# AN ELEMENTARY EXPOSÉ ON EQUISINGULARITIES

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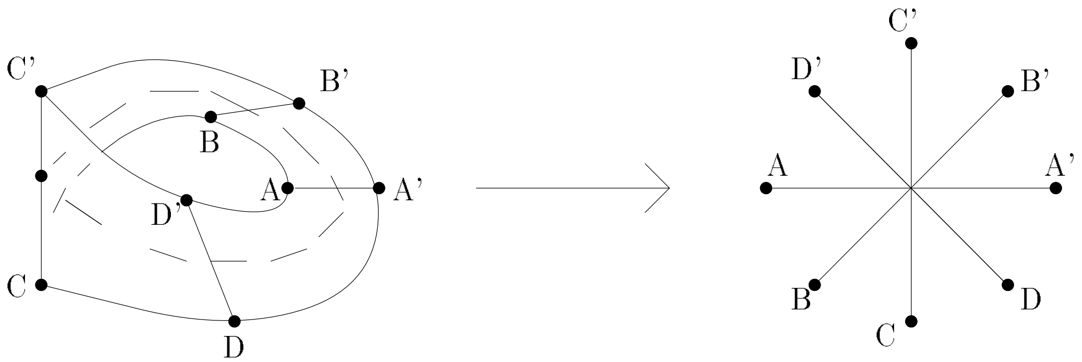
ABSTRACT. Take two convergent power series in $n$ variables. When should they be declared "equivalent"? This is the equisingularity problem. For the case of $n$ real variables the notion of blow-analytic equivalence has led to interesting results. The main ideas and results can be fully explained in the two variables case. This we do in §2. The notion of blow-up plays a vital rôle, hence we give an elementary description of it in §1. This notes can be understood by non-specialists.

## 1. BLOW-UP

Take a Möbius band, $\mathbf{M}$, as obtained by rotating a line around the centre circle. A natural map $\beta : \mathbf{M} \to \mathbb{R}$ can be visualized as follows.




Every generating line is mapped to a line through the origin $O$. The entire centre circle collapses to the origin; elsewhere, $\beta$ is bijective.

We call $\beta$ (or more properly $\beta^{-1}$) the blow-up of $\mathbb{R}^2$ at the origin $O$, or with centre $O$. A neighbourhood of $O$ is magnified, infinitely many times, to a neighbourhood of the centre circle. The precise definition is given below.

Consider the canonical line bundle over the real projective line

$$\pi_1 : \mathbf{E} \to \mathbb{R}P^1 \quad (l, x) \mapsto l, \qquad \mathbf{E} := \{(l, x) | x \in l \in \mathbb{R}P^1\}.$$

Recall that a point, $l$, of $\mathbb{R}P^1$ is just a line in $\mathbb{R}^2$ through $O$.

Topologically, $\mathbf{E}$ is a Möbius band, $\mathbb{R}P^1$ is a circle. The restriction to $\mathbf{E}$ of the second projection

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$$\beta := \pi_2|\mathbf{E} : \mathbf{E} \to \mathbb{R}^2, \qquad (l, x) \mapsto x,$$

is called the blow-up of $\mathbb{R}^2$ at $O$.

Note that $\beta^{-1}(O) = \mathbb{R}P^1 \times \{O\}$ is the zero cross-section of the bundle, outside of which $\beta$ is a bianalytic isomorphism.

Intuitively, $\mathbf{E}$ is obtained by separating lines in $\mathbb{R}^2$ through $O$ and spread them over $\mathbb{R}P^1$ in a natural manner; what $\beta$ does is just putting each line back to where it used to be in $\mathbb{R}^2$, as indicated earlier.

The above readily generalises to any 2-dimensional real analytic manifold $\mathbb{M}$. Take a point $P \in \mathbb{M}$. A neighbourhood of $P$ is a copy of $\mathbb{R}^2$ with $P = O$, whence we can define the blow-up of $\mathbb{M}$ at $P$:

$$\beta : (\mathbf{M}, \beta^{-1}(P)) \to (\mathbb{M}, P),$$

which is an analytic isomorphism outside $\beta^{-1}(P)$.

