On the homotopy types of closed 4-manifolds covered by $S^2 \times R^2$

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ABSTRACT

We show that if M is a closed 4-manifold such that $\pi_2(M) \cong Z$ and $\chi(M) \leq 0$ then its homotopy type is determined up to a finite ambiguity by $\pi = \pi_1(M)$. We also show that (for given π) the action of π on $\pi_2(M)$ and the orientation character $w_1(M)$ determine each other, and find strong constraints on the possible k-invariants when $\chi(M) = 0$. Finally we show that $Wh(\pi) = 0$.

AMS Subject Classification (1991): Primary 57N13. Secondary 55R10, 53C30.

Key words and phrases: 4-manifold, geometry, homotopy type, sphere bundle.

§0. Introduction

This paper considers the problem of characterizing the homotopy types of closed 4-manifolds admitting one of the geometries $S^2 \times E^2$ or $S^2 \times H^2$. Such manifolds have universal covering space homeomorphic to $S^2 \times R^2$; in particular the fundamental group is nontrivial and the second homotopy group is infinite cyclic. It seems plausible that the latter conditions might characterize such manifolds, up to homotopy.

For suppose that M is a closed 4-manifold with $\pi_2(M) \cong Z$, and let $\pi = \pi_1(M)$. If $\pi = 1$ then M is homotopy equivalent to CP^2. Otherwise π must be infinite, and the subgroup ρ which acts trivially on $\pi_2(M)$ is torsion free, by Theorem II.11 of [H]. If moreover $\chi(M) \leq 0$ then π is isomorphic to a discrete subgroup of isometries of $S^2 \times E^2$ or $S^2 \times H^2$ which acts freely and properly discontinously. Moreover the covering space corresponding to the subgroup ρ is homotopy equivalent to the total space of an S^2-

bundle over an aspherical closed surface, by Theorem VII.1 of [H]. Such bundle spaces are geometric, of type $S^2 \times E^2$ (if $\chi(M) = 0$) or $S^2 \times H^2$ (if $\chi(M) < 0$) [CH96].

We may put this problem in another context by recalling that if the universal covering space of a closed 4-manifold is homotopy finite then the manifold is either aspherical or has finite fundamental group or its universal covering space is homotopy equivalent to S^3 or S^2, by Theorem II.9 of [H]. The homotopy type of an aspherical space is determined by its fundamental group. Closed 4-manifolds with finite fundamental group have been studied intensively; if the fundamental group has odd order the quadratic 2-type is a complete homotopy invariant [HK], and this may be generally true in the orientable case. The fundamental group and first k-invariant provide complete homotopy invariants for 4-manifolds covered by S^3, by Theorem II.10 of [H]. Thus we consider here the remaining possibility.

We shall assume henceforth that $\chi(M) \leq 0$. (We do not know whether this follows from the conditions $\pi_1(M) \neq 1$ and $\pi_2(M) \cong Z$). The main result is that there are only finitely many homotopy types of such 4-manifolds with given fundamental group π. In particular, in the nonorientable cases there are at most two such manifolds with given π and first k-invariant. En route, we show that the action of π on $\pi_2(M)$ and the orientation character $w_1(M)$ determine each other. We also determine the second Stiefel-Whitney class $w_2(M)$ and find strong restrictions on the possible k-invariants when $\chi(M) = 0$. We also show that $Wh(\pi) = 0$ for all groups π arising as fundamental groups of such manifolds.

§1. Homotopy type

In this section we shall extend Theorem VII.9 of [H] to all the 4-manifolds satisfying our hypotheses.

Let Z^x denote the π-module with underlying group Z determined by a homomorphism $x : \pi \to Aut(Z) = Z^\times$.

