficients, the postmultiplier theory was crucial to the successful attack on F-VI and hence Chazy-X. In fact, Chazy-IX and Chazy-X also have a unit postmultiplier and an autonomous first integral (new in [25]), but in these cases the first integrals are formidable. The direct reduction to standard hyperelliptic form would be exceedingly difficult without detouring via the associated fourth-order equations F-V and F-VI.

In Section 4 below, we present the details of the postmultiplier method for the solution of the fourth-order equation F-III. This is the most instructive example. Mastery of the method for F-III is essential before looking at F-VI. Details of the method for F-VI will not be shown because the essential principles are drowned in a sea of algebra. (We used the computer algebra package REDUCE to solve F-VI and Chazy-X.)

By way of illustration, we apply the postmultiplier method to the integrable non-Painlevé equation,

$$y''' \;=\; yy'' - 2(y')^2 + 6y^2y'. \tag{1.29}$$

This equation has the same basic shape as the reduced equations in Chazy Classes IV–VII but it has the complex resonance numbers $3 \pm i\sqrt{3}$ and hence a relatively complicated singularity structure.

It is easily verified that (1.29) admits the postmultiplier,

$$I(y, y', y'') \;=\; (y'' + 2yy')^{1/3}, \tag{1.30}$$

and the first integral,

$$(y'' - yy' - y^3)^2 + (2y' + y^2)(y' - y^2)^2 \;=\; K, \tag{1.31}$$

where K is a constant of integration. Introduce the auxiliary variables,

$$w := y' + y^2, \qquad p := \frac{(w')^{2/3}}{2^{1/3}w}\,, \qquad q := \frac{yw'}{w^2}\,, \tag{1.32}$$

and rewrite (1.29) as a first-order system for the variables p, q, and K. Obviously $K' = 0$. A short calculation shows that

$$p' \;=\; \frac{2^{1/3}K^{1/6}(q - p^3)}{(p^6 - 3p^3q + p^3 + q^3)^{1/6}}\,, \tag{1.33}$$

$$q' \;=\; \frac{2^{1/3}K^{1/6}(p^3 - 2p^3q + q^2)}{p(p^6 - 3p^3q + p^3 + q^3)^{1/6}}\,. \tag{1.34}$$

With the aid of (1.27), a postmultiplier for the latter system is found to be

$$\bar{I}(p,q,K) \;=\; p(p^6-3p^3q+p^3+q^3)^{-5/6}, \tag{1.35}$$

whence we deduce the exact differential,

$$\frac{p^3-2p^3q+q^2}{p^6-3p^3q+p^3+q^3}\,dp+\frac{p(p^3-q)}{p^6-3p^3q+p^3+q^3}\,dq \;=\; 0. \tag{1.36}$$

Integration yields

$$\begin{aligned} 2\log(p^2+p+q)-\log(p^4-p^3-p^2q+p^2-pq+q^2)& \\ -\,2\sqrt{3}\,\tan^{-1}\left\{\frac{\sqrt{3}(p-q)}{2p^2-p-q}\right\} \;&=\; K_2\,, \end{aligned} \tag{1.37}$$

where K_2 is a second constant of integration. Equation (1.37) can be locally inverted to express q as a function of p, after which (1.33) is integrable by quadratures. Hence the functions $p(x)$ and $q(x)$, depending on three integration constants, are now known. In terms of these functions, the solution of the original equation (1.29) is given by

$$y \;=\; \frac{2^{-2/3}K^{1/6}q}{p(p^6-3p^3q+p^3+q^3)^{1/6}}\,. \tag{1.38}$$

While this example is not especially difficult, it illustrates the main ideas, and we think that the reader may find equation (1.29) somewhat less easy to solve without any knowledge of postmultiplier theory. The least straightforward step is the choice of auxiliary variables. In more complicated problems, this is usually done in several stages by studying the available integrals carefully.

2. Derivation of the fourth- and fifth-order equations

In this section, we carry out the Painlevé classification of all fourth-order differential equations of the form,

$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & A(x)yy''+B(x)(y')^2+C(x)y^3+D(x)y'''+E(x)yy' \\ & +F(x)y''+G(x)y^2+H(x)y'+I(x)y+J(x), \end{aligned} \tag{2.1}$$

and all fifth-order equations of the form,

$$\begin{aligned} y^{(\mathrm{v})} \;=\; & A(x)yy'''+B(x)y'y''+C(x)y^2y'+D(x)y^{(\mathrm{iv})} \\ & +E(x)yy''+F(x)(y')^2+G(x)y^3+H(x)y'''+I(x)yy' \\ & +J(x)y''+L(x)y^2+M(x)y'+N(x)y+P(x). \end{aligned} \tag{2.2}$$

The analysis of the open problems regarding Chazy Classes IX, X, and XI in [25] led us to particular Painlevé-type equations in these classes.

When $A(x)$, $B(x)$, and $C(x)$ are not all identically zero, equations (2.1) and (2.2) are the most general fourth- and fifth-order equations in the polynomial class having the Bureau symbol $P2$. The α-test,

$$x = x_0 + \alpha X, \qquad y = \alpha^{-2} Y, \qquad \alpha \to 0, \tag{2.3}$$

yields the corresponding $P2$-type reduced equations,

$$Y^{(\mathrm{iv})} = AYY'' + B(Y')^2 + CY^3, \tag{2.4}$$

$$Y^{(\mathrm{v})} = AYY''' + BY'Y'' + CY^2Y', \tag{2.5}$$

where A, B, and C are constants, not all zero. Because of scaling freedom, only the ratios B/A and C/A^2 are essential parameters (when $A \neq 0$).

For the sake of completeness, we consider here the case $A(x) = B(x) = C(x) = 0$, where the index is higher than 2. The conclusion will be that the only Painlevé-type equations in the respective classes are the linear differential equations of fourth and fifth order, the Bureau symbol being $P\infty$ in these cases. For finite $k > 2$, the possible Pk-type reduced equations are as follows:

$$P3: \qquad Y^{(\mathrm{iv})} = AYY', \tag{2.6}$$

$$P4: \qquad Y^{(\mathrm{iv})} = AY^2, \tag{2.7}$$

$$P\tfrac{5}{2}: \qquad Y^{(\mathrm{v})} = AY^3, \tag{2.8}$$

$$P3: \qquad Y^{(\mathrm{v})} = AYY'' + B(Y')^2, \tag{2.9}$$

$$P4: \qquad Y^{(\mathrm{v})} = AYY', \tag{2.10}$$

$$P5: \qquad Y^{(\mathrm{v})} = AY^2. \tag{2.11}$$

Except for (2.9), these can all be settled by inspection or by a short calculation to reveal noninteger resonances. (A fractional Bureau symbol as in (2.8) only needs to be taken seriously in higher-degree problems, where singular integrals are possible [8].) With somewhat greater difficulty, it can also be proved that there are no sixth-order equations in the polynomial class having any of the Bureau symbols $P3$–$P6$. (We remark that Bureau symbols different from $P1$, $P2$, and $P\infty$ are quite common in rational equations and polynomial-class equations of higher degree, a simple example with symbol $P3$ being equation SD-V in [6].)

In the case of (2.9), the cases $4A + 3B \neq 0$ and $4A + 3B = 0$ need to treated separately, A and B being not both zero. After scaling, the latter case becomes the special case, $m = 5$, $n = 1$, of a class of equations mentioned briefly in [25, (2.3)]. There, we showed that these equations admit two-parameter solutions, necessarily embedded in the general solution, that involve movable logarithmic branch points. Another way to investigate equations of this type is shown in [31, 32]. When $4A + 3B \neq 0$, resonance analysis is applicable. (See Subsection 2.2 for an example where resonance analysis is carried out in greater detail.) A short calculation shows that the resonance numbers n_1, n_2, n_3, and n_4 (not including -1) are the roots of the quartic equation,

$$(n-3)(n-12)(n^2-11n+70) + \frac{840A}{4A+3B}\, n \;=\; 0. \tag{2.12}$$

A necessary condition for (2.9) to have the Painlevé property is that all the resonance numbers be integers. (There are additional rules that we do not need here.) The product of the roots is 2520. Let n_1 be the root with least absolute value. Then $|n_1| \leq 6$ because the product $7(-7)8(-8)$ exceeds 2520. Exclude the values $n_1 = 0, -1$. This leaves 11 cases to consider. Each choice of n_1 determines the coefficient $840A/(4A{+}3B)$, after which the quartic polynomial can be divided by $n - n_1$ to yield a cubic polynomial with integer coefficients. In every case, the cubic equation has at most one integer root and so there are no equations of the form (2.9) having all integer resonance numbers. (Calculations of this sort are very well suited to a hand-held programmable calculator.)

The more difficult case of Bureau symbol $P1$ is discussed in paper II. The results of that calculation for the fourth-order problem are included in the Appendix below. The remainder of this section will be concerned with the case of Bureau symbol $P2$.

2.1. The reduced equations

Consider, first, the fourth-order case, where the appropriate $P2$-type reduced equation is (2.4).

The cases $C \neq 0$ and $C = 0$ need to be analysed separately. We begin with the latter case. The special case where $3A + 2B = 0$ is distinguished. After a scaling, we get the equation,

$$Y^{(\mathrm{iv})} \;=\; 2YY'' - 3(Y')^2. \tag{2.13}$$

This is another member of the aforementioned class of equations [25, (2.3)] that we know does not have the Painlevé property. When $3A + 2B \neq 0$, including the cases where one of A or B is zero, we apply resonance analysis. A short calculation shows that the resonance numbers are the roots of the cubic equation,

$$(n-2)(n^2-13n+60) - \frac{60A}{3A+2B}\, n \;=\; 0. \tag{2.14}$$

We want conditions on A and B that ensure that all of the roots are integers. The product of the roots is 120. Let n_1 be the root with least absolute value. Then $|n_1| \leq 4$. This leaves 7 candidates. In each case, the coefficient $60A/(3A+2B)$ is determined and the cubic can be divided by $n - n_1$ to yield a quadratic equation for the remaining roots. Three values of n_1 in this range yield integer values for the remaining roots, of which one is redundant. The values of n_1 giving distinct equations are found to be $n_1 = 4$, for which $A = B$, and $n_1 = -2$, for which $4A + 3B = 0$.

Hence we have found two distinct reduced equations of the form (2.4) when $C = 0$ having admissible integer resonance numbers. After a suitable scaling of the coefficient A, we arrive at the reduced versions of equations F-I and F-II. These are

F-I reduced: $$Y^{(\mathrm{iv})} = 12YY'' + 12(Y')^2; \tag{2.15}$$

F-II reduced: $$Y^{(\mathrm{iv})} = 3YY'' - 4(Y')^2. \tag{2.16}$$

Let the resonance numbers be assembled into an unordered triplet $\{n_1, n_2, n_3\}$. Then the resonance numbers for F-I are $\{4, 5, 6\}$ and the resonance numbers for F-II are $\{-2, -3, 20\}$.

Equation (2.15) is easily integrated to give the Painlevé-I equation,

$$Y'' = 6Y^2 + K_1 X + K_2\,, \tag{2.17}$$

and so (2.15) certainly has the Painlevé property.

Equation (2.16) is the Bureau barrier equation [10, p. 79, (2.9)]. Although it passes what may be called the traditional necessary conditions for the Painlevé property, it fails a more subtle test. The first proof that (2.16) admits solutions with movable branch points (or branched singular sets of undetermined character) was provided by Conte, Fordy, and Pickering [26]. An alternative proof based on a classical Painlevé α-expansion is given in the next section. There we also consider the problem of attaching recessive terms to this reduced equation. The actual nature of the branching has been determined by Littlewood [33] in a numerical study. Littlewood observed that the generic solution of (2.16) appears to have a natural barrier of fractal character and that the domain of analyticity inside the barrier overlaps itself for a limited range of initial conditions. It should be emphasized that the Runge-Kutta method for solving differential equations numerically can be an important diagnostic tool in Painlevé analysis.

Next we turn to the case $C \neq 0$ of equation (2.4). (Either or both of A and B may possibly be zero.) Here we show the resonance analysis in greater detail. The method is based on analysis of formal Laurent series or generalizations thereof, as practiced by Chazy [4] and popularized in recent times by Ablowitz et al. [34]. It is the most widely used method in Painlevé analysis. Except for the Bureau barrier equation, where more powerful methods must be invoked, the Laurent series methods are the most efficient for our present purposes.

In the first instance, useful information about most equations having Bureau symbol $P2$ can be obtained by examining a formal Laurent series solution with a double pole in the leading term. For equation (2.4) we try the following expansion,

$$Y \;=\; a_0(X - X_0)^{-2} + \ldots + a_n(X - X_0)^{n-2} + \ldots\,, \tag{2.18}$$

$a_0 \neq 0$, where X_0 is a constant of integration and the coefficients are to be determined.

We want to know if the remaining three constants of integration can be accommodated in this expansion. It is well known that one may have to augment the series with noninteger powers and/or logarithms (where resonance numbers are positive or have positive real part) or that the series may only be relevant to a class of solutions with fewer than four integration constants (because of negative resonances, for example). Either way, a necessary condition for the Painlevé property is that all resonance numbers be integers, positive or negative, and that no resonance numbers have multiplicity greater than 1. We always ignore a trivial resonance having the value -1, unless we are running a Conte-Fordy-Pickering (CFP) test [26]. A positive resonance having multiplicity greater than 1 implies movable logarithmic branch points. Similarly, a repeated negative resonance implies movable branch points that could be either algebraic or logarithmic. The latter includes the case of a resonance number -1 coinciding with the trivial resonance. Also a zero resonance number indicates movable logarithmic branch points except when the leading coefficient a_0 is an arbitrary constant of integration (which does not happen here). The rules we have just outlined apply to scalar ordinary or partial differential equations of the first degree. Modifications of these rules for systems of equations are discussed in [26] and for higher-degree equations where singular integrals may admit movable branch points in [8].

After substituting (2.18) into (2.4), we get a quadratic equation for the leading coefficient a_0, namely,

$$C(a_0)^2 + (6A + 4B)a_0 - 120 \;=\; 0. \tag{2.19}$$

Hence, in general, there will be two distinct values for a_0, say, a_{01} and a_{02}, and we must examine two separate Laurent series for integer resonances.

Next, for each choice of a_0, we determine three resonance numbers as follows. Substitution of the formal series (2.18) into the differential equation (2.4) reveals that the coefficients a_n satisfy a recurrence relation of the form,

$$(n + 1)(n - n_1)(n - n_2)(n - n_3)a_n \;=\; \phi_n(a_0, a_1, \ldots, a_{n-1}), \tag{2.20}$$

where the precise form of the right-hand side is not needed just yet. The numbers n_1, n_2, and n_3 are the resonance numbers associated with the leading coefficient a_0. We require them to be integers distinct from each other and from 0 and -1. There are two such sets of resonance numbers, one for each choice of a_0. Let us use the notation $\{n_1, n_2, n_3\}$ for the unordered triplet of resonance numbers associated with a_{01} and $\{m_1, m_2, m_3\}$ for the unordered triplet of resonance numbers associated with a_{02}. A short calculation shows that these triplets are the roots of the respective cubic equations,

$$n^3 - 15n^2 + (86 - Aa_{01})n - 240 + (6A + 4B)a_{01} = 0, \tag{2.21}$$

$$m^3 - 15m^2 + (86 - Aa_{02})m - 240 + (6A + 4B)a_{02} = 0. \tag{2.22}$$

We shall determine all cases where the six resonance numbers are nonrepeated integers distinct from 0 and -1. (If we broadened the search to allow repeated resonances and -1 resonances, we would not get any additional reduced equations. On the other hand, there is a one-parameter family of equations in the class considered having zero resonances, which we presume are nonintegrable.)

Let N_1 and N_2 denote the products of the roots of equations (2.21) and (2.22), respectively. Then N_1 and N_2 are necessarily nonzero integers. With the aid of (2.19), their values are seen to be

$$N_1 = 240 - (6A + 4B)a_{01} = 120 + C(a_{01})^2, \tag{2.23}$$

$$N_2 = 240 - (6A + 4B)a_{02} = 120 + C(a_{02})^2. \tag{2.24}$$

Since

$$a_{01}a_{02} = -120C^{-1}, \tag{2.25}$$

we immediately get the Diophantine equation,

$$(N_1 - 120)(N_2 - 120) = 14400, \tag{2.26}$$

which can be rearranged into the form,

$$\frac{1}{N_1} + \frac{1}{N_2} = \frac{1}{120}. \tag{2.27}$$

This result was obtained, but not pursued further, by Bureau [10, p. 80, (2.15)]. It was also obtained by Exton [11, p. 394] who found three solutions but rejected them all because of an error in his method of Painlevé testing. Martynov [12] found only two solutions. Harada and Oishi [13] restricted attention to positive-integer resonances and consequently they were able to solve the problem much more quickly by using the fact that the sum of the roots of equation (2.21) is 15 (but, of course, they missed the Bureau barrier equation.) As already mentioned, elementary number-theoretic calculations of this sort are very well suited to a hand-held pocket calculator with a little programming capability. (Such calculators became efficient and affordable in the early 1980's.) For the present problem, we will demonstrate a shortcut that makes it reasonably easy to carry out all the necessary calculations with pen and paper.

We remark that equation (2.26) has exactly 63 unordered pairs $\{N_1, N_2\}$ of nonzero solutions. These can be obtained easily by letting $N_1 - 120$ run through all the divisors of 14400 between -120 (exclusive) and $+120$ (inclusive). The values of N_1 in the stated range are

20, 24, 30, 40, 45, 48, 56, 60, 70, 72, 75, 80, 84,

88, 90, 95, 96, 100, 102, 104, 105, 108, 110, 111, 112,

114–119 inclusive, 121–126 inclusive, 128, 129, 130, 132,

135, 136, 138, 140, 144, 145, 150, 152, 156, 160, 165,

168, 170, 180, 184, 192, 195, 200, 210, 216, 220, 240.

Then either $-14280 \leq N_2 \leq -24$ or $240 \leq N_2 \leq 14520$. However, we can proceed without knowing these values of N_1. We just need to know that $0 < N_1 \leq 240$.

So far, we have established necessary conditions for the products $N_1 = n_1 n_2 n_3$ and $N_2 = m_1 m_2 m_3$ to be nonzero integers. We now want to force the resonance numbers to be integers separately. Write equations (2.21) and (2.22) as

$$n^3 - 15n^2 + (86 - Aa_{01})n - N_1 \;=\; 0, \tag{2.28}$$

$$m^3 - 15m^2 + (86 - Aa_{02})m - N_2 \;=\; 0. \tag{2.29}$$

We see at once that Aa_{01} and Aa_{02} must be integers. Arrange the roots of (2.28) in order of absolute value, i.e., $|n_1| \leq |n_2| \leq |n_3|$, with $n_1 \neq n_2$ and $n_2 \neq n_3$. Because $0 < n_1 n_2 n_3 \leq 240$ and $6(-6)(-7) > 240$, we get immediately that

$$-5 \leq n_1 \leq 5. \tag{2.30}$$

Equation (2.28) also implies $n_1 + n_2 + n_3 = 15$, which we use to eliminate n_3. Effective bounds can now be placed on n_2.

If $n_1 > 0$, the above inequalities can be rearranged to give

$$n_1 < n_2 < \tfrac{1}{2}(15 - n_1). \tag{2.31}$$

This gives 12 candidates for $\{n_1, n_2, n_3\}$, namely,

$$\begin{array}{cccc} \{1,2,12\}, & \{1,3,11\}, & \{1,4,10\}, & \{1,5,9\}, \\ \{1,6,8\}, & \{2,3,10\}, & \{2,4,9\}, & \{2,5,8\}, \\ \{2,6,7\}, & \{3,4,8\}, & \{3,5,7\}, & \{4,5,6\}. \end{array} \qquad (2.32)$$

Of these, we can discard $\{1,3,11\}$ because $n_1 n_2 n_3 - 120$ is not a divisor of 14400 and $\{4,5,6\}$ because it forces $C = 0$, that case yielding the reduced version of equation F-I. If $n_1 \leq -2$, we find that there are only three candidates, namely,

$$(-2,-3,20), \qquad (-2,-4,21), \qquad (-2,-5,22). \qquad (2.33)$$

Again, the first can be discarded because it forces $C = 0$ and yields the reduced version of equation F-II. Hence 12 candidates remain to be examined.