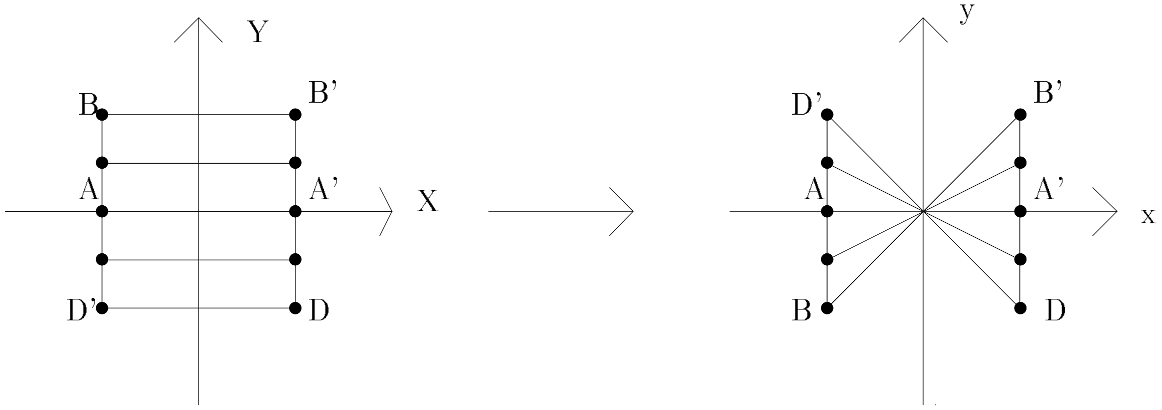
We can also compose them. Take $\beta_n : \mathbf{M}_n \to \mathbf{M}_{n-1}, 1 \leq n \leq N, \mathbf{M}_0 := \mathbb{M}$, and consider the composition:

$$\boldsymbol{\beta} = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_N : \mathbf{M}_N \to \mathbb{M}.$$

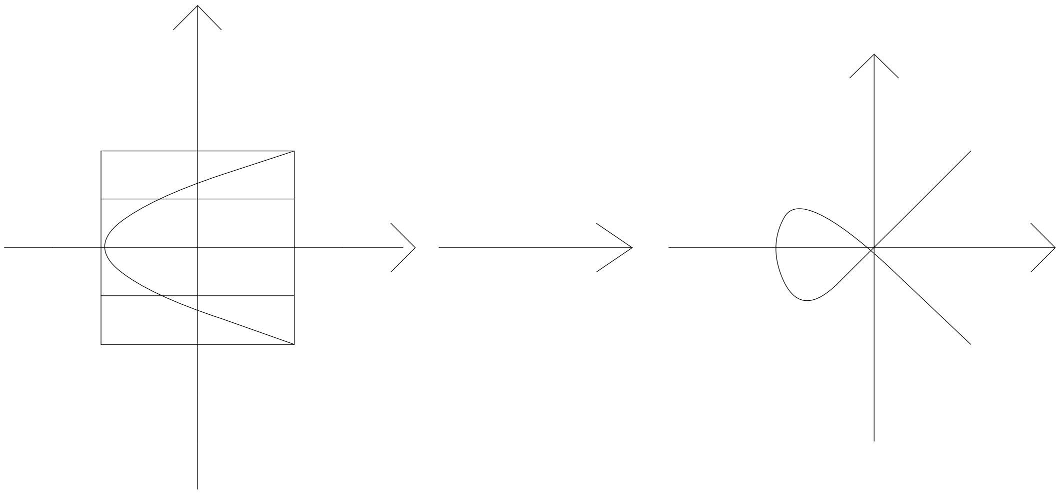
We shall discuss the generalisations to higher dimensional, and the complex cases in §3. Now consider the map

$$\beta_1^* : \mathbb{R}^2 \to \mathbb{R}^2, \quad (X, Y) \mapsto (x, y) = (X, XY).$$

