L^2-Homology and asphericity

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ABSTRACT

We use L^2 methods to show that if a group with a presentation of deficiency one is an extension of Z by a finitely generated normal subgroup then the 2-complex corresponding to any presentation of optimal deficiency is aspherical and to prove a converse of the Cheeger-Gromov-Gottlieb theorem relating Euler characteristic and asphericity. These results are applied to several open problems on knot groups and on the homotopy types of certain 4-manifolds.

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§0. Introduction

One of the applications of L^2-cohomology in [CG] was to show that if X is a finite aspherical complex such that $\pi_1(X)$ has an infinite amenable normal subgroup A then $\chi(X) = 0$. (This generalised a theorem of Gottlieb, who assumed that A was a central subgroup [Gt]). When A is an elementary amenable group this has also been proven by a localization argument, and there is then a converse: if X is a $[\pi, m]_f$-complex (a finite m-dimensional complex with $\pi_1(X) \cong \pi$ and with $(m-1)$-connected universal cover $\tilde{X}$) and π has a nontrivial torsion free elementary amenable group then X is aspherical if and only if $\chi(X) = 0$ ([Hi3] - see also [Hi1, Hi2, Li and Ro]). Here we shall show that the converse also holds when A is merely infinite and amenable, as another easy application of L^2 methods. We shall first give a similar but easier argument for groups π which are extensions of Z by a finitely generated normal subgroup which relates the conditions "π

has deficiency 1" and "there is an aspherical $[\pi, 2]_f$-complex". The result restricts further possible counterexamples to the Whitehead problem on subcomplexes of aspherical 2-complexes and has interesting consequences for knot theory. In the final section we shall apply the earlier work to the study of abelian subgroups of 2-knot groups and to the homotopy characterization of certain closed 4-manifolds.

If π is a group then $\zeta\pi$ and π' shall denote the centre and the commutator subgroup of π, respectively. If π is finitely presentable $def(\pi)$ shall denote its deficiency. A $PD_n^{(+)}$-group is an (orientable) Poincaré duality group of dimension n. A $[\pi, m]_f$-complex X is aspherical if and only if $\pi_m(X) = 0$. In that case we shall say that π has geometric dimension at most m, written $g.d.\pi \leq m$.

§1. Extensions of Z by finitely generated normal subgroups

The L^2-Betti numbers $\beta_i^{(2)}(X)$ of a finite complex X are defined in [At]. (See also [CG], [Ec2] and [Lü]). They are multiplicative in finite covers, and for $i = 0$ or 1 depend only on $\pi_1(X)$. In [CG] a limiting process is used to define L^2-Betti numbers $\beta_i^{(2)}(Y; \pi)$ for general actions of countable group π on a space Y, and it is shown that $\beta_i^{(2)}(\pi) = \beta_i^{(2)}(K_\pi; \pi)$, where K_π is (any) contractible complex on which π acts freely. If X is a finite Poincaré duality complex then these Betti numbers satisfy Poincaré duality. The alternating sum of the L^2-Betti numbers is the Euler characteristic $\chi(X)$ [At]. The usual Betti numbers of a space or group shall be denoted by $\beta_i(X; Q) = dim_Q H_i(X; Q)$.

Theorem 1. *Let π be a finitely presentable group such that $\beta_i^{(2)}(\pi) = 0$ for $i < m$, and let X be a $[\pi, m]_f$-complex. If $\chi(X) = 0$ then X is aspherical.*

Proof. Since X is $(m-1)$-connected $\beta_i^{(2)}(X) = \beta_i^{(2)}(\pi) = 0$ for $i < m$ and so $\chi(X) = (-1)^m \beta_m^{(2)}(X)$. Hence $\beta_m^{(2)}(X) = 0$ also, and so the L^2-homology of X is trivial. Since X is m-dimensional $\pi_m(X) = H_m(\tilde{X}; Z)$ is a subgroup of the m^{th} L^2-homology group of X. Therefore $\pi_m(X) = 0$ and so X is aspherical. //

If $X = S^1 \vee S^1$ then X is an aspherical $[F(2), 1]_f$-complex and $\beta_0^{(2)}(F(2)) = 0$, but

$\chi(X) = -1 \neq 0$. Thus the implication in the statement of this theorem cannot be reversed, in general.

