Remarks on the Milnor fibration at infinity

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1 Introduction

For $a \in (0, \infty)$, we denote: $D_a := \{ w \in \mathbf{C} \mid |w| \le a \}$, $B_a := \{ z \in \mathbf{C}^n \mid \|z\| \le a \}$, $S^1_a := \partial D_a$, $S_a := \partial B_a$ and $E_a := \mathbf{C}^n \setminus \overset{\circ}{B}_a$.

Let $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ be a polynomial function such that $f(0) = 0$. In [6] J. Milnor showed that if $R > 0$ is sufficiently small, then the map

$$\Phi : S_R \setminus f^{-1}(0) \longrightarrow S^1 \ , \quad \Phi(z) := \frac{f(z)}{|f(z)|} \tag{1}$$

is the projection map of a smooth fibre bundle. This fibration, called the Milnor fibration of f at 0, is an important tool in the study of singular points of complex analytic functions. Its properties have been intensively studied, starting with [6], especially in the case of isolated singular points.

Consider now a polynomial f, without the condition $f(0) = 0$. In general, for R sufficiently large, the map (1) is not a fibration. (Here and in the sequel, "for R sufficiently large" will mean that "there exists $R_0 \in (0, \infty)$ such that if $R \ge R_0$".) In [10] it was introduced a class of polynomials, the so-called semitame polynomials, for which the map (1) is the projection map of a smooth fibre bundle when R is sufficiently large. This fibration is called the Milnor fibration at infinity, and its fibres, $F_\theta := \Phi^{-1}(e^{i\theta})$, are called the Milnor fibres at infinity. In [10], the topological description of the fibre F_θ was done only for a subclass of semitame polynomials, namely for the so-called M-tame polynomials.

The aim of this paper is to continue the study started in [10] and to give a topological description of the Milnor fibre at infinity. We give a formula for the Euler-Poincaré characteristic of F_θ and we describe a class of polynomials f for which the Milnor fibre at infinity has the homotopy type of a bouquet $S^{n-1} \bigvee \ldots \bigvee S^{n-1}$, and we compute the number of spheres S^{n-1} in this bouquet.

As an application, we show that the links at infinity of some hypersurfaces diffeomorphic to affine spaces $\mathbf{C}^k$, given in [3], are knotted spheres. These results and their proofs are given in the next two Sections.

In [11] we investigated how some metric properties of polynomials are affected by the group of algebraic automorphisms (preserving the origin) of $\mathbf{C}^n$. For example, we found that, when $n \ge 3$, the property of satisfying Malgrange condition (see [12] for the definition and some properties) of a linear form may disappear under the action. In the last Section of this paper we give examples which show that the property of being M-tame also depends on the algebraic coordinate system of $\mathbf{C}^n$, when $n \geq 4$. In particular, when $n \geq 4$, our definition of M-tameness is not equivalent to the definition given in [8]. This is rather surprising and contrasts deeply to the case $n = 2$, where, by [16], being M-tame does not depend on the coordinate system.

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Dedication. *To Professor Takuo Fukuda at his sixtieth birthday anniversary.*

2 Main results

Let $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ be a polynomial and let denote by B_f the minimal set $B \subset \mathbf{C}$ such that the restriction $f : \mathbf{C}^n \setminus f^{-1}(B) \longrightarrow \mathbf{C} \setminus B$ is a locally trivial fibration. It is well-known, see for instance [15], [13], that B_f is finite. The elements of B_f are called *atypical values* of f. A *generic* fibre of f is a fibre $f^{-1}(a)$ for some $a \in \mathbf{C} \setminus B_f$. Let $\delta \in (0, \infty)$ be such that $\delta > |y|$ for all $y \in B_f$. The *global monodromy fibration* of f is the restriction

$$f : f^{-1}(S^1_\delta) \longrightarrow S^1_\delta \ . \tag{2}$$

The *total Milnor number* of f, denoted by $\mu(f)$, is the sum of the local Milnor numbers in all the singular points of f. If f has only isolated singularities, then

$$\mu(f) = \dim_{\mathbf{C}} \frac{\mathbf{C}[x_1, \ldots, x_n]}{\left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right)} \ .$$

The gradient of $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ is defined by

$$\operatorname{grad} f(x) := \left(\overline{\frac{\partial f}{\partial x_1}}(x), \ldots, \overline{\frac{\partial f}{\partial x_n}}(x)\right),$$

where $x = (x_1, \cdots, x_n)$ and the bar denotes the conjugation. We put:

$$M(f) := \{ \ x \in \mathbf{C}^n \ \mid \exists \lambda \in \mathbf{C} \ \text{ such that } \ \operatorname{grad} f(x) = \lambda \cdot x \ \} \tag{3}$$

Geometrically, a point $x \in M(f) \setminus \{0\}$ if and only if either x is a critical point of f, or x is not a critical point of f, but the hypersurface $f^{-1}(f(x))$ does not intersect transversally the sphere $S_{\|x\|}$ (at x) .