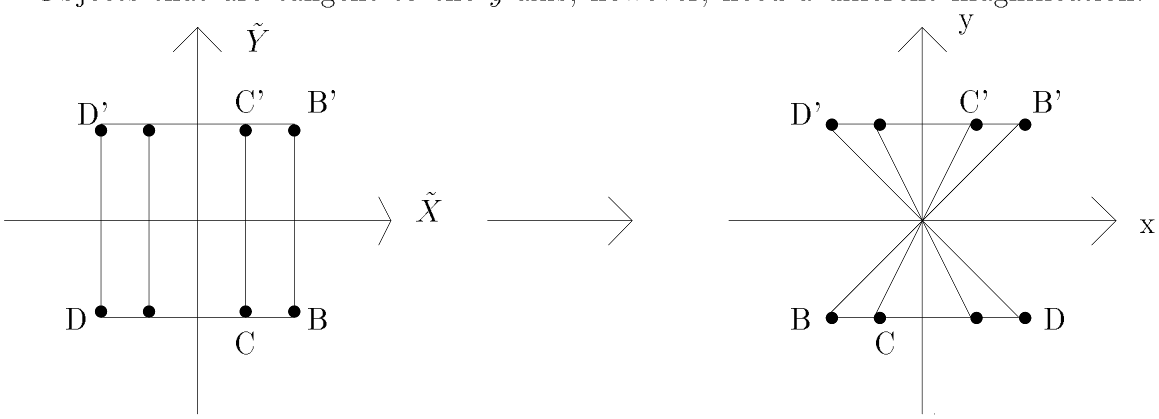
Each horizontal line $Y = c$, $c$ a constant, has image $y = cx$. The $Y$-axis $(X = 0)$, collapses to the origin, while $X = c, c \neq 0$, has image $x = c$. The rectangle in the $(X, Y)$-plane is mapped to a sector around the $x$-axis, as indicated below.




**Philosophy**: When an object lies in the sector, $(\beta_1^*)^{-1}$ would magnify it into an object in the rectangle, often making it simpler. For example, the cusp $y^2 = x^3$ is magnified to $X^2(Y^2 - X) = 0$, consisting of a parabola plus the $Y$-axis, as indicated below.



Objects that are tangent to the $y$-axis, however, need a different magnification:




$$\beta_2^*: \mathbb{R}^2 \to \mathbb{R}^2, \quad (\tilde{X}, \tilde{Y}) \mapsto (x, y) = (\tilde{X}\tilde{Y}, \tilde{Y}).$$

Gluing up the above two rectangles yields a neighbourhood of the centre circle of **E**. Using the language of Differential Topology, $\beta_1^*$, $\beta_2^*$, are two coordinate charts for the blow-up map $\beta$.

In order to study a given function $f(x, y)$, we often study $f \circ \beta_1^*$ and $f \circ \beta_2^*$, which can be simpler.

## 2. Blow-analytic equisingularity

Consider the familiar function of two real variables:

$$m(x, y) = \frac{x^2 y}{x^2 + y^2}, \qquad m(0, 0) = 0,$$

which is continuous, but not $C^1$.

Take the blow-up $\beta : \mathbf{M} \to \mathbb{R}^2$ introduced earlier, and then consider the composition $m \circ \beta$.

In either charts,

$$(m \circ \beta_1^*)(X, Y) = \frac{XY}{1 + Y^2}, \quad (m \circ \beta_2^*)(\tilde{X}, \tilde{Y}) = \frac{\tilde{X}^2 \tilde{Y}}{1 + \tilde{X}^2},$$

both are analytic functions.

Thus $m \circ \beta$ is an analytic function on $\mathbf{M}$.

A function $q : \mathbb{R}^2 \to \mathbb{R}$ is called *blow-analytic* if there exists a succession of blow-ups:

$$\boldsymbol{\beta} := \beta_1 \circ \cdots \circ \beta_N : \mathbf{M_N} \to \mathbf{M_{N-1}} \to \cdots \to \mathbf{M_1} \to \mathbb{R}^2,$$

such that the composition $q \circ \boldsymbol{\beta}$ is analytic on $\mathbf{M_N}$.

Let us also localize this notion.

Take a germ $q : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. We say $q$ is *blow-analytic* if there exists a composite (as above) $\boldsymbol{\beta}$, for which each $\beta_i$ is the blow-up of $\mathbf{M_{i-1}}$ with centre a point in $(\beta_1 \circ \beta_2 \cdots \circ \beta_{i-1})^{-1}(0)$, such that $q \circ \boldsymbol{\beta}$ is analytic in a neighbourhood of $\boldsymbol{\beta}^{-1}(0)$.

An important example of blow-analytic function, found in [12], is

$$Q(x, y) = \frac{xy^5}{x^4 + y^6}.$$

The readers are invited to find out for themselves how many blow-ups are needed to make $Q$ analytic.

We introduce a more effective way in the following.

According to [11], we say that $q$ is *arc-analytic* if for any two convergent power series $x(t), y(t)$,we have also that $q(x(t), y(t))$ is a convergent power series.