Theorem 2. *Let π be an infinite finitely presentable group. Then $def(\pi) \leq 1 + \beta_1^{(2)}(\pi)$, with equality only if $g.d.\pi \leq 2$.*

Proof. Let X be a finite 2-complex correponding to a presentation P for π. Then $def(P) = 1 - \chi(X) = 1 + \beta_1^{(2)}(\pi) - \beta_2^{(2)}(X) \leq 1$. If $def(P) = 1 + \beta_1^{(2)}(\pi)$ then $\beta_2^{(2)}(X) = 0$, so $\pi_2(X) = H_2(\tilde{X}; Z) = 0$ and X is aspherical. Hence $g.d.\pi \leq 2$. //

Let $G = F(2) \times F(2)$. Then $\beta_1^{(2)}(G) = 0$, by Proposition 2.7 of [CG]. Moreover $def(G) = 0$ and $g.d.G = 2$; in fact $< u, v, x, y | ux = xu, uy = yu, vx = xv, vy = yv >$ is an optimal presentation. Thus the implication in the final sentence of the statement of this theorem cannot be reversed, in general. (Compare the Corollary of Theorem 3 below).

The following lemma is part of Theorem 2.1 of [Lü] (proven there under slightly stronger hypotheses).

Lemma [Lück]. *If a finitely presentable group π is an extension of Z by a finitely generated normal subgroup N then $\beta_1^{(2)}(\pi) = 0$.*

Proof. Suppose that N is generated by g elements and let N_n be the preimage in π of the subgroup $nZ \leq Z$. Then $[\pi : N_n] = n$, so $\beta_1^{(2)}(N_n) = n\beta_1^{(2)}(\pi)$. But each N_n is also finitely presentable and is generated by $g + 1$ elements. Hence $\beta_1^{(2)}(N_n) \leq g + 1$, and so $\beta_1^{(2)}(\pi) = 0$. //

If the Whitehead conjecture is false then either there is a finite nonaspherical 2-complex X such that $X \cup_f D^2$ is contractible for some $f : S^1 \to X$ or there is an infinite ascending chain of nonaspherical 2-complexes whose union is contractible [Ho]. In the finite case $\chi(X) = 0$ and so $\pi = \pi_1(X)$ has deficiency 1; moreover, π has weight 1 since it is normally generated by the conjugay class represented by f. Such groups are 2-knot groups. Conversely, the exterior of a ribbon n-knot or of a ribbon concordance between classical knots is homotopy equivalent to such a 2-complex. (The asphericity of such ribbon exteriors has been raised in Question 2 of [Co] and Question 6.5 of [Go]). If π' is finitely generated then $\beta_1^{(2)}(\pi)=0$, by Lück's Lemma, and so X is aspherical, by Theorem 1.

A group is called knot-like if it has abelianization Z and deficiency 1 [Ra]. Rapaport asked whether the commutator subgroup of a knot-like group must be free if it is finitely generated, and established this in the 2-generator, 1-relator case [Ra]. Our next corollary provides a substantial partial answer to this question.

Corollary. *Let π be a finitely presentable group which is an extension of Z by a finitely generated normal subgoup N, and suppose that $\beta_2(\pi;Q)=\beta_1(\pi;Q)-1$. Then $def(\pi)=1$ if and only if $g.d.\pi\leq 2$. If $def(\pi)=1$ and N is almost finitely presentable then N is free.*

Proof. If $def(\pi)=1$ then $g.d.\pi\leq 2$ by Lück's Lemma and Theorem 2. Conversely, if X is a finite aspherical 2-complex with $\pi_1(X)\cong\pi$ then $\chi(X)=1-\beta_1(\pi;Q)+\beta_2(\pi;Q)=0$. After collapsing a maximal tree in X we may assume it has a single 0-cell, and then the presentation read off the 1- and 2-cells has deficiency 1. The final assertion then follows from Corollary 8.6 of [Bi]. //

In particular, if the group of a fibred 2-knot has a presentation of deficiency 1 then its commutator subgroup must be free. This settles Problem 14 of [Hi2]. Any 2-knot with such a group is s-concordant to a fibred homotopy ribbon knot, by Theorem VIII.7 of [Hi2]. Must it in fact be a ribbon knot? (This is Problem 13 of [Hi2]).

