Simple 4-knots

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ABSTRACT

We show that the isotopy type of a 1-simple n-knot K is determined by the Postnikov $(n-1)$-stage of its exterior $X(K)$, together with the homotopy class of the longitude $\lambda_K \in \pi_n(X(K))$. Moreover any pair (X,j) where X is a 4-dimensional homology circle with $\pi_1(X) \cong Z$ and $j : S^4 \times S^1 \to X$ is a map such that $(X,j) = (MCyl(j), S^4 \times S^1)$ is an orientable PD_6-pair is realizable by some simple 4-knot. As a consequence we are able to characterize completely the Artin spins of fibred simple 3-knots.

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Introduction.

In [HK97] we suggested a "homotopy triple" for simple 4-knots which was in some sense an unstable analogue of the Farber quintuple, which is a complete invariant of simple $2q$-knots with $q \geq 4$ [Fa83]. Here we shall show that a refinement of the homotopy triple provides a complete invariant for simple 4-knots. In the fibred case the homotopy triple determines the knot up to a finite ambiguity (proven under rather special additional hypotheses in [HK97]). If moreover π_2 is torsion free the homotopy invariant is already a complete invariant, and the longitude lies in a canonical quotient of $\pi_2 \otimes \pi_3$. Our realization result is not yet fully satisfactory, as we have not been able to translate the Poincaré duality condition into purely algebraic terms. However we are able to characterize completely the simple 4-knots which are Artin spins of fibred simple 3-knots.

Our approach is motivated by the fact that the exterior $X(K)$ of a 1-simple knot K is determined up to homeomorphism by the homotopy type of the pair $(X(K), \partial X(K))$ [LS69]. It is an open question whether every such knot is reflexive, that is, is determined by its exterior. This is true for all simple n-knots with $n \neq 4$ or 6, and in general at most two knots share a common exterior. Our argument does not address this issue, although it follows easily from the classification of simply connected 5-manifolds that simple 4-knots such that π_2 is torsion free are reflexive.

The arguments of §2 apply to all 1-simple knots, in any dimension. However the Postnikov data seems only accessible in the low dimensional cases. En route to our main result on simple 4-knots we shall sketch an approach to the classification of simple 3-knots which uses only homotopy theory and the (π, π)-theorem of Wall.

1. Generalities on knots.

An *n-knot* is a locally flat embedding $K : S^n \to S^{n+2}$. If we assume that the spheres have been oriented once and for all, the knot is determined up to ambient isotopy by its image as an oriented submanifold. Composing K with orientation reversing homeomorphisms of range or domain gives the reflected and inverted knots rK, $K\rho$ and $-K = rK\rho$. (See Chapter 1 of [H]). The *exterior* of K is $X(K) = S^{n+2} \backslash intN$, where $N \cong S^n \times D^2$ is a product neighbourhood of the image of K, and the *group* of K is $\pi K = \pi_1(X(K))$.

The orientations of the spheres determine a conjugacy class of *meridians*, represented by the boundary of an oriented transverse disc, and hence an isomorphism $\pi K/\pi K' = H_1(X(K); \mathbb{Z}) = Z$. Let $t = t_K$ be such a meridian. The n-knot K is *simple* if $\pi K = Z$ and $\pi_i = 0$ for $1 < i \leq (n-1)/2$. (This concept is only of interest if $n \geq 3$, for if $n = 1$ or 2 any such knot is trivial).

Every n-knot bounds a parallelizable $(n+1)$-dimensional submanifold of S^{n+1}; such a submanifold V is called a *Seifert hypersurface* for $K = \partial V$. If K is simple then we may assume that V is $[(n-1)/2]$-connected and that the natural homomorphism from $\pi_{[(n+1)/2]}(V)$ to $\pi_{[(n+1)/2]}(X(K))$ is injective [Le65]. The knot K is *fibred* if $X(K)$ fibres over S^1, in which case the fibre is a Seifert hypersurface for K. The exterior is then the mapping torus of the monodromy of the fibre F, and its infinite cyclic covering space $X(K)'$ is homeomorphic to $F \times R$.

The *sum* of two n-knots K_1 and K_2 may be defined (up to isotopy) as the n-knot $K_1 \sharp K_2$ obtained as follows. Let $D^n(\pm)$ denote the upper and lower hemispheres of S^n. We may isotope K_1 and K_2 so that each $K_i(D^n(\pm))$ is contained in $D^{n+2}(\pm)$, $K_1(D^n(+))$ is a trivial n-disc in $D^{n+2}(+)$, $K_2(D^{n+2}(-))$ is a trivial n-disc in $D^{n+2}(-)$ and $K_1|_{S^{n+1}} = K_2|_{S^{n+1}}$. Then $K_1 \sharp K_2 = K_1|_{D^n(-)} \cup K_2|_{D^n(+)}$. It is not hard to see that $X(K_1 \sharp K_2)' \sim X(K_1)' \vee X(K_2)'$; in particular, the sum of two simple knots is again simple.