Lemma 1. *The group $H^2(\pi; Z[\pi])$ is infinite cyclic, and $u = w_1(M) + v$, where $u : \pi \to$*

$Aut(\pi_2(M)) = Z/2Z$ *and* $v : \pi \to Aut(H^2(\pi; Z[\pi])) = Z/2Z$ *are the natural actions.*

Proof. Since π is infinite $Hom_{Z[\pi]}(\pi_2(M), Z[\pi]) = 0$ and so $\overline{H^2(\pi; Z[\pi])} \cong \pi_2(M)$, by Lemma II.4 of [H]. Now $\overline{H^2(\pi; Z[\pi])} \cong H^2(\pi; Z[\pi]) \otimes Z^{w_1(M)}$, (where the tensor product is over Z and has the diagonal π-action). Hence $Z^u = Z^v \otimes Z^{w_1(M)}$ and so $u = w_1(M) + v$. //

This lemma provides the natural explanation of the *ad hoc* observations of [H] relating the action of π on $\pi_2(M)$ to $w_1(M)$ in the $\chi(M) = 0$ cases. Note that u and $w_1(M)$ are constrained by the further conditions that $\rho = \mathrm{Ker} u$ is torsion free and $\mathrm{Ker} w_1(M)$ has infinite abelianization if $\chi(M) \leq 0$.

Theorem 1. *Let M be a closed 4-manifold such that $\pi_2(M) \cong Z$ and $\chi(M) \leq 0$. Then there are only finitely many homotopy types of such manifolds with fundamental group π and orientation character $w_1(M)$. If $w_1(M) \neq 0$ there are at most two such homotopy types with given first k-invariant.*

Proof. By the lemma, the action of π on $\pi_2(M)$ is determined by $w_1(M)$. As $c.d.\rho = 2$, a calculation with the LHS spectral sequence shows that $H^3(\pi; \pi_2(M))$ is finite, so there are only finitely many possible k-invariants. The action and the first k-invariant $k_1(M)$ determine $P = P_2(M)$, the second stage of the Postnikov tower for M. Let $\tilde{P} \simeq K(Z, 2)$ denote the universal covering space of P.

As $f_M : M \to P$ is 3-connected we may define a class w in $H^1(P; Z/2Z)$ by $f_M^* w = w_1(M)$. Let $S_4^{PD}(P)$ be the set of "polarized" PD_4-complexes (X, f) where $f : X \to P$ is 3-connected and $w_1(X) = f^* w$, modulo homotopy equivalence over P. (Note that as π is one-ended the universal cover of X is homotopy equivalent to S^2). Let $[X]$ be the fundamental class of X in $H_4(X; Z^w)$. It follows as in Lemma 1.3 of [HK88] that given two such polarized complexes (X, f) and (Y, g) there is a map $h : X \to Y$ with $gh = f$ if and only if $f_*[X] = g_*[Y]$ in $H_4(P; Z^w)$. Since $\tilde{X} \simeq \tilde{Y} \simeq S^2$ and f and g are 3-connected such a map h must be a homotopy equivalence.

From the Cartan-Leray homology spectral sequence for the classifying map $c_P : P \to K = K(\pi, 1)$ we see that there is an exact sequence

$$0 \to H_2(\pi; H_2(\tilde{P}) \otimes Z^w)/im(d^2_{5,0}) \to H_4(P; Z^w)/J \to H_4(\pi; Z^w),$$

where $J = H_0(\pi; H_4(\tilde{P}; Z) \otimes Z^w)/im(d^2_{3,2} + d^4_{5,0})$ is the image of $H_4(\tilde{P}; Z) \otimes Z^w$ in $H_4(P; Z^w)$. On comparing this spectral sequence with that for c_X we see that f induces an isomorphism from $H_4(X; Z^w)$ to $H_4(P; Z^w)/J$. We also see that $H_3(f; Z^w)$ is an isomorphism. Hence the cokernel of $H_4(f; Z^w)$ is $H_4(P, X; Z^w) \cong H_0(\pi; H_4(\tilde{P}, \tilde{X}; Z) \otimes Z^w)$, by the exact sequence of homology with coefficients Z^w for the pair (P, X). Since $H_4(\tilde{P}, \tilde{X}; Z) \cong Z$ as a π-module this cokernel is Z if $w = 0$ and $Z/2Z$ otherwise. Hence $J \cong \mathrm{Coker} H_4(f; Z^w)$. Thus if $w \neq 0$ there are at most two possible values for $f_*[X]$, up to sign. If $w = 0$ we shall show that there are only finitely many orbits of fundamental classes of such polarized complexes under the action of the group G of (based) self homotopy equivalences of P which induce the identity on π and $\pi_2(P)$.