For a sequence $\{y^k\} \subseteq M(f)$ we consider the conditions

$$\lim_{k\to\infty} \|y^k\| = \infty \quad \text{and} \quad \lim_{k\to\infty} f(y^k) = c. \tag{4}$$

We denote by Σ_f the set of critical values of f and we put:

$S_f := \{ c \in \mathbf{C} \mid$ there exists a sequence $\{y^k\} \subseteq M(f)$ such that (4) is fulfilled $\}$.

In general $\Sigma_f \subseteq B_f$ and it may happen that $B_f \neq \Sigma_f$. In [9] it is shown that $B_f \subseteq \Sigma_f \cup S_f$. When $n = 2$, it follows from from [16] that $B_f = \Sigma_f \cup S_f$. In the last Section we give an example, with $n = 4$, which shows that in general $B_f \neq \Sigma_f \cup S_f$.

The polynomial $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ is called *semitame* if $S_f \subseteq \{0\}$. The polynomial f is called *M-tame* if $S_f = \emptyset$. For examples and some counterexamples, we refer to [9], [11].

Let $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ be a semitame polynomial function. From the definition, it follows easily:

(i) If z is a non-isolated critical point of f, then $f(z) = 0$. If f is M-tame, then f has only isolated singularities.

(ii) Let $\alpha > 0$. For R sufficiently large, we have: $M(f) \cap E_R \cap f^{-1}(\mathbf{C} \setminus D_\alpha) = \emptyset$.

We recall from [10] some properties of semitame and M-tame polynomials.

(iii) For R sufficiently large, the Milnor map at infinity

$$\Phi = \frac{f}{|f|} : S_R \setminus f^{-1}(0) \longrightarrow S^1 \tag{5}$$

is the projection map of a smooth fibre bundle with fibres $F_\theta := \Phi^{-1}(e^{i\theta})$.

Assume that $\delta \in (0, \infty)$ is such that $\delta > |y|$ for all $y \in \Sigma_f$.

(iv) For R sufficiently large, the fibrations (2) *and*

$$\Phi : S_R \setminus f^{-1}(D_\delta) \longrightarrow S^1 \tag{6}$$

are diffeomorphic. Moreover, if f is M-tame, then the fibrations (2) *and* (5) *are diffeomorphic and the Milnor fibre at infinity has the homotopy type of a bouquet $S^{n-1} \bigvee \ldots \bigvee S^{n-1}$, the number of spheres S^{n-1} being equal to $\mu(f)$.*

(v) Let $D', D'' \subseteq \mathbf{C}$ be closed discs centered in 0, such that $D' \subseteq D''$ and $D' \neq D''$. For R sufficiently large, and if $z \in S_R \setminus f^{-1}(0)$ satisfies $\operatorname{grad} \log f(z) = \lambda z$, *then we have: either $f(z) \in D'$, or $f(z) \notin D''$. In the first case,* $\arg(-\lambda) \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$*; in the second case,* $\arg(\lambda) \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$.

By [6], Corollary 2.6, the real algebraic sets $M(f)$ and $f^{-1}(0)$ can be expressed as finite disjoint unions, $M(f) = M_1 \cup \ldots \cup M_p$ and $f^{-1}(0) = M_{p+1} \cup \ldots \cup M_q$, where each M_j is a smooth manifold with only finitely many components. By [6], Corollary 2.8, we can suppose that

(vi) For R sufficiently large, the sphere S_r is transverse to M_j, for all j and for all $r \geq R$.

Our first result describes the fibration $\Phi : S_R \cap f^{-1}(\mathring{D}_\delta \setminus \{0\}) \longrightarrow S^1$. Together with (iv), this give us a better picture of the Milnor fibration at infinity.

