On 4-dimensional mapping tori and product geometries

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ABSTRACT

We give simple necessary and sufficient conditions for an infinite cyclic covering space of a closed 4-manifold to be homotopy equivalent to the mapping torus of a self homotopy equivalence of a PD_3-complex. This is a homotopy analogue of the Stallings and Farrell fibration theorems available in other dimensions. We consider also 4-manifolds which admit a geometry of euclidean factor type.

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Introduction

If M is a 4-dimensional mapping torus then its fundamental group must be an extension of Z by a finitely generated normal subgroup and its Euler characteristic $\chi(M)$ must be 0. In [H] we gave several theorems characterizing spaces with the homotopy properties of such mapping tori, under restrictive assumptions on the fundamental group. The main result of this note is a general criterion characterizing PD_4-complexes (finitely dominated 4-dimensional Poincaré duality complexes) which are homotopy equivalent to mapping tori of self homotopy equivalences of PD_3-complexes in terms of the fundamental group and the second homotopy group. Although our criteria are straightforward and natural, it remains possible that the necessary conditions in the first sentence above are also sufficient. We consider first the aspherical case; our argument in Theorem 1 applies in all dimensions and gives something new even in dimension 3. It also leads to a new characterization of surface bundles over the torus or Klein bottle (Theorem 2). Theorems 1 and 3 (the main result) rely on the use of coinduced modules to transfer arguments about subgroups and covering spaces to contexts where Poincaré duality applies. In the second section these results are applied to mapping tori of homeomorphisms of geometric 3-manifolds.

§1. Poincaré duality for infinite cyclic covering spaces

If R is a subring of S, A is a left R-module and C is a left S-module then the abelian groups $Hom_R(C|_R, A)$ and $Hom_S(C, Hom_R(S|_R, A))$ are naturally isomorphic, where $C|_R$ and $S|_R$ are the R-modules underlying C and S respectively. (The maps I and J defined by $I(f)(c)(s) = f(sc)$ and $J(\theta)(c) = \theta(c)(1)$ for $f : C \to A$ and $\theta : C \to Hom_R(S, A)$ are mutually inverse isomorphisms). When K is a subgroup of π and $R = Z[K]$ and $S = Z[\pi]$ these isomorphisms give rise to Shapiro's lemma. In our applications π/K shall be infinite cyclic and then S is a twisted Laurent extension of R.

We shall use an overbar to convert left modules over group rings into right modules, and vice-versa. Note that most of the homology and cohomology groups defined below do not have natural module structures, and so the Poincaré duality isomorphisms are isomorphisms of abelian groups only.

Theorem 1. *Let π be a PD_n-group which is an extension of Z by a normal subgroup K which is $FP_{[n/2]}$. Then K is a PD_{n-1}-group.*

Proof. It is sufficient to show that K is FP_{n-1}, by Theorem 9.11 of [B]. This in turn is true if $Lim_I H^q(K;M_i)=0$ for any exact direct system $\{M_i\}_{i\in I}$ with limit 0 and for all $q\leq n-1$ [Br 75]. Since K is $FP_{[n/2]}$ we may assume $q>n/2$. We have $H^q(K;M_i)\cong H^q(\pi;W_i)\cong H_{n-q}(\pi;\bar{W}_i)$, where $W_i=Hom_{Z[K]}(Z[\pi],M_i)$, by Shapiro's lemma and Poincaré duality. Since $\pi/K\cong Z$ the LHSSS for π as an extension of Z by K reduces to short exact sequences $0\to H_0(\pi/K;H_s(K;\bar{W}_i))\to H_s(\pi;\bar{W}_i)\to H_1(\pi/K;H_{s-1}(K;\bar{W}_i))\to 0$. As a $Z[K]$-module $W_i\cong(M_i)^{\pi/K}$ (the direct product of countably many copies of M_i), with the action of π/K shifting the entries. Since K is $FP_{[n/2]}$ homology commutes with direct products in this range, and so $H_s(K;\bar{W}_i)=H_s(K;\bar{M}_i)^{\pi/K}$ if $s\leq n/2$. Therefore $H_s(\pi;\bar{W}_i)\cong H_{s-1}(K;\bar{M}_i)$ if $s\leq n/2$, and the result now follows easily. //

In particular, if the fundamental group of a PD_3-complex X is an extension of Z by a finitely generated normal subgroup then X is aspherical and is homotopy equivalent to the total space of a surface bundle over S^1. (If X is a 3-manifold this is essentially Stalling's fibration theorem. Note also that finitely generated subgroups of 3-manifold groups are always finitely presentable).