The kernel of the homomorphism from $F(2)\times F(2)=|<u,v,x,y|ux=xu,uy=yu,vx=xv,vy=yv>|$ to Z which sends u and y to 0 and v and x to a generator is generated by u, vx^{-1} and y, but is not free, as u and y generate a rank two abelian subgroup. (Thus this kernel is finitely generated but not almost finitely presentable. See page 119 of [Bi]). Silver has given examples of high-dimensional knot groups whose commutators subgroups are finitely generated but not finitely presentable [Si1]. He has also suggested that every knot-like group should have a finitely presentable HNN base. If this were true the Corollary would settle Rapaport's question completely, for if the commutator subgroup is finitely generated then it is the unique HNN base [Si2]. (See also Problem 22 of [Hi2]).

§2. Infinite amenable normal subgroups

The following result is stated without proof on page 226 of [Gr]. I am grateful to Peter Linnell for explaining how it follows from the argument of §3 of [CG].

Theorem [Gromov]. *Let π be a group with an infinite amenable normal subgroup A. Then $\beta_i^{(2)}(\pi) = 0$ for all i.*

Proof. Since $K_{\pi/A} \times K_\pi$ (with the diagonal π-action) is π-freely homotopy equivalent to K_π we have $\beta_i^{(2)}(\pi) = \beta_i^{(2)}(K_{\pi/A} \times K_\pi; \pi)$, for all i. This is in turn equal to $\beta_i^{(2)}(K_{\pi/A}; \pi)$, by Proposition 2.2 of [CG]. Now the cell-stabilizers of the action of π on $K_{\pi/A}$ are all A, and by Theorem 0.2 of [CG], $\beta_i^{(2)}(A) = 0$, for all i. Since $\Sigma\beta_i^{(2)}(K_{\pi/A}; \pi) \leq \Sigma\beta_i^{(2)}(A)$, by Theorem 0.1 of [CG], it follows that $\beta_i^{(2)}(\pi) = \beta_i^{(2)}(K_{\pi/A}; \pi) = 0$, for all i. //

Since the amenability of A is used only to ensure that $\beta_i^{(2)}(A) = 0$ for all i, it is sufficient to assume that A be subnormal in π and $\beta_i^{(2)}(A) = 0$ for all i. (Note however that if a group has an infinite amenable subnormal subgroup then it has an infinite amenable normal subgroup).

The next result gives a converse to the Cheeger-Gromov extension of Gottlieb's Theorem.

Theorem 3. *Let X be a $[\pi, m]_f$-complex and suppose that π has an infinite amenable normal subgroup. Then X is aspherical if and only if $\chi(X) = 0$.*

Proof. Since $\chi(X) = (-1)^m \beta_m^{(2)}(X)$ this follows immediately from Gromov's Theorem and Theorem 1. //

Corollary. *Let π be a finitely presentable group with an infinite amenable normal subgroup A. Then $def(\pi) = 1$ if and only if $g.d.\pi \leq 2$.*

Proof. If X is an aspherical $[\pi, 2]_f$-complex then $\chi(X) = 0$ by Gromov's Theorem, so $def(\pi) = 1 - \chi(X) = 1$. Conversely if X is the finite 2-complex corresponding to a presentation of deficiency 1 then $\chi(X) = 0$ and so X is aspherical by Theorem 3.//

In [Hi3] it is shown that if $def(\pi) = 1$ and the subgroup A is elementary amenable then either $A \cong Z$ or π is metabelian. Is this true in general? (If the Tits alternative holds for groups of finite cohomological dimension this would be so).

§3. Applications to 4-manifolds and 2-knots.

If $L : \mu S^2 \to S^4$ is a 2-link $M(L)$ shall denote the closed orientable 4-manifold obtained by surgery on the components of L. The link group is then $\pi L = \pi_1(M(L))$. If K is a 2-knot ($\mu = 1$) $M(K)'$ shall denote the infinite cyclic covering space, with $\pi_1(M(K)') = \pi K'$.

The arguments of §1 enable us to show that no 2-knot group π with deficiency 1 can be a PD_4-group (thereby removing the hypothesis $\pi' \neq \pi''$ from Theorem III.9 of [Hi2]). For if X is a finite 2-complex corresponding to a deficiency 1 presentation of π then $\beta_2^{(2)}(\pi) - \beta_1^{(2)}(\pi) \leq \beta_2^{(2)}(X) - \beta_1^{(2)}(\pi) = \chi(X) = 0$. If π were a PD_4-group then we would also have $0 = \chi(\pi) = \beta_2^{(2)}(\pi) - 2\beta_1^{(2)}(\pi)$. Hence all these L^2-Betti numbers are 0 and so X is aspherical, and so $c.d.\pi = 2 \neq 4$.