Theorem 1. *Let $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ be a semitame polynomial function and let δ be as above. Let $R > 0$ be a large number such that above conditions $(ii)\ldots(vi)$ hold, with $\alpha = \delta/2$ in (ii) and $D' = D_\delta$ in (iv). Let $\varepsilon \in (0, \delta) \setminus \Sigma_f$ be such that the sphere S_R is transverse to $f^{-1}(c)$ for all $c \in D_\varepsilon$ (transversality to $f^{-1}(0)$ in a stratified sense, as in (vi)). Then the fibration*

$$\Phi : S_R \cap f^{-1}(\mathring{D}_\delta \setminus \{0\}) \longrightarrow S^1 \tag{7}$$

is diffeomorphic to

$$f : f^{-1}(S^1_\varepsilon) \cap \mathring{E}_R \longrightarrow S^1_\varepsilon \ . \tag{8}$$

Corollary 2. *With the hypothesis of Theorem 1, we have:*

$$\chi(F_\theta) = \chi(f^{-1}(\varepsilon)) + \chi(f^{-1}(\varepsilon) \cap E_R) \ . \tag{9}$$

If 0 is a regular value of f, then

$$\chi(F_\theta) = 2\chi(f^{-1}(\varepsilon)) - \chi(f^{-1}(0)) \ . \tag{10}$$

In particular, if 0 is a regular value of f and $\chi(f^{-1}(\varepsilon)) \neq \chi(f^{-1}(0))$, then F_θ is not diffeomorphic to $f^{-1}(\varepsilon)$.

Let $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ be a polynomial and let $B_f = \{b_1, \ldots, b_s\}$ denote its atypical values. Let Δ_i be a small closed disc centered at b_i. Assume that these discs are pairwise disjoint, and let $a_i \in \Delta \setminus \{b_i\}$. In [2], (proof of) Theorem 1.1, Broughton shows that for R sufficiently large and for $a \in \mathbf{C} \setminus B_f$,

$$H_{n-1}(f^{-1}(a)) \cong \mathbf{Z}^{\mu(f)} \oplus \bigoplus_{i=1}^{s} H_n(f^{-1}(\Delta_i) \cap E_R,\ f^{-1}(a_i) \cap E_R) \ .$$

We denote by $\lambda(f)$ the rank of $\bigoplus_{i=1}^{s} H_n(f^{-1}(\Delta_i) \cap E_R,\ f^{-1}(a_i) \cap E_R)$.

With this notation, we have the following:

Theorem 3. *Let $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ be a semitame polynomial. Suppose that f has only isolated singularities and that its generic fibre has the homotopy type of a bouquet $S^{n-1} \vee \ldots \vee S^{n-1}$ with $\mu(f) + \lambda(f)$ spheres. Then the Milnor fibre at infinity has also the homotopy type of a bouquet $S^{n-1} \vee \ldots \vee S^{n-1}$ with $\mu(f) + 2\lambda(f)$ spheres.*

3 Proofs

Proof of Theorem 1. We define $\psi : \mathbf{C}^n \longrightarrow \mathbf{R}^2$ by: $\psi(z) := (|f(z)|, \|z\|)$. By semitameness, the set $B := M(f) \cap f^{-1}(\overline{D_\delta \setminus D_\varepsilon})$ is compact. Therefore, the set $Y := \psi(B)$ is also compact. By the choice of δ, we have: $Y \cap (\{\delta\} \times [R, \infty)) =$

$\emptyset$. Therefore, there exists a line $L \subset \mathbf{R}^2$, with slope equal to $-b_0$, for some $b_0 \in (0, \infty)$, such that $L \cap Y = \emptyset$, Y is situated under L and $(\delta, R) \in L$. Let U be a neighborhood of B such that $\psi(U)$ is situated under L and such that $L \cap \psi(U) = \emptyset$.

The set $A := E_R \cap B$ is compact, hence, by (i), there exists $\sigma > 0$ such that for all $z \in A$, if λ satisfies grad $\log f(z) = \lambda z$, then $|\lambda| \geq \sigma$. Therefore, for a suitable large $b' > b_0$, we have:

$$\sup \left\{ Re \frac{-\lambda}{|\lambda|^2} \;\middle|\; z \in A \text{ and } \operatorname{grad} \log f(z) = \lambda z \right\} < b' .$$

Let $b > b'$ and let $R_1 := R + b(\delta - \varepsilon)$. Then the segment Δ, which joins the points (ε, R_1) and (δ, R) in $\mathbf{R}^2$, is above L, and the fibrations (8) and

$$f : \mathring{B}_{R_1} \cap f^{-1}(S^1_\varepsilon) \cap \mathring{E}_R \longrightarrow S^1_\varepsilon \tag{11}$$

are diffeomorphic.

We will construct a vector field v in a neighborhood X of $E_R \cap f^{-1}(\overline{D_\delta \setminus D_\varepsilon})$, satisfying the following properties:

$(*)$ For all $z \in X$, $\langle v(z), \operatorname{grad} \log f(z) \rangle = -1$.

$(**)$ For all $z \in X \setminus U$, $\langle v(z), z \rangle = b$.

$(***)$ For all $z \in X \cap U$, $Re\langle v(z), z \rangle \in (0, b]$.