We shall write $H_q(X(K);\Lambda)$ for $H_q(X(K)';\mathbb{Z})$, considered as modules over $\Lambda = \mathbb{Z}[t,t^{-1}] = \mathbb{Z}[\pi K/\pi K']$ with respect to the natural action of the covering group on $X(K)'$. These are finitely generated as Λ-modules, and $t-1$ acts invertibly; if $\pi K = Z$ then K is fibred if and only if they are finitely generated as abelian groups [BL66]. If $H_q(X(K);\Lambda)$ is $\mathbb{Z}$-torsion free or finite (for all $q > 0$) the knot is *torsion free* or *finite*, respectively. If N is a Λ-module let $T(N)$ be its $\mathbb{Z}$-torsion submodule. The conjugate Λ-module $\overline{N}$ has the same underlying abelian group and the Λ-structure obtained by inverting the action of t. Let $e^i(N) = \overline{Ext^i_\Lambda(N,\Lambda)}$. Then $e^1(N) \cong \overline{Hom_\Lambda(N,\mathbb{Q}(t)/\Lambda)}$.

2. Postnikov approximations.

Let K be a 1-simple n-knot and let $\pi_i = \pi_i(X(K))$, considered as modules over Λ, for $i \geq 2$. We shall write $\tilde{\pi}_i$ when such groups appear as local coefficient systems. If X is any cell complex we may construct the Postnikov q-stage $P_q(X)$ by adding cells of dimension $\geq q+2$ to X, so that $\pi_j(P_q(X)) = 0$ for $j > q$. The inclusion of X into $P_q(X)$ is then $(q+1)$-connected. Let $HAut_+(X)$ denote the group of self homotopy equivalences of X which induce the identity on π_1. If X is n-dimensional $HAut_+(X)$ is isomorphic to $HAut_+(P_n(X))$, and maps onto $HAut_+(P_{n-1}(X))$.

Lemma 1. *Let P be a cell complex with finite n-skeleton such that $\pi_1(P) \cong Z$ and $\pi_q(P) = 0$ for all $q \geq n$. Suppose also that $p.d._\Lambda H_{n-1}(P;\Lambda) \leq 1$ and*

$H_n(P;\Lambda)=0$. Then there is a finite n-dimensional complex X with an n-connected map to P and such that $H_n(X;\Lambda)=0$.

Proof. Let $U=P^{[n]}$. Then the inclusion of U into P is n-connected, and $H_n(U;\Lambda)$ is projective, by Schanuel's lemma, and hence is free. The relative Hurewicz homomorphism from $\pi_{n+1}(P,U)$ to $H_{n+1}(P,U;\Lambda)$ is an isomorphism and so the Hurewicz homomorphism from $\pi_n(U)$ to $H_n(U;\Lambda)$ is onto. We may add $(n+1)$-cells to U along representatives for a basis for $H_n(U;\Lambda)$ to obtain a finite complex V with $H_n(V;\Lambda)=H_{n+1}(V;\Lambda)=0$. Since $H^q(V;\Lambda)=0$ for all $q>n$ this complex is homotopy equivalent to a finite n-dimensional complex X [Wa65]. □

Lemma 2. *Let P be as above. If $f:X\to P$ and $f_1:X_1\to P$ are n-connected maps where X and X_1 are n-dimensional complexes such that $H_n(X;\Lambda)=H_n(X_1;\Lambda)=0$ then there is a homotopy equivalence $h:X\to X_1$ such that $f\sim f_1h$.*

Proof. Up to homotopy we may assume that f_1 is an inclusion of a subcomplex. There are no obstructions to homotoping f to a map $h:X\to X_1$ (so that $f\sim f_1h$). Since $\pi_q(h)$ is an isomorphism for all $q<n$ and $H_n(X;\Lambda)=H_n(X_1;\Lambda)=0$ the lift of h to the universal covers is a homotopy equivalence, by Whitehead's theorem, and so h itself is a homotopy equivalence. □

If $K:S^n\to S^{n+2}$ is an n-knot with $n>1$ then its normal bundle has an essentially unique framing. Let j_K be the homotopy class in $[S^n\times S^1,X]$ of the homeomorphism onto ∂X determined by such a framing. The *longitude* λ_K of K is the image of j_K in π_n under restriction to $S^n\times\{*\}$.