The cohomology spectral sequence for c_P gives rise to an exact sequence

$$0 \to H^2(\pi; Z^u) \to H^2(P; Z^u) \to H^0(\pi; H^2(\tilde{P}; Z) \otimes Z^u) \cong Z \to H^3(\pi; Z^u).$$

Note that $H^2(\pi; Z^u) \cong Z$ modulo 2-torsion (since $w = 0$), $H^2(\tilde{P}; Z) \cong Z^u$ and $Z^u \otimes Z^u \cong Z$ as π-modules. Moreover the right hand map is the transgression, with image generated by $k_1(M)$. There is a parallel exact sequence with rational coefficients

$$0 \to H^2(\pi; Q^u) \cong Q \to H^2(P; Q^u) \to H^0(\pi; H^2(\tilde{P}; Z) \otimes Q^u) \cong Q \to 0.$$

Thus $H^2(P; Q^u)$ has a Q-basis t, z in the image of $H^2(P; Z^u)$ such that t is the image of a generator of $H^2(\pi; Z^u)/(torsion)$ and z has nonzero restriction to $H^2(\tilde{P}; Z)$. The spectral sequence also gives an exact sequence

$$0 \to H^2(\pi; H^2(\tilde{P}; Q)) \cong H^2(\pi; Q^u) \to H^4(P; Q) \to H^0(\pi; H^4(\tilde{P}; Q)) \cong Q \to 0.$$

Since $cd_Q\pi = 2$ we have $t^2 = 0$ in $H^4(P; Q^u \otimes Q^u) = H^4(P; Q)$; since $\tilde{P} \simeq K(Z, 2)$ we have $z^2 \neq 0$. Thus tz, z^2 is a Q-basis for $H^4(P; Q)$. A self homotopy h in G induces the identity on π, and its lift to a self map of $\tilde{P}$ is homotopic to the identity. Hence $h^*t = t$ and $h^*z \equiv z$ modulo Qt. Nevertheless we shall see that the action of G on $H^2(P; Q^u)$ is nontrivial.

Suppose first that $u = 0$, so π is an orientable surface group and $k_1(M) = 0$. Then $P \simeq K(\pi, 1) \times K(Z, 2)$ and $G \cong [K(\pi, 1), K(Z, 2)]$. Let $f : K(\pi, 1) \to K(Z, 2)$ be a map which induces an isomorphism on H^2 and fix a generator ζ for $H^2(K(Z, 2); Z)$. Then $t = pr_1^* f^* \zeta$ and $z = pr_2^* \zeta$ freely generate $H^2(P; Z)$, and so tz, z^2 freely generate $H^4(P; Z)$. Each $g \in [K(\pi, 1), K(Z, 2)]$ determines a self homotopy equivalence $\tilde{g} : P \to P$ by $\tilde{g}(k, n) = (k, g(k).n)$, where $K(Z, 2) = \Omega K(Z, 3)$ has the natural loop multiplication. Clearly $\tilde{g}$ is in G, and all elements of G are of this form [Ts80]. Let $d : G \to Z$ be the isomorphism determined by the equation $g^*\zeta = d(G) f^*\zeta$. Then $\tilde{g}^*t = (f pr_1 \tilde{g})^*\zeta = t$ and $\tilde{g}^*z = (pr_2\tilde{g})^*\zeta = (g pr_1)^*\zeta + pr_2^*\zeta = pr_1^*(g^*\zeta) + z = z + d(g)t$. On taking cup products we have $h^*(tz) = tz$ and $h^*(z^2) = z^2 + 2d(g)tz$. On passing to homology we see that there are two G-orbits of elements in $H_4(P; Z)$ whose images generate $H_4(P; Z)/J$.