We take $X = E_r \cap f^{-1}(D_{\delta'} \setminus D_{\varepsilon'})$, for some $r < R$ close to R, some $\delta' > \delta$ close to δ and some $\varepsilon' < \varepsilon$ close to ε. For any $z \in X \setminus M(f)$, the vectors z and grad $\log f(z)$ are linearly independent, hence there exists a neighborhood $W_z \subseteq X \setminus M(f)$ of z such that for any $u \in W_z$, the equations $\langle w, \operatorname{grad} \log f(u) \rangle = -1$ and $\langle w, u \rangle = b$ have a common solution, $w = w_z(u)$, which depends smoothly on u.

If $z \in X \cap M(f)$, the vectors z and grad $\log f(z)$ are linearly dependent, hence there exists $\lambda \in \mathbf{C}$ such that grad $\log f(z) = \lambda z$. By hypothesis, we have $\arg(-\lambda) \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$. Thus, the vector $w = \frac{-\lambda z}{\|\lambda z\|^2}$ satisfy $\langle w, \operatorname{grad} \log f(z) \rangle = -1$ and $Re\langle w, z \rangle = Re\frac{-\lambda}{|\lambda|^2} \in (0, b')$. We can extend the vector w to a a vector field w_z, defined in a small ball U_z centered at z, such that conditions $\langle w_z(u), \operatorname{grad} \log f(u) \rangle = -1$ and $Re\langle w_z(u), u \rangle \in (0, b')$ are satisfied for all $u \in U_z$. Moreover, we can assume that the radius of U_z is small enough, such that $U_z \subseteq U$.

Next, using a partition of the unity, we glue together the vector fields w_z, for all $z \in X$, and we obtain the desired vector field v.

Let $p(s)$ be an integral curve of the vector field v. Property $(*)$ tells us that the argument of $f(p(s))$ is constant and that $|f(p(s))|$ is strictly decreasing as a function of s. Properties $(**)$ and $(***)$ guarantee that $\|p(s)\|$ is a strictly increasing function of s. Thus, starting at any point $z \in F_\theta \cap f^{-1}(D)$ and following the trajectory through z, we travel in X, "away" from the origin, in a direction of decreasing $|f|$, until we reach a point $z' \in f^{-1}(\varepsilon e^{i\theta}) \cap E_R$. Note also that starting at a point $z \in f^{-1}(\delta e^{i\theta}) \cap S_R$ and following the trajectory of the vector field v, the point $\psi(p(s))$ stays in Δ, hence we rich a point in $f^{-1}(\varepsilon e^{i\theta}) \cap S_{R_1}$. Therefore, the correspondence $z \longmapsto z'$ gives us that the fibrations (11) and

$$\Phi : S_R \cap f^{-1}(\overset{\circ}{D}_\delta \setminus D_\varepsilon) \longrightarrow S^1 \tag{12}$$

are diffeomorphic. To complete the proof, we note that the choice of ε guarantees that the fibrations (12) and (7) are diffeomorphic.

$\diamondsuit$

Proof of Corollary 2. Let $\delta' > \delta$ be close to δ. Relation (9) follows from the Mayer-Vietoris sequence associated to $F_\theta = (F_\theta \cap f^{-1}(D_{\delta'})) \cup (F_\theta \cap f^{-1}(\mathbf{C} \setminus D_\delta))$ and from the fact that $\chi(f^{-1}(\delta) \cap S_R) = 0$.

When 0 is a regular value of f, relation (10) follows from (9), since $f^{-1}(0)$ is isomorphic to $f^{-1}(\varepsilon) \cap \overset{\circ}{B}_R$ and $\chi(f^{-1}(0) \cap S_R) = 0$.

$\diamondsuit$

Proof of Theorem 3. By [10], Lemma 15, the closure $\overline{F_\theta}$ is embedded in S_R in such a way as to have the same homotopy type as its complement $S_R \setminus \overline{F_\theta}$. By Alexander duality, the reduced homology group $\widetilde{H}_i(F_\theta) = \widetilde{H}_i(S_R \setminus \overline{F_\theta})$ is isomorphic to the reduced cohomology group $\widetilde{H}^{2n-i}(\overline{F_\theta})$. By [10], Corollary 14, F_θ is obtained from a generic fibre $f^{-1}(a)$ by attaching cells of dimension at least $n-1$. Since $f^{-1}(a)$ has the homotopy type of a bouquet $S^{n-1} \bigvee \ldots \bigvee S^{n-1}$, it follows that F_θ is $(n-2)$-connected. And now, the argument in [6], page 58, shows that F_θ has the homotopy type of a bouquet $S^{n-1} \bigvee \ldots \bigvee S^{n-1}$. Assume that the number of spheres in this bouquet is $\mu(f) + \lambda(f) + \nu$. We have to show that $\nu = \lambda(f)$. We will take $\varepsilon > 0$ very small. Relation (9) implies that