A similar argument shows that if $\phi:\pi\to Z$ is any epimorphism then $c.d.\mathrm{Ker}\phi<n$. (This weak version of Strebel's theorem in [St77] is sufficient for many applications in [H]).

Corollary. *If a PD_n-group π is an extension of a virtually poly-Z group Q by an $FP_{[n/2]}$ normal subgroup K then K is a PD_m-group, where $m=n-h(Q)$. //*

This idea is used in the next result.

Theorem 2. *A closed 4-manifold M is homotopy equivalent to the total space of a surface bundle over the torus or Klein bottle if and only if $\pi=\pi_1(M)$ is an extension of Z^2 or $Z\tilde{\times}Z$ (respectively) by a FP_2 normal subgroup G and $\pi_2(M)$ is finitely generated as a $Z[G]$-module.*

Proof. The conditions are clearly necessary. Suppose that they hold. It is easily seen that π has one end. If G is finite then $\pi_2(M)$ is finitely generated as an abelian group. Therefore $\tilde{M}$ is homotopy equivalent to a finite complex, since it is simply connected and has finitely generated homology. Since π is virtually Z^2 it has one end and is not a PD_4-group, and so $\tilde{M}$ is neither homotopy equivalent to S^3 nor contractible. Therefore $\tilde{M}$ must be homotopy equivalent to S^2, by Theorem II.9 of [H], and the result follows from Theorem IV.3 of [H].

If G is infinite then $H^s(\pi;Z[\pi])=0$ for $s\leq 2$ and $Hom_{Z[\pi]}(\pi_2(M),Z[\pi])=0$, so M is aspherical by Theorem II.6 of [H]. By the Corollary to Theorem 1, G is a PD_2-group, and the result then follows from Theorem IV.1 of [H]. //

This result should be compared with Theorems IV.1, IV.3 and VIII.4 of [H].

Theorem 3. *Let M be a PD_4-complex whose fundamental group π is an extension of Z by a normal subgroup K. Then the infinite cyclic covering space M' with fundamental group K is a PD_3-complex if and only if*

(i) K is FP_2;

(ii) $\Pi=\pi_2(M)$ is finitely generated as a $Z[K]$-module; and

(iii) the map from $Hom_{Z[K]}(\Pi|_K,Z[K])$ to $Ext^1_{Z[\pi]}(\Pi,Z[\pi])$ induced by cup product with the generator η of $Ext^1_{Z[\pi/K]}(Z,Z[\pi/K])$ is a monomorphism.

Proof. If M' is a PD_3-complex then $K=\pi_1(M')$ is FP_2. Hence K is accessible and so $H^1(K;Z[K])$ is finitely generated as a $Z[K]$-module. Since $\Pi=\pi_2(M')\cong H^1(K;Z[K])$ it follows that K is in fact FP_3. It follows from the UCSS that $H^2(K;Z[K])\leq H^2(M';Z[K])$, which is 0 by Poincaré duality. The necessity of the final condition follows on considering the map between the UCSSs for $H^*(M';Z[K])$ and $H^{*+1}(M;Z[\pi])$ induced by cup product with η. (Note that since K is in fact FP_3 the map from $H^3(K;Z[K])$ to $H^4(\pi;Z[\pi])$ is a monomorphism).

Suppose that these conditions hold. Applying the isomorphisms above to the cellular chain complex C_* of the universal covering space $\tilde{M}$, we see that $H^q(M';A)\cong H^q(M;Hom_{Z[K]}(Z[\pi],A))\cong H_{4-q}(M;\overline{Hom_{Z[K]}(Z[\pi],A)})$ for any local coefficient system of left $Z[K]$-modules A on M'. Applying the Wang sequence which relates the homology of M and M', we see that $H^4(M';A)=0$ for any local coefficient system W, and so M' is homotopy equivalent to a 3-dimensional complex.