We get additional constraints when we force the roots $\{m_1, m_2, m_3\}$ of (2.22) (or, equivalently, (2.29)) to be integers. By a scaling of the y variable, we normalize $a_{01} = 1$. Then, from (2.21) and (2.23)–(2.28), we read off

$$\begin{aligned} A &= 86 - (n_1 n_2 + n_1 n_3 + n_2 n_3), \\ 6A + 4B &= 240 - n_1 n_2 n_3, \\ C &= n_1 n_2 n_3 - 120, \\ a_{02} &= -120 C^{-1}. \end{aligned}$$

The coefficients of the cubic equation (2.22) are now known in terms of $\{n_1, n_2, n_3\}$, and its roots can therefore be examined in each of the 12 surviving cases so far. We find that (2.22) has integer roots in just four cases. For these cases, the triplet $\{n_1, n_2, n_3\}$ takes the values (in no particular order),

$$\{3,5,7\}, \qquad \{2,3,10\}, \qquad \{2,5,8\}, \qquad \{3,4,8\}. \qquad (2.34)$$

The corresponding pairs (N_1, N_2) satisfying the Diophantine equation (2.27) are

$$(105,-840), \qquad (60,-120), \qquad (80,-240), \qquad (96,-480). \qquad (2.35)$$

The corresponding reduced differential equations of the form (2.4) do indeed have the Painlevé property. They are the reduced versions of equations F-III, F-IV, F-V, and F-VI. The canonical forms of these four reduced equations, together with their resonance data, are as follows:

$$
\begin{aligned}
&\text{F-III reduced:} && Y^{(\mathrm{iv})} \;=\; 15YY'' + \tfrac{45}{4}(Y')^2 - 15Y^3; && (2.36)\\
&& a_{01} = 1, \quad & \{n_1, n_2, n_3\} \;=\; \{3, 5, 7\};\\
&& a_{02} = 8, \quad & \{m_1, m_2, m_3\} \;=\; \{-7, 10, 12\};\\
&\text{F-IV reduced:} && Y^{(\mathrm{iv})} \;=\; 30YY'' - 60Y^3; && (2.37)\\
&& a_{01} = 1, \quad & \{n_1, n_2, n_3\} \;=\; \{2, 3, 10\};\\
&& a_{02} = 2, \quad & \{m_1, m_2, m_3\} \;=\; \{-2, 5, 12\};\\
&\text{F-V reduced:} && Y^{(\mathrm{iv})} \;=\; 20YY'' + 10(Y')^2 - 40Y^3; && (2.38)\\
&& a_{01} = 1, \quad & \{n_1, n_2, n_3\} \;=\; \{2, 5, 8\};\\
&& a_{02} = 3, \quad & \{m_1, m_2, m_3\} \;=\; \{-3, 8, 10\};\\
&\text{F-VI reduced:} && Y^{(\mathrm{iv})} \;=\; 18YY'' + 9(Y')^2 - 24Y^3; && (2.39)\\
&& a_{01} = 1, \quad & \{n_1, n_2, n_3\} \;=\; \{3, 4, 8\};\\
&& a_{02} = 5, \quad & \{m_1, m_2, m_3\} \;=\; \{-5, 8, 12\}.
\end{aligned}
$$

These equations are solved in Sections 4–7, where they are treated as special cases of the corresponding full equations. Equations (2.36), (2.37), and (2.38) are solved in terms of hyperelliptic functions of genus 2. Equation (2.39) is solved in terms of two elliptic functions with different periods. The reduced equations F-I, F-II, F-IV, and F-V appear on Martynov's list [12] as equations (XXI), (XXII), (XIX), and (XX), respectively. The reduced equations F-III and F-VI do not appear on Martynov's list. The reduced version of F-VI (or, more precisely, its derivative given by (2.44) below) is mentioned briefly by Harada and Oishi [13] as an exceptional case that does not arise from a KdV5-like partial differential equation. We continue discussion of the history of the other equations after the full versions are presented below.

Almost no additional work is needed to classify the fifth-order reduced equations of the form (2.5). This is because (2.5) has a first integral,

$$Y^{(\mathrm{iv})} \;=\; AYY'' + \tfrac{1}{2}(B-A)(Y')^2 + \tfrac{1}{3}CY^3 + K, \tag{2.40}$$

which is of the form (2.4), which we have just classified, with an additive constant on the right-hand side. The fifth-order equations have the same resonance numbers as the corresponding fourth-order equations, except that the former have an additional resonance number $n_4 = 6$ (in all six cases) and $m_4 = 6$ (in the last four cases). Because

F-I has resonance numbers $\{4,5,6\}$, its derivative has resonance numbers $\{4,5,6,6\}$. The repeated positive resonance implies the existence of movable logarithmic branch points. So there is no fifth-order Painlevé equation corresponding to F-I. In fact, the reduced F-I with an additive constant can be integrated three times to give $Y'' = 6Y^2 + S(X)$, where $S(X)$ is a quadratic polynomial, which is well known to not have the Painlevé property. In the next section, we present an argument that equation F-II is also not able to support an additive constant, and so there is again no corresponding fifth-order equation.

The remaining four fifth-order reduced equations have the Painlevé property and can be solved in terms of hyperelliptic functions. These equations are

Fif-I reduced: $$Y^{(\mathrm{v})} = 15YY''' + \tfrac{75}{2}Y'Y'' - 45Y^2Y'; \tag{2.41}$$

Fif-II reduced: $$Y^{(\mathrm{v})} = 30YY''' + 30Y'Y'' - 180Y^2Y'; \tag{2.42}$$

Fif-III reduced: $$Y^{(\mathrm{v})} = 20YY''' + 40Y'Y'' - 120Y^2Y'; \tag{2.43}$$

Fif-IV reduced: $$Y^{(\mathrm{v})} = 18YY''' + 36Y'Y'' - 72Y^2Y'. \tag{2.44}$$

If we differentiate the reduced equations one more time, we lose the derivatives of Fif-I and Fif-II. The former has resonance numbers $\{3,5,6,7,7\}$ in which one is repeated, and the latter has resonance numbers $\{2,3,6,7,10\}$ in which the condition for the absence of logarithms at the last resonance is not satisfied. Thus the derivatives of Fif-I and Fif-II do not have the Painlevé property, and we do not know whether or not they are integrable. On the other hand, the derivatives of the reduced Fif-III and Fif-IV equations do have the Painlevé property. However, the corresponding full equations are just derivatives of the full versions of Fif-III and Fif-IV, and so the sixth-order versions do not increase the generality.

At the sixth and seventh orders, there are additional $P2$-type reduced equations having integer resonances that are not just derivatives of fifth-order equations. One example of each order can be obtained by group-invariant reduction of the KdV7 equation. The corresponding full equations define new transcendents, and the autonomous sixth-order subcase appears in the Drach hierarchy [14] and can be solved in terms of hyperelliptic functions of genus 3. Two more examples of each order can be obtained by group-invariant reduction of the next members of the Kaup-Kuperschmidt and Sawada-Kotera hierarchies. There are also two sixth-order equations, (3.20) and (3.21) below, of similar character to the Bureau barrier equation, the latter being the next member of a hierarchy of such equations.

The condition for nonrepeated integer resonances distinct from 0 and -1 has led us uniquely to six fourth-order reduced equations and four fifth-order reduced equations in the polynomial class having Bureau symbol $P2$. These are the equations (2.15), (2.16), (2.36)–(2.39), and (2.41)–(2.44). Of these, all have the Painlevé property except the Bureau barrier equation F-II, which we leave on the list because it is very interesting from the point of view of Painlevé analysis. All solutions (except F-II) are given below in Sections 4–7.

2.2. Corresponding full equations

In this subsection, we complete the determination of the remaining coefficients in the full equations (2.1) and (2.2). Since, under the α-test (2.3), the constants A, B, and C in the reduced equations are equal, respectively, to the functions $A(x)$, $B(x)$, and $C(x)$ in the full equations evaluated at $x = x_0$ and x_0 is arbitrary, then in the full equations the ratios $B(x)/A(x)$ and $C(x)/A^2(x)$ are constants (or zero) and take the same values as in the reduced equations. Then a suitable gauge change, $y = \lambda(x)\bar{y}$, can be employed to make $A(x)$, $B(x)$, and $C(x)$ exactly equal to the corresponding constant values in the canonical forms of the reduced equations.

The techniques for attaching recessive terms are well established in the cases where there are sufficiently many positive-integer resonances. So only F-II gives trouble, and we postpone discussion of this equation till the next section. In the remaining cases, the only difficulties that could arise are tedious calculations caused by high resonance numbers or poor choice of gauge, but often there are shortcuts available. We show details of the calculation only for F-III, since the other cases are similar. Readers who are familiar with this technique can skip forward to the lists beginning at equation (2.63).

We begin with the fourth-order equations, which now take the form,

$$\begin{aligned} y^{(\mathrm{iv})} \; &= \; Ayy'' + B(y')^2 + Cy^3 + D(x)y''' + E(x)yy' \\ &\quad + F(x)y'' + G(x)y^2 + H(x)y' + I(x)y + J(x), \end{aligned} \tag{2.45}$$

where the ordered triplets $\{A, B, C\}$ take the values $\{12, 12, 0\}$, $\{3, -4, 0\}$, $\{15, 45/4, -15\}$, $\{30, 0, -60\}$, $\{20, 10, -40\}$, and $\{18, 9, -24\}$ for F-I–F-VI, respectively.

The first step is to use gauge freedom to simplify the problem. The form of the equation (2.45) is preserved under changes of variables of the form,

$$y \; = \; \lambda(x)\bar{y} + \mu(x), \qquad \bar{x} \; = \; \phi(x), \qquad \lambda(x) \; = \; \{\phi'(x)\}^2. \tag{2.46}$$

The coefficients A, B, and C are not changed, while the coefficients D, E, F, and G change according to

$$\bar{D} \; = \; \frac{1}{\phi'}\left\{D - 14\frac{\phi''}{\phi'}\right\}, \tag{2.47}$$

$$\bar{E} = \frac{1}{\phi'}\left\{E + (5A+4B)\frac{\phi''}{\phi'}\right\}, \tag{2.48}$$

$$\bar{F} = \frac{1}{(\phi')^2}\left\{F + A\mu + 9D\frac{\phi''}{\phi'} - 16\frac{\phi'''}{\phi'} - 39\frac{(\phi'')^2}{(\phi')^2}\right\}, \tag{2.49}$$

$$\bar{G} = \frac{1}{(\phi')^2}\left\{G + 3C\mu + 2E\frac{\phi''}{\phi'} + 2A\frac{\phi'''}{\phi'} + (2A+4B)\frac{(\phi'')^2}{(\phi')^2}\right\}. \tag{2.50}$$

For each case except F-II, we choose the normalizations,

$$E(x) = -12D(x), \qquad F(x) = 0. \tag{2.51}$$

This can always be done with a suitable choice of $\lambda(x)$ and $\mu(x)$ because the quantities $5A+4B-168$ and A are never zero. Hence the class of equations (2.45) simplifies to

$$\begin{aligned} y^{(\mathrm{iv})} = {} & Ayy'' + B(y')^2 + Cy^3 + D(x)(y''' - 12yy') \\ & + G(x)y^2 + H(x)y' + I(x)y + J(x). \end{aligned} \tag{2.52}$$

Next, in F-I and F-III–F-VI, we substitute a trial Laurent expansion of the form,

$$\begin{aligned} y = {} & a_0(x-x_0)^{-2} + a_1(x-x_0)^{-1} + a_2 + a_3(x-x_0) \\ & + \ldots + a_n(x-x_0)^{n-2} + \ldots, \end{aligned} \tag{2.53}$$

into (2.52). For F-I, $a_0 = 1$ only. For F-III–F-VI, a_0 takes the pairs of values $\{1,8\}$, $\{1,2\}$, $\{1,3\}$, and $\{1,5\}$, respectively. For the expansions having $a_0 = 1$, the first gauge condition (2.51) guarantees that $a_1 = 0$. Later coefficients are determined by a recurrence relation of the form,

$$(n+1)(n-n_1)(n-n_2)(n-n_3)a_n = F_n(a_0, a_1, \ldots, a_{n-1}), \tag{2.54}$$

where n_1, n_2, and n_3 are the resonance numbers, there being one set for each a_0.

When n coincides with a positive resonance number, the expression F_n must vanish identically after taking into account all constraints among the earlier coefficients, otherwise the expansion (2.53) must be supplemented by additional terms involving powers of $\log(x-x_0)$, which have movable branch points. The vanishing of the F_n at the positive resonances places differential constraints on the coefficients $D(x)$, etc., in equation (2.52). Then those values of a_n become arbitrary constants of integration. If all three resonances are positive integers, then the three corresponding values of a_n together with the position x_0 of the movable pole comprise a complete set of integration constants.

If one or more resonances are negative, the expansion represents a nongeneric solution. Take F-III as an illustration, the reduced equation being (2.36). The generic solution has an infinity of double poles of strength 1 and a three-parameter solution has double poles of strength 8. When the latter solution is given a small perturbation, each double pole of strength 8 splits into 8 double poles of strength 1 with 7 poles arranged in a regular heptagon of small diameter while the eighth pole resides at the center. A different behavior is seen in F-IV, F-V, and F-VI. In these cases, there is a nongeneric solution having a double pole of strength m, where $m = 2$, 3, and 5, respectively. When this solution is given a small perturbation, the pole splits into m double poles of strength 1 arranged in a regular m-gon of small diameter.

We now show the details of the construction of the full equation F-III, which, with our choice of gauge, will take the form,

$$\begin{aligned} y^{(\mathrm{iv})} &= 15yy'' + \tfrac{45}{4}(y')^2 - 15y^3 + D(x)(y''' - 12yy') \\ &\qquad + G(x)y^2 + H(x)y' + I(x)y + J(x). \end{aligned} \tag{2.55}$$

Begin with the expansion (2.53) having $a_0 = 1$. The resonance numbers are $\{3, 5, 7\}$. If none of the conditions for freedom from logarithms are met, the expansion would need to be supplemented with logarithmic terms and would take the form,

$$\begin{aligned} y &= (x - x_0)^{-2} + a_2 + \{a_3 + b_3 \log(x - x_0)\}(x - x_0) \\ &\quad + \{a_4 + b_4 \log(x - x_0)\}(x - x_0)^2 + \{a_5 + b_5 \log(x - x_0) \\ &\quad + c_5 \log^2(x - x_0)\}(x - x_0)^3 \\ &\quad + \{a_6 + b_6 \log(x - x_0) + c_6 \log^2(x - x_0)\}(x - x_0)^4 \\ &\quad + \{a_7 + b_7 \log(x - x_0) + c_7 \log^2(x - x_0) \\ &\quad + d_7 \log^3(x - x_0)\}(x - x_0)^5 + \ldots . \end{aligned} \tag{2.56}$$

The coefficients a_3, a_5, and a_7 are arbitrary constants of integration (x_0 being the fourth), while all other coefficients are uniquely determined after substituting the series into equation (2.55). If b_3, b_5, and b_7 were all zero, then all of the other logarithmic terms in the expansion would vanish.

The usual way to proceed is to assume a pure Laurent expansion and determine constraints on the coefficient functions for consistency at the resonances. These constraints are equivalent to requiring the coefficients b_3, b_5, and b_7 to be zero in turn. The vanishing of b_3 automatically implies the vanishing of b_4, c_5, c_6, and d_7, as well as an infinity of logarithmic terms further along in the series. After that, the vanishing of b_5 implies the vanishing of b_6 and c_7 and later terms. Finally, the vanishing of b_7 ensures that there are no logarithm terms anywhere in the expansion.

There are advantages in looking at the logarithm term b_7 first, without calculating it in full. A reasonably short calculation shows that b_7 and b_5 must take the form,

$$b_7 \;=\; \tfrac{1}{32}(G - 3D')a_5 - \tfrac{3}{8}D(a_3)^2 + \big\{\;\ldots\;\big\}\,a_3 + \big\{\;\ldots\;\big\}, \tag{2.57}$$

$$b_5 \;=\; -\tfrac{1}{6}(G + 3D')a_3 + \big\{\;\ldots\;\big\}, \tag{2.58}$$

where the quantities in braces do not depend on a_3, a_5, or a_7, and the functions D, G, etc., are to be evaluated at $x = x_0$. Because a_3 and a_5 are integration constants, the vanishing of b_7 gives four constraints. Among them, the coefficient of $(a_3)^2$ gives

$$D(x) \;=\; 0, \tag{2.59}$$

which permits a major simplification. (This result cannot be deduced from the earlier resonances; they leave $D(x)$ undetermined and express the remaining coefficient functions in terms of $D(x)$ and its derivatives.) Next, the coefficient of a_3 on the right-hand side of (2.58) gives

$$G(x) \;=\; 0. \tag{2.60}$$

With the coefficients $D(x)$ and $G(x)$ known to vanish, it is a simple matter to calculate the complete forms of the logarithm terms b_3, b_5, and b_7. They are

$$\begin{aligned}
b_3 \;&=\; -\tfrac{1}{16}H, \\
b_5 \;&=\; \tfrac{1}{24}(H'' - I'), \\
b_7 \;&=\; \tfrac{3}{64}H'a_3 - \tfrac{1}{768}(H^{(\mathrm{iv})} - \tfrac{643}{160}HH') + \tfrac{1}{384}(I''' - \tfrac{7}{10}HI) + \tfrac{1}{64}J',
\end{aligned}$$

where the functions are to be evaluated at $x = x_0$. The vanishing of these logarithm terms now gives immediately,

$$H(x) \;=\; 0, \qquad I(x) \;=\; \alpha, \qquad J(x) \;=\; \beta, \tag{2.61}$$

where α and β are arbitrary constants.

Next we must examine the positive-integer resonances in the second Laurent expansion with leading coefficient $a_0 = 8$. The resonance numbers are $\{-7, 10, 12\}$. With the coefficient functions in the equation (2.55) taking the simple forms (2.59)–(2.61), the calculation is reasonably short despite the high resonance numbers. The result is that the freedom-from-logarithms conditions at both positive resonances are satisfied identically. There are no more tests to run.

The final form of equation F-III is

$$y^{(\mathrm{iv})} \;=\; 15yy'' + \tfrac{45}{4}(y')^2 - 15y^3 + \alpha y + \beta. \tag{2.62}$$

It is solved in Section 4 in terms of hyperelliptic functions.

A similar calculation and choice of gauge can be used to construct the full equations F-I, F-IV, F-V, and F-VI from their reduced counterparts given by (2.15) and (2.37)–(2.39). The full version of F-II, in a particular gauge, is the same as the reduced version.

The complete list of fourth-order equations of the type considered is as follows:

$$\text{F-I:} \qquad y^{(\mathrm{iv})} = 12yy'' + 12(y')^2 + (\alpha x + \beta)y' + 2\alpha y - \tfrac{1}{6}(\alpha x + \beta)^2; \tag{2.63}$$

$$\text{F-II:} \qquad y^{(\mathrm{iv})} = 3yy'' - 4(y')^2; \tag{2.64}$$

$$\text{F-III:} \qquad y^{(\mathrm{iv})} = 15yy'' + \tfrac{45}{4}(y')^2 - 15y^3 + \alpha y + \beta; \tag{2.65}$$

$$\text{F-IV:} \qquad y^{(\mathrm{iv})} = 30yy'' - 60y^3 + \alpha y + \beta; \tag{2.66}$$

$$\text{F-V:} \qquad y^{(\mathrm{iv})} = 20yy'' + 10(y')^2 - 40y^3 + \alpha y + \kappa x + \beta; \tag{2.67}$$

$$\text{F-VI:} \qquad y^{(\mathrm{iv})} = 18yy'' + 9(y')^2 - 24y^3 + \alpha y^2 + \tfrac{1}{9}\alpha^2 y + \kappa x + \beta. \tag{2.68}$$

In Sections 4–7, we construct the general solutions of F-III and F-IV and the autonomous subcases $\kappa = 0$ of F-V and F-VI. All these cases involve hyperelliptic functions of genus 2.

Of the above equations F-I and F-V are well known, being similarity reductions of the KP and KdV5 equations, respectively. The autonomous subcase of F-V is one of Drach's equations [14], and was solved by Drach in terms of hyperelliptic functions. Lax [48] made a qualitative study of this equation (in a slightly different gauge) and looked, in particular, for double solitary waves. Bureau [27, p. 258] also obtained the autonomous subcase of F-V as a constraint on a coefficient function appearing in a certain second-order second-degree equation. Equations F-I and F-II were also obtained by Bureau [10]. Equation F-I with $\alpha \neq 0$ can be solved in terms of Painlevé-IV functions, while the cases with $\alpha = 0$ can be solved with Painlevé-II or Painlevé-I functions. These results for F-I are presented at the end of this section.

As already mentioned, the Bureau barrier equation F-II was used as an important test case by Conte et al. [26]. The integrability status of this equation is unknown. In the next section, we present what we know about this interesting equation and its associated hierarchy.

Equation F-IV was known to Chazy [4, p. 380], who supplied a three-parameter solution in the form of a sum of two elliptic functions with different periods. Exton [11, p. 395] obtained the reduced version of F-IV but then gave an incorrect proof that this equation was unstable. Equation F-III is a first integral of the autonomous subcase of Fif-I, which was first found by Chazy (see below).

As far as we know, the full version of equation F-VI is new, and the solution of F-VI with $\kappa \neq 0$ is a new fourth-order Painlevé transcendent. This equation is the subject of Section 7 below. Exton [11, p. 394] obtained the resonance numbers for F-VI, but derived the reduced equation incorrectly and rejected it immediately. Harada and Oishi [13] also obtained the resonance numbers of F-VI and the coefficients of an equation equivalent under scaling to (2.44) but did not pursue the matter further.

The nonautonomous case $\kappa \neq 0$ of F-V is a member of the so-called Painlevé-I hierarchy and has been denoted ${}_4P_1$ by recent authors [35, 36, 37]. Equation F-V is related to equation F-XVII in the appendix below, the latter equation being the corresponding member ${}_4P_2$ of the Painlevé-II hierarchy, which has been known for some time [38, 39, 40]. Several authors have conjectured that F-V (or one of its relatives) defines a new higher-order transcendent when $\kappa \neq 0$. By this it is understood that the general solution of F-V cannot be built out of known transcendents, which include the six Painlevé transcendents, elliptic, hyperelliptic, abelian, and automorphic functions, for example, in any finite number of steps that involve algebraic operations, nesting of functions and functional inversions, differentiation, integration, or solutions of linear differential equations. (Note that solutions of linear integral equations are not counted among these processes.)