Lemma 3. *Let X be an n-dimensional homology circle with $\pi=\pi_1(X)\cong Z$ and such that $H_n(X;\Lambda)=0$. If two maps $j_1,j_2:S^n\times S^1\to X$ induce isomorphisms on π and agree on $Y=S^n\vee S^1$ then there is a self homotopy equivalence $f:X\to X$ such that $fj_1\sim j_2$.*

Proof. Let $[S^n\times S^1,X]_{S^1}$ be the set of homotopy classes of maps from $S^n\times S^1$ to X which induce the isomorphism on π which sends the inclusion of S^1 as $\{*\}\times S^1$ to $t\in\pi$. As $S^n\times S^1$ is the cofibre of the Whitehead product $w=[\iota_n,\iota_1]:S^n\to Y=S^n\vee S^1$ and Σw is trivial (see page 485 of [Wh]), there is an exact sequence of pointed sets $Z\otimes_\Lambda \pi_{n+1}\to[S^n\times S^1,X]_{S^1}\to\pi_n^\pi\to 0$, where the right hand map is induced by restriction to $S^n\times\{1\}$, and maps onto the subgroup of π_n fixed by the action of $\pi=Z$.

The inclusion of X into (X,Y) induces isomorphisms of $H_q(X;\Lambda)$ with $H_q(X,Y;\Lambda)$ for $q\leq n$, while $H_{n+1}(X,Y;\Lambda)\cong H_n(Y;\Lambda)\cong\Lambda$. Hence $H^{n+1}(X,Y;\tilde\pi_{n+1})\cong\pi_{n+1}$. For any $\alpha\in\pi_{n+1}$ there is a self map $f_\alpha:X\to X$ which is the identity on the (relative) n-skeleton of (X,Y) and such that $d(f_\alpha,id_X)=\alpha\in H^{n+1}(X,Y;\tilde\pi_{n+1})$. Clearly f is a self homotopy equivalence of X. Moreover $d(f_\alpha j,j)$ is the image of α in $H^{n+1}(S^n\times S^1,Y;\tilde\pi_{n+1})\cong Z\otimes_\Lambda\pi_{n+1}$, and so composition with such self homotopy equivalences induces the action of $Z\otimes_\Lambda\pi_{n+1}$ in the above sequence. □

If $j:S^n\times S^1\to X$ is a map let (X,j) denote the pair $(MCyl(j),S^n\times S^1)$, where $MCyl(j)$ is the mapping cylinder of j. A self homotopy equivalence $h:X\to X$ induces a homotopy equivalence of pairs $(X,j)\simeq(X,hj)$.

Lemma 4. *Let X be an homology circle with $\pi_1(X)\cong Z$ and $j:S^n\times S^1\to X$ be a map such that $\pi_1(j)$ is an isomorphism and (X,j) is an orientable PD_{n+2}-pair whose Spivak normal bundle admits a TOP reduction. Then there is a 1-simple n-knot K such that $(X(K),j_K)\simeq(X,j)$.*

Proof. We may assume that $n\geq 3$, since otherwise $X\simeq S^1$ and we may take K to be a trivial knot. Since the Spivak normal bundle has a TOP reduction there is a degree 1 normal map $F:(W,\partial W)\to(X,j)$, where W is a compact $(n+2)$-manifold. Since $\pi_1(j)$ is an isomorphism we may surger the map F to obtain a homotopy equivalence of pairs $F_1:(W_1,\partial W_1)\to(X,j)$, by Theorem 3.3 of [W]. The map $j^{-1}F_1|_\partial$ is homotopic to a homeomorphism, h say. Hence $\Sigma=W_1\cup_h(S^n\times D^2)$ is then a simply connected homology sphere, and so is homeomorphic to S^{n+2}. The submanifold $S^n\times\{0\}$ is an n-knot with exterior W_1. □

Moreover the knot K is unique up to isotopy.

Theorem 5. *Two 1-simple n-knots K_1 and K_2 are isotopic if and only if there is a homotopy equivalence $f:X(K_1)\simeq X(K_2)$ such that $f_*t_{K_1}=t_{K_2}$ and $f\lambda_{K_1}\sim\lambda_{K_2}$. Moreover the following are equivalent:*

(i) $X(K_1)$ *is homotopy equivalent to* $X(K_2)$*;*

(ii) there is an $(n-1)$*-connected map from* $X(K_1)$ *to* $X(K_2)$*;*

(iii) $P_{n-1}(X(K_1)) \simeq P_{n-1}(X(K_2))$.

Proof. The first assertion follows from Lemma 3 and [LS69]. (Note that j_K determines the attachment of $S^n \times D^2$).

If $f : X(K_1) \to X(K_2)$ is a homotopy equivalence it is $(n-1)$-connected, and determines a homotopy equivalence of the Postnikov $(n-1)$-stages. Conversely, if f is an $(n-1)$-connected map the modules $H_{n-1}(X(K_1);\Lambda)$ and $H_{n-1}(X(K_2);\Lambda)$ are isomorphic (as they are each isomorphic to $e^1(\pi_2)$). As these modules are noetherian and $H_{n-1}(f;\Lambda)$ is an epimorphism it is an isomorphism. Hence f is a homotopy equivalence, by Whitehead's theorem. An isomorphism of the Postnikov data determines a homotopy equivalence $g : P_{n-1}(X(K_1)) \simeq P_{n-1}(X(K_2))$, and so $X(K_1)$ and $X(K_2)$ are homotopy equivalent, by Lemma 2. $\square$