The readers are invited to check, quite easily, that $Q$ is arc analytic.

Consider a meromorphic function, i.e. a quotient of analytic functions,

$$m(x, y) = \frac{n(x, y)}{d(x, y)}.$$

An important fact is that $m(x, y)$ is arc-analytic if and only if it is blow-analytic.

This follows immediately from the celebrated Desingularization Theorem of Hironaka. It is also a special case of the Bierstone-Milman theorem on arc-analytic functions [1].

Following [9], [10], we call two germs of analytic functions $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *blow-analytically equivalent* if there exists a local homeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, such that $g \circ h = f$, and the components of $h$ and $h^{-1}$ are (germs of) blow-analytic functions.

In 1965, Whitney [13] discovered a family of germs,

$$w_t(x, y) = xy(x + y)(x - ty), \quad 0 < t < \infty,$$

parametrized by $t$, which consists of infinitely many different $C^1$-types.

This family, however, consists of only one blow-analytic type [9].

In 1998, Fukui [2], proved that if $f$ and $g$ are blow-analytically equivalent, then they have the same multiplicity. His proof is very short and elegant, making use of the Fukui invariant set $A(f)$. An effective method of computing the Fukui set has recently been found in [6].

In 1999, Fukui and Paunescu [5], have proved that a $t$-parametrised family of weighted homogeneous forms $w_t(x,y)$, having an isolated singularity at $(x,y) = (0,0)$, consists of only one blow-analytic equivalence class. A typical example is:

$$x^2 + txy^2 + y^4, \quad t > 2.$$

## 3. Higher dimensional case

It is rather straightforward to generalize the notion of blow-up.

Take $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Consider $\mathbb{K}^n$ and the canonical $\mathbb{K}$-line bundle over the $\mathbb{K}$-projective space $\mathbb{K}P^{n-1}$:

$$\pi : \mathbf{E} \to \mathbb{K}P^{n-1}, \quad \pi(l,x) = l, \quad \mathbf{E} := \{(l,x) \in \mathbb{K}P^{n-1} \times \mathbb{K}^n \,|x \in l\}.$$

The restriction to $\mathbf{E}$ of the second projection

$$\beta := \pi_2|\mathbf{E} : \mathbf{E} \to \mathbb{K}^n, \qquad \beta(l,x) = x,$$

is called the blow-up of $\mathbb{K}^n$ at the origin O.

Now, take a $\mathbb{K}$-analytic manifold, $\mathbf{M}$, say of dimension $m$, and a closed, smooth submanifold, $\mathbf{C}$, say of codimension $n$. Consider a tubular neighbourhood of $\mathbf{C}$, its normal sections can be considered as copies of $\mathbb{K}^n$. We can blow-up each copy at its origin to yield

$$\beta : (\tilde{\mathbf{M}}, \beta^{-1}(\mathbf{C})) \to (\mathbf{M}, \mathbf{C}),$$

which we call the blow-up of $\mathbf{M}$ with centre $\mathbf{C}$.

The blow-analytic equivalence readily generalises to the higher dimensional real analytic case ($\mathbb{K} = \mathbb{R}$), but not to the complex case.

For further details see [3].

A number of puzzling examples have been discovered in recent years.

The first was by Koike in [8]. He discovered a blow-analytic homeomorphism on $\mathbb{R}^3$ which is not Lipschitz.

In [12] the following simpler example is constructed. Consider

$$h : \mathbb{R}^3 \to \mathbb{R}^3, \qquad h(X,Y,Z) = (X, Y, Z - 2\frac{XY^5}{X^4 + Y^6}),$$

whose inverse is

$$h^{-1} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad h^{-1}(x,y,z) = (x, y, z + 2\frac{xy^5}{x^4 + y^6}).$$

The component functions are arc-analytic, whence blow-analytic.

Take a smooth arc $\lambda(t) = (t^3, t^2, t)$. Then $h \circ \lambda(t) = (t^3, t^2, 0)$ is a cusp. Hence $h$ cannot be Lipschitz.

In [7], some blow-analytic homeomorphisms on $\mathbb{R}^2$ are constructed which can transform singular curves to smooth ones; in particular, they are not Lipschitz.

In a forthcoming paper [4], a more systematic construction of this kind of homeomorphisms is developed.

Much research remains to be accomplished in this dirrection.

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