The corresponding full versions of the fifth-order equations Fif-I–Fif-IV can be derived from the reduced equations (2.41)–(2.44) in a similar fashion. The results are given in the following list:

Fif-I: $$y^{(\mathrm{v})} = 15yy''' + \tfrac{75}{2}y'y'' - 45y^2y' + (\lambda x + \alpha)y' + 2\lambda y; \tag{2.69}$$

Fif-II: $$y^{(\mathrm{v})} = 30yy''' + 30y'y'' - 180y^2y' + (\lambda x + \alpha)y' + 2\lambda y; \tag{2.70}$$

Fif-III: $$y^{(\mathrm{v})} = 20yy''' + 40y'y'' - 120y^2y' + (\lambda x + \alpha)y' + 2\lambda y + \kappa; \tag{2.71}$$

Fif-III.b: $$\begin{aligned} y^{(\mathrm{v})} = {} & 20yy''' + 40y'y'' - 120y^2y' + \alpha y' + (1/x)\Big\{y^{(\mathrm{iv})} \\ & - 20yy'' - 10(y')^2 + 40y^3 - \alpha y - \beta\Big\}; \end{aligned} \tag{2.72}$$

Fif-IV: $$\begin{aligned} y^{(\mathrm{v})} = {} & 18yy''' + 36y'y'' - 72y^2y' + 3\lambda y'' + (1/x)\Big\{y^{(\mathrm{iv})} \\ & - 18yy'' - 9(y')^2 + 24y^3 - 3\lambda y' + \kappa\Big\} \\ & + \tfrac{1}{2}\lambda x(5y''' - 36yy') - \tfrac{1}{2}\lambda^2 x(2xy' + y); \end{aligned} \tag{2.73}$$

Fif-IV.b: $$y^{(\mathrm{v})} = 18yy''' + 36y'y'' - 72y^2y' + 2\alpha yy' + \tfrac{1}{9}\alpha^2y' + \kappa; \tag{2.74}$$

$$\text{Fif-IV.c:} \qquad y^{(\mathrm{v})} = 18yy''' + 36y'y'' - 72y^2y' + 2\alpha yy' + \tfrac{1}{9}\alpha^2 y' + (1/x)\left\{ y^{(\mathrm{iv})} - 18yy'' - 9(y')^2 + 24y^3 - \alpha y^2 - \tfrac{1}{9}\alpha^2 y - \beta \right\}. \qquad (2.75)$$

All of these equations admit first integrals, not necessarily of polynomial type. We believe that the members of this list with $\lambda \neq 0$ define fourth-order Painlevé transcendents, three of which are distinct and one of which (Fif-IV) is new. Some of these admit a four-parameter exact solution in terms of classical Painlevé transcendents. Details are given in Sections 4–7. The cases with $\lambda = 0$ are easily seen to be derivatives of F-III–F-VI. Similarly, equation Fif-III.b is a derivative of F-V and equations Fif-IV.b and Fif-IV.c are derivatives of F-VI.

Equations Fif-I, Fif-II, and Fif-III are, respectively, group-invariant reductions of the Kaup-Kuperschmidt equation [41, 42, 43],

$$u_{xxxxx} = 15uu_{xxx} + \tfrac{75}{2}u_x u_{xx} - 45u^2 u_x + u_t\,, \qquad (2.76)$$

the Sawada-Kotera equation [44, 45, 46, 47],

$$u_{xxxxx} = 30uu_{xxx} + 30u_x u_{xx} - 180u^2 u_x + u_t\,, \qquad (2.77)$$

and the Lax KdV5 equation [48],

$$u_{xxxxx} = 20uu_{xxx} + 40u_x u_{xx} - 120u^2 u_x + u_t + R'(t)\,, \qquad (2.78)$$

subscripts denoting partial differentiation. Equation Fif-I was known to Chazy [4, p. 345], it being the differential equation for the coefficient function in the case $N = 7$ of Chazy-XI. These equations were first derived from the point of view of Painlevé classification by Harada and Oishi [13]. They looked for evolution equations with constant coefficients of the above form and restricted their search to cases where one set of resonances is positive. They also observed that the reduced form of Fif-IV could be added to the list if the u_t term is dropped.

The Kaup-Kuperschmidt and Sawada-Kotera equations admit the same modified equation, found by Fordy and Gibbons [42], namely,

$$w_{xxxxx} = -5(w_x - w^2)w_{xxx} - 5(w_{xx})^2 + 20ww_x w_{xx} + 5(w_x)^3 - 5w^4 w_x + w_t\,. \qquad (2.79)$$

If $w(x,t)$ satisfies (2.79), then

$$u(x,t) = -\tfrac{1}{3}(2w_x - w^2) \qquad (2.80)$$

satisfies the Kaup-Kuperschmidt equation and

$$u(x,t) \;=\; \tfrac{1}{6}(w_x + w^2) \tag{2.81}$$

satisfies the Sawada-Kotera equation. (An alternative presentation of the Miura transformation connecting (2.76) and (2.77) appears in [49].) It follows that the transcendents defined by Fif-I and Fif-II are not independent of each other, the relationship being given in Section 5.

Similarly, the KdV5 equation has the modified equation,

$$\begin{aligned} w_{xxxxx} \;=\; & 10\{w^2 - 2R(t)\}w_{xxx} + 40ww_xw_{xx} + 10(w_x)^3 \\ & - 30\{w^2 - 2R(t)\}^2 w_x + w_t\,. \end{aligned} \tag{2.82}$$

If $w(x,t)$ satisfies (2.82), then

$$u(x,t) \;=\; -\tfrac{1}{2}\{w_x - w^2 + 2R(t)\} \tag{2.83}$$

satisfies KdV5. Each of the Miura-like transformations (2.80), (2.81), and (2.83) can be supplemented by an expression for w_t in terms of w, u, and x-derivatives of u, thereby making them Bäcklund transformations. The equations F-XVII and F-XVIII in the Appendix are group invariant reductions of the modified equations (2.82) and (2.79), respectively.

Ito [50, (2.15)] formed a partial differential equation equivalent under scaling to

$$u_{xxxxx} \;=\; 18uu_{xxx} + 36u_xu_{xx} - 72u^2u_x + u_t\,, \tag{2.84}$$

which has the same reduced part as Fif-IV. This equation does not have the Painlevé property and it appears that Ito added the term u_t artificially. The rest of the equation was deduced from the functional derivative of the seventh conserved quantity of the Sawada-Kotera equation. We have not found an integrable soliton equation that admits F-VI or Fif-IV as a group-invariant reduction in the usual sense.

The techniques for adding recessive terms in the case of partial differential equations with the Painlevé property are also well established [2, 3, 51, 52, 53, 54]. Starting with the reduced equations (2.41)–(2.44) and choosing gauge appropriately, we are led uniquely to the three KdV5-like equations (2.76)–(2.78). In fact we have proved these equations unique under much more general hypotheses (details to appear in a future publication). Up to gauge, they are the unique semilinear Painlevé-type partial differential equations of the fifth order in two independent variables in the polynomial class having the Bureau symbol $P2$, namely,

$$\begin{aligned}
A_1u_{xxxxx} + A_2u_{xxxxt} + A_3u_{xxxtt} + A_4u_{xxttt} + A_5u_{xtttt} + A_6u_{ttttt} \\
= B_1uu_{xxx} + B_2uu_{xxt} + B_3uu_{xtt} + B_4uu_{ttt} \\
+ C_1u_xu_{xx} + C_2u_xu_{xt} + C_3u_xu_{tt} + C_4u_tu_{xx} \\
+ C_5u_tu_{xt} + C_6u_tu_{tt} + D_1u^2u_x + D_2u^2u_t \\
+ E_1u_{xxxx} + E_2u_{xxxt} + E_3u_{xxtt} + E_4u_{xttt} + E_5u_{tttt} \\
+ F_1uu_{xx} + F_2uu_{xt} + F_3uu_{tt} \\
+ G_1(u_x)^2 + G_2u_xu_t + G_3(u_t)^2 + Hu^3 \\
+ I_1u_{xxx} + I_2u_{xxt} + I_3u_{xtt} + I_4u_{ttt} \\
+ J_1uu_x + J_2uu_t + K_1u_{xx} + K_2u_{xt} + K_3u_{tt} \\
+ Lu^2 + M_1u_x + M_2u_t + Nu + P,
\end{aligned} \tag{2.85}$$

where each of the 44 coefficients $A_1, \ldots, P$ are locally analytic functions of x and t. Similarly, we have proved that the KP equation,

$$u_{xxxx} = 12uu_{xx} + 12(u_x)^2 + \alpha u_{xt} + \beta u_{yy}\,, \tag{2.86}$$

is, up to gauge, the unique semilinear Painlevé-type partial differential equation of the fourth order in three independent variables in the polynomial class having the Bureau symbol $P2$ (53 coefficients), substantially generalizing the results in [55, 56]. These calculations were carried out with the assistance of the computer algebra package Reduce.

2.3. Solution of F-I

For completeness, we include here the solution of the fourth-order equation F-I. The other differential equations derived above in this section will be treated in some detail in the next five sections.

Equation F-I, namely,

$$y^{(\mathrm{iv})} = 12yy'' + 12(y')^2 + (\alpha x + \beta)y' + 2\alpha y - \tfrac{1}{6}(\alpha x + \beta)^2, \tag{2.87}$$

is well known, being obtained by a nonclassical reduction of the KP equation (2.86) and the Boussinesq equation [57]. It can be integrated once to give a member of Chazy Class XIII. When $\alpha \neq 0$, it can be integrated again to give a second-order second-degree equation known to Bureau [27, (19.7)]. When $\alpha = 0$, the second integral is Ince-XXXIV [18, p. 340].

In the generic case $\alpha \neq 0$, the solution of F-I is

$$y = \tfrac{1}{2}\alpha_1 w' - \tfrac{1}{2}\alpha w^2 - \tfrac{1}{2}(\alpha x + \beta)w - \tfrac{1}{6}(\tfrac{1}{2}\alpha x^2 + \beta x + K_1 - \alpha_1), \tag{2.88}$$

where $\alpha_1 := \pm\sqrt{-\alpha}$ and $w(x)$ satisfies the Painlevé-IV equation,

$$w'' = \frac{1}{2w}(w')^2 - \tfrac{3}{2}\alpha w^3 - 2(\alpha x + \beta)w^2 - (\tfrac{1}{2}\alpha x^2 + \beta x + K_1)w + \frac{K_2}{w}, \tag{2.89}$$

K_1 and K_2 being integration constants.

When $\alpha = 0$ and $\beta \neq 0$, the solution of F-I is

$$y = \tfrac{1}{2}(\epsilon_1 w' + w^2) + \tfrac{1}{12}(\beta x + K_1), \tag{2.90}$$

where $\epsilon_1 := \pm 1$ and $w(x)$ satisfies the Painlevé-II equation,

$$w'' = 2w^3 + (\beta x + K_1)w + K_2\,. \tag{2.91}$$

In the case $\alpha = \beta = 0$, equation F-I integrates directly up to the Painlevé-I equation,

$$y'' = 6y^2 + K_1 x + K_2\,. \tag{2.92}$$

We make no further mention of equation F-I.

In the following, the symbols K, K_1, K_2, etc., will be reserved for constants of integration. When we speak of an n-parameter solution of a differential equation, we mean a solution containing n independent constants of integration and we are not counting parameters appearing in the coefficients of the differential equation.

3. The Bureau barrier equation F-II

The Bureau barrier equation (equation F-II) is the fourth-order differential equation,

$$y^{(\mathrm{iv})} = 3yy'' - 4(y')^2. \tag{3.1}$$

It was discovered by Bureau [10, p. 79, (2.9)], during his investigation of fourth-order equations which may possibly have the Painlevé property. (Bureau wrote the equation with an incorrect coefficient due to an anomalous factor of 2 appearing in the calculation immediately preceding.)

This equation has integer resonance numbers $\{-2, -3, 20\}$ and passes certain other routine Painlevé tests but it does not, in fact, have the Painlevé property. We do not know whether or not it is integrable in any sense. Nevertheless, it is a very interesting equation from the point of view of Painlevé analysis and has attracted attention from recent researchers in the field (for example, [26, 58]). The problem of defining the Painlevé property or determining integrability for a differential equation is less straightforward when natural barriers are involved, and we believe that the Bureau barrier equation will have a continuing role to play in this study. There is a superficial resemblance to the Chazy-III equation, $y''' = 2yy'' - 3(y')^2$, which has also attracted a considerable amount of attention on account of its natural barrier. Equation (3.1) is also the first nontrivial member of a hierarchy of equations, all having a similar structure (see equation (3.23) below).

The only exact solution of (3.1) known is the rational solution,

$$y = \frac{60}{(x-x_0)^2} + \frac{C}{(x-x_0)^3}, \tag{3.2}$$

where x_0 and C are arbitrary complex constants. This is a special case of a three-parameter solution defined by a convergent descending power series [12],

$$y = 60/x^2 + C_1/x^3 + C_2/x^4 + C_3/x^5 + C_4/x^6 + \dots, \tag{3.3}$$

where C_1, C_2, and C_3 are arbitrary complex constants and

$$\begin{aligned} C_4 &= \tfrac{1}{72}\{3C_1C_3 - 2(C_2)^2\}, \\ C_5 &= \tfrac{1}{2400}\{3(C_1)^2C_3 - 2C_1(C_2)^2 - 40C_2C_3\}. \end{aligned}$$

(C_1 can be set to zero by a translation of the origin.) The C_n satisfy a simple recurrence relation, and it is a straightforward exercise to show by induction that

$$|C_n| \le (72/5)(R_0)^n \tag{3.4}$$

for $n \ge 1$, where

$$R_0 = \max\left\{\tfrac{5}{72}|C_1|, \left(\tfrac{5}{72}|C_2|\right)^{1/2}, \left(\tfrac{5}{72}|C_3|\right)^{1/3}\right\}. \tag{3.5}$$

(This result is not best possible.) Hence the series (3.3) converges to an analytic function in some domain $|x| > R$, where $0 \le R \le R_0$. A numerical study by Littlewood [33] appears to confirm that the singular region inside the circle of convergence is, in general, a region of positive planar measure bounded by a natural barrier with an apparent fractal structure. The series converges (outside a circle) to the rational solution (3.2) when

$$16200(C_3)^2 - 1080C_1C_2C_3 + 9(C_1)^3C_3 + 640(C_2)^3 - 6(C_1)^2(C_2)^2 = 0. \tag{3.6}$$

The Bureau barrier equation also admits a convergent ascending power series solution. It is

$$y \;=\; 60(x-x_0)^{-2} + C(x-x_0)^{18} + \sum_{n=2}^{\infty} a_n C^n (x-x_0)^{20n-2}\,, \tag{3.7}$$

where x_0 and C are arbitrary complex constants and

$$\begin{aligned} a_2 &= -\tfrac{9}{35{,}260}\,, \\ a_3 &= \tfrac{3}{333{,}383{,}300}\,, \\ a_4 &= -\tfrac{9{,}339}{3{,}902{,}691{,}592{,}456{,}000}\,, \\ a_5 &= -\tfrac{1{,}692{,}063}{13{,}803{,}898{,}216{,}348{,}721{,}120{,}000}\,. \end{aligned}$$

Again, it is a straightforward exercise to prove that

$$|a_n| \leq (9/35260)^{n-1}, \tag{3.8}$$

and so the radius of convergence of the series (3.7) is at least $(35260/9)^{1/20}|C|^{-1/20}$.

With the aid of the computer algebra package MATHEMATICA, we calculated over 100 (nonzero) terms of the series (3.7) exactly and over 2000 terms to high numerical precision. From these results we deduced the approximate disc of convergence,

$$0 \;<\; |x-x_0| \;<\; (1.625...)|C|^{-1/20}. \tag{3.9}$$

(The last digit shown is not reliable.) This radius was confirmed by a different numerical study by Littlewood [33]. He first evaluated $y(x)$ and its first three derivatives at the point $x=0$, having selected $x_0=-1$ and $C=1$. These values were then used as input for a study of the equation in the complex plane with Runge-Kutta techniques. He concluded that the particular solution (3.7) analytically continues up to a natural barrier with a beautiful fractal structure which can be inscribed in a regular polygon with 20 sides. The precise details of the fractal barrier could not be determined. (The method could not, for example, reveal if there were spiral structures of the sort observed in the barriers of certain Kleinian functions.)

The first proof that the Bureau barrier equation does not have the Painlevé property was given by Conte et al. [26]. This proof appears among several instructive examples showing the power of their new perturbative Painlevé test. A shorter and more direct proof was provided later by Musette and Conte [58]. The latter proof is essentially an application of Painlevé's original α-expansion method [15, pp. 224–225] (reviewed by Ince [18, pp. 329–330]). Painlevé applied this method to the class of equations $y'' = 6y^2 + S(x)$ by expanding the solution in a series of elliptic and related functions rather than, as is usually done, in a Laurent series (augmented, when necessary, by logarithm terms).

The Laurent series approach was favored by Chazy [4] and was popularized more recently by Ablowitz et al. [34], after which there was a flood of applications of the method in the literature. Bureau's method [59, 60] was a further refinement in the efficiency of the Laurent series method. It seems likely that many researchers presumed that the Laurent series methods in skilled hands could tackle the same problems as Painlevé's α-expansions, but that is not true. The Laurent series methods could not handle negative-integer resonances until Conte et al. [26] substantially generalized those methods to arrive at their CFP test. On the other hand, α-expansions can bypass resonance analysis and give direct information about singularities of many different types, including essential singularities and natural barriers. Kruskal [61] has emphasized the need for greater flexibility in the running of Painlevé tests. Other fundamental contributions to the Painlevé analysis of differential equations that go substantially beyond the Laurent series methods in different directions have been provided, for example, by Joshi and Kruskal [62, 63] and Kruskal and Clarkson [64].

To see the non-Painlevé character of the Bureau barrier equation (3.1), apply the following Painlevé α-test,

$$\begin{aligned} x &= x_0 + \alpha X, \\ y &= \alpha^{-2} Y_0(X) + \alpha^{-1} Y_1(X) + Y_2(X) + \alpha Y_3(X) + \ldots . \end{aligned} \tag{3.10}$$

The reduced equation satisfied by $Y_0(X)$ is the Bureau barrier equation itself. One way to proceed is to let $Y_0(X)$ be the two-parameter rational solution,

$$Y_0(X) = \frac{60}{(X - X_0)^2} + \frac{C}{(X - X_0)^3}, \tag{3.11}$$

with $C \neq 0$. Then $Y_1(X)$ satisfies a fourth-order linear equation which is easily integrated twice to give

$$Y_1(X) = \tilde{Y}(X) + \frac{K_1}{(X - X_0)^3} + \frac{K_2}{(X - X_0)^4}, \tag{3.12}$$

where $\tilde{Y}(X)$ satisfies the second-order linear equation,

$$\tilde{Y}'' - \frac{14}{X - X_0} \tilde{Y}' - \left\{ \frac{54}{(X - X_0)^2} + \frac{3C}{(X - X_0)^3} \right\} \tilde{Y} = 0, \tag{3.13}$$

K_1 and K_2 being integration constants. The latter equation is easily transformed into Bessel's equation of order 21. The general solution of (3.13) is

$$\begin{aligned}\tilde{Y}(X) \;=\; (X-X_0)^{15/2}\Big\{ & K_3\, J_{21}\left(\sqrt{-12C/(X-X_0)}\right) \\ & + K_4\, Y_{21}\left(\sqrt{-12C/(X-X_0)}\right)\Big\}\,, \end{aligned} \tag{3.14}$$

where K_3 and K_4 are integration constants and $J_{21}(z)$ and $Y_{21}(z)$ are the Bessel functions of the first and second kinds of order 21. (Be careful not to confuse the latter function with the 22nd term in the series (3.10).)

Observe that the term containing the J_{21} function in (3.14) has a movable isolated (and unbranched) essential singularity at $X = X_0$ while the term containing the Y_{21} function has a movable logarithmically branched essential singularity at the same location. Hence $Y_1(X)$ has a movable branch point when $K_4 \neq 0$. It follows that the Bureau barrier equation admits some violation of the Painlevé property, the precise nature of which is to be determined.

Another way to proceed is to let $Y_0(X)$ be the two-parameter Laurent series solution,

$$Y_0(X) \;=\; 60(X-X_0)^{-2} \;+\; C(X-X_0)^{18} \;-\; \tfrac{9}{35260}C^2(X-X_0)^{38} + \ldots\,, \tag{3.15}$$

with $C \neq 0$, of which only the first two terms on the right-hand side are needed to arrive at the first Painlevé violation. The simultaneous use of Laurent series and α-expansions is the essence of the CFP method. The linear equations defining $Y_1(X)$, $Y_2(X)$, etc., can be solved in turn by means of Laurent series, augmented when necessary by logarithm terms, there being four constants of integration at each step corresponding to the resonance numbers -3, -2, -1, and 20.

After translating the movable singularity to the origin, so that $X_0 = 0$, we find that the first perturbation term takes the form,

$$\begin{aligned} Y_1(X) \;=\; & b_{-3}X^{-5} + b_{-2}X^{-4} + b_{-1}X^{-3} + b_{17}X^{15} + b_{18}X^{16} \\ & + b_{19}X^{17} + b_{20}X^{18} + b_{37}X^{35} + b_{38}X^{36} \\ & + b_{39}X^{37} + b_{40}X^{38} + b_{57}X^{55} + \ldots\,, \end{aligned} \tag{3.16}$$

where the coefficients b_{-3}, b_{-2}, b_{-1}, and b_{20} are arbitrary, and

$$b_{17} \;=\; -\tfrac{8}{95}Cb_{-3}\,, \qquad b_{18} \;=\; -\tfrac{37}{380}Cb_{-2}\,, \qquad b_{19} \;=\; -\tfrac{3}{20}Cb_{-1}\,,$$

$$b_{37} \;=\; \tfrac{2{,}794{,}167}{56{,}261{,}561{,}200}C^2b_{-3}\,, \qquad b_{38} \;=\; \tfrac{9{,}599}{174{,}184{,}400}C^2b_{-2}\,,$$

$$b_{39} \;=\; \tfrac{57}{705{,}200}C^2b_{-1}\,, \qquad b_{40} \;=\; -\tfrac{9}{17{,}630}Cb_{20}\,.$$

Similarly, $Y_m(X)$ begins with a term, (const.)$(b_{-3})^m X^{-3m-2}$. For $m \le 6$, $Y_m(X)$ is a pure Laurent series consisting of a initial block of $3m$ nonzero terms followed by $20-3m$ zero terms, after which each block of 20 consecutive terms begins with $3m+1$ nonzero terms and ends with $19-3m$ zero terms.