In low dimensions the homotopy type of the Postnikov $(n-1)$-stage and hence of X may be described in algebraic terms. We shall consider below the cases $n = 3$ and $n = 4$. We recall here the "certain exact sequence" of J.H.C.Whitehead for a simply-connected complex Y:

$$\cdots \to \pi_4(Y) \to H_4(Y;\mathbb{Z}) \to \Gamma(\pi_2(Y)) \to \pi_3(Y) \to H_3(Y;\mathbb{Z}) \to 0.$$

Here the extreme maps are Hurewicz homomorphisms and Γ is the universal quadratic functor of Whitehead. In particular, $H_3(K(A,2);\mathbb{Z}) = 0$ and $H_4(K(A,2);\mathbb{Z}) \cong \Gamma(A)$. (See Chapter 1 of [Ba]).

3. Simple 3-knots.

It is well known that simple odd-dimensional knots may be classified by means of their Blanchfield pairings [Ke75, Hi84]. We shall sketch here a homotopy-theoretic approach to this classification when $n = 3$, the lowest dimension in which there are nontrivial simple knots. In this case $X = X(K)$ is homotopy equivalent to a 3-dimensional complex and $X' \simeq M(\pi_2, 2)$, the Moore space with nontrivial homology in degree 2 only. In particular, $\pi_3 \cong \Gamma(\pi_2)$. Therefore the homotopy type of X is determined by the Λ-module

$\pi_2(X)$, and $HAut_+(X) \cong HAut_+(P_3(X))$ maps onto $Aut_\Lambda(\pi_2)$. Poincaré duality gives rise to a "skew-hermitean" Blanchfield pairing, which is an isomorphism $\beta_\lambda : e^1(\pi_2) \to \pi_2$ such that $e^1(\beta_\lambda) = -\beta_\lambda$. Hence $p.d._\Lambda \pi_2 \leq 1$.

Let $M = X \cup D^4 \times S^1$ be the closed 5-manifold obtained by surgery on the knot. Then $M' \simeq X' \cup_\lambda D^4$ has a fundamental class $[M'] \in H_4(M';\mathbb{Z})$, and is an orientable PD_4-complex over $\mathbb{Q}$. Cup product on $H^2(M';\mathbb{Q})$ defines a pairing $B(x,y) = (x \cup y) \cap [M']$, for which t acts isometrically, and the pairing and isometry together are essentially equivalent to the rational Blanchfield pairing.

The relative Hurewicz homomorphism from $\pi_4(M',X')$ to $H_4(M',X';\mathbb{Z})$ is an isomorphism. The composition of the homomorphism induced by the inclusion of M' into (M',X') with the inverse of this Hurewicz isomorphism and the connecting map in the homotopy exact sequence of this pair gives a homomorphism from $H_4(M';\mathbb{Q})$ to $\pi_3 \otimes \mathbb{Q}$ which carries $[M']$ to the image of λ. Considering instead the pair $(K(\pi_2,2),X')$ we see that the corresponding map from $H_4(K(\pi_2,2);\mathbb{Q})$ to $\pi_3 \otimes \mathbb{Q}$ is an isomorphism, since $X' \simeq M(\pi_2,2)$ is a Moore space. Let $c : M' \to K(\pi_2,2)$ be the classifying map. As $H_4(K(A,2);\mathbb{Q}) \cong \Gamma(A) \otimes \mathbb{Q}$ is isomorphic to the space of symmetric bilinear forms : $A^* \times A^* \to \mathbb{Q}$ the Blanchfield pairing determines $c_*[M']$. Since π_3 is $\mathbb{Z}$-torsion free this in turn determines λ.

As $HAut_+(X)$ maps onto $Aut_\Lambda(\pi_2)$ an isometry of pairings on π_2 corresponding to longitudes λ_i is induced by a self homotopy equivalence $f : X \to X$ such that $\beta_{f_*\lambda_1} = f_*\beta_{\lambda_1} f^* = \beta_{\lambda_2}$. Thus the orbit of the longitude under $HAut_+(X)$ is determined by the isometry class of the Blanchfield pairing, and so the isotopy type of the knot is determined by the isomorphism class of the pair (π_2, β_λ).