The following theorem is implicit in the addendum to [Ec].

Theorem [Eckmann]. *Let M be a finite PD_4-complex with $\chi(M) = 0$ and let $\pi = \pi_1(M)$. If $\beta_1^{(2)}(\pi) = 0$ then the natural map from $H^2(\pi; Z[\pi])$ to $H^2(M; Z[\pi])$ is an isomorphism. In particular, if moreover $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$ then M is aspherical.*

Proof. Since M is a PD_4-complex $\chi(M) = 2\beta_0^{(2)}(\pi) - 2\beta_1^{(2)}(\pi) + \beta_2^{(2)}(M)$. Since π is infinite $\beta_0^{(2)}(\pi) = 0$, and $\beta_1^{(2)}(\pi) = 0$ by hypothesis. Hence $\beta_2^{(2)}(M) = \chi(M) = 0$ also. It now follows from Proposition 1.2 of [Ec2] (the natural map from *un*reduced L^2-cohomology to ordinary cohomology factors through reduced L^2-cohomology) and diagram (4) on page 504 of [Ec2] that the natural map from $H^2(M; Z[\pi])$ to $H^2(\tilde{M}; Z)$ is 0 (where $\tilde{M}$ is the universal cover of M) and so the map from $H^2(\pi; Z[\pi])$ to $H^2(M; Z[\pi])$ is an isomorphism. The final assertion follows by equivariant Poincaré duality in the universal covering space. //

We shall use the theorems of Eckmann and Gromov in conjunction to obtain strong restrictions on the possible abelian normal subgroups of 2-knot groups.

Theorem 4. *Let $L : \mu S^2 \to S^4$ be a 2-link with group $\pi = \pi L$. If $\mu \geq 2$ then $\zeta\pi$ is finite. If $\mu = 1$ and $\zeta\pi$ is infinite, then either π is a PD_4-group (and so $\zeta\pi$ is torsion free of rank 1 or 2) or $H^2(\pi; Z[\pi]) \neq 0$ and $\zeta\pi$ is finitely generated of rank 1 or is a torsion group.*

Proof. We have $\beta_2^{(2)}(\pi) - 2\beta_1^{(2)}(\pi) = \chi(M(L)) = 2(1-\mu)$. Hence if $\zeta\pi$ is infinite, so $\beta_1^{(2)}(\pi) = 0$ by Gromov's Theorem, we must have $\mu = 1$, and so L is a 2-knot. Moreover π has one end, i.e., $H^s(\pi; Z[\pi]) = 0$ for $s \leq 1$. Hence if also $H^2(\pi; Z[\pi]) = 0$ then $M(L)$ is aspherical, by Eckmann's Theorem. (In this case the centre is torsion free, and is of rank at most 2: see Chapter VI of [Hi2]). If $\zeta\pi$ has an infinite cyclic subgroup A such that $\zeta\pi/A$ is infinite then this cohomological condition holds, for then $\zeta(\pi/A)$ is infinite, so π/A has one end, and so π is simply connected at ∞, by Theorem 1 of [Mi]. Thus if $H^2(\pi; Z[\pi]) \neq 0$ then $\zeta\pi$ is finitely generated of rank 1 or is a torsion group. //

This provides a substantial partial answer to Problem 16 of [Hi2], which asked whether the centre of a 2-knot group must be finite cyclic, Z, $Z \oplus (Z/2Z)$ or Z^2 and whether the centre of the group of a 2-link with more than one component must be trivial. Most of the results in the case $\mu = 1$ follow also via the localization arguments of [Hi2], on observing that π/A cannot have two ends (since a knot group cannot be virtually Z^2) and has finite centre if it has infinitely many ends; however Gromov's Theorem is needed to consider the possibility that $\zeta\pi$ may be an infinite torsion group. No examples of the latter type are known. Note also that the argument used in the rank 1 case settles Problem 19 of [Hi2], for if the group π of a 2-knot K has a rank one abelian normal subgroup A such that π/A has one end then $\beta_1^{(2)}(\pi) = 0$ and $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$, and so $M(K)$ is aspherical.

The theorems of Gromov and Eckmann together settle Problem 11 of [Hi5]. We may similarly simplify the hypotheses of some theorems in Chapter VI of [Hi5]. In the next result h shall denote the Hirsch length, a natural measure of the size of an elementary amenable group. (See [Hi3,5] for details).