The first opportunity for logarithm terms to occur is in $Y_7(X)$, and indeed they do occur there at the positions corresponding to the resonances -1 and 20, as observed by Conte et al. [26]. The series for $Y_7(X)$ takes the form,

$$\begin{aligned} Y_7(X) \;=\; & \sum_{j=-21}^{\infty} h_j X^{j-2} + \Big\{ s_{-1} X^{-3} + s_{19} X^{17} \\ & + s_{20} X^{18} + s_{39} X^{37} + s_{40} X^{38} + \ldots \Big\} \log X, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} h_{-21} &= -\tfrac{1{,}459{,}961}{74{,}466{,}222{,}397{,}670{,}051{,}020{,}800} (b_{-3})^7 \,, \\ s_{-1} &= -\tfrac{15{,}910{,}425}{12{,}096{,}018{,}049{,}024} C (b_{-3})^7 \,, \\ s_{20} &= -\tfrac{1{,}674{,}906{,}693{,}579{,}834{,}993{,}639}{6{,}155{,}562{,}352{,}883{,}651{,}129{,}144{,}968{,}960} C^2 b_{-2} (b_{-3})^6 \,, \end{aligned}$$

$$s_{19} = -\tfrac{3}{20} C s_{-1} \,, \qquad s_{39} = \tfrac{57}{705{,}200} C^2 s_{-1} \,, \qquad s_{40} = -\tfrac{9}{17{,}630} C s_{20} \,.$$

While the CFP method appears to be slower in this instance, one can see certain attractive features. First, it identifies which resonances, positive or negative, can take the blame for specific Painlevé violations. Secondly, its effectiveness does not depend on knowing an exact solution of the reduced equation which gives an immediate or early violation.

The precise nature of the singularities admitted by the Bureau barrier equation is difficult to determine by analytical means alone. Littlewood [33] tried various random initial data and obtained numerical evidence for a natural barrier with an apparent fractal structure on every attempt. On most occasions, the apparent barrier was just a Jordan curve in the complex plane and the solution was analytic everywhere on the inside. In these cases, there was no branching. He also looked for solutions that were analytic on the outside of a barrier to see if a branch cut joined the barrier to infinity, for example, but such solutions are difficult to analyse numerically and no such branch cuts were observed.

So where is the branching implied by the Painlevé analysis? It can be found by examining solutions that are sufficiently close to the rational solution (3.2), but not too close to the family of solutions (3.3). For such solutions, it was found that there was an inner barrier close to the pole as well as an outer barrier. The solutions were unstable near the barriers, and the location of the outer barrier was difficult to determine accurately when the solution was very close to the rational solution. On the annulus between the barriers, the solution did not, in general, return to its starting value after a revolution, implying that the annulus continued onto a new Riemann sheet. After following a path around several sheets of a Riemann surface, the outer and inner barriers moved towards each other and eventually joined. (As a check on the accuracy of the numerical integration, we integrated back along several paths that returned to the initial point on the initial sheet.) We concluded that the domain of analyticity was the region inside a fractal Jordan curve drawn on a Riemann surface of one or more sheets. When the solution was very close to the rational solution, the number of sheets was high. In the limit when the solution approached the rational solution, we conjectured that the number of sheets tended to infinity while the radius of the outer and inner barriers tended, respectively, to infinity and zero. This is consistent with the results of the Painlevé analysis. On the other hand, as the solution was allowed to move further away from the rational solution and (3.3), the number of Riemann sheets decreased until eventually only one was needed. We found numerical solutions defined in horseshoe-shaped domains with varying amounts of overlap at the ends (case of two Riemann sheets) and also with no overlap (case of one sheet).

A natural question to ask is whether or not the Bureau barrier equation admits single-valued solutions when certain offending parameters are set to zero. Clearly, the CFP violations in $Y_7(X)$ (given by (3.17)) do not occur if $b_{-3} = 0$. But if $b_{-3} = 0$ and $b_{-2} \neq 0$, a corresponding violation (at the position corresponding to resonance number 20 only) occurs in $Y_{10}(X)$. If $b_{-3} = b_{-2} = 0$, then logarithms can be avoided by restricting to perturbations of $Y_0(X)$ (given by (3.15)) that just vary its internal parameters X_0 and C. Hence, no nontrivial perturbation of the ascending series solution gives a single-valued solution.

A better result is obtained in the case where $Y_0(X)$ is the rational solution (3.11) and where the Bessel function of the second kind is suppressed from $Y_1(X)$ (given by (3.12) and (3.14)) by setting $K_4 = 0$. It is then found that, with integration constants suitably chosen, all higher terms in the α-expansion (3.10) have an unbranched essential singularity at $X = X_0$. This is easily proved by examining the nature of the fixed point $X = \infty$, since the point X_0 is branched if and only if ∞ is branched. In fact, for $m \geq 2$, each $Y_m(X)$ can be expressed by a descending power series with infinite radius of convergence, beginning with the power $(X - X_0)^{-m-4}$. After renaming variables and parameters, the full solution can be identified with the three-parameter solution (3.3), which is analytic outside a bounded region.

In view of the preceding paragraph, it is now possible to give a sensible answer to the question of whether or not the Bureau barrier equation can support recessive terms when the Bessel functions of the second kind are suppressed from the α-expansion. In other words, we want to know if an equation of the form,

$$\begin{aligned} y^{(\mathrm{iv})} &= 3yy'' - 4(y')^2 + A(x)y''' + B(x)yy' \\ &\qquad + C(x)y'' + D(x)y^2 + E(x)y' + F(x)y + G(x), \end{aligned} \tag{3.18}$$

can admit a three-parameter solution with no movable branch points and no movable barriers that run onto two or more Riemann sheets. First, use the gauge freedom (2.46) to set $A(x) = 0$ and $C(x) = 0$. Next, apply the Painlevé α-test (3.10) with the simple leading term $Y_0(X) = 60/(X - X_0)^2$. This leading term is common to both (3.11) and (3.15), the Bessel functions now degenerating to rational functions, and we may proceed with the CFP test. (Note that this CFP test has been deliberately weakened so that it cannot detect the Painlevé violation in the reduced equation.)

The fourth-order linear differential equation for $Y_1(X)$ has the rational solution,

$$\begin{aligned} Y_1(X) &= b_{-3}(X - X_0)^{-5} + b_{-2}(X - X_0)^{-4} + b_{-1}(X - X_0)^{-3} \\ &\qquad + \tfrac{300}{19}B(x_0)(X - X_0)^{-1} + b_{20}(X - X_0)^{18}, \end{aligned} \tag{3.19}$$

where b_{-3}, b_{-2}, b_{-1}, and b_{20} are constants of integration. We must set $b_{20} = 0$ for otherwise we would be keeping (a degenerate form of) the Bessel function of the second kind. In fact, we only need to keep b_{-3} to get the result at which we are aiming. Thence, $Y_2(X)$ is found to have a logarithmic branch point at X_0 unless $B(x_0) = 0$. Since x_0 is arbitrary, this forces the coefficient function $B(x)$ to vanish. (If gauge freedom were not used, the condition for the absence of logarithms would be $A(x) = 14B(x)$. This is precisely the condition for a gauge transformation to exist such that $A(x)$ and $B(x)$ can be made to vanish simultaneously.)

So far, with $A(x) = B(x) = 0$, integration constants can be chosen so that

$$\begin{aligned} Y_1(X) &= b_{-3}(X - X_0)^{-5}, \\ Y_2(X) &= -\tfrac{1}{156}(b_{-3})^2(X - X_0)^{-8} - \tfrac{10}{3}D(x_0). \end{aligned}$$

Next, it is found that $Y_3(X)$ contains logarithms unless $D(x) = 0$. So, we now have $C(x) = D(x) = 0$. Then $Y_4(X)$ is free of logarithms. $Y_5(X)$ contains logarithms unless $E(x) = 0$. Having set $E(x) = 0$, it is found that $Y_6(X)$ contains logarithms unless $F(x) = 0$. Then $Y_7(X)$ and $Y_8(X)$ are free of logarithms. Finally $Y_9(X)$ contains logarithms unless $G(x) = 0$.

It follows that, under the assumed hypotheses, the Bureau barrier equation cannot support any recessive terms except those induced by a gauge change. It cannot even support an additive constant. That is why we did not include the derivative of the

Bureau barrier equation among the fifth-order reduced equations (2.41)–(2.44). (By comparison, the second derivative of the Bureau barrier equation, having resonance numbers $\{-3,-2,6,7,20\}$, fails a standard Painlevé test at the last resonance.)

Among the sixth-order equations with Bureau symbol $P2$, there are two with properties similar to the Bureau barrier equation. These are

$$y^{(\mathrm{vi})} = 13yy^{(\mathrm{iv})} + 10y'y''' - 45(y'')^2, \tag{3.20}$$

$$y^{(\mathrm{vi})} = 2yy^{(\mathrm{iv})} - 16y'y''' + 15(y'')^2. \tag{3.21}$$

The resonance numbers are $\{-3,-2,4,14,15\}$ and $\{-5,-4,-3,-2,42\}$, respectively. Both admit rational solutions and have convergent descending and ascending series solutions analogous to (3.3) and (3.7).

If the α-test (3.10) is applied to (3.20), with

$$Y_0(X) = \frac{12}{(X-X_0)^2} + \frac{C}{(X-X_0)^3}, \tag{3.22}$$

then $Y_1(X)$ satisfies a sixth-order linear equation which is easily integrated three times to give a third-order equation solvable in terms of confluent hypergeometric functions of ${}_1F_2$ type. In general, the movable essential singularity at $X = X_0$ is logarithmically branched.

Equation (3.21) is the second nontrivial member of a hierarchy of Bureau barrier equations. This class of equations is given by

$$y^{(2m)} = \frac{1}{2D_m} \sum_{r=1}^{2m-1} A_{m,r}\, y^{(r-1)} y^{(2m-r-1)}, \tag{3.23}$$

$m \geq 2$, where $y^{(k)}$ denotes $d^k y/dx^k$, D_m is an inessential scaling parameter, and, for $1 \leq r \leq 2m-1$,

$$A_{m,r} = A_{m,2m-r} = \frac{(-1)^{r-1}(2m)!(2m+2)!}{(r-1)!(r+1)!(2m-r-1)!(2m-r+1)!}. \tag{3.24}$$

Je n'ai pas intégré ces équations. If these equations are nonintegrable, then they have a surprising amount of structure. They definitely deserve further study.

When $m = 1$, equation (3.23) is simply $y'' = (6/D_1)y^2$, solvable with elliptic functions. When $m = 2$, the equation becomes the Bureau barrier equation. It takes the standard form (3.1) if we choose $D_2 = 60$. When $m = 3$ (and $D_3 = 420$), equation (3.23) becomes the sixth-order Bureau barrier equation (3.21). The resonance numbers (including the trivial resonance) run through the consecutive negative integers from -1 to $-(2m-1)$. There is also a single large positive resonance with value $2m(2m+1)$.

All of the following claims about the hierarchy (3.23) can be proved by elementary methods involving hypergeometric sums of ${}_2F_1(1)$ and ${}_3F_2(1)$ type. We leave the proofs as an exercise for the interested reader.

First, (3.23) admits the exact m-parameter rational solution,

$$y \;=\; \sum_{k=0}^{m-1} \frac{C_k}{(x-x_0)^{k+2}}\,, \tag{3.25}$$

where C_1, C_2, $\ldots$, C_{m-1}, and x_0 are arbitrary and $C_0 = D_m$. This can be generalized to a $(2m-1)$-parameter solution analytic outside a bounded region. It is given by the convergent (outside a circle) descending series,

$$y \;=\; \sum_{k=0}^{\infty} C_k x^{-k-2}, \tag{3.26}$$

where C_1, C_2, $\ldots$, C_{2m-1} are arbitrary and $C_0 = D_m$. The next coefficient is

$$C_{2m} \;=\; \frac{(2m+1)!}{2D_m} \sum_{j=1}^{2m-1} (-1)^{j-1} \frac{C_j C_{2m-j}}{(j+1)!(2m-j+1)!}\,. \tag{3.27}$$

Equation (3.23) also admits a two-parameter solution defined by a convergent (inside a circle) ascending power series,

$$y \;=\; \sum_{k=0}^{\infty} a_k C^k (x-x_0)^{2km(2m+1)-2}, \tag{3.28}$$

where x_0 and C are arbitrary, $a_0 = D_m$, and $a_1 = 1$. The next coefficient is

$$a_2 \;=\; \frac{(-1)^{m-1}(2m)!(4m^2+3m-1)!(8m^2+4m-1)!}{D_m(4m^2+2m-1)(m!)^2(4m^2+m-1)!(8m^2+6m-1)!}\,. \tag{3.29}$$

Next, consider the effect of the α-expansion (3.10) on (3.23). $Y_0(X)$ satisfies the original equation (3.23) with X instead of x and $Y_1(X)$ satisfies the $(2m)$th-order linear equation,

$$Y_1^{(2m)} \;=\; \frac{1}{D_m} \sum_{r=1}^{2m-1} A_{m,r}\, Y_0^{(r-1)} Y_1^{(2m-r-1)}. \tag{3.30}$$

If we let $Y_0(X)$ be the m-parameter rational solution, then (3.30) can be integrated m times by elementary integrating factors. We can integrate further if we keep fewer parameters in the rational solution. Let

$$Y_0(X) \;=\; \sum_{k=0}^{q-1} C_k (X-X_0)^{-k-2}, \tag{3.31}$$

where $2 \le q \le m$ and $C_0 = D_m$. Then (3.30) possesses the integrating factor $(X - X_0)^p$ for all integers p in the range $2m+q+1 \le p \le 4m$. Hence (3.30) can be integrated $2m-q$ times to yield a linear equation of order q.

The special case $q = 2$ yields Bessel functions of order one greater than the positive resonance. Use the first integrating factor $(X - X_0)^{2m+3}$. Then the differential equation (3.30) can be written,

$$\frac{d^{2m-2}}{dX^{2m-2}}\left\{(X - X_0)^{2m+3}\left[Y_1'' - \frac{2(m-1)(2m+3)}{X - X_0}Y_1' - \left(\frac{6(m+1)(2m-1)}{(X-X_0)^2} + \frac{A_{m,1}C_1}{D_m(X-X_0)^3}\right)Y_1\right]\right\} = 0, \quad (3.32)$$

where $A_{m,1} = 2m(m+1)(2m-1)(2m+1)$ in accordance with (3.24). The general solution is

$$Y_1(X) = \sum_{j=1}^{2m-2} K_j(X - X_0)^{-j-2} + \tilde{Y}(X), \quad (3.33)$$

where

$$\tilde{Y}(X) = (X - X_0)^{2m^2+m-(5/2)} \times \left\{K_{2m-1}\, J_{4m^2+2m+1}(z) + K_{2m}\, Y_{4m^2+2m+1}(z)\right\}, \quad (3.34)$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions of the first and second kinds, order $\nu = 4m^2+2m+1$, whose argument is

$$z = \sqrt{-4A_{m,1}C_1/D_m}\,(X - X_0)^{-1/2}. \quad (3.35)$$

The K_j, $1 \le j \le 2m$, are integration constants. The presence of the Bessel function of the second kind indicates that $Y_1(X)$ has a movable branched essential singularity, just as in the case of the original Bureau barrier equation.

The CFP test uses the α-expansion (3.10) with $Y_0(X)$ being the ascending power series solution. Let the movable singularity at $X = X_0$ be translated to the origin. Then

$$Y_0(X) = B_m X^{-2} + CX^{4m^2+2m-2} + a_2C^2X^{8m^2+4m-2} + \ldots, \quad (3.36)$$

where a_2 is given by (3.29), and

$$Y_1(X) = \sum_{j=-(2m-1)}^{\infty} b_j X^{j-2}, \quad (3.37)$$

the coefficient b_j being arbitrary whenever j coincides with one of the resonance numbers. In particular, the leading coefficient $b_{-(2m-1)}$ is arbitrary. The later terms,

$Y_2(X)$, $Y_3(X)$, etc., in the α-expansion each satisfy an inhomogeneous version of the linear equation (3.30). Each can be shown to be expressible by a pure Laurent series until the term $Y_{2m+3}(X)$ is reached. We conjecture that the series for $Y_{2m+3}(X)$ needs to be augmented by logarithms at the position corresponding to every resonance except the lowest two negative resonances.

4. Solutions of F-III and Fif-I

In this and the next three sections, we present complete solutions to all of the fourth- and fifth-order equations derived in Section 2 that can be solved in terms of hyperelliptic functions of genus 2. Most of these solutions are new. We also present auxiliary variables that allow a neater presentation of the first integrals and which play an important role in the postmultiplier theory.

In the case of those equations that we believe define higher-order Painlevé transcendents, some new and some previously known, we present first integrals and special solutions in terms of classical Painlevé transcendents, if any are known.

In this section, we will construct the general solution of equation F-III,

$$y^{(\mathrm{iv})} = 15yy'' + \tfrac{45}{4}(y')^2 - 15y^3 + \alpha y + \beta, \tag{4.1}$$

in terms of hyperelliptic functions using postmultiplier theory and present a first integral and a four-parameter solution of equation Fif-I,

$$y^{(\mathrm{v})} = 15yy''' + \tfrac{75}{2}y'y'' - 45y^2y' + (\lambda x + \alpha)y' + 2\lambda y, \tag{4.2}$$

in terms of Painlevé-I functions. We believe that all of the solutions in this section are new.

Equation Fif-I defines the coefficient function in Chazy Class XI when $N = 7$. It is also obtained by group-invariant reduction of the Kaup-Kuperschmidt equation (2.76). We believe that Fif-I defines a new higher-order transcendent (of the fourth order because Fif-I admits a first integral) when $\lambda \neq 0$. This transcendent is related to the corresponding transcendent defined by Fif-II by the transformations (5.19)–(5.22) below. When $\lambda = 0$, Fif-I admits F-III as a first integral. We have not seen F-III appearing in any previous attempt on the Painlevé classification of fourth-order equations (see especially [12]), and so it is new in that context.

As already mentioned at the end of Section 1, if an autonomous differential equation of order n has a known postmultiplier and $n-2$ autonomous first integrals, then it can be integrated by quadratures. If the penultimate derivative $y^{(n-1)}$ is absent from the equation, then the equation admits the unit postmultiplier $I = 1$. These conditions are met by all autonomous members of F-III–F-VI and Fif-I–Fif-IV. Hence we know at once that all such equations are integrable by quadratures. All of the effort in the application of the method occurs in the presentation of these solutions in terms of known special functions, namely, elliptic and hyperelliptic functions (or abelian functions in related higher-order problems).

The first step in the solution of F-III is to find first integrals. After a little experimentation with simple ansätzen, we found the following two integrals of the third order and second degree:

$$\begin{aligned}
\{y''' - 12yy'\}^2 - 3y(y'')^2 + \{\tfrac{3}{2}(y')^2 + 30y^3\}y'' \\
- 9y^2(y')^2 - 72y^5 - \alpha\{2yy'' - (y')^2 - 8y^3\} \\
- 2\beta\{y'' - 6y^2\} - \tfrac{4}{3}\alpha\beta \;&=\; K_1\,, \qquad (4.3)\\
y(y''')^2 - \{y'' + 18y^2\}y'y''' + \tfrac{1}{3}(y'')^3 - 6y^2(y'')^2 \\
+ \{\tfrac{27}{2}y(y')^2 + 30y^4\}y'' - \tfrac{9}{16}(y')^4 + \tfrac{135}{2}y^3(y')^2 - 45y^6 \\
- \alpha\{\tfrac{2}{3}y'y''' - \tfrac{1}{3}(y'')^2 + 2y^2y'' - \tfrac{15}{2}y(y')^2 - 2y^4\} \\
- \beta\{2yy'' - \tfrac{3}{2}(y')^2 - 6y^3\} + \tfrac{1}{3}\alpha^2y^2 \\
+ \tfrac{2}{3}\alpha\beta y - \tfrac{4}{81}\alpha^3 - \beta^2 \;&=\; K_2\,. \qquad (4.4)
\end{aligned}$$

K_1 and K_2 are the constants of integration. Eliminating y''' yields a differential equation for y of the second order and sixth degree. Readers who do not wish to study the method of solution should note the auxiliary variables in the next paragraph and then skip forward to the paragraph containing equation (4.26).

The second step is to search for combinations of y and its derivatives that simplify the presentation of these integrals as well as provide convenient variables for a first-order system equivalent to F-III. This step is, of course, subjective. We chose the following auxiliary variables:

$$H \;:=\; y'' - 6y^2 + \tfrac{2}{3}\alpha, \qquad (4.5)$$

$$J \;:=\; yy'' - \tfrac{1}{2}(y')^2 - 4y^3 + \tfrac{2}{3}\beta, \qquad (4.6)$$

$$H_1 \;:=\; y''' - 12yy'. \qquad (4.7)$$

These three variables, together with y, satisfy the system of first-order equations,

$$H' \;=\; H_1, \qquad (4.8)$$

$$J' \;=\; yH_1, \qquad (4.9)$$

$$H_1' \;=\; \tfrac{3}{2}(yH + J), \tag{4.10}$$

$$y' \;=\; \left\{ 2(yH - J) + 4y^3 - \tfrac{4}{3}(\alpha y - \beta) \right\}^{1/2}. \tag{4.11}$$

(All square roots appearing take both signs.) According to the change of variable rule (1.27), a postmultiplier for this system is

$$I_1(H, J, H_1, y) \;=\; \left\{ 2(yH - J) + 4y^3 - \tfrac{4}{3}(\alpha y - \beta) \right\}^{-1/2}, \tag{4.12}$$

which is easy to verify directly. In terms of these auxiliary variables, the above first integrals simplify to

$$(H_1)^2 - 3HJ \;=\; K_1\,, \tag{4.13}$$

$$y(H_1)^2 - y'HH_1 - \tfrac{9}{4}J^2 + \tfrac{3}{2}yHJ + \tfrac{1}{3}H^3 + (\tfrac{3}{4}y^2 - \tfrac{1}{3}\alpha)H^2 \;=\; K_2\,, \tag{4.14}$$

where y' is given by (4.11). Eliminating H_1 yields a polynomial equation of degree 6 in H and of degree 4 in each of y and J.