Conversely, every pair (A,b) where A is a finitely generated Λ-module such that $p.d._\Lambda A \leq 1$ and $(t-1)A = A$ and $b : e^1(A) \to A$ is an isomorphism such that $e^1(b) = -b$ arises from some TOP locally flat 3-knot [Hi84]. (In the fibred case this can be seen as follows. There is a Postnikov 2-stage P with $\pi_1(P) \cong Z$ and $\pi_2(P) \cong A$. Since $p.d._\Lambda A \leq 1$ and $H_3(P;\Lambda) = H_3(K(A,2);\mathbb{Z}) = 0$ there is a finite 3-dimensional homology circle X with $\pi_1(X) \cong Z$ and $\pi_2(X) \cong A$, by Lemma 1. The isomorphism b determines a longitude λ in $\pi_3(X)$ such that $(t-1)\lambda=0$, by the integral version of the above argument. Hence it determines an orientable PD_5-pair (X,j). The only obstruction to a TOP reduction of the Spivak normal bundle ν is in $H^i(X,j;\pi_5(BG/TOP))\cong Z$, and its image in $H^5(X,j;\mathbb{Q})$ is $(\theta\nu)^*k_5$, where $\theta:BSG\to BG/TOP$ is the canonical map and k_5 generates $H^5(BG/TOP;\mathbb{Q})$. Since $H^5(BSG;\mathbb{Q})=0$ the obstruction is 0, and so the pair (A,b) is realized by some simple 3-knot. Can this argument be made to work in general?)

4. Simple 4-knots.

When $n=4$ the homotopy type of $X=X(K)$ is determined by the equivalence class of its "Postnikov triple" (π_2,π_3,κ), where κ is the k-invariant in $H^4(P_2(X);\tilde{\pi}_3))$, and the Postnikov 2-stage $P_2(X)$ is determined by the Λ-module π_2. Two such triples (π_2,π_3,κ) and $(\pi_2^\circ,\pi_3^\circ,\kappa^\circ)$ are equivalent if there are isomorphisms $\theta_2:\pi_2\to\pi_2^\circ$ and $\theta_3:\pi_3\to\pi_3^\circ$ such that $\theta_2^*(\kappa^\circ)=\theta_{3\sharp}(\kappa)$, where $\theta_{3\sharp}:H^4(P_2(X);\pi_3)\to H^4(P_2(X);\pi_3^\circ)$ is the change of coefficients homomorphism.

The Wang sequence for the classifying map from $P_2(X)$ to $P_1(X)=S^1$ gives rise to an exact sequence

$$0\to H^3(K(\pi_2,2);\pi_3)/(t-1)\to H^4(P_2(X);\tilde{\pi}_3)\to H^4(K(\pi_2,2);\pi_3)^{\langle t\rangle}\to 0$$

in which the right hand map is onto the subgroup of $H^4(K(\pi_2,2);\pi_3)$ fixed by the action of $\pi K=\langle t\rangle$. Since $H_3(K(A,2);\mathbb{Z})=0$ and $H_4(K(A,2);\mathbb{Z})\cong\Gamma(A)$ we have isomorphisms $H^3(K(A;2);B)\cong Ext_{\mathbb{Z}}(A,B)$ and $H^4(K(A,2);B)\cong Hom_{\mathbb{Z}}(\Gamma(A),B)$. The image of κ is the first nontrivial k-invariant of the universal cover X', and corresponds to the inclusion of $\Gamma(\pi_2)$ into π_3 under the latter isomorphism (see Chapter 1 of [Ba]). Thus κ determines the Whitehead product and Hopf map η on the truncated homotopy algebra $L=\mathbb{Z}\oplus\pi_2\oplus\pi_3$, which was part of the "homotopy triple" invariant $(L,\eta,[\lambda])$ of [HK97]. Every such algebra (L,η) with an automorphism t such that L_1 and L_2 are finitely generated as Λ-modules and $t-1$ acts invertibly on L_1 and $L_2/L_1^2+\eta(L_1)$ determines a k-invariant in $H^4(K(L_1,2);L_2)$ which is invariant under t, and so may be lifted to such a Postnikov triple.

A presentation for π_4 in terms of π_2 and π_3 may be extracted from the discussion in Part III.§2 of [Ba85]. In particular, the subgroup $[\pi_2,\pi_3]$ generated by Whitehead products is isomorphic to the quotient of $\pi_2\otimes\pi_3$ by the relations determined by the Jacobi identities, commutativity and the vanishing of $[\alpha\circ\eta,\alpha]$ for all $\alpha\in\pi_2$. (This uses the fact that $\Gamma(\pi_2)$ is a subgroup of π_3). The longitude is in the kernel of the suspension homomorphism $E:\pi_4=\pi_4(X')\to\pi_4^{st}(X')$ [Su92]). In the torsion free case $kerE=[\pi_2,\pi_3]$ (see page 533 of [Wh]), and it is easy to see directly that $\lambda_K\in[\pi_2,\pi_3]$. Note that $HAut_+(X)$ acts naturally on $[\pi_2,\pi_3]$ through $HAut_+(P_3(X))$, which is the subgroup of $Aut_\Lambda(\pi_2)\times Aut_\Lambda(\pi_3)$ that fixes κ.

Since X is 4-dimensional every automorphism of the Postnikov triple is realizable by a self homotopy equivalence of X. The equivalence class of the Postnikov triple and the orbit of λ_K under the action of $HAut_+(X)$ together determine the pair $(X(K),j_K)$, and hence the knot, by Theorem 5.