In general let P^ρ denote the covering space corresponding to the subgroup ρ, and let G^ρ be the image of G in the group of self homotopy equivalences of P^ρ. Then lifting self homotopy equivalences defines a homomorphism from G to G^ρ, which by [Ts80] may be identified with the restriction from $H^2(\pi; Z^u)$ to $H^2(\rho; Z) \cong Z$, which has image of index ≤ 2. Moreover the projection induces an isomorphism from $H^4(P; Q)$ to $H^4(P^\rho; Q)$. Hence the action of G on $H_4(P; Z)/(torsion) \cong Z^2$ is nontrivial, and so there are only finitely many G-orbits of elements whose images generate $H_4(P; Z)/J$. This proves the theorem. //

Let β^u be the Bockstein operator associated with the exact sequence of coefficients $0 \to Z^u \to Z^u \to Z/2Z \to 0$, and let $\overline{\beta^u}$ be the composition with reduction modulo (2).

In general β^u is NOT the Bockstein operator for the untwisted sequence $0 \to Z \to Z \to Z/2Z \to 0$, and $\overline{\beta^u}$ is not Sq^1, as can be seen already for cohomology of the group $Z/2Z$ acting nontrivially on Z.

Lemma 2. *The Bockstein operator $\beta^u : H^2(\pi; Z/2Z) \to H^3(\pi; Z^u)$ is onto, and reduction modulo 2 from $H^3(\pi; Z^u)$ to $H^3(\pi; Z/2Z)$ is a monomorphism. If moreover π has nontrivial torsion then $k_1(M)$ has nonzero restriction to each subgroup of order 2.*

Proof. The group $H^3(\pi; Z^u)$ has exponent dividing 2, since the composition of restriction to $H^3(\rho; Z) = 0$ with the corestriction back to $H^3(\pi; Z^u)$ is multiplication by the index $[\pi : \rho]$. The first assertion then follows from the long exact sequence associated to the coefficient sequence. If $f : Z/2Z \to \pi$ is a monomorphism then $f^* k_1(M)$ is the first k-invariant of $\tilde{M}/f(Z/2Z) \simeq RP^2$, which is the nonzero element of $H^3(Z/2Z; \pi_2(M))$. //

As a consequence of this lemma we may assume that the cohomology class z in the above Theorem restricts to 2 times a generator of $H^2(\tilde{P}; Z)$, if $k_1(M) \neq 0$. A closer study of the action of G on $H^2(P; Z^u)$ suggests that in general there are at most 4 homotopy types with given π, w_1 and k-invariant. However we have not succeeded in proving this.

Significant features of the duality pairing such as the second Stiefel-Whitney class w_2 are not reflected in the Postnikov 2-stage. If π is torsion free $k_1(M) = 0$ and w_2 is the only other invariant needed. For then π is a surface group and each such manifold is homotopy equivalent to the total space of an S^2-bundle. There are two such bundle spaces for each group and orientation character, distinguished by the value of $w_2(M)$.

For RP^2-bundles $\pi \cong \rho \times (Z/2Z)$ and $H^3(\pi; Z^u) \cong (Z/2Z)^2$. There is an unique element of order 2 in π and $Aut(\pi) \cong Hom(\rho, Z/2Z) \tilde{\times} Aut(\rho)$. It is not hard to compute the action of $Aut(\pi)$ on $H^1(\pi; Z/2Z)$ and hence on $\beta^u(H^2(\pi; Z/2Z))$. There is a single $Aut(\pi)$-orbit of elements in $H^3(\pi; Z^u)$ which restrict nontrivially to $H^3(Z/2Z; Z^u)$. The orientation character is unique up to composition with an automorphism of π and there are two such bundle spaces for each surface group ρ, again distinguished by $w_2(M)$. All

RP^2-bundles over surfaces are geometric [CH].

If $w_1(M) = 0$, $u \neq 0$, $H^3(u; Z^u) = 0$ and $w_2(M)$ restricts to 0 in $H^2(\rho; Z/2Z)$ then M is homotopy equivalent to the total space of a surface bundle over RP^2. (These are all geometric, by Lemma IV.9 and Theorem VII.1 of [H]).