The essence of the postmultiplier method is to choose the constants K_1 and K_2 as new variables and choose two other suitable variables that will eventually appear in an exact differential of the form (1.28). We also want the integral of this exact differential to take the same form as the hyperelliptic integrals in (1.19), or else something simpler involving elliptic integrals. In particular, algebraic equations that we meet must be solvable with square roots at worst, and the final variables must be separable so that only a single variable appears under each square root in the exact differential.

With the present auxiliary variables, it is not possible to eliminate two variables in favor of K_1 and K_2 in (4.13) and (4.14) without solving an algebraic equation of degree 4 or higher. In fact, in the original variables, it is possible to eliminate y' and y''' from (4.3) and (4.4) by solving a quartic equation, so we do not appear to be better off in the new variables. However, a major simplification occurs if we replace J by

$$J_1 \;:=\; \tfrac{1}{3}(J + yH). \tag{4.15}$$

Then elimination of H_1 from (4.13) and (4.14) yields a quadratic equation for y in terms of H and J_1, so we keep the latter two variables.

Let the aforementioned quadratic equation for y be written,

$$Ey^2 + Fy + G \;=\; 0, \tag{4.16}$$

where

$$E \;=\; -4K_2H^2 + 2K_1HJ_1 + (K_1)^2, \tag{4.17}$$

and F and G are simple polynomial expressions in the variables H and J_1 which we do not need to write out in full.

In general, the formula for y in terms of H and J_1 involves a square root and H_1 involves two nested square roots. Now it is well known that, if p and q are rational numbers or rational functions of one or more variables, then a nested square root $\sqrt{p+\sqrt{q}}$ can be separated into the form $\sqrt{r}+\sqrt{s}$ whenever p^2-q is a square. When we apply this criterion to H_1, we find that p^2-q is a square divided by E. The problem now proceeds along different lines according as $K_1 \neq 0$ or $K_1 = 0$. We will show the details for the former case and just give results for the easier latter case.

When $K_1 \neq 0$, E is not a square in the current variables, but we can rename variables to make it a square. Eliminate J_1 in favor of a new variable L, where

$$(K_1)^2L^2 \;:=\; E \;=\; -4K_2H^2+2K_1HJ_1+(K_1)^2. \tag{4.18}$$

Now H_1 is expressible as a sum of square roots and y and y' are also expressible in terms of the same square roots. The expressions obtained so far are

$$H_1 \;=\; \frac{\sqrt{D_1}\;-\;\sqrt{D_2}}{8\sqrt{3}\,(K_1)^{5/2}L}\,, \tag{4.19}$$

$$y \;=\; \frac{F_1-\sqrt{D_1}\,\sqrt{D_2}}{96(K_1)^5H^2L^2}\,, \tag{4.20}$$

where

$$\begin{aligned} D_1 :=\; & 3\Big\{(K_1)^2(1-L)^2+4K_2H^2\Big\}^3+32(K_1)^4H^5(1-L) \\ & -16\alpha(K_1)^2H^4\Big\{(K_1)^2(1-L)^2+4K_2H^2\Big\} \\ & +64\beta(K_1)^3H^6, \end{aligned} \tag{4.21}$$

$$\begin{aligned} D_2 :=\; & 3\Big\{(K_1)^2(1+L)^2+4K_2H^2\Big\}^3+32(K_1)^4H^5(1+L) \\ & -16\alpha(K_1)^2H^4\Big\{(K_1)^2(1+L)^2+4K_2H^2\Big\} \\ & +64\beta(K_1)^3H^6, \end{aligned} \tag{4.22}$$

$$\begin{aligned} F_1 :=\; & -48(K_1)^3H^2F \\ =\; & 3(K_1)^6(1-L^2)^2(1+L^2)+12(K_1)^4K_2H^2(3L^4+2L^2+3) \\ & +144(K_1)^2(K_2)^2H^4(1+L^2)+32(K_1)^4H^5 \\ & +192(K_2)^3H^6-16\alpha(K_1)^4H^4(1+L^2) \\ & -64\alpha(K_1)^2K_2H^6+64\beta(K_1)^3H^6. \end{aligned} \tag{4.23}$$

In these variables, the system of first-order equations consists of two trivial equations $K_1' = 0$ and $K_2' = 0$, the equation $H' = H_1$, the right-hand side being given by (4.19), and

$$L' \;=\; \frac{(1+L)\sqrt{D_1} + (1-L)\sqrt{D_2}}{8\sqrt{3}\,(K_1)^{5/2} H L}\,. \tag{4.24}$$

According to (1.17), a postmultiplier for this system may be calculated, the result being

$$I_2(H, L, K_1, K_2) \;=\; \frac{H^2 L}{\sqrt{D_1}\sqrt{D_2}}\,, \tag{4.25}$$

It is now a straightforward matter to construct an exact differential and complete the integration by quadratures.

The last step is separation of variables. For the case $K_1 \neq 0$, rename the constants of integration according to

$$K_1 \;=\; 9B^2, \qquad\qquad K_2 \;=\; 9AB^2. \tag{4.26}$$

Next, choose new variables u and v so that

$$H \;=\; \frac{3B}{u+v}\,, \qquad L \;=\; -\frac{u-v}{u+v}\,. \tag{4.27}$$

Define the sextic (6th-degree) polynomial,

$$P(t) \;:=\; (t^2+A)^3 - \tfrac{1}{3}\alpha(t^2+A) + Bt + \tfrac{1}{3}\beta. \tag{4.28}$$

We find that

$$D_1 \;=\; \frac{192(3B)^{12}}{(u+v)^6}\,P(u), \qquad D_2 \;=\; \frac{192(3B)^{12}}{(u+v)^6}\,P(v), \tag{4.29}$$

and that u and v satisfy the differential equations,

$$u' \;=\; \frac{\sqrt{P(u)}}{u-v}\,, \qquad v' \;=\; -\frac{\sqrt{P(v)}}{u-v}\,, \tag{4.30}$$

with obvious postmultiplier $I_3 = (u-v)/\sqrt{P(u)P(v)}$. These equations can be solved by inversion of the hyperelliptic integrals,

$$\int_\infty^{u(x)} \frac{dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{dt}{\sqrt{P(t)}} \;=\; K_3\,, \tag{4.31}$$

$$\int_\infty^{u(x)} \frac{t\,dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{t\,dt}{\sqrt{P(t)}} \;=\; x + K_4\,, \tag{4.32}$$

where K_3 and K_4 are the third and fourth integration constants. By taking a suitable Möbius transformation in t, it is possible to replace the sextic polynomial $P(t)$ with a quintic (5th-degree) polynomial, which would be more standard, but the presentation of the solution would be much less elegant.

In terms of u and v, the solution of equation F-III for the case $K_1 \neq 0$ is

$$y = \frac{(u' + v')^2}{(u + v)^2} - u^2 - v^2 - 2A. \tag{4.33}$$

This solution is globally meromorphic. The auxiliary variables H, J, and H_1 are given, respectively, by (4.27) above,

$$J = \frac{B(y - 2uv + 2A)}{u + v}, \tag{4.34}$$

and $H_1 = H'$. It is not difficult to verify directly that y satisfies F-III and that these auxiliary variables satisfy the first-order system (4.8)–(4.11) and the integrals (4.13) and (4.14).

Now, it appears that we have written up the solution in such a way that the postmultipliers were never actually used. This is possible with the advantage of hindsight. Postmultipliers were very important in the initial discovery of the solution, especially in determining the broad shape of the solution, and played an important role in the choice of auxiliary variables and the separation of variables. Only after the method of solution was understood was it possible to simplify the solution so that postmultipliers no longer seemed essential. We remark that Kowalevski's analysis of the motion of a spinning top about a fixed point [28, 29, 30] is perhaps the most celebrated application of postmultipliers to a system of differential equations, but even there the explicit values of the postmultipliers were not used. It was enough to know four first integrals explicitly and the fact that an elementary postmultiplier existed.

There are a number of cases where the hyperelliptic functions degenerate to elliptic or simpler functions. The most obvious cases are when the sextic polynomial $P(t)$ has only even powers or when it has a double root. Degeneracy also occurs when the sextic polynomial can be made even after applying a Möbius transformation in t. In these cases, the hyperelliptic functions are expressible, in general, in terms of two elliptic functions with different periods. When the degeneracy is due to a double root, the hyperelliptic function is expressible in terms of a single elliptic function and a quadrature. There are other cases for which we do not have a quick explanation, the solution (6.15)–(6.18) below of F-V being one such example. Note, however, that the reduced F-III equation, for which $\alpha = \beta = 0$, is not one of these degenerate cases. Its solution involves true hyperelliptic functions.

We will present only two degenerate cases, namely, $K_1 = 0$ and $K_1 = K_2 = 0$. These cases are missed by the above solution and can be recovered from it only by taking nontrivial limits. In fact, the above solution collapses to a single Weierstrass function when $K_1 = 0$ on account of elliptic addition theorems. The best way to get these limits is to start again from first principles and follow the above procedure. The solution of F-III when $K_1 = 0$ and $K_2 \neq 0$, including the auxiliary variables, is

$$y = \frac{(u'-v')^2}{(u-v)^2} - 4(u+v), \tag{4.35}$$

$$H = \tfrac{1}{2}K_{21}(u-v)^{-1}, \tag{4.36}$$

$$J = \tfrac{1}{6}K_{21}\,\frac{y+4u+4v}{u-v}, \tag{4.37}$$

and $H_1 = H'$, where $K_{21} := \sqrt{-K_2}$ and $u(x)$ and $v(x)$ are the Weierstrass elliptic functions defined by the differential equations,

$$(u')^2 = 4u^3 - \tfrac{1}{12}\alpha u + \tfrac{1}{48}\left(\beta + K_{21}\right), \tag{4.38}$$

$$(v')^2 = 4v^3 - \tfrac{1}{12}\alpha v + \tfrac{1}{48}\left(\beta - K_{21}\right), \tag{4.39}$$

Thus the solution involves two elliptic functions with different periods. The remaining two constants of integration are the phases of these elliptic functions.

The solution (4.35) again collapses to a single Weierstrass function when $K_2 = 0$, and hence a more careful limit is needed. When $K_1 = K_2 = 0$, the solution of F-III is

$$y = \frac{(qu'u''+1)^2}{q^2(u')^4} - 8u, \tag{4.40}$$

$$H = \frac{24}{qu'}, \tag{4.41}$$

$$J = \tfrac{1}{72}qu'(H')^2, \tag{4.42}$$

where $u(x)$ is the elliptic function defined by

$$(u')^2 = 4u^3 - \tfrac{1}{12}\alpha u + \tfrac{1}{48}\beta, \tag{4.43}$$

and $q(x)$ is given by

$$q = \int \left(4u^3 - \tfrac{1}{12}\alpha u + \tfrac{1}{48}\beta\right)^{-1} dx + K_3\,. \tag{4.44}$$

The latter integral can be evaluated in terms of the Weierstrass zeta function.

In the fully degenerate case where $\alpha = \beta = K_1 = K_2 = 0$, the solution is rational:

$$y \;=\; \frac{8(x-x_0)^{14} + 40C(x-x_0)^7 + C^2}{(x-x_0)^2\{(x-x_0)^7 - C\}^2}\,. \tag{4.45}$$

Next, we turn to the associated fifth-order equation Fif-I, given by (4.2). When $\lambda = 0$, this equation is easily integrated to give F-III, and so the general solution involves hyperelliptic functions in this case. When λ is unrestricted, Fif-I admits the first integral,

$$2HH'' - (H')^2 - 3yH^2 + K_1 \;=\; 0, \tag{4.46}$$

where H is the auxiliary variable,

$$H \;:=\; y'' - 6y^2 + \tfrac{2}{3}(\lambda x + \alpha). \tag{4.47}$$

(When $\lambda = 0$, the quantities H and K_1 can be identified with the corresponding quantities associated with F-III.)

A four-parameter solution of Fif-I having $K_1 = \lambda^2$ is

$$y \;=\; \frac{(u'-v')^2}{(u-v)^2} \;-\; 4(u+v), \tag{4.48}$$

$$H \;=\; 12\frac{(u')^2 - (v')^2}{u-v} \;-\; 48(u^2+uv+v^2) + \lambda x + \alpha, \tag{4.49}$$

where $u(x)$ and $v(x)$ are independent Painlevé-I functions satisfying the differential equation,

$$w'' \;=\; 6w^2 - \tfrac{1}{24}(\lambda x + \alpha). \tag{4.50}$$

This solution is clearly a generalization of the degenerate solution (4.35) of F-III. An additional two-parameter solution involving Painlevé-I functions is given by setting $K_1 = 0$ and $H = 0$.

We will not attempt to prove that the generic solution of equation Fif-I is a new fourth-order Painlevé transcendent. Such a proof would require the full force of differential Galois theory. However, the evidence in favor of that hypothesis is the following. Equation Fif-I admits a four-parameter solution depending on Painlevé-I functions. In the limit that yields the reduced equation $\lambda = \alpha = 0$, the general solution involves hyperelliptic functions. A naive application of the Boutroux ansatz to determine large-x asymptotics of the solution of Fif-I when $\lambda \neq 0$ also leads to hyperelliptic functions. We do not believe that a function with all of these reductions can be expressible in terms of classical transcendents.

5. Solutions of F-IV and Fif-II

In this section, we will write down the general solution of equation F-IV,

$$y^{(\mathrm{iv})} = 30yy'' - 60y^3 + \alpha y + \beta, \tag{5.1}$$

in terms of hyperelliptic functions and present a first integral and special solutions of equation Fif-II,

$$y^{(\mathrm{v})} = 30yy''' + 30y'y'' - 180y^2y' + (\lambda x + \alpha)y' + 2\lambda y, \tag{5.2}$$

in terms of Painlevé-I functions. We will also present a Bäcklund transformation connecting Fif-I and Fif-II (which induces a transformation between F-III and F-IV).

Equation F-IV was known to Chazy [4, p. 380], who also knew the degenerate solution (5.12) below. It also appears on Martynov's list [12]. Exton [11, p. 395] gave an incorrect proof that this equation was unstable. Equation Fif-II can be obtained by group-invariant reduction of the Sawada-Kotera equation (2.77). When $\lambda \neq 0$, equation Fif-II defines a fourth-order Painlevé transcendent which is related in a simple way to the new transcendent defined by Fif-I. When $\lambda = 0$, Fif-II admits F-IV as a first integral. We believe that all solutions in this section, except for the aforementioned solution of Chazy, are new.

For F-IV, suitable auxiliary variables are

$$H := y'' - 3y^2 + \tfrac{1}{12}\alpha, \tag{5.3}$$

$$J := yy'' - \tfrac{1}{2}(y')^2 - 2y^3 + \tfrac{1}{12}\beta, \tag{5.4}$$

and $H_1 := H'$. The variables H and J satisfy

$$J' = yH', \qquad H'' = 12(yH + J). \tag{5.5}$$

Equation F-IV possesses two first integrals of the third order and second degree. In terms of the auxiliary variables, these integrals are

$$(H')^2 - 24HJ = K_1, \tag{5.6}$$

$$\begin{aligned} y(H')^2 - y'H'(H + 9y^2 - \tfrac{1}{4}\alpha) + 12yHJ - 36J^2 \\ + 3J(12y^3 - \alpha y + \beta) + \tfrac{1}{3}H(H + 9y^2 - \tfrac{1}{4}\alpha)^2 = K_2. \end{aligned} \tag{5.7}$$

Elimination of y''' yields a differential equation for y of the second order and sixth degree.

When $K_1 \neq 0$, rename the constants of integration according to

$$K_1 = \tfrac{9}{16}B^2, \qquad K_2 = -\tfrac{9}{64}AB^2. \tag{5.8}$$

The general solution of F-IV when $K_1 \neq 0$ is given by

$$y = \tfrac{1}{2}(u' + v' + u^2 + uv + v^2 + A), \tag{5.9}$$

$$H = \tfrac{3}{2}(u+v)\left\{uu' + vv' + (u+v)(u^2 + v^2 + \tfrac{3}{2}A)\right\}, \tag{5.10}$$

$$J = \tfrac{1}{48}(4H' - 3B)\left(\frac{u' + v'}{u+v} + u + v\right), \tag{5.11}$$

where $u(x)$ and $v(x)$ are defined by inversion of the hyperelliptic integrals (4.31) and (4.32) above and the sixth-degree polynomial $P(t)$ appearing therein is given by (4.28) above. This solution is globally meromorphic. The solution of the reduced equation $\alpha = \beta = 0$ also requires true hyperelliptic functions. There are several cases of degeneracy to simpler functions.

When $K_1 = 0$, we obtain Chazy's three-parameter solution,

$$y = u + v, \qquad H = 3(u-v)^2, \qquad J = \tfrac{1}{2}(u' - v')^2, \tag{5.12}$$

where $u(x)$ and $v(x)$ are elliptic functions defined by the differential equations,

$$(u')^2 = 4u^3 - \tfrac{1}{12}\alpha u + \tfrac{1}{24}\beta + \tfrac{1}{6}\sqrt{K_2}\,, \tag{5.13}$$

$$(v')^2 = 4v^3 - \tfrac{1}{12}\alpha v + \tfrac{1}{24}\beta - \tfrac{1}{6}\sqrt{K_2}\,. \tag{5.14}$$

This solution is a simple superposition of elliptic functions with different periods and arbitrary phases. It includes the case $K_1 = K_2 = 0$, where the elliptic functions have the same periods but independent phases. When $\alpha = \beta = 0$, the latter solution degenerates to a rational solution.

The associated fifth-order equation Fif-II admits the first integral,

$$2HH'' - (H')^2 - 24yH^2 + K_1 = 0, \tag{5.15}$$

where H is the auxiliary variable,

$$H := y'' - 3y^2 + \tfrac{1}{12}(\lambda x + \alpha). \tag{5.16}$$

When $\lambda = 0$, the quantities H and K_1 can be identified with the corresponding quantities associated with F-IV, which is an obvious first integral in this case.

A four-parameter solution of Fif-II is

$$y = u + v, \qquad H = 3(u-v)^2, \tag{5.17}$$

where $u(x)$ and $v(x)$ are independent Painlevé-I functions satisfying the differential equation (4.50) above. Substitution into the first integral identifies $K_1 = 0$. This simple superposition of Painlevé-I functions is an obvious generalization of Chazy's solution (5.12).

The fourth-order Painlevé transcendent defined by Fif-II when $\lambda \neq 0$ is not distinct from the transcendent defined by Fif-I. Let $y(x)$ satisfy Fif-II as above with $K_1 = (9/16)B^2$ and let $\bar{y}(x)$ satisfy Fif-I with parameters,

$$\bar{\lambda} = \lambda, \qquad \bar{\alpha} = \alpha, \qquad \bar{K}_1 = 9\bar{B}^2 = (3B+\lambda)^2, \tag{5.18}$$

and auxiliary variable $\bar{H}(x)$. Then the Fif-I transcendent is given in terms of the Fif-II transcendent by

$$\bar{y} = \frac{(4H'+3B)^2}{64H^2} - 4y, \tag{5.19}$$

$$\bar{H} = \frac{3y'(4H'+3B)}{2H} - 4H - 36y^2 + \lambda x + \alpha. \tag{5.20}$$

The inverse relations are

$$y = \tfrac{1}{4}\bar{H}^{-2}(\bar{H}' - 3B - \lambda)^2 - \tfrac{1}{4}\bar{y}, \tag{5.21}$$

$$\begin{aligned} H = {} & \tfrac{3}{16}\bar{H}^{-4}(\bar{H}' - 3B - \lambda)^4 - \tfrac{9}{8}\bar{y}\bar{H}^{-2}(\bar{H}' - 3B - \lambda)^2 \\ & + \tfrac{3}{4}\bar{y}'\bar{H}^{-1}(\bar{H}' - 3B - \lambda) - \tfrac{1}{4}\bar{H} - \tfrac{9}{16}\bar{y}^2 + \tfrac{1}{4}(\lambda x + \alpha). \end{aligned} \tag{5.22}$$

When $\lambda = 0$, the additional parameters in F-III and F-IV are related according to

$$\bar{\beta} = 2\beta, \qquad \bar{K}_2 = -64K_2, \qquad \bar{A} = A. \tag{5.23}$$

These transformations are induced by the Bäcklund transformation relating the Kaup-Kuperschmidt equation (2.76) to the Sawada-Kotera equation (2.77) [42, 49], given by the Miura transformations (2.80) and (2.81) (supplemented by easily obtained expressions for w_t). Similarly, the transcendent defined by equation F-XVIII in the Appendix is obtained by group-invariant reduction of the modified Sawada-Kotera equation (2.79) and is simply related to the transcendents defined by Fif-I and Fif-II. The direct transformation from the Kaup-Kuperschmidt equation to the Sawada-Kotera equation requires the solution of a Riccati equation in either direction. In contrast, the above transformations between the transcendents $y(x)$ and $\bar{y}(x)$ are explicit in both directions.

6. Solutions of F-V and Fif-III

In this section, we will write down the general solution of equation F-V,

$$y^{(\mathrm{iv})} = 20yy'' + 10(y')^2 - 40y^3 + \alpha y + \kappa x + \beta, \tag{6.1}$$

in terms of hyperelliptic functions when $\kappa = 0$ and present a first integral and a special solution of equation Fif-III,

$$y^{(\mathrm{v})} = 20yy''' + 40y'y'' - 120y^2y' + (\lambda x + \alpha)y' + 2\lambda y + \kappa, \tag{6.2}$$

in terms of Painlevé-I functions. When $\kappa \neq 0$, we do not know any exact solutions of F-V. It is possible that F-V with $\kappa \neq 0$ has no elementary solutions expressible in terms of classical transcendents or Painlevé transcendents, as has been shown to hold under certain hypotheses for the Painlevé-I equation [65, 66, 67, 68]. The special case $\kappa = \beta = 0$ of F-V is needed for the solution of the Chazy Class IX equation [25].