$$(-1)^{n-1}\nu = \chi(f^{-1}(\varepsilon) \cap E_R) \ . \tag{13}$$

The exact sequence of the pair $(f^{-1}(D_\varepsilon) \cap E_R\, ,\ f^{-1}(\varepsilon) \cap E_R)$ implies that

$$\chi(f^{-1}(\varepsilon) \cap E_R) = \chi(f^{-1}(D_\varepsilon) \cap E_R) - \chi(f^{-1}(D_\varepsilon) \cap E_R\, ,\ f^{-1}(\varepsilon) \cap E_R) \ . \tag{14}$$

If ε is sufficiently small, we have (see for instance [1], page 320)

$$\chi(f^{-1}(D_\varepsilon)) = \chi(f^{-1}(0)) \ .$$

This implies, for R sufficiently large, that $\chi(f^{-1}(D_\varepsilon) \cap E_R) = 0$. Thus, the relations (13) and (14) show that the equality $\nu = \lambda(f)$ is equivalent to

$$\chi(f^{-1}(D_\varepsilon) \cap E_R\, ,\ f^{-1}(\varepsilon) \cap E_R) = (-1)^n \lambda(f) \ .$$

And now, this equality follows easily since the generic fiber of f has the homotopy type of a bouquet $S^{n-1} \bigvee \ldots \bigvee S^{n-1}$, by using semitameness and [2], relation (1.2) and pages 232-233.

$\diamondsuit$

Remark. The monodromy of the fibration (2) was studied in some papers, see for instance [7], [5], [14] and the references quoted there. When f is a semitame polynomial, one can compute the zeta-function of the monodromy associated with the Milnor fibration at infinity of f by combining Theorems 1 and 4 in [5] with our Theorem 1.

Example. As in [3], Theorem 5, let m, d and a be integers such that $m \geq 2$, $0 < a < d-1$ and $(a,d) = (a, d-1) = 1$. Let $X_{d,a} \subset \mathbf{C}^{2m}$ be the hypersurface defined by the equation $f(x) = 0$, where

$$f(x_0, ...x_{2m-1}) = x_0^a x_1^{d-a} + x_1 x_2^{d-1} + ... + x_{2m-3} x_{2m-2}^{d-1} + x_{2m-2} + x_{2m-1}^d \ .$$

The polynomial f is quasihomogeneous (with integer weights), hence, by [10], f is semitame. Moreover, $\mu(f) = 0$. In [3], page 174, it is shown that the link at infinity of $X_{d,a}$ is diffeomorphic to a sphere S^{2m-3}. Moreover, Theorem 5 and its proof in [3], Theorem 5.2 in [2] and the argument in [6], pag. 58, show that the generic fibre of f has the homotopy type of a bouquet $S^{2m-1} \bigvee \ldots \bigvee S^{2m-1}$ and that, for $a > 1$, we have $\lambda(f) > 0$. Thus, when $a > 1$, the Milnor fibre at infinity is not contractible. Now, the argument in [6], page 61, shows that, for $a > 1$, the link at infinity of $X_{d,a}$ is a knotted sphere.

Note also that when $a = 1$, the polynomial f is M-tame.

4 M-tameness and the coordinate system of $\mathbf{C}^n$

The problem of describing the atypical values of a polynomial $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ was considered by many authors, see for instance [16], [9], [12], [4] and the references quoted there. One approach to this problem was made by considering the behavior of the gradient of f near infinity. In [9] it is shown that $B_f \subseteq \Sigma_f \cup S_f$. When $n = 2$, it follows from from [16] that $B_f = \Sigma_f \cup S_f$. We give an example, with $n = 4$, which shows that in general $B_f \neq \Sigma_f \cup S_f$.

Theorem 4. *Let $f : \mathbf{C}^4 \longrightarrow \mathbf{C}$ be defined by*

$$f(x, y, z, u) = x + y - 2x^2y^3 + x^3y^6 + zy^3 - z^2y^5 + uy^5 \ .$$

Then we have:

(1) *There exists a polynomial automorphism $\varphi : \mathbf{C}^4 \longrightarrow \mathbf{C}^4$ such that $\varphi = (f, \varphi_2, \varphi_3, \varphi_4)$. In particular, $B_f = \emptyset$.*

(2) *The polynomial f is not M-tame. More precisely, $S_f = \{0\}$, hence f is semitame.*

Remark. By Theorem 3, the Milnor fibre at infinity of the above polynomial is still contractible, despite the fact that it is obtained by adding cells of dimension at least 3 to the generic fibre (which is diffeomorphic to $\mathbf{C}^3$).