We also have a realization theorem, given appropriate boundary data.

Theorem 6. *Let (Π_2,Π_3,κ) be a Postnikov triple such that κ induces a monomorphism from $\Gamma(\Pi_2)$ to Π_3, Π_2 and $H_3=\Pi_3/\Gamma(\Pi_2)$ are finitely generated as Λ-modules, $t-1$ acts invertibly on these and $p.d._\Lambda H_3\leq 1$. Then there is a finite 4-dimensional homology circle X with $\pi_1(X)\cong Z$ realizing this Postnikov triple. If $j:S^4\times S^1\to X$ is a map such that (X,j) is an orientable PD_6-pair there is a simple 4-knot K such that $(X(K),j_K)\simeq(X,j)$.*

Proof. The Postnikov triple may be realized by a complex P with $\pi_1(P)\cong Z$ and $\pi_q(P)=0$ for $q>3$. Since $H_q(P;\Lambda)$ is finitely generated for all q we may assume that P has finite skeleta. Since the map from $\Gamma(\Pi_2)\cong H_4(K(\Pi_2,2);\mathbb{Z})$ to $\Pi_3=H_3(K(\Pi_3,3;\mathbb{Z})$ induced by κ is a monomorphism $H_4(P;\Lambda)=H_4(P';\mathbb{Z})=0$, by the Leray-Serre spectral sequence for the classifying map from P' to $K(\Pi_2,2)$. Hence there is a finite 4-dimensional cell complex X such that $H_4(X;\Lambda)=0$ which realizes this data, by Lemma 1. It is an immediate consequence of the Wang sequence for the classifying map $X\to S^1$ that X is an homology circle.

Since $S^4\times S^1$ is s-parallelizable the obstructions to trivializing the Spivak normal fibration $\nu:X\to BSG$ lie in $H^q(X,j;\pi_q(BSG))$. (The coefficients are simple as BSG is an H-space). By duality, we need consider only $q=5$ or 6. Since $\pi_5(BSG)=\pi_6(BSG)=0$ there are no obstructions. The result now follows from Lemma 3. $\square$

We shall see in the next section that the duality condition can be described more explicitly in the fibred case.

If π_2 is $\mathbb{Z}$-torsion free K has a 1-connected Seifert hypersurface with torsion free homology. As this hypersurface is parallelisable and has boundary S^4, it is homeomorphic to a once-punctured connected sum of copies of $S^2\times S^3$ [Sm62]. Hence it admits S^1-actions with nonempty, codimension-2 fixed point set, and so K is reflexive (see §17 of [Gl62]). The homotopy class of the inclusion of S^4 as the boundary of $S^2\times S^3\backslash intD^5$ is the basic Whitehead product $[\iota_2,\iota_3]$. Therefore λ_K is in $[\pi_2,\pi_3]$. We shall modify the homotopy triple $q(K)=(L,\eta,[\lambda_K])$ for such knots by taking L as the commutative, graded ring with $L_0=Z$, $L_1=\pi_2$, $L_2=\pi_3$ and $L_3=[\pi_2,\pi_3]$ (the quotient of $L_1\otimes L_2$ by the relations stated above), and $[\lambda_K]$ to be the orbit of λ_K under the group $\{(\theta_1,\theta_2)\in Aut_\Lambda(L_1)\times Aut_\Lambda(L_2)\mid \theta_2|_{\Gamma(L_1)}=\Gamma(\theta_1)\}$ (rather than the group $G(X'))$. The quadratic function $\eta: L_1\to L_2$ determined by composition with the Hopf map in $\pi_3(S^2)$, and the automorphism induced by $t\in\pi K$ are the other ingredients of this invariant.

The *sum* of two homotopy triples is $q\sharp q^\circ=(L\sharp L^\circ,\eta\sharp\eta^\circ,[\lambda+\lambda^\circ])$, where $L\sharp L^\circ=L\otimes L^\circ/(degree>3)$ is the truncation of the graded tensor product (with the diagonal action of the group $\langle t\rangle$), and $\eta\sharp\eta^\circ(x,x^\circ)=\eta(x)+x\otimes x^\circ+\eta^\circ(x^\circ)$. The orbit $[\lambda+\lambda^\circ]$ is well defined, and so the sum is well defined.

Theorem 7. *Let K and K° be torsion free simple 4-knots. Then*
(i) $q(K\sharp K^\circ)=q(K)\sharp q(K^\circ)$;
(ii) $q(rK)=(\bar{L},\bar{\eta},[\lambda])$, $q(K\rho)=(\bar{L},\bar{\eta},[-\lambda])$ *and* $q(-K)=(L,\eta,[-\lambda])$.