A second fifth-order equation revealed by the Painlevé analysis, namely, equation Fif-III.b given by (2.72), admits F-V as a first integral and so does not require separate study. When $\lambda = 0$, Fif-III also admits F-V as a first integral. Fif-III can be solved in terms of hyperelliptic functions when $\kappa = \lambda = 0$.

Equation F-V is certainly the best known of the equations in this paper after F-I. The autonomous case $\kappa = 0$ and its associated linear problem were recognized to be part of a hierarchy solvable in terms of hyperelliptic functions of every genus by Drach [14]. The same hierarchy was recognized to be a reduction of the KdV hierarchy by Lax [48], who was not apparently aware of their solvability in terms of hyperelliptic functions at the time. The transcendental case $\kappa \neq 0$ and the modified version F-XVII in the Appendix appear on Martynov's list [12]. The modified version was recognized to be a member of a hierarchy, sometimes called the Painlevé-II hierarchy, that could be solved by inverse scattering techniques by Ablowitz and Segur [38] and by isomonodromy techniques by Flashka and Newell [40] and Kitaev [35, 36]. The hierarchy containing F-V, obtained by reduction of the KdV hierarchy, is sometimes called the Painlevé-I hierarchy. The two hierarchies are related by a Miura transformation together with a limiting contraction. Some authors use the notation ${}_4P_1$ for F-V and ${}_4P_2$ for F-XVII and similarly for other members of the corresponding hierarchies. (Further discussion of Painlevé hierarchies may be found in [37, 39].)

When $\kappa = 0$, F-V admits the easy first integral,

$$y'y''' = \tfrac{1}{2}(y'')^2 + 10y(y')^2 - 10y^4 + \tfrac{1}{2}\alpha y^2 + \beta y + \gamma, \tag{6.3}$$

where γ is a constant of integration. Introduce the auxiliary variables,

$$H := y'' - 6y^2 + \tfrac{1}{4}\alpha, \tag{6.4}$$

$$J := yy'' - \tfrac{1}{2}(y')^2 - 4y^3 + \tfrac{1}{4}\beta. \tag{6.5}$$

These variables satisfy

$$J' = yH', \qquad H'' = 4(yH + J). \tag{6.6}$$

In terms of these variables, the above first integral becomes

$$y'H' - 4yJ - \tfrac{1}{2}H^2 - (2y^2 - \tfrac{1}{4}\alpha)H = K_1\,, \tag{6.7}$$

where $K_1 = \gamma + (1/32)\alpha^2$. Equation F-V also admits the second-degree integral,

$$(H')^2 - 8HJ = K_2\,. \tag{6.8}$$

Both these integrals were known to Lax [48].

Elimination of y''' produces a differential equation of the second order and fourth degree. Without making use of auxiliary variables, this latter integral can be solved for y' using square roots only and so the solution of F-V by postmultiplier theory is fairly simple and direct. In fact, it is too simple to be a useful guide to the other equations of hyperelliptic type appearing in this paper.

The case $\kappa = 0$ of F-V can be solved in terms of hyperelliptic functions of genus 2 as follows. Let $P(t)$ denote the quintic polynomial,

$$P(t) := t^5 - 2\alpha t^3 + 8\beta t^2 + 32K_1 t + 16K_2\,. \tag{6.9}$$

Let $u(x)$ and $v(x)$ be defined by inversion of the hyperelliptic integrals,

$$\int_\infty^{u(x)} \frac{dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{dt}{\sqrt{P(t)}} = K_3\,, \tag{6.10}$$

$$\int_\infty^{u(x)} \frac{t\,dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{t\,dt}{\sqrt{P(t)}} = x + K_4\,. \tag{6.11}$$

Then

$$y = \tfrac{1}{4}(u + v), \tag{6.12}$$

$$H = -\tfrac{1}{4}uv, \tag{6.13}$$

$$J = -\tfrac{1}{32}(u' + v')^2 + \tfrac{1}{32}(u + v)(u^2 + v^2 - 2\alpha) + \tfrac{1}{4}\beta. \tag{6.14}$$

This solution is globally meromorphic. It is the same hyperelliptic function that Kowalevski encountered in her analysis of the motion of a spinning top about a fixed point [28, 29, 30]. The reduced form of F-V, for which $\alpha = \beta = \kappa = 0$, also requires true hyperelliptic functions for its solution.

There are several cases of degeneracy to simpler functions. If $P(t)$ has a double root, then F-V can be solved by elliptic functions and quadratures. This solution is not doubly periodic in general. Another case of degeneracy occurs when $P(t)$ is divisible by t and $P(t)/t$ is a palindromic quartic, or can be put in this form by a linear transformation in t. Here, the solution involves two elliptic functions with different periods. The case where $P(t)$ is an odd polynomial is included in this category. The simple ansatz $(y')^2 = Q(y)$, where $Q(y)$ is a cubic polynomial, yields pure elliptic solutions.

As mentioned in Section 4, there are cases of degeneracy that we find less easy to explain on the basis of symmetries of $P(t)$. Consider the case,

$$K_1 = \tfrac{3}{98}\alpha^2, \qquad K_2 = -\tfrac{4}{7}\alpha\beta, \tag{6.15}$$

where the quintic factorizes as follows:

$$P(t) = \left(t^2 - \tfrac{8}{7}\alpha\right)\left(t^3 - \tfrac{6}{7}\alpha t + 8\beta\right). \tag{6.16}$$

In general, this is not one of the aforementioned cases of degeneracy. It is perhaps surprising that the hyperelliptic integrals (6.10) and (6.11) can be inverted to yield a doubly periodic solution. Let $w(x)$ be the Weierstrass elliptic function,

$$w = \wp(x - x_0) = \wp(x - x_0, g_2, g_3),$$

where x_0 is arbitrary, $g_2 = (1/42)\alpha$, and $g_3 = (1/54)\beta$. Then w satisfies

$$(w')^2 = 4w^3 - \tfrac{1}{42}\alpha w - \tfrac{1}{54}\beta. \tag{6.17}$$

Let c be arbitrary and let d be chosen such that $\wp'(d) = -\wp'(c)$. This implies

$$\alpha = 168\left\{\wp^2(c) + \wp(c)\wp(d) + \wp^2(d)\right\}.$$

Then the corresponding two-parameter solution of F-V is

$$y = \wp(x - x_0) + \wp(x + c - x_0) + \wp(x + d - x_0). \tag{6.18}$$

This is an elliptic function of order 6 having three double poles in each period parallelogram. A number of other doubly periodic solutions can be constructed by postulating a sum of Weierstrass functions.

When $\alpha = \beta = K_1 = K_2 = 0$, we obtain the following rational solution:

$$y = \frac{3(x - x_0)\{(x - x_0)^3 + 2C\}}{\{(x - x_0)^3 - C\}^2}. \tag{6.19}$$

When $\lambda \neq 0$, the corresponding fifth-order equation Fif-III admits the first integral,

$$2\tilde{H}\tilde{H}'' - (\tilde{H}')^2 - (8y + 4\kappa\lambda^{-1})\tilde{H}^2 + \tilde{K}_2 \;=\; 0, \tag{6.20}$$

where $\tilde{H}$ is the auxiliary variable,

$$\tilde{H} \;:=\; y'' - 6y^2 + 4(\kappa/\lambda)y + \tfrac{1}{4}(\lambda x + \alpha) - 4(\kappa/\lambda)^2. \tag{6.21}$$

When $\tilde{K}_2 = 0$, an obvious particular solution of Fif-III can be obtained by setting $\tilde{H} = 0$. This solution is

$$y \;=\; w + \tfrac{1}{3}(\kappa/\lambda), \tag{6.22}$$

where $w(x)$ satisfies the Painlevé-I equation,

$$w'' \;=\; 6w^2 - \tfrac{1}{4}(\lambda x + \alpha) + \tfrac{10}{3}(\kappa/\lambda)^2. \tag{6.23}$$

This solution was known to Airault [39] in its disguise as a particular similarity reduction of KdV5. We did not succeed in finding a solution involving two independent Painlevé transcendents like (4.48) and (5.17) above or (7.54) below.

It is possible to take the limit $\lambda \to 0$, $\kappa \to 0$, in such a way that $\delta := \kappa/\lambda$ remains an arbitrary constant. In that case, the first integral (6.20) contains three separate integrals, namely, F-V, (6.7), and (6.8). We obtain the identification,

$$\tilde{H} \;=\; H + 4\delta y - 4\delta^2, \tag{6.24}$$

$$\tilde{K}_2 \;=\; K_2 + 8\delta(K_1 + 8\delta^4 - \alpha\delta^2 + \beta\delta). \tag{6.25}$$

One can also artificially introduce the parameter δ to J and K_1 as follows:

$$\tilde{J} \;:=\; J + \tfrac{1}{2}\delta(y'' - 2y^2 + 4\delta y + 4\delta^2 - \tfrac{1}{4}\alpha), \tag{6.26}$$

$$\tilde{K}_1 \;:=\; K_1 + \delta(40\delta^3 - 3\alpha\delta + 2\beta). \tag{6.27}$$

Then the following identities analogous to (6.6)–(6.8) hold:

$$\tilde{J}' \;=\; (y + \tfrac{1}{2}\delta)\tilde{H}', \tag{6.28}$$

$$\tilde{H}'' \;=\; 2(2y + \delta)\tilde{H} + 4\tilde{J}, \tag{6.29}$$

$$\begin{aligned}\tilde{K}_1 \;=&\; y'\tilde{H}' - 4(y - 2\delta)\tilde{J} - \tfrac{1}{2}\tilde{H}^2 \\ &- 2(y + \tfrac{1}{2}\delta)^2\tilde{H} + \tfrac{1}{4}(\alpha - 30\delta^2)\tilde{H},\end{aligned} \tag{6.30}$$

$$\tilde{K}_2 \;=\; (\tilde{H}')^2 - 8\tilde{H}\tilde{J}. \tag{6.31}$$

These formulas are important in the study of equation F-XVII, because δ is an essential parameter in that equation whether or not λ is zero.

7. Solutions of F-VI and Fif-IV

In this section, we will write down the general solution of our new equation F-VI,

$$y^{(\mathrm{iv})} = 18yy'' + 9(y')^2 - 24y^3 + \alpha y^2 + \tfrac{1}{9}\alpha^2 y + \kappa x + \beta, \tag{7.1}$$

in terms of hyperelliptic functions when $\kappa = 0$ and present a first integral of the fourth order and third degree and several special solutions of equation Fif-IV,

$$\begin{aligned} y^{(\mathrm{v})} = {} & 18yy''' + 36y'y'' - 72y^2y' + 3\lambda y'' \\ & + (1/x)\left\{ y^{(\mathrm{iv})} - 18yy'' - 9(y')^2 + 24y^3 - 3\lambda y' + \kappa \right\} \\ & + \tfrac{1}{2}\lambda x(5y''' - 36yy') - \tfrac{1}{2}\lambda^2 x(2xy' + y), \end{aligned} \tag{7.2}$$

which is also new, in terms of Painlevé-II functions. The reduced version of Fif-IV given by (2.44) above was known to Ito [50, (2.15)] and Harada and Oishi [13]. In contrast to the other equations of hyperelliptic type discussed in this paper, the reduced form of F-VI with $\alpha = \beta = \kappa = 0$ can be solved in terms of a pair of elliptic functions. The special case $\kappa = 0$ and $\beta = -(13/7776)\alpha^3$ of F-VI is needed for the solution of the Chazy Class X equation [25].

When $\kappa \neq 0$, equation F-VI defines a new fourth-order Painlevé transcendent. We do not know any exact solutions of F-VI when $\kappa \neq 0$. We conjecture that F-VI with $\kappa \neq 0$ has no elementary exact solutions whatsoever expressible in terms of classical transcendents or Painlevé transcendents, under hypotheses similar those for the absence of elementary solutions of the Painlevé-I equation [65, 66, 67, 68]. Equation Fif-IV defines a further generalization of that transcendent, but still of the fourth order because Fif-IV admits a first integral, albeit complicated. We know four exact solutions of Fif-IV in terms of Painlevé-II functions.

The Boutroux ansatz, $y = x^{1/3}\bar{y}$, $\bar{x} = (6/7)x^{7/6}$, shows that, at least on a formal basis, equation F-VI with $\kappa \neq 0$ has hyperelliptic asymptotics in Stokes sectors of angle $2\pi/7$ with trigonometric asymptotics along the seven Stokes rays forming the sector boundaries. When κ is real and positive (which can always be arranged by scaling variables), then the Stokes rays are $\arg x = \pm\pi/7$, $\pm 3\pi/7$, $\pm 5\pi/7$, and π.

We have not found an integrable soliton equation that admits F-VI or Fif-IV as a group-invariant reduction. Ito's equation (2.84) does not have the Painlevé property. Ito deduced equation (2.84) (without the u_t term) from the functional derivative of one of the conserved quantities of the Sawada-Kotera equation. Similarly, we do not know an associated linear problem for these equations or how to construct their hierarchies. Also, the first integral of Fif-IV is a differential equation of the fourth order and third degree and we have not found a change of variable that transforms this equation to the first degree in the fourth derivative. We believe that these open questions are all related and the answering of one will lead to answers to all of them. Until then, these attractive equations stand apart from the familiar infrastructure surrounding other transcendents and soliton equations.

Two further fifth-order equations revealed by the Painlevé analysis, namely, equations Fif-IV.b and Fif-IV.c given by (2.74) and (2.75), each admit F-VI as a first integral and so do not require separate study. When $\lambda = 0$, Fif-IV admits the case $\alpha = 0$ of F-VI as a first integral. The case with α unrestricted can be obtained by first renaming $x = \bar{x} + (\alpha/\lambda)$, $y = \bar{y} - (5/36)\alpha$, and $\kappa = (\alpha/\lambda)\bar{\kappa}$ before taking the limit $\lambda \to 0$. The reduction of Fif-IV to the autonomous version of F-VI is discussed below.

When $\kappa = 0$, F-VI admits the easy first integral,

$$\begin{aligned} y'y''' \;=\; & \tfrac{1}{2}(y'')^2 + 9y(y')^2 - 6y^4 \\ & + \tfrac{1}{3}\alpha y^3 + \tfrac{1}{18}\alpha^2 y^2 + \beta y + \gamma, \end{aligned} \tag{7.3}$$

where γ is a constant of integration. Introduce the auxiliary variables,

$$H := y'' - 6y^2 - \tfrac{1}{3}\alpha y + \tfrac{1}{27}\alpha^2, \tag{7.4}$$

$$J := (y - \tfrac{1}{18}\alpha)y'' - \tfrac{1}{2}(y')^2 - 4y^3 + \tfrac{1}{6}(\alpha y^2 + \tfrac{1}{9}\alpha^2 y + \beta). \tag{7.5}$$

These variables satisfy

$$J' = (y - \tfrac{1}{18}\alpha)H', \qquad H'' = 6J. \tag{7.6}$$

In terms of these variables, the above first integral becomes

$$y'H' - \tfrac{1}{2}H^2 - 6(y + \tfrac{1}{9}\alpha)J = K_1\,, \tag{7.7}$$

where $K_1 = \gamma - (1/9)\alpha\beta - (1/1458)\alpha^4$. Equation F-VI also admits the second-degree integral,

$$(y - \tfrac{5}{36}\alpha)(H')^2 - y'HH' - 9J^2 + \tfrac{1}{3}H^3 + 6(y + \tfrac{1}{9}\alpha)HJ = K_2. \tag{7.8}$$

Elimination of y''' produces a differential equation of the second order and fourth degree.

Equation F-VI with $\kappa = 0$ can be solved in terms of hyperelliptic functions of genus 2. The derivation of these results was carried out by a method that is not too different from the method shown above for F-III, but the algebraic expressions encountered are substantially longer. We did the calculation using the computer algebra package REDUCE. Here, we present just the results of that calculation. It is straightforward to verify these results by direct substitution with the aid of a computer. (We found REDUCE's "evenp" command very useful for handling square roots.)

First, rename the coefficient parameters α and β and the constants of integration K_1 (or γ) and K_2 as follows:

$$\alpha = \mu^2 A, \tag{7.9}$$

$$\beta = \mu^6 \left(\tfrac{1}{4}AF + \tfrac{3}{16}E^2 - \tfrac{8}{243}A^3 \right), \tag{7.10}$$

$$K_1 = \tfrac{1}{32}\mu^8 (3CE^2 - 4F^2), \tag{7.11}$$

$$K_2 = \tfrac{1}{192}\mu^{12} F(9CE^2 - 8F^2). \tag{7.12}$$

This renaming is valid provided $E \neq 0$, which we assume for the present. The case $E = 0$ is degenerate and is discussed separately below. The parameter μ is an inessential scaling parameter, and we could set $\mu = 1$ in the following. It is included because it will allow a simplification of the solution of Chazy-X given in the accompanying paper [25].

Define the octic (8th-degree) polynomials,

$$\begin{aligned} P(s,t) &:= (s^2 - C)^2(t^2 - C)^2 - 2A(s^2 + t^2)(s^2t^2 + C^2) \\ &\qquad + 4Est(s+t)(st + C) + 8(AC + 2F)s^2t^2, \end{aligned} \tag{7.13}$$

$$\begin{aligned} P(t) &:= P(t,t) \\ &= (t^2 - C)^4 - 4At^2(t^2 - C)^2 + 8Et^3(t^2 + C) + 16Ft^4. \end{aligned} \tag{7.14}$$

Define functions $u(x)$ and $v(x)$ by inversion of the hyperelliptic integrals,

$$\int_\infty^{u(x)} \frac{t\,dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{t\,dt}{\sqrt{P(t)}} = \tfrac{1}{4}K_3\,, \tag{7.15}$$

$$\int_\infty^{u(x)} \frac{(t^2 + C)\,dt}{\sqrt{P(t)}} + \int_\infty^{v(x)} \frac{(t^2 + C)\,dt}{\sqrt{P(t)}} = \tfrac{1}{2}\mu(x + K_4). \tag{7.16}$$

Note that these hyperelliptic integrals have genus 2 despite the polynomial $P(t)$ being octic. There is a quadratic transformation that transforms these integrals to the standard form with a sextic polynomial under the square root (see below), but calculations are easier in the present variables. The octic version was suggested by the postmultiplier theory.

The solution of F-VI with $\kappa = 0$ and $E \neq 0$ is

$$y = \tfrac{1}{8}\mu^2 \, \frac{P(u,v) + \sqrt{P(u)}\,\sqrt{P(v)}}{(u+v)^2(uv - C)^2} + \tfrac{5}{36}\alpha. \tag{7.17}$$

The auxiliary variables are

$$H \;=\; -\tfrac{3}{4}\mu^4 E\frac{uv+C}{u+v} \;-\; \tfrac{1}{2}\mu^4 F, \tag{7.18}$$

$$J \;=\; -\tfrac{1}{4}\mu^4 E(y+\tfrac{1}{9}\alpha)\,\frac{uv+C}{u+v} \;+\; \tfrac{1}{8}\mu^6 E^2\frac{uv}{(u+v)^2}\,. \tag{7.19}$$

A short calculation shows that the movable algebraic branch points in $u(x)$ and $v(x)$ cancel out and therefore y is globally meromorphic.

To get a more standard presentation of the hyperelliptic functions appearing in the above solution, apply the quadratic transformation,

$$\bar{u} \;=\; \frac{u^2+C}{2u}\,, \qquad \sqrt{\bar{u}^2-C} \;=\; \frac{u^2-C}{2u}\,, \tag{7.20}$$

and similarly for the variables v and $\bar{v}$. Then $\bar{u}(x)$ and $\bar{v}(x)$ are defined by inversion of the hyperelliptic integrals,

$$\int_\infty^{\bar{u}(x)} \frac{dt}{\sqrt{\bar{P}(t)}} \;+\; \int_\infty^{\bar{v}(x)} \frac{dt}{\sqrt{\bar{P}(t)}} \;=\; K_3\,, \tag{7.21}$$

$$\int_\infty^{\bar{u}(x)} \frac{t\,dt}{\sqrt{\bar{P}(t)}} \;+\; \int_\infty^{\bar{v}(x)} \frac{t\,dt}{\sqrt{\bar{P}(t)}} \;=\; \mu(x+K_4)\,, \tag{7.22}$$

where $\bar{P}(t)$ is the sextic polynomial,

$$\bar{P}(t) \;=\; (t^2-C)Q(t), \tag{7.23}$$

where

$$Q(t) \;:=\; (t^2-C)^2 - A(t^2-C) + Et + F. \tag{7.24}$$

Define also

$$Q(s,t) \;:=\; (s^2-C)(t^2-C) - \tfrac{1}{2}A(s^2+t^2-2C) + \tfrac{1}{2}E(s+t) + F. \tag{7.25}$$

In these variables, the solution of F-VI with $\kappa=0$ and $E\neq 0$ is

$$y \;=\; \tfrac{1}{2}\mu^2\,\frac{Q(\bar{u},\bar{v})+\sqrt{Q(\bar{u})}\,\sqrt{Q(\bar{v})}}{\{\sqrt{\bar{u}^2-C}+\sqrt{\bar{v}^2-C}\,\}^2} \;+\; \tfrac{5}{36}\alpha, \tag{7.26}$$

$$H \;=\; -\tfrac{3}{4}\mu^4 E\,\frac{\bar{u}\bar{v}+C+\sqrt{\bar{u}^2-C}\,\sqrt{\bar{v}^2-C}}{\bar{u}+\bar{v}} \;-\; \tfrac{1}{2}\mu^4 F, \tag{7.27}$$

$$J \;=\; \tfrac{1}{3}(H+\tfrac{1}{2}\mu^4 F)\Big\{y+\tfrac{1}{9}\alpha-\tfrac{1}{4}\mu^2 E(\bar{u}+\bar{v})^{-1}\Big\}\,. \tag{7.28}$$

These formulas can be verified directly, but it is easier to transform back to the previous version with the octic $P(t)$ before doing calculations with this solution.