Corollary 5. *Let $f : \mathbf{C}^4 \longrightarrow \mathbf{C}$ be as in Theorem 4, let $g : \mathbf{C} \longrightarrow \mathbf{C}$ be any polynomial and let $h : \mathbf{C}^5 \longrightarrow \mathbf{C}$ be defined by $h(x, y, z, u, v) = f(x, y, z, u) + g(v)$. Then h is a component of a polynomial automorphism of $\mathbf{C}^5$ and has the property that $S_h = \Sigma_g$. In particular, if g has at least two critical values, then h is not semitame.*

Proof of Theorem 4. The coordinate change $U = u - z^2$ gives us $f(x, y, z, U) = x + y - 2x^2y^3 + x^3y^6 + zy^3 + Uy^5$. Next, the coordinate change $Z = z - 2x^2 + x^3y^3 + Uy^2$ gives us $f(x, y, Z, U) = x + y + Zy^3$, and now the first point follows.

We prove now that f is semi–tame. Using Curve Selection Lemma at infinity, see for instance [10], we can replace the sequences in the definition of S_f by functions, $p : (0, \epsilon) \longrightarrow \mathbf{C}^4$ and $\lambda : (0, \epsilon) \longrightarrow \mathbf{C}$, given by Laurent power series, such that the following conditions hold:

(a) $\lim_{s \to 0} \|p(s)\| = \infty$.

(b) $\lim_{s \to 0} f(p(s)) \in \mathbf{C}$.

(c) The relation $\operatorname{grad} f(p(s)) = \lambda(s) \cdot p(s)$ is satisfied for all $s \in (0, \epsilon)$.

We have to show that that the limit in (b) is equal to 0. Let us write $p(s) = (x(s), y(s), z(s), u(s))$ and let us denote by f_x, f_y, f_z and f_u the partial derivative of f with respect to x, y, z and respectively u. Let $\operatorname{ord}(\lambda)$ be the order of the Laurent power series λ. If $\varphi : \mathbf{C}^4 \longrightarrow \mathbf{C}$ is a polynomial, $\operatorname{ord}(\varphi)$ will denote the order of $\varphi(p(s))$.

First we note that the condition (b) implies that $\lim_{s \to 0} \operatorname{grad} f(p(s)) = 0$. Considering f_u, it follows that $\lim_{s \to 0} y(s) = 0$. Next, considering f_x, we obtain that $\lim_{s \to 0} x(s)y(s)^3$ is a root of the equation $1 - 4T + 3T^2 = 0$, hence it can be equal to $\dfrac{1}{3}$ or 1. Thus, $\lim_{s \to 0} \|x(s)\| = \infty$ and $\operatorname{ord}(x) = -3\operatorname{ord}(y) < 0$.

Suppose now that $\lim_{s \to 0} x(s)y(s)^3 = \dfrac{1}{3}$.

Since $5f - yf_y = x(5 + xy^3)(1 - xy^3) + 2y^3z + 4y$ tends to a finite value along the curve $p(s)$, when s tends to 0, and since $\operatorname{ord}(x(5 + xy^3)(1 - xy^3)) = \operatorname{ord}(x) < 0$, we must have $\operatorname{ord}(y^3z) = \operatorname{ord}(x)$, hence $\operatorname{ord}(z) = -6\operatorname{ord}(y)$. This implies that $\operatorname{ord}(f_z) = -\operatorname{ord}(y) < 0$, hence $\lim_{s \to 0} f_z(p(s) \neq 0$, a contradiction.

It remains that $\lim_{s \to 0} x(s)y(s)^3 = 1$. Let us put $a = \operatorname{ord}(1 - xy^3)$. Then $0 < a = \operatorname{ord}(f_x) = \operatorname{ord}(\overline{\lambda x})$, hence $\operatorname{ord}(\lambda) = a + 3\operatorname{ord}(y)$. Now it follows that $\operatorname{ord}(u) = 2\operatorname{ord}(y) - a$ and $\operatorname{ord}(f_y) = a + 4\operatorname{ord}(y)$.

Let us put $b = \operatorname{ord}(z)$. Then $\operatorname{ord}(f_z) = \operatorname{ord}(\overline{\lambda z}) = a + 3\operatorname{ord}(y) + b$, hence $\operatorname{ord}(1 - 2zy^2) = a + b$. Since $f_z = y^3 - 2zy^5$ tends to 0 along $p(s)$, it follows that $b > -5\operatorname{ord}(y)$.