We follow the strategy and notation used in Sections III.2 and III.3 of [H]. We must show that the cup product maps $\eta_p:H^p(M';Z[K])\to H^{p+1}(M;Z[\pi])$ are isomorphisms, for $0\leq p\leq 4$. If $p=0$ or 4 then all the groups are 0, and so η_0 and η_4 are isomorphisms. Since Π is finitely generated as a $Z[K]$-module $Hom_\pi(\Pi,Z[\pi])=0$. Therefore $H^2(M;Z[\pi])\cong H^2(\pi;Z[\pi])$ and so η_1 is an isomorphism. Moreover $H^3(\pi;Z[\pi])$is a subgroup of $H^3(M;Z[\pi])$, which is 0 by Poincaré duality. Therefore the Wang sequence map $t-1:H^2(K;Z[\pi])\to H^2(K;Z[\pi])$ is onto. Since K is FP_2 this cohomology group is isomorphic to $H^2(K;Z[K])\otimes_Z Z[\pi/K]$, where $Z[\pi/K]\cong Z[t,t^{-1}]$ acts diagonally. It is easily seen that if $H^2(K;Z[K])$ has a nonzero element h then $h\otimes 1$ is not divisible by $t-1$. Hence $H^2(K;Z[K])=0$. The UCSS and condition (iii) imply that $H^2(M';Z[K])=0$, and so η_2 is an isomorphism. It remains only to check that $H^3(M';Z[K])\cong Z$ and that η_3 is onto.

We have $H^3(M';Z[K])\cong H_1(M;Hom_{Z[K]}(Z[\pi],Z[K]))=H_1(\pi;Z[K]^{\pi/K})$. The homology LHSSS for π as an extension of Z by K gives a short exact sequence $0\to H_0(\pi/K;H_1(K;Z[K]^{\pi/K}))\to H_1(\pi;Z[K]^{\pi/K})\to H_1(\pi/K;H_0(K;Z[K]^{\pi/K}))\to 0$. Since K is finitely generated homology commutes with direct products in this range, and so

$H_1(\pi; Z[K]^{\pi/K}) \cong H_1(\pi/K; Z^{\pi/K})$. Since $\pi/K \cong Z$ and acts by translation on the index set this homology group is Z. The homomorphisms from $H^3(M'; Z[K])$ to $H^3(M'; Z)$ and from $H^4(M; Z[\pi])$ to $H^4(M; Z)$ induced by the augmentation homomomorphisms are epimorphisms and hence isomorphisms, since M' and M are homotopy equivalent to 3- and 4-dimensional complexes, respectively. These isomorphisms form the vertical sides of a commutative square whose upper horizontal side is η_3 and whose lower horizontal side is $-\cup\eta : H^3(M'; Z) \to H^4(M; Z)$. Since $H_*(M'; Z)$ is finite dimensional M' the latter map is an isomorphism [Ba80]. Therefore η_3 is also an isomorphism. //

(An alternative strategy would be to show that $Lim_I H^q(M'; A_i) = 0$ for any exact direct system with limit 0. We could then conclude that the cellular chain complex of $\tilde{M} = \tilde{M}'$ is chain homotopy equivalent to a finite complex of finitely generated projective $Z[K]$-modules, and hence that M' is finitely dominated. Since K is FP_2 the above argument applies easily when $q = 0$, 1, 3 or 4, but something else is needed when $q = 2$).

Can condition (*i*) be weakened to "K is finitely generated"? If K is finite then (*ii*) implies (*iii*) [Ba80]. Is this true in general? If M' satisfies Poincaré duality with constant coefficients then $\chi(M) = 0$. In general, does $\chi(M) = 0$ imply that $Hom_\pi(\Pi, Z[\pi]) = 0$?