Two degenerate cases having $E = 0$ are missed by the above solution. In fact, when $E = 0$, the above solution collapses to a single Weierstrass elliptic function. The identity,

$$\begin{aligned}(\beta + \tfrac{8}{243}\alpha^3)^3 + \tfrac{3}{2}\alpha^2(\beta + \tfrac{8}{243}\alpha^3)K_1 - \tfrac{3}{4}\alpha^3 K_2 \\ = \tfrac{27}{4096}\,\mu^{18}E^4(4A^2C + 4AF + E^2),\end{aligned} \tag{7.29}$$

shows that the vanishing of the left-hand side is a necessary condition for degeneracy. An essentially equivalent condition for degeneracy is

$$\beta \;=\; \tfrac{1}{4}\mu^4\alpha F - \tfrac{8}{243}\alpha^3. \tag{7.30}$$

The cases $\alpha \neq 0$ and $\alpha = 0$ need to be treated separately.

When $\alpha \neq 0$, the degeneracy condition gives a linear constraint between the constants of integration K_1 and K_2. Write these constants in terms of a single constant K according to

$$K_1 \;=\; \mu^8(24K^2 - \tfrac{1}{8}F^2), \qquad K_2 \;=\; \mu^{12}F(12K^2 - \tfrac{1}{24}F^2). \tag{7.31}$$

Also define the constants A and F in terms of α and β by (7.9) and (7.30). The solution of F-VI in the degenerate case with $\alpha \neq 0$ is

$$y \;=\; \frac{(u'-v')^2}{(u-v)^2} - \mu^2(u+v)^2 + \tfrac{5}{36}\alpha, \tag{7.32}$$

$$H \;=\; 12\mu^4 K\,\frac{u+v}{u-v} \;-\; \tfrac{1}{2}\mu^4 F, \tag{7.33}$$

$$J \;=\; 4\mu^4 K(y + \tfrac{1}{9}\alpha)\,\frac{u+v}{u-v} \;-\; \frac{8\mu^6K^2}{(u-v)^2}\,, \tag{7.34}$$

where $u(x)$ and $v(x)$ are elliptic functions defined by

$$(u')^2 \;=\; \mu^2\left\{u^4 - \tfrac{1}{4}Au^2 + \tfrac{1}{16}F + K\right\}, \tag{7.35}$$

$$(v')^2 \;=\; \mu^2\left\{v^4 - \tfrac{1}{4}Av^2 + \tfrac{1}{16}F - K\right\}. \tag{7.36}$$

The remaining two integration constants are the phases of these elliptic functions. The case $K = 0$ can be constructed by setting $v = u - Kr$ before taking the limit as $K \to 0$, the function $r(x)$ being expressible in terms of $u(x)$ and a quadrature.

When $\alpha = 0$, the degeneracy condition forces $\beta = 0$ but leaves all of the integration constants unrestricted. This case is the reduced equation. These results will be used in [25] to solve the reduced form of Chazy-X. The general solution of the reduced form of F-VI is

$$y = \frac{(u'-v')^2}{(u-v)^2} - \mu^2(u+v)^2, \tag{7.37}$$

$$H = 2\mu^4 \frac{C(u+5v) - D(5u+v)}{u-v}, \tag{7.38}$$

$$J = 2\mu^4 y \frac{(C-D)(u+v)}{u-v} - 2\mu^6 \frac{(C-D)^2}{(u-v)^2}, \tag{7.39}$$

where $u(x)$ and $v(x)$ are elliptic functions defined by

$$(u')^2 = \mu^2(u^4 + C), \qquad (v')^2 = \mu^2(v^4 + D), \tag{7.40}$$

and C and D are integration constants. The phases of the elliptic functions are also constants of integration. The constants K_1 and K_2 are given by

$$K_1 = -2\mu^8(C^2 + 14CD + D^2), \tag{7.41}$$

$$K_2 = \tfrac{8}{3}\mu^{12}(C+D)(C^2 - 34CD + D^2). \tag{7.42}$$

The limiting case $D = C$ is given by

$$y = (r'/r)^2 - 4\mu^2u^2, \qquad H = 8\mu^4 C(6u - r)/r, \tag{7.43}$$

where $u(x)$ is defined by (7.40) and

$$r(x) = (x - K_3)u' + u, \tag{7.44}$$

K_3 being a constant of integration. The fully degenerate solution with $\alpha = \beta = K_1 = K_2 = 0$ is the following rational solution:

$$y = \frac{5(x-x_0)^3\{(x-x_0)^5 + 4C\}}{\{(x-x_0)^5 - C\}^2}. \tag{7.45}$$

The corresponding fifth-order equation Fif-IV (with $\lambda \neq 0$) admits a first integral of a different character to the integrals (4.46), (5.15), and (6.20) above. First, introduce the auxiliary variables,

$$H := y'' - 6y^2 - 2\lambda xy - \tfrac{1}{8}\lambda^2x^2, \tag{7.46}$$

$$\begin{aligned} J := {} & (y + \tfrac{1}{12}\lambda x)H - \tfrac{1}{2}(y')^2 + 2y^3 - \tfrac{1}{6}\lambda y' \\ & + \lambda xy^2 + \tfrac{1}{8}\lambda^2x^2y - \tfrac{1}{24}\lambda^2 - \tfrac{1}{6}\kappa, \end{aligned} \tag{7.47}$$

$$L := H'' - 6J. \tag{7.48}$$

These variables satisfy

$$J' = (y + \tfrac{1}{12}\lambda x)H' - \tfrac{1}{12}\lambda(H + \lambda xy + \tfrac{1}{4}\lambda^2 x^2), \tag{7.49}$$

$$L' = (1/x)L + \tfrac{1}{2}\lambda^2 x(y + \tfrac{1}{4}\lambda x), \tag{7.50}$$

the latter being a reformulation of Fif-IV. The first integral of Fif-IV is

$$\begin{aligned} L^3 - \tfrac{3}{2}\lambda^2 x^2 L \Big\{ (y + \tfrac{1}{4}\lambda x)H'' - y'H' + \tfrac{1}{2}H^2 + \tfrac{1}{4}\lambda^2 y \Big\} \\ + \tfrac{3}{16}\lambda^3 x^3 \Big\{ (H'')^2 - 4(y + \tfrac{1}{4}\lambda x)HH'' - 4y(H')^2 \\ + 4y'HH' - \tfrac{4}{3}H^3 \Big\} + \tfrac{3}{8}\lambda^4 x^3 \Big\{ y'H'' - 4y(y + \tfrac{1}{4}\lambda x)H' \\ + \lambda y H - \tfrac{1}{4}\lambda(y')^2 + \lambda y(y + \tfrac{1}{4}\lambda x)^2 \Big\} = -\tfrac{3}{4}K_2 x^3. \end{aligned} \tag{7.51}$$

The fourth derivative $y^{(\mathrm{iv})}$ is the leading term in L. It follows that (7.51) is a differential equation for y of the fourth order and third degree. We have here a new fourth-order Painlevé transcendent defined by a differential equation not of the first degree. We expect that a change of variable will be found that reduces this equation to the fourth order and first degree.

The autonomous version of F-VI can be recovered from Fif-IV by making the change of variables,

$$x = \bar{x} + (\alpha/\lambda), \qquad y = \bar{y} - \tfrac{5}{36}\alpha, \tag{7.52}$$

and parameters,

$$\kappa = -(\beta + \tfrac{8}{243}\alpha^3), \qquad K_2 = \lambda^3 \bar{K}_2\,, \tag{7.53}$$

taking the limit $\lambda \to 0$, and carrying out an easy integration. Also, with κ so chosen, the auxiliary variables H and J reduce to the corresponding variables for F-VI (with bars on x and y), and the variable L vanishes in the limit. As already mentioned, the choice $\kappa = (\alpha/\lambda)\bar{\kappa}$ together with (7.52) yields the full nonautonomous version of F-VI in the limit $\lambda \to 0$. In the latter case, the third-degree integral (7.51) collapses to F-VI itself.

Equation Fif-IV admits a four-parameter solution involving two Painlevé-II functions when $\kappa = -(3/8)\lambda^2$. This solution is

$$y = \frac{(u' - v')^2}{(u - v)^2} - (u + v)^2, \tag{7.54}$$

$$\begin{aligned} H = (u - v)^{-1} \Big\{ 2(u + 5v)(u')^2 - 2(5u + v)(v')^2 - \tfrac{1}{2}\lambda(u' - v') \Big\} \\ - 2(u^4 + 6u^3 v + 6u^2 v^2 + 6uv^3 + v^4) \\ + \tfrac{1}{2}\lambda x(u^2 + 6uv + v^2), \end{aligned} \tag{7.55}$$

$$L = 2\lambda x \left\{ (u')^2 + (v')^2 - u^4 - v^4 \right\} \\ - \tfrac{1}{2}\lambda^2 x \left\{ \frac{u' - v'}{u - v} - x(u^2 + v^2) \right\}, \tag{7.56}$$

where $u(x)$ and $v(x)$ are Painlevé-II functions that satisfy the differential equation,

$$w'' = 2w^3 - \tfrac{1}{4}\lambda x w. \tag{7.57}$$

The expression for J is a little longer but can be calculated without difficulty from (7.47). Substitution into the third-degree integral identifies $K_2 = -(1/256)\lambda^7$.

We also found three solutions of Fif-IV in terms of single Painlevé-II functions. First, when

$$\kappa = -\tfrac{3}{64}(5\lambda^2 + 8\lambda\nu + 16\nu^2), \qquad K_2 = \tfrac{1}{256}\lambda^5(16\kappa + 5\lambda^2),$$

we have the solution

$$y = \tfrac{1}{2}(w' + w^2) - \tfrac{1}{8}\lambda x, \tag{7.58}$$

$$H = -\tfrac{1}{2}\left\{ (w')^2 - w^4 \right\} + \tfrac{1}{4}\lambda x w^2 + \tfrac{1}{4}(\lambda + 4\nu)w + \tfrac{1}{32}\lambda^2 x^2, \tag{7.59}$$

$$L = \lambda x H, \tag{7.60}$$

where $w(x)$ satisfies

$$w'' = 2w^3 + \tfrac{1}{2}\lambda x w + \nu. \tag{7.61}$$

Secondly, when

$$\kappa = -\tfrac{3}{16}(\lambda - 2\nu)^2, \qquad K_2 = -\tfrac{1}{64}\lambda^5(8\kappa + \lambda^2),$$

we have the solution,

$$y = \tfrac{1}{2}(w' + w^2) - \tfrac{1}{4}\lambda x, \tag{7.62}$$

$$H = -\tfrac{1}{2}\left\{ (w')^2 - w^4 + \lambda x w^2 + (\lambda - 2\nu)w \right\}, \tag{7.63}$$

$$L = -\tfrac{1}{2}\lambda x H, \tag{7.64}$$

where $w(x)$ satisfies

$$w'' = 2w^3 - \lambda x w + \nu. \tag{7.65}$$

Equations (7.60) and (7.64) are both fourth-order equations for y having the same reduced part as F-VI. However, they are not equivalent to F-VI and they violate the freedom-from-logarithms condition at the $n = 8$ resonance. Thirdly, when

$$\kappa = -6\nu^2, \qquad K_2 = -\tfrac{1}{64}\lambda^3(4\kappa + \lambda^2)^2,$$

we have the solution,

$$y = w^2, \tag{7.66}$$

$$H = 2(w')^2 - 2w^4 - \lambda x w^2 + 2\nu w - \tfrac{1}{8}\lambda^2 x^2, \tag{7.67}$$

$$L = -\tfrac{1}{2}\lambda x(H - 6\nu w), \tag{7.68}$$

where $w(x)$ satisfies

$$w'' = 2w^3 + \tfrac{1}{2}\lambda x w + \nu. \tag{7.69}$$

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Appendix. List of fourth-order Painlevé-type equations in the polynomial class

We present here the results of the Painlevé classification of all fourth-order differential equations in the polynomial class. We also include a few brief remarks on their solutions and the relationship between F-XVII and F-XVIII and some of the higher-order transcendents appearing in the body of this paper. The proof of uniqueness, as well as full details of their solutions, will appear in the sequel (paper II).

The Painlevé classification carried out in Section 2 and the analysis of the results in Sections 3–7 above apply to the case of Bureau symbol $P2$. We found that there were six fourth-order equations of this type, denoted F-I, F-II, …, F-VI, including the (non-Painlevé) Bureau barrier equation, and four fifth-order equations, denoted Fif-I, Fif-II, Fif-III, and Fif-IV, some of which split into canonical subcases. They appear above in the two lists, (2.63)–(2.68) and (2.69)–(2.75). Some define higher-order Painlevé transcendents which are known or can be deduced by group-invariant reduction of the three KdV5-like soliton equations (2.76)–(2.78). Special cases of these, as well as the other equations (excluding F-II), can be integrated in terms of hyperelliptic functions, Painlevé transcendents, or simpler functions. The equations F-VI and Fif-IV are new and define new higher-order transcendents.

The case of Bureau symbol $P1$ is a more difficult Painlevé classification problem. Here, the class of equations under investigation is

$$\begin{aligned} y^{(\mathrm{iv})} &= Ayy''' + By'y'' + Cy^2y'' + Dy(y')^2 + Ey^3y' + Fy^5 \\ &\qquad + G(x)y''' + H(x)yy'' + I(x)(y')^2 + J(x)y^2y' \\ &\qquad + K(x)y^4 + L(x)y'' + M(x)yy' + N(x)y^3 \\ &\qquad + O(x)y' + P(x)y^2 + Q(x)y + R(x). \end{aligned} \tag{A.1}$$

Elementary Painlevé analysis, together with a choice of gauge, shows that the coefficients $A, \ldots, F$ can be taken as constant. According to our analysis, there are 12 equations of Painlevé type having Bureau symbol $P1$, which we denote F-VII, F-VIII, ..., F-XVIII. Only 11 of the corresponding reduced equations are distinct because F-X and F-XI are built upon the same reduced equation.

The case $F \neq 0$ presents the greatest difficulties. As Chazy pointed out [4, p. 379], the problem can be made to depend on the Diophantine equation,

$$\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} + \frac{1}{N_4} = \frac{1}{24}. \tag{A.2}$$

Each N_i is a product of three resonance numbers, the four sets of resonance numbers corresponding to the four values of a_0 that occur in a Laurent series expansion beginning with the term $a_0/(x-x_0)$. Equation (A.2) has a large number of solutions, including infinite families. Nevertheless, with modern computers, it is a straightforward matter to search through the finite families for admissible cases. The infinite family with $N_2 = -N_1$ is relatively easy to analyse. The case $N_1 = 24$ is, however, intractable by traditional methods because there is no apparent way to put bounds on the other N_i using conditions for integer resonances. We managed to solve the so-called intractable case by examining the 11 admissible factorizations $n_1n_2n_3 = 24$ for the possibility of logarithms. This calculation used the Conte-Fordy-Pickering test [26] and shows that this test has an important role to play in certain Painlevé classification problems.

Earlier work on the classification of fourth-order equations has been carried out by Chazy [4, pp. 378–385], Bureau [9, 10, pp. 74–116], Exton [11], and Martynov [12]. Chazy found F-I and F-VII and observed that these were just derivatives of Chazy-XIII and Chazy-I, respectively. He also found F-IV and its three-parameter solution (5.12). Bureau found and studied in detail equations F-I, F-VII, F-VIII, and F-IX. These are his Classes I–IV, respectively. He also discovered F-II, but did not pursue it further.

Martynov [12] claims to have a complete list of reduced equations, but he missed equations F-III, F-VI, F-XV, and F-XVIII. Exton [11] studied those cases not previously examined by Bureau. He obtained F-XI (with incorrect restrictions), F-XII, F-XIII, and F-XVI, rejected the reduced equations defining F-III, F-IV, F-VI, and F-XIV (F-III and F-VI having incorrect coefficients), and missed F-V, F-X, F-XV, F-XVII, and F-XVIII. Exton had a fundamental error in his method of analysing equations for the Painlevé property. He also rejected the third-order equations Chazy-IX and Chazy-X as unstable. Since both authors were looking for reduced equations having integer resonances, which were then to be subjected to further tests, they should have also run across the equation,

$$y^{(\mathrm{iv})} = 2yy''' - 17y'y'' + 11y^2y'' - 15y(y')^2 + 4y^3y' - y^5. \tag{A.3}$$

This equation has admissible integer resonances but fails a condition for the absence of logarithms. At this stage, we do not know if this is an interesting equation worthy of further study.

The list of canonical representatives of the fourth-order equations in the polynomial class having the Painlevé property, plus the Bureau barrier equation, is as follows:

F-I: $$y^{(\mathrm{iv})} = 12yy'' + 12(y')^2 + (\alpha x + \beta)y' + 2\alpha y - \tfrac{1}{6}(\alpha x + \beta)^2; \tag{A.4}$$

F-II: $$y^{(\mathrm{iv})} = 3yy'' - 4(y')^2; \tag{A.5}$$

F-III: $$y^{(\mathrm{iv})} = 15yy'' + \tfrac{45}{4}(y')^2 - 15y^3 + \alpha y + \beta; \tag{A.6}$$

F-IV: $$y^{(\mathrm{iv})} = 30yy'' - 60y^3 + \alpha y + \beta; \tag{A.7}$$

F-V: $$y^{(\mathrm{iv})} = 20yy'' + 10(y')^2 - 40y^3 + \alpha y + \kappa x + \beta; \tag{A.8}$$

F-VI: $$y^{(\mathrm{iv})} = 18yy'' + 9(y')^2 - 24y^3 + \alpha y^2 + \tfrac{1}{9}\alpha^2 y + \kappa x + \beta; \tag{A.9}$$

F-VII: $$\begin{aligned} y^{(\mathrm{iv})} = {} & \frac{fg' - 3f'g}{fg}\,y''' - \frac{ff''g + (f')^2g - ff'g'}{f^2g}\,y'' \\ & - \frac{2}{f^2}\left\{\frac{d}{dx} - \frac{g'}{g}\right\}\Big\{3c_1x(xy'-y)^2 \\ & + c_2(xy'-y)(3xy'-y) + c_3y'(3xy'-2y) \\ & + 3c_4(y')^2 + 2c_5x(xy'-y) + c_6(2xy'-y) \\ & + 2c_7y' + c_8x + c_9\Big\}, \end{aligned} \tag{A.10}$$

where $f(x) := c_1x^3 + c_2x^2 + c_3x + c_4$ and $g(x) := d_1x + d_2$;

F-VIII:
$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & -\frac{d^3}{dx^3}\Big\{y^2 + P(x)y + Q(x)\Big\} \\ & + A(x)\frac{d^2}{dx^2}\Big\{y' + y^2 + P(x)y + Q(x)\Big\} \\ & + B(x)\frac{d}{dx}\Big\{y' + y^2 + P(x)y + Q(x)\Big\} \\ & + C(x)\Big\{y' + y^2 + P(x)y + Q(x)\Big\} + D(x); \end{aligned} \tag{A.11}$$

F-IX:
$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & -\frac{d^2}{dx^2}\Big\{3yy' + y^3 + P(x)(y' + y^2) + Q(x)y + R(x)\Big\} \\ & + A(x)\frac{d}{dx}\Big\{y'' + 3yy' + y^3 + P(x)(y' + y^2) \\ & + Q(x)y + R(x)\Big\} + B(x)\Big\{y'' + 3yy' + y^3 \\ & + P(x)(y' + y^2) + Q(x)y + R(x)\Big\} + C(x); \end{aligned} \tag{A.12}$$

F-X:
$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & \frac{d}{dx}R(x, y, y', y'') + \frac{1}{x}\Big\{y''' - R(x, y, y', y'')\Big\} \\ & + \lambda x\{y' + y^2 - g(x)\} - \tfrac{1}{96}(N^2 - 1)\lambda^2 x^3; \end{aligned} \tag{A.13}$$

F-XI:
$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & \frac{d}{dx}R(x, y, y', y'') \\ & + \lambda\{y' + y^2 - g(x)\} - \tfrac{1}{24}(N^2 - 1)\lambda^2 x, \end{aligned} \tag{A.14}$$

where

$$\begin{aligned} R(x, y, y', y'') :=\; & -2yy'' - 2(y')^2 + g''(x) + \tfrac{1}{4}(N^2 - 1)(\alpha x + \beta) \\ & + [24/(N^2-1)]\{y' + y^2 - g(x)\}^2; \end{aligned}$$

F-XII:
$$\begin{aligned} y^{(\mathrm{iv})} \;=\; & \frac{d}{dx}\Big\{2yy'' - 3(y')^2 + \tfrac{1}{8}(6y' - y^2)^2\Big\} \\ & + A(x)\Big\{y''' - 2yy'' + 3(y')^2 - \tfrac{1}{8}(6y' - y^2)^2\Big\} \\ & + B(x)y' + \{B'(x) - A(x)B(x)\}y + C(x); \end{aligned} \tag{A.15}$$

$$\text{F-XIII:} \qquad y^{(\mathrm{iv})} = \frac{d}{dx}\left\{2yy'' - 3(y')^2 + \tfrac{4}{27}(6y' - y^2)^2\right\}$$
$$+ A(x)\left\{y''' - 2yy'' + 3(y')^2 - \tfrac{4}{27}(6y' - y^2)^2\right\}$$
$$+ C(x); \tag{A.16}$$

$$\text{F-XIV:} \qquad y^{(\mathrm{iv})} = \frac{d}{dx}\left\{12(y')^2 + 72y^2y' + 54y^4\right\}; \tag{A.17}$$

$$\text{F-XV:} \qquad y^{(\mathrm{iv})} = -2yy''' - 11y'y'' - 2y^2y'' - 7y(y')^2 - y^3y'$$
$$+ 3R(x)(y'' + yy') + R'(x)(4y' + y^2)$$
$$+ R'''(x) - 2R(x)R'(x); \tag{A.18}$$

$$\text{F-XVI:} \qquad y^{(\mathrm{iv})} = -5yy''' - 10y'y'' - 10y^2y'' - 15y(y')^2 - 10y^3y' - y^5$$
$$+ A(x)\{y''' + 4yy'' + 3(y')^2 + 6y^2y' + y^4)$$
$$+ B(x)(y'' + 3yy' + y^3)$$
$$+ C(x)(y' + y^2) + D(x)y + E(x); \tag{A.19}$$

$$\text{F-XVII:} \qquad y^{(\mathrm{iv})} = 10y^2y'' + 10y(y')^2 - 6y^5$$
$$- 10\delta(y'' - 2y^3) + (\lambda x + \mu)y + \gamma; \tag{A.20}$$

$$\text{F-XVIII:} \qquad y^{(\mathrm{iv})} = -5y'y'' + 5y^2y'' + 5y(y')^2 - y^5 + (\lambda x + \alpha)y + \gamma. \tag{A.21}$$

We conclude with a few brief remarks about equations F-VII–F-XVIII; full details will appear in paper II. Equations F-VII, F-VIII, F-IX, F-XII, F-XIII, and F-XIV are, respectively, derivatives of Chazy equations I, II, IV, XII ($N = 2$), XII ($N = 3$), and IX, respectively. (The full list of Chazy equations is presented in [25, Appendix].) These fourth-order equations do not admit additional terms that would generalize these Chazy equations. Hence, in the generic case, F-VII is solvable with Painlevé-VI functions, F-XIV with hyperelliptic functions of genus 2 (see [25] for the solution of its integral Chazy-IX), and the other four equations are linearizable. (As already mentioned at the end of Section 2, equation F-I is the derivative of Chazy-XIII.) The derivatives of Chazy equations V, VII, VIII, and X do not have the Painlevé property because the resonance number 4 is repeated. The derivatives of Chazy equations III, VI, and XII ($N \neq 2,\ 3$) do not have the Painlevé property because the reduced versions of these Chazy equations cannot take an additive constant.