In general, we may view the classifying map $c_M : M \to K(\pi, 1)$ as a fibration with fibre S^2. The orientation character of this fibration is $w_1(c_M) = u$, and $k_1(M)$ is the (twisted) Euler class, so the reduction modulo (2) of $k_1(M)$ is $w_3(c_M) \in H^3(\pi; Z/2Z)$. (Conversely $k_1(M)$ is determined by $w_3(c_M)$, by Lemma 2). The covering space associated to the subgroup $\rho \cap \mathrm{Ker} w_1(M) \leq \pi$ is homotopy equivalent to $S^2 \times F$, where F is a closed orientable surface. Hence $w_2(M)$ restricts to 0 in $H^2(\tilde{M}; Z/2Z)$ and so is induced from π. The Wu formulae for M then imply that the total Stiefel-Whitney class $w(M)$ is induced from π. It can be shown that $c_M^*(w(c_M))$ is determined by $w(M)$ and π; unfortunately as $c_M^*(w_3(c_M)) = 0$ (by exactness of the Gysin sequence for c_M) we do not know whether $k_1(M)$ is also determined by these invariants.

§2. The spherical-euclidean case ($\chi = 0$).

If $\chi(M) = 0$ and π/π' is infinite there are six possible groups, and the homotopy type is also determined by π and $w(M)$, by Theorem VII.7 of [H]. (All possible homotopy types are realized by mapping tori of self homeomorphisms of $S^2 \times E^1$-manifolds). There are three other possible groups, $D *_Z D$, $D *_Z (Z \oplus (Z/2Z))$ and $Z *_Z D$, which have finite abelianization and are realizable only by nonorientable 4-manifolds. (Here $D = (Z/2Z) * (Z/2Z)$ is the infinite dihedral group. See pages 115-116 of [H] for a discussion of orientation characters for these groups and $S^2 \times E^2$-manifolds realizing them).

Let T and Kb denote the torus and Klein bottle, respectively. If M is a closed 4-manifold $v_2(M)$ shall denote the Wu class in $H^2(M; Z/2Z)$.

Theorem 2. *Let M be a closed 4-manifold such that $\pi_2(M) \cong Z$ and $\chi(M) = 0$, and let $\pi = \pi_1(M)$. Let U be the cohomology class in $H^1(\pi; Z/2Z)$ corresponding to the* *homomorphism* $u : \pi \to Aut(\pi_2(M))$. *Then*

(*i*) *if* $\pi \cong D *_Z D$ *then* $k_1(M) = \beta^u(v_2(M))$;

(*ii*) *if* $\pi \cong D *_Z (Z \oplus (Z/2Z))$ *then* $k_1(M) = \beta^u(v_2(M))$;

(*iii*) *if* $\pi \cong Z *_Z D$ *there are at most two* $Aut(\pi)$*-orbits of possible* k*-invariants.*

In all cases $v_2(M) = U^2$. *The double covering space associated to* $\rho = Keru$ *is homeomorphic to* $S^2 \times T$, $S^2 \times Kb$ *and to the total space of the* S^2*-bundle* ξ *over* Kb *with* $w_2(\xi) = w_1(\xi)^2 \neq 0$ *in cases (i), (ii) and (iii), respectively.*

Proof. (*i*) The group $\pi = D *_Z D$ is isomorphic to the semidirect product $Z^2 \times_{-I} (Z/2Z)$, and has a presentation $\langle s, t, x \mid x^2 = 1, xsx = s^{-1}, xtx = t^{-1}, st = ts\rangle$. Let S, T, U be the basis for $H^1(\pi; Z/2Z)$ determined by the equations $S(t) = S(x) = T(s) = T(x) = U(s) = U(t) = 0$. Since the LHS spectral sequence for π as an extension of $Z/2Z$ by $\rho = \langle s, t\rangle \cong Z^2$ is a spectral sequence of bigraded algebras $H^2(\pi; Z/2Z)$ is generated by $\{U^2, US, UT, ST\}$. We may check that these classes are linearly independent, and so form a basis, by restriction to ρ and to the subgroups isomorphic to D generated by $\{x, xs\}$ and $\{x, xt\}$. In particular, the natural inclusion $H^2(\pi; Z/2Z) \to H^2(M; Z/2Z)$ is an isomorphism. An element of π has order 2 if and only if it is of the form xs^mt^n for some $(m, n) \in Z^2$. It is easy to check that the only linear combination $aU^2 + bUS + cUT + dST$ which has nonzero restriction to all subgroups of order 2 is U^2. Hence $k_1(M) = \beta^u(U^2)$.