We may compare the hypotheses of the theorem with the special cases considered in [Hi89] and Chapter III of [H]. Theorem III.2 gives criteria (involving also the orientation character and first k-invariant) for M' to be homotopy equivalent to a *particular* PD_3-complex N. Corollary C of that theorem has as its hypotheses that K is a finitely generated free group and $\chi(M) = 0$. Then K is FP_2 and $\Pi \cong H^2(\pi; Z[\pi]) \cong H^1(K; Z[K])$, and so is finitely generated as a $Z[K]$-module, while $Hom_K(\Pi, Z[K]) = Ext^1_{Z[\pi]}(\Pi, Z[\pi]) = 0$ and so condition (*iii*) holds. Theorem III.3 has as its hypotheses that π is virtually $K \times Z$ and $\chi(M) = 0$. Then K is FP_2 and $Hom_{Z[\pi]}(\Pi, Z[\pi]) = 0$, and it follows from the UCSS for $H^*(M; Z[\pi])$ that $\Pi \cong H^2(\pi; Z[\pi]) \cong H^1(K; Z[K])$ and so is finitely generated as a $Z[K]$-module. Condition (*iii*) then follows algebraically as in [Ba80]. The case considered in Theorem III.4 of [H] is covered by Theorem 1 above. (It also follows directly from Corollary 3 of [St 76], on taking groups $G = K$, $N < G$ infinite cyclic and coefficient ring $K = Z$).

Is there a corresponding simple characterization of cartesian products with S^1? We must also require that the orientation character be trivial on the S^1 factor and that the Pin^- condition $w_2 = w_1^2$ hold. These are the only new conditions required when the other factor is a 3-manifold with torsion free fundamental group, by Theorem III.5 of [H]. However there are examples of orientable 4-manifolds homotopy equivalent to products $N \times S^1$ where $K = \pi_1(N)$ is *not* a 3-manifold group. (See Chapter VIII of [H]).

§2. Applications to geometric manifolds

We shall now combine the above criteria with more geometric hypotheses.

Theorem 4. *Let M be a closed 4-manifold with fundamental group π. Then M is homotopy equivalent to the mapping torus of a self homeomorphism of a closed 3-manifold* *with one of the geometries E^3, Nil^3, Sol^3, $H^2 \times E^1$, $\tilde{SL}$ or $S^2 \times E^1$ if and only if*

(i) $\chi(M) = 0$;

(ii) π *is an extension of* Z *by an* FP_2 *normal subgroup* K*; and*

(iii) K *has subgroups* $1 \neq A \leq C$ *such that* A *is torsion free abelian and normal in* π, C *has finite index in* K *and* C/C' *is infinite.*

Proof. The conditions are clearly necessary. Since K has an infinite abelian normal subgroup it has one or two ends. If K has one end then M is aspherical and so K is a PD_3-group by Theorem 1. Condition (*iii*) then implies that M' is homotopy equivalent to a closed 3-manifold with one of the first five of the geometries listed above. (See the Appendix of [H]). If K has two ends then M' is homotopy equivalent to $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, $RP^2 \times S^1$ or $RP^3 \sharp RP^3$, by the Corollary to Theorem III.3 of [H]. As every self homotopy equivalence of each of these manifolds is homotopic to a self homeomorphism the theorem follows. //

If π is torsion free and solvable then M is in fact homeomorphic to such a mapping torus, by Theorem V.9 of [H]. Mapping tori of self homeomorphisms of hyperbolic and spherical 3-manifolds satisfy conditions (*i*) and (*ii*). In the hyperbolic case there are additional conditions

(*iii-H*) K has one end and no noncyclic abelian subgroup; and

(*iv-H*) $\pi_2(M) = 0$.

Do these four conditions characterize such mapping tori?

In the spherical case the appropriate conditions are

(*iii-S*) K is a fixed point free finite subgroup of $SO(4)$ and (if K is not cyclic) the characteristic automorphism of K determining π is realized by an isometry of S^3/K; and

(*iv-S*) the first nontrivial k-invariant of M is "linear".

The list of fixed point free finite subgroups of $SO(4)$ is well known. (See Chapter VIII of [H]). If K is cyclic or $Q \times Z/p^j Z$ for some odd prime p or T_k^* then the second part of (*iii-S*) and (*iv-S*) are redundant, but the general picture is not yet clear [HM86].