Proof. Since $X(K\sharp K^\circ)'\sim X(K)'\vee X(K^\circ)'$ it follows from the Hilton-Milnor theorem that $L(X(K\sharp K^\circ)')=L(X(K)')\sharp L(X(K^\circ)')$ and $\eta_{X(K\sharp K^\circ)'}=\eta_{X(K)'}\sharp\eta_{X(K^\circ)'}$. If V and V° are Seifert hypersurfaces for K and K° respectively then $V\natural V^\circ$ is a Seifert hypersurface for $K\sharp K^\circ$. Hence the sum of the longitudes is the longitude of the sum. This proves the first assertion. The second assertion follows from the facts that if the orientation of the knot is changed the longitude is inverted, while if the transverse orientation of the knot is changed the meridian is inverted. □

How should one define the sum of Postnikov triples in order to obtain an additive invariant in general?

Can we compute this invariant from Seifert data or handlebody presentations of K?

5. Fibred simple 4-knots.

A simple 4-knot K is fibred if and only if π_2 is finitely generated as an abelian group, for then $H_3(X';Z)$ is also finitely generated, and the higher homology groups are 0, by Poincaré duality. In this case there are natural isomorphisms $H^q(X';\mathbb{Z}) \cong H^{q+1}(X;\Lambda)$, and so the duality isomorphism $D: H^4(X;\Lambda) \to H_2(X;\Lambda)$ is detected by the corresponding duality for X'.

Let $\beta^*: H^2(X';\mathbb{Q}/\mathbb{Z}) \to H^3(X';\mathbb{Z})$ be the Bockstein homomorphism and let $S\eta \in \pi_4(S^3)$ be the suspension of the Hopf map. Let $W = [\pi_2, \pi_2] \circ S\eta + [\pi_2, [\pi_2, \pi_2]]$ and $F = \{f \in Hom_{\mathbb{Z}}(H^3, H_2) \mid \langle x, f(\beta^*(x))\rangle = 0 \ \forall x \in H^2(X';\mathbb{Q}/\mathbb{Z})\}$. We may identify W with $[\pi_2, \Gamma(\pi_2)]$ by means of the Barcus-Barratt formula $[\alpha,\beta] \circ S\eta = [\alpha \circ \eta, \beta] + [[\alpha,\beta],\alpha]$ [BB58]. There is an exact sequence $0 \to W \to kerE \to F \to 0$, by Proposition 7.2 of [St82]. Then (X',λ) is a PD_5-pair (and so (X,j) is a PD_6-pair) if and only if the image $D(\lambda)$ of λ in F is an isomorphism.

If $f: X \to X$ is a self homotopy equivalence then $D(\pi_1(f)(\lambda)) = f_* D(\lambda) f^*$, where f_* and f^* are the induced homomorphisms on (co)homology. In general, we shall say that two isomorphisms $D_i : H^3(X';\mathbb{Z}) \cong H^4(X;\Lambda) \to H_2(X;\Lambda) = H_2(X';\mathbb{Z})$ (for $i = 1,2$) are *equivalent* if $D_2 = f_* D_1 f^*$ for some self homotopy equivalence f. If $D(\lambda_1) = D(\lambda_2)$ there is a self homotopy equivalence $f_u : X' \to X'$ which induces the identity on homology and such that $f_{u*}\lambda_1 = \lambda_2$, by Proposition 7.13 of [St82]. Here $u \in H^3(X';\Gamma(\pi_2))$. As $t\lambda_1 = \lambda_1$ and $t\lambda_2 = \lambda_2$ we may assume that $tu = u$ and hence that $\pi_3(f_u)$ is a Λ-homomorphism. Hence f_u is the lift of a self homotopy equivalence $f : X \to X$ such that $H_*(f;\Lambda) = id$ and $fj_1|_{S^4 \vee S^1} \sim j_2|_{S^4 \vee S^1}$. Thus the orbit of the longitude and the equivalence class of the duality isomorphism determine each other.

The truncated homotopy algebra and the orbit of the longitude determine the isotopy class of the knot up to a finite ambiguity, for as π_2 and π_3 are finitely generated $Ext_{\mathbb{Z}}(\pi_2, \pi_3) \cong Ext_{\mathbb{Z}}(T(\pi_2), \pi_3)$ and is finite. The homotopy triple (as defined in §4) is a complete invariant for torsion free fibred simple 4-knots, and is additive with respect to sum of such knots.

The duality isomorphism gives rise to the Farber-Levine pairing, which is a nondegenerate skew-symmetric bilinear pairing $\ell_K : T(\pi_2) \times T(\pi_2) \to \mathbb{Q}/\mathbb{Z}$ on which t acts isometrically. If π_2 is finite ℓ_K determines the duality isomorphism and the homotopy triple, and every such pairing is realized by some finite simple 4-knot. In dimensions $2q \geq 8$ the analogous pairing is a complete invariant for finite simple $2q$-knots with $|\pi_2|$ odd [Ko79,Fa83]. However this may not be correct for such simple 4-knots. For instance, if $T = (\Lambda/(5, t-2)) \oplus (\Lambda/(5, t-3))$ then T^2 supports a skew-symmetric Farber-Levine pairing, and so may be realized by some odd finite simple 4-knot. However it may be shown that $t-1$ does not act invertibly on $Ext_{\mathbb{Z}}(T^2, \Gamma(T^2))$.