Suppose that $a + b = 0$. Then $a = -b = 2\operatorname{ord}(y)$ (since $\operatorname{ord}(1 - 2zy^2) = a + b = 0$). It follows that $\operatorname{ord}(5f - yf_y) = -\operatorname{ord}(y) < 0$, in contradiction to the fact that $5f - yf_y$ tends to a finite value along $p(s)$.

Suppose now that $0 > a+b$. Using the inequality $b > -5\mathrm{ord}\,(y)$, it follows that $5\mathrm{ord}\,(y) > a$, hence $\mathrm{ord}\,(y^4u) = 6\mathrm{ord}\,(y) - a > 0$. Note also that $0 > a+b = \mathrm{ord}\,(1-2zy^2) = \mathrm{ord}\,(zy^2) = b + 2\mathrm{ord}\,(y)$, hence $a = 2\mathrm{ord}\,(y)$ and $\mathrm{ord}\,(y^2z(3-5y^2z)) = 2b + 4\mathrm{ord}\,(y) < 0$.

We consider now $f_y = -6x^2y^2(1-xy^3) - y^2z(3-5y^2z) + 5y^4u + 1$. Since f_y tends to 0 along $p(s)$, we must have $\mathrm{ord}\,(x^2y^2(1-xy^3)) = 2b + 4\mathrm{ord}\,(y)$, hence $-2\mathrm{ord}\,(y) = 2b + 4\mathrm{ord}\,(y)$, i.e. $b = -3\mathrm{ord}\,(y)$. And now it follows that $\mathrm{ord}\,(5f - yf_y) = -\mathrm{ord}\,(y) < 0$, a contradiction.

It remains to consider the case when $0 < a+b$. It follows that $0 < a+b = \mathrm{ord}\,(1-2zy^2)$, hence $b = \mathrm{ord}\,(z) = -2\mathrm{ord}\,(y)$ and y^2z tends to $\dfrac{1}{2}$ along the curve $p(s)$. And now, since $\mathrm{ord}\,(x^2y^2(1-xy^3)) = a - 4\mathrm{ord}\,(y)$ and $\mathrm{ord}\,(y^4u) = 6\mathrm{ord}\,(y) - a$, the condition that f_y tends to infinity along $p(s)$ implies that $a = 4\mathrm{ord}\,(y)$ or $a = 6\mathrm{ord}\,(y)$. Both these values for a implies that $5f - yf_y$ tends to 0 along $p(s)$, hence $S_f \subseteq \{0\}$.

To prove that f is not M-tame, we show that there exist differentiable functions $p : (0,\epsilon) \longrightarrow \mathbf{C}^4$ and $\lambda : (0,\epsilon) \longrightarrow \mathbf{C}$ such that the above conditions (a), (b) and (c) hold. We take

$$\overline{\lambda(s)} = s^7 \cdot \frac{\varrho(s)\cdot(2+3s^4\varrho(s))}{1+s^4\overline{\varrho(s)}}$$

and $p(s) = (x(s), y(s), z(s), u(s))$, where

$$x(s) = s^{-3} + s\cdot\varrho(s)\ , \quad y(s) = s\ , \quad z(s) = \frac{1}{2}s^{-2} - \eta(s)$$

and

$$\overline{u(s)} = s^{-2}\cdot\frac{1+s^4\overline{\varrho(s)}}{\varrho(s)\cdot(2+3s^4\varrho(s))}\ .$$

We will choose the functions $\varrho, \eta : [0,\epsilon) \longrightarrow \mathbf{C}$ such that conditions (a), (b) and (c) hold.

First, we note that for any continuous functions $\varrho, \eta : [0,\epsilon) \longrightarrow \mathbf{C}$, the conditions (a) and (b) hold, if $\varrho(0) \neq 0$.

Next, we observe that

$$\frac{\partial f}{\partial x}(p(s)) = s^4\varrho(s)(2+3s^4\varrho(s)) = \overline{\lambda(s)\cdot x(s)}$$

and

$$\frac{\partial f}{\partial u}(p(s)) = s^5 = \overline{\lambda(s)\cdot u(s)}\ .$$

Thus, condition (c) will be satisfied if

$$\frac{\partial f}{\partial y}(p(s)) = \overline{\lambda(s)\cdot y(s)} \tag{15}$$

and

$$\frac{\partial f}{\partial z}(p(s)) = \overline{\lambda(s) \cdot z(s)} \; . \tag{16}$$