Theorem 5. *Let* M *be a closed geometric 4-manifold with* $\chi(M) = 0$ *and such that* $\pi = \pi_1(M)$ *is an extension of* Z *by a finitely generated normal subgroup* K*. Then* K *is the fundamental group of a geometric 3-manifold.*

Proof. Since $\chi(M) = 0$ the geometry must be either an infrasolvmanifold geometry or a product geometry $X \times E^1$, where X is one of the 3-dimensional geometries S^3, $S^2 \times E^1$, H^3, $H^2 \times E^1$ or $\tilde{SL}$. If M is an infrasolvmanifold then π is torsion free and virtually poly-Z of Hirsch length 4, so K is torsion free and virtually poly-Z of Hirsch length 3, and the result is clear.

If $X = S^3$ then π is a discrete uniform subgroup of $O(4) \times Isom(R)$. Since π maps onto Z it must in fact be a subgroup of $O(4) \times R$, and K is a finite subgroup of $O(4)$. Since π acts freely on $S^3 \times R$ the subgroup K acts freely on S^3, and so K is the fundamental group of an S^3-manifold. If $X = S^2 \times E^1$ it follows from the Corollary to Theorem III.4

of [H] that $K \cong Z$, $Z \oplus (Z/2Z)$ or $D = (Z/2Z) * (Z/2Z)$, and so K is the fundamental group of an $S^2 \times E^1$-manifold.

In the remaining cases X is of aspherical type. The key point here is that a discrete cocompact subgroup of the isometry group $Isom(X \times E^1)$ must meet the radical of this group in a lattice subgroup. Suppose first that $X = H^3$. After passing to a subgroup of finite index if necessary, we may assume that $\pi \cong H \times Z < PSL(2,C) \times R$, where H is a discrete uniform subgroup of $PSL(2,C)$. If $K \cap (\{1\} \times R) = 1$ then K is commensurate with H, and hence is the fundamental group of an X-manifold. Otherwise the subgroup generated by $K \cap H = K \cap PSL(2,C)$ and $K \cap (\{1\} \times R)$ has finite index in K and is isomorphic to $(K \cap H) \times Z$. Since K is finitely generated so is $K \cap H$, and hence it is finitely presentable, since H is a 3-manifold group. Therefore $K \cap H$ is a PD_2-group and so K is the fundamental group of a $H^2 \times E^1$-manifold.

If $X = H^2 \times E^1$ then we may assume that $\pi \cong H \times Z^2 < PSL(2,R) \times R^2$, where H is a discrete uniform subgroup of $PSL(2,R)$. Since such groups do not admit nontrivial maps to Z with finitely generated kernel $K \cap H$ must be commensurate with H, and we again see that K is the fundamental group of an $H^2 \times E^1$-manifold.

A similar argument applies if $X = \tilde{SL}$. We may assume that $\pi \cong H \times Z$ where H is a discrete uniform subgroup of $Isom(\tilde{SL})$. Since such groups H do not admit nontrivial maps to Z with finitely generated kernel K must be commensurate with H and so is the fundamental group of an $\tilde{SL}$-manifold. //

Corollary. *Suppose that M has a product geometry $X \times E^1$. If $X = E^3$, S^3, $S^2 \times E^1$, $\tilde{SL}$ or $H^2 \times E^1$ then M is the mapping torus of an isometry of an X-manifold with fundamental group K. (In the latter case we must assume that M is orientable). If $X = Nil^3$ or Sol^3 then K is the fundamental group of an X-manifold or of an E^3-manifold. If $X = H^3$ then K is the fundamental group of a H^3- or $H^2 \times E^1$-manifold.*

Proof. In all cases π is a semidirect product $K \times_\theta Z$ and may be realised by the mapping torus of a self homeomorphism of a closed 3-manifold with fundamental group K. If this manifold is an X-manifold then the outer automorphism class of θ is finite (see Chapter VI of [H]) and θ may then be realized by an isometry of an X-manifold. Infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups. This is also true of $S^2 \times E^2$- and $S^3 \times E^1$-manifolds [Oh90], provided K is not finite cyclic, and for orientable $\tilde{SL} \times E^1$- and $H^2 \times E^2$-manifolds [Ue90,91]. (Note that $\tilde{SL}$-manifolds are orientable and self homeomorphisms of such manifolds are orientation preserving [NR78]). When K is finite cyclic it is still true that every such $S^3 \times E^1$-manifold is a mapping torus of an isometry of a suitable lens space [Oh90]. Thus if M is an $X \times E^1$-manifold and K is the fundamental group of an X-manifold M is the mapping torus of an isometry of an X-manifold with fundamental group K. //

Does the Corollary remain true for nonorientable $H^2 \times E^2$-manifolds?