Theorem 5. *Let M be a closed 4-manifold with $\chi(M) = 0$. Suppose that $\pi = \pi_1(M)$ has an elementary amenable normal subgroup ρ with $h(\rho) \geq 2$ and $H^2(\pi; Z[\pi]) = 0$. Then*

M is aspherical. If $h(\rho) = 2$ then ρ is virtually abelian, while if $h(\rho) \geq 3$ then M is homeomorphic to an infrasolvmanifold.

Proof. Since $\beta_i^{(2)}(\pi) = 0$ for all i, by the theorems of Eckmann and Gromov, M is aspherical. Hence ρ must be torsion free and the theorem follows from Theorems VI.2 and VI.11 of [Hi5]. //

If ρ is torsion free, of infinite index and $h(\rho) = 2$ then the hypothesis $H^2(\pi; Z[\pi]) = 0$ follows from [Mi]. (See Theorem VI.11 of [Hi5]). The hypothesis that the index be infinite is necessary; every group with a presentation of the form $< a, t | tat^{-1} = a^n >$ is torsion free and solvable of Hirsch length 2, and is the fundamental group of some closed orientable 4-manifold with Euler characteristic 0. If M is a closed orientable 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is amenable, has one end and $H^2(\pi; Z[\pi]) \neq 0$ must π be one of these groups? (Corollary 4.2 of [Ec2] implies that Problem 17 of [Hi2] may be reduced to a special case of this question). If $h(\rho) \geq 3$ can the hypothesis on $H^2(\pi; Z[\pi])$ be dropped completely? Can the hypotheses of this theorem be rephrased in terms of amenable normal subgroups, using homological dimension over Q rather than Hirsch length as a measure of the size of such groups? (This is Problem 12 of [Hi5], and would follow from a Tits alternative for groups of finite cohomological dimension).

Corollary. *Let K be a 2-knot whose group $\pi = \pi K$ has an abelian normal subgroup A of rank > 1. Then $M(K)$ is aspherical and so A is torsion free.*

Proof. An easy LHS spectral sequence argument shows that $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$. Hence $M(K)$ is aspherical, by Theorem 5, and so π is a PD_4^+-group and A is torsion free.//

In fact either π' is a PD_3^+-group with centre $A \cap \pi'$ of rank 1 but not finitely generated (which seems unlikely) or A is free abelian, of rank 2, 3 or 4 [Hi2].

If π' is almost finitely presentable (FP_2) we can do better. The strategy of the next result is adapted from that of [Hi4].

Theorem 6. *Let M be a closed 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is an extension of Z by an almost finitely presentable infinite normal subgroup N with* *a nontrivial finite normal subgroup F. Then M is homotopy equivalent to the mapping torus of a self homeomorphism of $RP^2 \times S^1$.*

Proof. Let $\tilde{M}$ be the universal covering space of M. Since N is infinite and finitely generated π has one end, and so $H_i(\tilde{M}; Z) = 0$ for $i \neq 0$ or 2. Let $\Pi = \pi_2(M) = H_2(\tilde{M}; Z)$. We wish to show that $\Pi \cong Z$, and that $w = w_1(M)$ maps F isomorphically onto $Z^\times = \{\pm 1\}$. Since $\beta_1^{(2)}(\pi) = 0$ by Lück's Lemma, Poincaré duality and Eckmann's Theorem together give an isomorphism of left $Z[\pi]$-modules $\Pi \cong \overline{H^2(\pi; Z[\pi])}$. An application of the LHS spectral sequence for π as an extension of Z by N then gives $\Pi \cong \overline{H^1(N; Z[N])}$, which is a free abelian group.

Let f be a nontrivial element of F. Since F is normal in π the centralizer $C_\pi(f)$ of f has finite index in π, and we may assume without loss of generality that F is generated by f and is central in π. It follows from the spectral sequence for the projection of $\tilde{M}$ onto $\tilde{M}/F$ that there are isomorphisms $H_{s+3}(F; Z) \cong H_s(F; \Pi)$ for all $s \geq 4$, since $\tilde{M}/F$ is a 4-dimensional complex. Here F acts trivially on Z, but we must determine its action on Π.