Relation (16) implies

$$2\eta(s) = \frac{\varrho(s)(2 + 3s^4\varrho(s))}{1 + s^4\overline{\varrho(s)}} \cdot \left(\frac{1}{2} - s^2\overline{\eta(s)}\right) \tag{17}$$

and relation (15) implies

$$\frac{5}{4} + 6\varrho(s) \cdot \left(1 + s^4\varrho(s)\right)^2 + s^2\left[2\eta(s) + \frac{5(1 + s^4\varrho(s))}{\overline{\varrho}(s)(2 + 3s^4\overline{\varrho}(s))}\right] - 5s^4(\eta(s))^2 =$$

$$= s^8 \cdot \frac{\varrho(s) \cdot (2 + 3s^4\varrho(s))}{1 + s^4\overline{\varrho(s)}} \tag{18}$$

We consider now in $\mathbf{R} \times \mathbf{C}^2$, with coordinates (s, ϱ, η), the real algebraic set X described by the equations which corresponds to (17) and (18), namely:

$$2\eta = \frac{\varrho(2 + 3s^4\varrho)}{1 + s^4\overline{\varrho}} \cdot \left(\frac{1}{2} - s^2\overline{\eta}\right)$$

and

$$\frac{5}{4} + 6\varrho \cdot \left(1 + s^4\varrho\right)^2 + s^2\left[2\eta + \frac{5(1 + s^4\varrho)}{\overline{\varrho}(2 + 3s^4\overline{\varrho})}\right] - 5s^4\eta^2 = s^8 \cdot \frac{\varrho \cdot (2 + 3s^4\varrho)}{1 + s^4\overline{\varrho}}$$

It is easy to see that the point $P := (0, \frac{-5}{24}, \frac{-5}{48}) \in X$ and that X is smooth in P. Moreover, X is transversal to the hyperplane $\{ s = 0 \}$ in P. Therefore there exists a C^∞ parametrization of X near P, hence there exists C^∞ functions $\varrho, \eta : [0, \epsilon) \longrightarrow \mathbf{C}$ such that (15) and (16) hold and $\varrho(0) = \frac{-5}{24}$.

$\diamondsuit$

Proof of Corollary 5. Follows easily from the definitions.

$\diamondsuit$

References

[1] E. Artal Bartolo, P. Cassou–Noguès and A. Dimca, *Sur la topologie des polynômes complexes*, Singularities (Oberwolfach, 1996), 317–343, Progr. Math., **162**, Birkhäuser, Basel, 1998.

[2] S.A. Broughton, *Milnor numbers and the topology of polynomial hypersurfaces*, Invent. math., **92** (1988), 217-241.

[3] A.D.R. Choudary and A.Dimca, *Complex Hypersurfaces Diffeomorphic To Affine Spaces*, Kodai Math. J., **17** (1994), 171-178.

[4] Durfee, A.H. *Five definitions of critical point at infinity*, Singularities (Oberwolfach, 1996), 345–360, Progr. Math., **162**, Birkhäuser, Basel, 1998.

[5] S.M. Gusein-Zade, I. Luengo and A. Melle-Hernandez, *On the zeta-function of a polynomial at infinity*, preprint.

[6] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, Princeton University Press, Princeton, 1968.

[7] A. Némethi, *On the Seifert form at infinity associated with polynomial maps*, J. Math. Soc. Japan **51** (1999), 63–70.

[8] A. Némethi and C. Sabbah, *Semicontinuity of the spectrum at infinity*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, *to appear.*

[9] A. Némethi and A. Zaharia, *On the bifurcation set of a polynomial function and Newton boundary*, Publ. of RIMS, Kyoto Univ. **26** (1990), 681–689.

[10] A. Némethi and A. Zaharia, *Milnor fibration at infinity*, Indag. Mathem. new series, **3** (1992), 323–335.

[11] L. Păunescu and A. Zaharia, *On the Lojasiewicz exponent at infinity for polynomial functions*, Kodai Mathematical Journal **20** (1997), 269–274.

[12] A. Parusiński, *A note on singularities at infinity of complex polynomials*, Banach Center Publ. **39** (1997), 131-141.

[13] F. Pham, *Vanishing homologies and the n variables saddlepoint method*, Proc. A. M. S. Symp. in Pure Math., Vol. 40, Part 2, 1983, 319–333.

[14] D. Siersma and M. Tibăr, *Singularities at infinity and their vanishing cycles, II. Monodromy*, preprint.

[15] R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. **75** (1969), 240–284.

[16] H. H. Vui, *Nombres de Lojasiewicz et singularités à l'infini des polynômes de deux variables complexes*, C. R. Acad. Sci. Paris, Série I, **311** (1990), 429–432.