What can be said about the factorization of finite simple 4-knots? (See [Hi86] for the factorization of finite simple $2q$-knots with $q \geq 4$).

6. Artin spins of fibred simple 3-knots.

The Artin spin σk of a simple 3-knot k is simple, and is torsion free, as $\pi_2(X(\sigma k)) \cong \pi_2(X(k))$ (see [Ke84]). Thus such spun simple knots are determined by their homotopy triples. In this case an additional feature enables us to give a purely algebraic characterization of such knots.

Lemma 8. *If π_2 is finitely generated and $\mathbb{Z}$-torsion free and the inclusion of $\Gamma(\pi_2)$ into π_3 splits as a Λ-homomorphism then any automorphism θ of $H_3(X;\Lambda)$ is induced by a self homotopy equivalence $f : X \to X$ such that $\pi_i(f) = id$ for $i \leq 2$. In particular, any two duality isomorphisms $D_i : H^4(X;\Lambda) \to H_2(X;\Lambda)$ are equivalent.*

Proof. Since the inclusion splits, θ extends to a Λ-automorphism of π_3 which is the identity on $\Gamma(\pi_2)$. Since π_2 is torsion free κ is determined by the inclusion of $\Gamma(\pi_2)$ into π_3, and so (id_{π_2}, θ) determines an isomorphism of the Postnikov data. Therefore there is a self homotopy equivalence $f : X \to X$ such that $\pi_i(f) = id$ for $i \leq 2$ and $\pi_3(f) = \theta$.

Now suppose that $D_i : H^4(X;\Lambda) \to H_2(X;\Lambda)$ are two isomorphisms. Then $D_2^{-1}D_1$ is an automorphism of $H^4(X;\Lambda) \cong H^3(X';\mathbb{Z})$. Let Θ be the transposed automorphism of $H_3(X;\Lambda) = H_3(X';\mathbb{Z})$. Hence there is a self homotopy equivalence $f : X \to X$ such that $H_2(f;\Lambda) = id$ and $H^4(f;\Lambda) = D_2^{-1}D_1$. Hence $D_1 = f_* D_2 f^*$. $\square$

Theorem 9. *Let K be a simple 4-knot. Then K is the Artin spin of a fibred simple 3-knot if and only if π_2 is finitely generated, $\mathbb{Z}$-torsion free, admits a skew-hermitean Blanchfield pairing and the inclusion of $\Gamma(\pi_2)$ into π_3 splits over Λ. Moreover K is determined up to isotopy by π_2.*

Proof. Let k be a simple 3-knot; then the spun knot σk is a simple 4-knot, and $\pi_2(X(k)) \cong \pi_2(X(\sigma k))$, so σk is torsion free and admits a skew-hermitean Blanchfield pairing. As $X(\sigma k) \simeq (X(k) \times S^1) \cup_{S^1 \times S^1} (S^1 \times D^2)$ [Ke84] and projection onto the first factor induces a retraction of the latter space onto $X(k)$, the inclusion of $\Gamma(\pi_2)$ into $\pi_3(X(\sigma k))$ splits over Λ. Conversely, if π_2 is finitely generated and torsion free, admits a skew-hermitean Blanchfield pairing and the inclusion of $\Gamma(\pi_2)$ into π_3 splits then K is the spin of any simple 3-knot k with $\pi_2(X(k)) \cong \pi_2$. $\square$

Let $\Delta(t) = t^3 - at^2 + (a+1)t - 1$ and let $A = \Lambda/(\Delta(t))$. Then $e^1(A) \cong \overline{Hom_\Lambda(A, \mathbb{Q}(t)/\Lambda)} \cong \overline{A} = \Lambda/(\Delta(t^{-1}))$. As Δ is not palindromic (up to sign), $\overline{A}$ is not isomorphic to A, and so A does not admit a Blanchfield pairing. Therefore no simple 4-knot with $\pi_2 \cong A$ can be a spun 3-knot.

Is it generally true that Artin spins of simple 3-knots are determined by their knot modules π_2 (cf. [Ke84])?

All spun knots are DNC (doubly null concordant). If a simple 4-knot is DNC then its homotopy algebra admits maps $i_\pm$ to algebras $W_\pm$ such that (i_+, i_-) induces isomorphisms $\pi_2 \cong W_{+2} \oplus W_{-2}$ and $H_3 \cong H_{+3} \oplus H_{-3}$, where $H_{\pm 3} = W_{\pm 3}/\Gamma(W_{\pm 2})$ and the duality behaves appropriately. Does this condition characterize DNC simple 4-knots, at least in the torsion free case? (A related criterion was established for simple $2q$-knots with $q \geq 4$ in [Ke84']).

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