Equation F-VII can be presented in a number of alternative ways. First, it is invariant under the same 6-parameter group of point transformations as its integrals Chazy-I and SD-I [6, 25], and so various normalizations are possible. The obvious normalization $g(x)=1$ is legal but not recommended because it takes away the freedom to normalize $f(x)$. To take advantage of the explicit reductions to the six Painlevé transcendents in [6], one needs to be able to normalize $f(x)$ to $x(x-1)$ in the generic case when $f(x)$ has no multiple roots and to x or 1 is the remaining cases. Equation F-VII can also be expressed in a much more compact form using the same gauge as the alternative form [25, (A.4)] of Chazy-I, but again this is less well suited to the direct reduction to the six Painlevé transcendents.

Equations F-X and F-XI are generalizations of Chazy-XI involving, respectively, the fourth and second Painlevé transcendents when $\lambda \neq 0$ and reduce to derivatives of Chazy-XI when $\lambda = 0$. Like Chazy-XI, they are integrable for unrestricted values of the parameters and function $g(x)$, but have movable logarithms except in a small number of "interesting" cases ($N = 2,\ 3$, and 7 for F-X, $N = 2,\ 3,\ 5$, and 7 for F-XI) and cases of F-XI solvable with Lamé functions for every odd N.

Equation F-XV appears to be a new equation. It is solvable in terms of the Riccati equation,

$$y' \;=\; -\,y^2 + 3(w'/w)y + R(x) - 3w''/w, \tag{A.22}$$

where $w(x)$ satisfies the third-order linear equation,

$$w''' \;=\; R(x)w' + Kw. \tag{A.23}$$

Equation F-XVI, found by Martynov, is the fourth-order analogue of Ince-VI [18] and Chazy-IV and has an obvious extension to all orders of differentiation. It is linearizable by the substitution, $y = u'/u$.

Equations F-XVII and F-XVIII define fourth-order Painlevé transcendents when $\lambda \neq 0$ and can be solved in terms of hyperelliptic functions when $\lambda = 0$. Equation F-XVII was known to Martynov [12] and is a member of the so-called Painlevé-II hierarchy found by Ablowitz and Segur [38]. The explicit form of F-XVII was given by Flashka and Newell [40, (A.15b)] and the case with $\delta = 0$ was given by Airault [39]. Recent authors have been using the notation ${}_{2n}P_1$ and ${}_{2n}P_2$ for the members of the Painlevé-I and Painlevé-II hierarchies, respectively. According to this scheme, ${}_4P_1$ denotes F-V and ${}_4P_2$ denotes F-XVII.

Equation F-XVIII is a new equation in the context of Painlevé classification but would be known to anybody who has examined the group-invariant reductions of the modified Sawada-Kotera equation (2.79) above.

Equation F-XVII can be transformed into the fifth-order equation Fif-III, given by (6.2), or its first integral (6.20). The transformation is the reduction of the Miura transformation relating KdV5 to modified KdV5, with the difference that it is explicit in both directions. Let $y(x)$ satisfy F-XVII and $\bar{y}(x)$ satisfy Fif-III and (6.20), with auxiliary variable $\tilde{H}$ defined by (6.21). Then

$$y = -\frac{4\tilde{H}' + 2\gamma - \lambda}{8\tilde{H}}, \tag{A.24}$$

$$\tilde{H} = \bar{y}'' - 6\bar{y}^2 + 4\delta\bar{y} + \tfrac{1}{4}(\lambda x + \mu) + \tfrac{7}{2}\delta^2, \tag{A.25}$$

$$\bar{y} = -\tfrac{1}{2}(y' - y^2 + \delta), \tag{A.26}$$

where the parameters in Fif-III and F-XVII are related according to

$$\lambda = \lambda, \qquad \kappa = \lambda\delta, \qquad \alpha = \mu + 30\delta^2, \tag{A.27}$$

and the constant of integration in (6.20) is given by

$$\tilde{K}_2 = \tfrac{1}{16}(2\gamma - \lambda)^2. \tag{A.28}$$

The case $\lambda = 0$, $\kappa \neq 0$, of Fif-III, which admits F-V as a first integral, is excluded from this transformation, but, of course, it can be included by a simple limiting process. Equation (A.24) collapses to 0/0 when $\bar{y}(x)$ is the particular solution (6.22). In that case $y(x)$ is not explicitly determined in terms of a Painlevé-I function but is determined by solving the Riccati equation (A.26). It also solves the case $N = 3$ of Chazy Class XI [25].

When $\lambda = 0$, equation F-XVII can be solved with hyperelliptic functions. Note that δ remains an essential parameter. The auxiliary variables appearing in (6.24)–(6.31) are useful in the study of this case. Let $u(x)$ and $v(x)$ be defined by (6.9)–(6.11). The constants β and K_1 appearing in (6.9) are constants of integration for F-XVII. The constant α is given by (A.27) above and the constant K_2 is given by

$$K_2 = \tfrac{1}{4}\gamma^2 - 8\delta(K_1 + 8\delta^4 - \alpha\delta^2 + \beta\delta). \tag{A.29}$$

The general solution of F-XVII when $\lambda = 0$ is

$$y = -\frac{1}{2}\left\{\frac{u'}{u - 4\delta} + \frac{v'}{v - 4\delta}\right\} + \frac{K_1 - \frac{1}{32}(\mu + 30\delta^2)^2}{(u - 4\delta)(v - 4\delta)}. \tag{A.30}$$

Equation F-XVIII can be transformed to both of the fifth-order equations Fif-I and Fif-II, given by (4.2) and (5.2), respectively, or their first integrals (4.46) and (5.15). Suppose $y(x)$ satisfies F-XVIII while $\bar{y}(x)$ satisfies Fif-I and (4.46) (with a bar on K_1) and $\tilde{y}(x)$ satisfies Fif-II and (5.15) (with a tilde on K_1). The relationship between Fif-I and Fif-II is given in Section 5. The parameters λ and α appearing in F-XVIII can be identified with the corresponding parameters in Fif-I and Fif-II. The relationship between F-XVIII and Fif-I is given by

$$\bar{y} = -\tfrac{1}{3}(2y' - y^2), \qquad y = -\frac{3\bar{H}' + 2(\gamma - \lambda)}{3\bar{H}}, \tag{A.31}$$

where

$$\bar{H} = \bar{y}'' - 6\bar{y}^2 + \tfrac{2}{3}(\lambda x + \alpha), \tag{A.32}$$

$$\bar{K}_1 = \tfrac{4}{9}(\gamma - \lambda)^2. \tag{A.33}$$

The relationship between F-XVIII and Fif-II is given by

$$\tilde{y} = \tfrac{1}{6}(y' + y^2), \qquad y = \frac{12\tilde{H}' - 2\gamma - \lambda}{24\tilde{H}}, \tag{A.34}$$

where

$$\tilde{H} = \tilde{y}'' - 3\tilde{y}^2 + \tfrac{1}{12}(\lambda x + \alpha), \tag{A.35}$$

$$\tilde{K}_1 = \tfrac{1}{144}(2\gamma + \lambda)^2. \tag{A.36}$$

When $\gamma = -(1/2)\lambda$, equation F-XVIII can be solved in terms of two Painlevé-I functions (elliptic if $\lambda = 0$). The result is

$$y = \frac{u' - v'}{u - v}, \tag{A.37}$$

where $u(x)$ and $v(x)$ are any two solutions of the Painlevé-I equation,

$$u'' = 6u^2 - \tfrac{1}{24}(\lambda x + \alpha). \tag{A.38}$$

If just one of $u(x)$ and $v(x)$ is eliminated, y satisfies the Ince-X equation (found by Painlevé [69]),

$$y'' = -yy' + y^3 - 12u(x)y + 12u'(x), \tag{A.39}$$

and the same equation with $v(x)$ replacing $u(x)$.

When $\lambda = 0$ and $\gamma \neq 0$, F-XVIII can be solved in terms of hyperelliptic functions. Let $u(x)$ and $v(x)$ be defined by (4.28), (4.31), and (4.32). The constants A and β appearing in (4.28) are integration constants for F-XVIII, while $B = -(2/9)\gamma$. The general solution of F-XVIII in this case is

$$y = \frac{u' + v'}{u + v} + u + v. \tag{A.40}$$

References

[1] F. Bureau, Differential equations with fixed critical points, *Annali di Matematica* 64:229–364 (1964).

[2] C. M. Cosgrove, Painlevé classification of all semilinear partial differential equations of the second order I. Hyperbolic equations in two independent variables, *Stud. Appl. Math.* 89:1–61 (1993).

[3] C. M. Cosgrove, Painlevé classification of all semilinear partial differential equations of the second order II. Parabolic and higher-dimensional equations, *Stud. Appl. Math.* 89:95–151 (1993).

[4] J. Chazy, Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, *Acta Mathematica* 34:317–385 (1911).

[5] C. M. Cosgrove, Higher-order Painlevé equations in the polynomial class II. Bureau symbol $P1$, in preparation (paper II).

[6] C. M. Cosgrove and G. Scoufis, Painlevé classification of a class of differential equations of the second order and second degree, *Stud. Appl. Math.* 88:25–87 (1993).

[7] C. M. Cosgrove, All binomial-type Painlevé equations of the second order and degree three or higher, *Stud. Appl. Math.* 90:119–187 (1993).

[8] C. M. Cosgrove, Painlevé classification problems featuring essential singularities, *Stud. Appl. Math.* 98:355–433 (1997).

[9] F. Bureau, Sur les équations différentielles du quatrième ordre dont l'intégrale générale est à points critiques fixes, *Comptes Rendus de l'Académie des Sciences, Paris* 258:38–40 (1964).

[10] F. Bureau, Differential equations with fixed critical points, *Annali di Matematica* 66:1–116 (1964).

[11] H. Exton, On nonlinear ordinary differential equations with fixed critical points, *Rendiconti di Matematica* 4:385–448 (1971).

[12] I. P. Martynov, Differential equations with stationary critical singularities, *Differential Equations* 9:1368–1376 (1973) [*Differentsial'nye Uravneniya* 9:1780–1791 (1973)].

[13] H. Harada and S. Oishi, A new approach to completely integrable partial differential equations by means of the singularity analysis, *J. Phys. Soc. Japan* 54:51–56 (1985).

[14] J. Drach, Sur l'intégration par quadratures de l'équation $d^2y/dx^2 = [\phi(x) + h]y$, *Comptes Rendus de l'Académie des Sciences, Paris* 168:337–340 (1919) [Erratum: 168:532 (1919)] [reprinted in D. V. Chudnovsky and G. V. Chudnovsky, eds., *Classical and Quantum Models and Arithmetic Problems,* Lecture Notes in Pure and Applied Mathematics 92:455–458, Marcel Dekker, New York, 1984].

[15] P. Painlevé, Mémoire sur les équations différentielles dont l'intégrale générale est uniforme, *Bulletin de la Société Mathématique de France* 28:201–261 (1900).

[16] P. Painlevé, Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, *Acta Math.* 25:1–85 (1902).

[17] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes, *Acta Math.* 33:1–55 (1910).

[18] E. L. Ince, *Ordinary Differential Equations,* Dover Publications, New York, 1956 (originally published by Longmans and Green, London and New York, 1926).

[19] C. M. Cosgrove, *The Painlevé-Gambier Class of Differential Equations,* monograph in preparation.

[20] P. Painlevé, Démonstration de l'irréductibilité absolue de l'équation $y'' = 6y^2 + x$, *Comptes Rendus de l'Académie des Sciences, Paris* 135:641–647 (1902).

[21] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes, *Comptes Rendus de l'Académie des Sciences, Paris* 144:827–830 (1907).

[22] N. A. Lukashevich, Theory of the fourth Painlevé equation, *Differential Equations* 3:395–399 (1967) [*Differentsial'nye Uravneniya* 3:771–780 (1967)].

[23] A. R. Forsyth, *Theory of Differential Equations,* Vol. 6, Cambridge Univ. Press, Cambridge (UK), 1906 [reprinted by Dover, New York, 1959].

[24] R. Garnier, Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, *Annales Scientifiques de l'École Normale Supérieure, Paris, Sér. 3* 29:1–126 (1912).

[25] C. M. Cosgrove, Chazy Classes IX–XII of third-order differential equations, *Stud. Appl. Math.* 00:000–000 (1999).

[26] R. Conte, A. P. Fordy, and A. Pickering, A perturbative Painlevé approach to nonlinear differential equations, *Physica D* 69:33–58 (1993).

[27] F. Bureau, Équations différentielles du second ordre en Y et du second degré en $\ddot{Y}$ dont l'intégrale générale est à points critiques fixes, *Annali di Matematica* 91:163–281 (1972).

[28] V. V. Golubev, *Lectures on the Integration of the Equations of Motion of a Rigid Body About a Fixed Point,* GITTL, Moscow, 1953 [English translation publ. by NSF Israel Program for Scientific Translations, Washington, 1960].

[29] S. Kowalevski, Sur le problème de la rotation d'un corps solide autour d'un point fixe, *Acta Mathematica* 12:177–232 (1889).

[30] S. Kowalevski, Sur une propriété du système d'équations différentielles qui définit la rotation d'un corps solide autour d'un point fixe, *Acta Mathematica* 14:81–93 (1890).

[31] J. Chazy, Sur la limitation du degré des coëfficients des équations différentielles algébriques à points critiques fixes, *Acta Mathematica* 41:29–69 (1918).

[32] G. Valiron, Fonctions entières vérifiant une classe d'équations différentielles, *Bulletin de la Société Mathématique de France* 51:33–45 (1923).

[33] R. Littlewood, Private communications (1996–1998).

[34] M. J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of p-type I, *J. Math. Phys.* 21:715–721 (1980).

[35] A. V. Kitaev, Turning points of linear systems and double asymptotics of the Painlevé transcendents, in *Painlevé Transcendents, Their Asymptotics and Physical Applications*, eds. P. Winternitz and D. Levi, *NATO ASI Series B: Physics* 278:81–96, Plenum, New York, 1992.

[36] A. V. Kitaev, Caustics in $1+1$ integrable systems, *J. Math. Phys.* 35:2934–2954 (1994).

[37] N. A. Kudryashov, The first and second Painlevé equations of higher order and some relations between them, *Phys. Lett. A* 224:353–360 (1997).

[38] M. J. Ablowitz and H. Segur, Exact linearization of a Painlevé transcendent, *Phys. Rev. Lett.* 38:1103–1106 (1977).

[39] H. Airault, Rational solutions of Painlevé equations, *Stud. Appl. Math.* 61:31–53 (1979).

[40] H. Flaschka and A. C. Newell, Monodromy and spectrum preserving deformations I, *Commun. Math. Phys.* 76:65–116 (1980).

[41] D. J. Kaup, On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, *Stud. Appl. Math.* 62:189–216 (1980).

[42] A. P. Fordy and J. Gibbons, Some remarkable nonlinear transformations, *Phys. Lett. A* 75:325–325 (1980).

[43] J. D. Gibbon, A survey of the origins and physical importance of soliton equations, *Phil. Trans. Roy. Soc. London A* 315:335–365 (1985).

[44] K. Sawada and T. Kotera, A method for finding N-soliton solutions of the KdV equation and KdV-like equation, *Prog. Theor. Phys.* 51:1355–1367 (1974).

[45] P. J. Caudrey, R. K. Dodd, and J. D. Gibbon, A new hierarchy of Korteweg-de Vries equations, *Proc. Roy. Soc. London A* 351:407–422 (1976).

[46] R. K. Dodd and J. D. Gibbon, The prolongation structure of a higher order Korteweg-de Vries equation, *Proc. Roy. Soc. London A*, 358:287–296 (1977).

[47] J. Satsuma and D. J. Kaup, A Bäcklund transformation for a higher order Korteweg-de Vries equation, *J. Phys. Soc. Japan* 43:692–697 (1978).

[48] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Commun. Pure Appl. Math.* 21:467–490 (1968).

[49] R. Hirota and A. Ramani, The Miura transformations of Kaup's equations and of Mikhailov's equation, *Phys. Lett. A* 76:95–96 (1980).

[50] M. Ito, An extension of nonlinear evolution equations of the K-dV (mK-dV) type to higher orders, *J. Phys. Soc. Japan* 49:771–778 (1980).

[51] L. Hlavatý, On the Painlevé classification of partial differential equations, *J. Math. Phys.* 31:605–609 (1990).

[52] L. Hlavatý, On the Painlevé classification of partial differential equations II. Rational equations with one pole, *Czech. J. Phys.* 42:765–781 (1992).

[53] L. Hlavatý, On the Painlevé classification of partial differential equations III. Rational equations with several poles, *J. Math. Phys.* 33:888–894 (1992).

[54] J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, *J. Math. Phys.* 24:522–526 (1983).

[55] T. Brugarino and A. M. Greco, Painlevé analysis and reducibility to the canonical form for the generalized Kadomtsev-Petviashvili equation, *J. Math. Phys.* 32:69–71 (1991).

[56] P. A. Clarkson, Painlevé analysis and the complete integrability of a generalized variable-coefficient Kadomtsev-Petviashvili equation, *IMA J. Appl. Math.* 44:27–53 (1990).

[57] P. A. Clarkson and M. D. Kruskal, New similarity solutions of the Boussinesq equation, *J. Math. Phys.* 30:2201–2213 (1989).

[58] M. Musette and R. Conte, Non-Fuchsian extension to the Painlevé test, *Phys. Lett. A* 206:340–346 (1995).

[59] F. Bureau, Sur la recherche des équations différentielles du second ordre dont l'intégrale générale est à points critiques fixes, *Académie Royale de Belgique, Bulletin de la Classe des Sciences, Bruxelles, Sér. 5* 25:51–68 (1939).

[60] F. Bureau, Sur les équations différentielles dont l'intégrale générale est à points critiques fixes, *Comptes Rendus de l'Académie des Sciences, Paris* 257:1215–1217 (1963).

[61] M. D. Kruskal, Flexibility in applying the Painlevé test, in *Painlevé Transcendents, Their Asymptotics and Physical Applications*, eds. P. Winternitz and D. Levi, *NATO ASI Series B: Physics* 278:187–195, Plenum, New York, 1992.

[62] N. Joshi and M. D. Kruskal, A direct proof that solutions of the first Painlevé equation have no movable singularities except poles, in *Nonlinear Evolution Equations and Dynamical Systems,* edited by M. Boiti, L. Martina and F. Pempinelli, 310–317 (World Scientific, Singapore, 1992).

[63] N. Joshi and M. D. Kruskal, A direct proof that solutions of the six Painlevé equations have no [movable] singularities except poles, *Stud. Appl. Math.* 93:187–207 (1994).

[64] M. D. Kruskal and P. A. Clarkson, The Painlevé-Kowalevski and poly-Painlevé tests for integrability, *Stud. Appl. Math.* 86:87–165 (1992).

[65] K. Nishioka, A note on the transcendency of Painlevé's first transcendent, *Nagoya Math. Journal* 109:63–67 (1988).

[66] H. Umemura, On the irreducibility of the first differential equation of Painlevé, in *Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, Vol. 2,* eds. H.Hijikata, H. Hironaka, M. Maruyama, H. Matsumura, M. Miyanishi, T. Oda, and K. Ueno, Kinokuniya, Tokyo, pp. 771–789, 1987.

[67] H. Umemura, Second proof of the irreducibility of the first differential equation of Painlevé, *Nagoya Math. Journal* 117:125–171 (1990).

[68] H. Umemura, Birational automorphism groups and differential equations, *Nagoya Math. Journal* 119:1–80 (1990).

[69] P. Painlevé, Sur les équations différentielles du second ordre à points critiques fixes, *Comptes Rendus de l'Académie des Sciences, Paris* 126:1697–1700 (1898).