The squares S^2 and T^2 must be linear combinations of the above basis elements. On restricting such linear combinations to subgroups as above we find that $S^2 = US$ and $T^2 = UT$. Hence $(US)^2 = USU^2$, $(UT)^2 = UTU^2$ and $(ST)^2 = STU^2$, so U^2 must be the Wu class $v_2(M)$ in $H^2(M; Z/2Z) \cong H^2(\pi; Z/2Z)$. Now $w_1(M) = U$ (by Lemma 1, or by page 115 of [H]) and so $w_2(M) = v_2(M) + w_1(M)^2 = 0$. As the covering space associated to $\rho \cong Z^2$ has trivial Stiefel-Whitney classes it is homotopy equivalent to $S^2 \times T$, and therefore homeomorphic to it, by Theorem V.8 of [H].

(*ii*) The group $D *_Z (Z \oplus (Z/2Z))$ has a presentation $\langle s, t, x \mid x^2 = 1, xsx = s^{-1}, xtx = t^{-1}, tst^{-1} = s^{-1}\rangle$. Let S, T, U be the basis for $H^1(\pi; Z/2Z)$ determined by the equations

$S(t) = S(x) = T(s) = T(x) = U(s) = U(t) = 0$. In the LHS spectral sequence all differentials ending on the bottom row must be 0, since $Z/2Z$ is a retract of π. Hence $E_\infty^{1,1} = E_2^{1,1} \cong (Z/2Z)^2$. Moreover ST has nonzero restriction to $\rho = \langle s,t\rangle \cong \pi_1(Kb)$, so $E_\infty^{0,2} \neq 0$. We again see that $\{U^2, US, UT, ST\}$ forms a basis for $H^2(\pi; Z/2Z)$, and that $H^2(M; Z/2Z) \cong H^2(\pi; Z/2Z)$. An element of $D *_Z (Z \oplus (Z/2Z))$ has order 2 if and only if it is of the form $xs^m t^n$ for some $(m,n) \in Z^2$, with either $m = 0$ or n even. Hence U^2 and $U^2 + ST$ are the only elements of $H^2(\pi; Z/2Z)$ with nonzero restriction to all subgroups of order 2. Now $H^1(\pi; Z^u) \cong Z \oplus (Z/2Z)$, while $H^1(\pi; Z/2Z) \cong (Z/2Z)^3$. Since U and S are the reductions modulo (2) of classes from $H^1(\pi; Z^u)$ and since $H^2(\pi; Z^u)$ has exponent 2 it follows that $H^2(\pi; Z^u)$ is generated by the image of S. Moreover $\mathrm{Ker}\beta^u$ is generated by the reduction modulo (2) of this image, $\overline{\beta^u}(S)$, say. We may check that $\overline{\beta^u}(S) = ST$, by restriction to the subgroups generated by $\{x, xs\}$, $\{x, xt\}$ and ρ. Hence $k_1(M) = \beta^u(U^2) = \beta^u(U^2 + ST)$.

We again find that $S^2 = US$ and $T^2 = UT$, and so $v_2(M) = U^2$. In this case $w_1(M) = U + T$ and $\mathrm{Ker}(w_1(M)) \cong Z \times D$. Hence $w_2(M) = U^2 + w_1(M)^2 = T^2$. Since $w_1(M)|_\rho = w_1(\rho)$ and has square 0, $w_2(M)|_\rho = 0$ and the covering space associated to ρ is homeomorphic to $S^2 \times Kb$, by Theorem V.8 of [H].