Now central elements n of N act trivially on $H^1(N; Z[N])$ and hence via $w(n)$ on Π. (See [Hi4]). Thus if $w(f) = 1$ the sequence $0 \to Z/|f|Z \to \Pi \to \Pi \to 0$ is exact, where the right hand homomorphism is multiplication by $|f|$. As Π is torsion free this contradicts $f \neq 1$. Therefore if f is nontrivial it has order 2 and $w(f) = -1$. Hence $w : F \to Z^\times$ is an isomorphism and there is an exact sequence $0 \to \Pi \to \Pi \to Z/2Z \to 0$, where the left hand homomorphism is multiplication by 2. Since Π is a free abelian group it must be infinite cyclic, and so $\tilde{M} \simeq S^2$. The theorem now follows from Theorems VII.4 and VII.7 of [Hi5]. //

Corollary. *Let K be a 2-knot whose group $\pi = \pi K$ has a nontrivial abelian normal subgroup A, and suppose that π' is finitely presentable. Then either π' is finite or $A \cap \pi' = 1$ and $A \cong Z$ or $M(K)$ is aspherical and A is torsion free. In all cases $M(K)'$ is an orientable PD_3-complex.*

Proof. If $A \cap \pi'$ is finite then either $A \cap \pi' = 1$ or π' is finite, by Theorem 6. Hence we may assume that $A \cap \pi'$ is infinite. But then π' has one end and so $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$, by an LHS spectral sequence argument. Hence $M(K)$ is aspherical, by Eckmann's Theorem. The final assertion follows from [Hi6]. //

If we assume only that π' is finitely generated then this Corollary and Theorem IV.5 of [Hi2] together imply that either π' is finite or $A \cap \pi' = 1$ or $M(K)$ is aspherical or A is a torsion group. There are no known examples of 2-knot groups π with π' finitely generated but not finitely presentable. (See [Si1] for higher dimensional examples).

L^2-Cohomology is used in [Hi6] to give a simple characterization of PD_4-complexes which are homotopy equivalent to mapping tori, which largely settles Problem 8 of [Hi2] and reduces Problem 7 of [Hi5] to that of characterizing PD_3-complexes which are homotopy equivalent to 3-manifolds. In particular, a closed 4-manifold M is homotopy equivalent to the total space of a surface bundle over a torus if and only if $\chi(M) = 0$ and $\pi = \pi_1(M)$ is an extension of Z^2 by a finitely presentable normal subgroup. The next result is an alternative characterization of 4-manifolds covered by such bundle spaces, and settles Problem 14 of [Hi5].

Theorem 7. *Let M be a closed 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ has a normal subgroup G of infinite index which is a PD_2-group. Then M is aspherical. If $\zeta G = 1$ then M is finitely covered by a manifold which is simple homotopy equivalent to the total space of a surface bundle over the torus.*

Proof. It follows easily from the LHS spectral sequence for π as an extension of $H = \pi/G$ by G that $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$. After passing to a finite covering space if necessary, we may assume that the image of H in $Out(G)$ is torsion free, since $Out(G)$ is virtually of finite cohomological dimension. The image of H in $Out(G)$ is isomorphic to $\pi/G.C_\pi(G)$ and H is an extension of this group by $G.C_\pi(G)/G \cong C_\pi(G)/\zeta G$.

If $\zeta G \neq 1$ then it is an infinite (elementary) amenable normal subgroup of π and so M is aspherical, by the theorems of Eckmann and Gromov. (The asphericity also follows from Theorem II.6 of [Hi5], since $Z[\pi]$ has a safe extension). If $\zeta G = 1$ and H has an element of infinite order then $\beta_1^{(2)}(\pi) = 0$ by Theorem 3.1 of [Lü] and so M is aspherical by Eckmann's Lemma. Moreover $v.c.d.H \leq v.c.d.K + c.d.C_\pi(G) < \infty$ and so H is virtually a PD_2-group, by Theorem 9.11 of [Bi]. In fact H must be virtually Z^2, since we then have $0 = \chi(\pi) = \chi(G)\chi_{virt}(H)$. If H is torsion then its image in $Out(G)$ is finite, hence trivial, so $\pi = G.C_\pi(G)$ and $H \cong C_\pi(G)/\zeta(G) \cong C_\pi(G)$. Since $G \cap C_\pi(G) = \zeta G$ is trivial $\pi \cong G \times C_\pi(G) \cong G \times H$. But then $\beta_1^{(2)}(G \times H) = 0$, by Proposition 2.7 of [CG], and so $G \times H$ is a PD_4-group, hence torsion free, contrary to the assumption that H is an infinite torsion group. This completes the argument. //

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