There are (orientable) $Nil \times E^1$ and $Sol \times E^1$ manifolds which are mapping tori of self homeomorphisms of flat 3-manifolds, but which are not mapping tori of self homeomorphisms of Nil- or Sol-manifolds. (See pages 96 and 99 of [H], respectively). We may construct analogous examples when $X = H^3$ as follows. Let F be a closed hyperbolic surface and $\alpha : F \to F$ a pseudo-Anasov diffeomorphism. Let $\Theta(f,z) = (\alpha(f), \bar{z})$ for all (f,z) in $F \times S^1$. Then the mapping torus of Θ is an $H^3 \times E^1$-manifold which is not a mapping torus of any self-homeomorphism of an H^3-manifold.

A closed $H^3 \times E^1$-manifold is either a mapping torus of a self homeomorphism of a hyperbolic 3-manifold or is the union of two twisted I-bundles over hyperbolic bases. (See the discussion of $H^2 \times E^1$-manifolds on page 460 of [Sc83]).

Let N be an aspherical closed X-manifold where $X = H^3$, $\tilde{SL}$ or $H^2 \times E^1$, and suppose that $\beta_1(N) > 0$ but N is not a surface bundle over the circle. Choose an epimorphism $\lambda : \pi_1(N) \to Z$ and let $\hat{N}$ be the 2-fold covering space associated to the subgroup $\lambda^{-1}(2Z)$. If $\nu : \hat{N} \to \hat{N}$ is the covering involution then $\mu(n,z) = (\nu(n), \bar{z})$ defines a free involution on $N \times S^1$, and the orbit space M is a closed $X \times E^1$-manifold with $\beta_1(M) > 0$ which is not a mapping torus. (However all manifolds with product geometries are virtually cartesian products).

References

[Ba80] Barge, J. Dualité dans les revêtements galoisiens, Invent. Math. 58 (1980), 101-106.

[B] Bieri, R *Homological Dimension of Discrete Groups*, Queen Mary College Lecture Notes, London (1976).

[Br75] Brown, K.S. A homological criterion for finiteness, Comment. Math. Helvetici 50 (1975), 129-135.

[HM86] Hambleton, I. and Madsen, I. Actions of finite groups on R^{n+k} with fixed point set R^k, Canadian Math. J. 38 (1986), 781-860.

[H] Hillman, J.A. *The Algebraic Characterization of Geometric 4-manifolds*, London Mathematical Society Lecture Series 198, Cambridge University Press, Cambridge - New York - Melbourne (1994).

[NR78] Neumann, W.D. and Raymond, F. Seifert manifolds, plumbing, μ-invariant and orientation reversing homeomorphisms, in *Algebraic and Geometric Topology* (edited by K.C.Millet), Lecture Notes in Mathematics 664, Springer-Verlag, Berlin - Heidelberg - New York (1978), 163-196.

[Oh90] Ohba, K. On fiber bundles over S^1 having small Seifert manifolds as fibres, J. Fac. Sci. Tokyo 37 (1990), 659-702.

[Sc83] Scott, G.P. The geometries of 3-manifolds, Bull. London Math. Soc. 15 (1983), 401-487.

[St76] Strebel, R. A homological finiteness condition, Math. Z. 151 (1976), 263-275.

[St77] Strebel, R. A remark on subgroups of infinite index in Poincaré duality groups, Comment. Math. Helvetici 52 (1977), 317-324.

[Wa65] Wall, C.T.C. Finiteness conditions for CW complexes,

Ann. Math. 81 (1965), 56-69.
[Ue90] Ue, M. Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds. I, J. Math. Soc. Japan 42 (1990), 511-540.
[Ue91] Ue, M. Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds. II, J. Math. Soc. Japan 43 (1991), 149-183.