(*iii*) The group $\pi \cong Z *_Z D$ has a presentation $\langle r, s, x \mid x^2 = 1, xrx = r^{-1}, xsx = rs^{-1}, srs^{-1} = r^{-1}\rangle$. Hence $\rho = \langle r, s\rangle \cong \pi_1(Kb)$ and $\pi/\pi' \cong (Z/4Z) \oplus (Z/2Z)$. Let S, U be the basis for $H^1(\pi; Z/2Z)$ determined by the equations $S(x) = U(s) = 0$. In the LHS spectral sequence all differentials ending on the bottom row must be 0, since $Z/2Z$ is a retract of π. Since $H^p(Z/2Z; H^1(\rho; Z/2Z)) = 0$ for all $p > 0$, it follows that $H^n(\pi; Z/2Z) \cong (Z/2Z)^2$, for all $n \geq 1$. In particular, $H^2(M; Z/2Z) \cong H^2(\pi; Z/2Z)$ and has a basis $\{U^2, V\}$, where $V|_\rho$ generates $H^2(\rho; Z/2Z)$. Moreover $H^4(M; Z/2Z)$ is a quotient of $H^4(\pi; Z/2Z)$. An element of π has order 2 if and only if it is of the form $xr^m s^{2n}$ for some $(m,n) \in Z^2$. Hence U^2 and $U^2 + V$ are the only elements of $H^2(\pi; Z/2Z)$ with nonzero restriction to all subgroups of order 2. In this case reduction modulo (2) from

$H^1(\pi; Z^u)$ to $H^1(\pi; Z/2Z)$ is onto, so multiplication by 2 is injective on $H^2(\pi; Z^u)$. Since this group has exponent 2 it must be 0, and so β^u is injective. Since every automorphism of π must fix U there remain two classes of possible k-invariants under the action of $Aut(\pi)$.

If M is an $S^2 \times E^2$-manifold with $\pi_1(M) \cong Z *_Z D$ then $v_2(M) = U^2$ and $U^4 = 0$ in $H^4(M; Z/2Z)$. (See below in this section). Hence $H^4(M; Z/2Z) \cong H^4(\pi; Z/2Z)/\langle U^4 \rangle$. Since $V^2 = U^2V \neq 0$ in $H^4(M; Z/2Z)$ it follows that $\{U^4, U^2V\}$ is a basis for $H^4(\pi; Z/2Z)$ and $V^2 = U^2V + mU^4$ in $H^4(\pi; Z/2Z)$, for some $m = 0$ or 1. Let $\sigma : Z/2Z \to \pi$ be the inclusion of the subgroup $\langle x \rangle$, which splits the projection onto π/ρ. Then $\sigma^*(V) = \sigma^*(U^2)$ or 0, while $\sigma^* U^4 \neq 0$. Hence $m\sigma^*(U^4) = \sigma^*(V^2 + U^2V) = 0$ and so $V^2 = U^2V$ in $H^4(\pi; Z/2Z)$. It can be shown (by direct calculation with cochains) that $S^2 = US = 0$ in $H^2(Z *_Z D; Z/2Z)$. Therefore if M is any closed 4-manifold with $\pi_1(M) \cong Z *_Z D$ and $\chi(M) = 0$ we must have $U^3 = 0$ in $H^3(M; Z/2Z)$, and hence $v_2(M) = U^2$, by Poincaré duality.

The covering space associated to ρ is homeomorphic to the total space of an S^2-bundle over KB, by Theorem V.8 of [H]. Since $w_1(M)|_\rho^2 \neq 0$ and $v_2(M)|_\rho = 0$ this is the bundle ξ with $w_2(\xi) = w_1(\xi)^2 \neq 0$. //

Each of the groups $D *_Z D$, $D *_Z (Z \oplus (Z/2Z))$ and $Z *_Z D$ may be embedded into $Isom(S^2 \times E^2)$, as a discrete subgroup which acts freely on $S^2 \times E^2$. For the first two of these groups the embedding is unique up to conjugacy in $O(3) \times Aff(2)$, where $Aff(2)$ is the group of affine motions of the euclidean plane. By Theorems 1 and 2 there is in each case at most one other homotopy type of closed 4-manifold M with this fundamental group and with $\pi_2(M) \cong Z$.

There are two $S^2 \times E^2$-manifolds with fundamental group $Z *_Z D$, up to "affine" diffeomorphism. They may be distinguished geometrically by the images $pr_1(\pi)$ of their groups in $O(3)$. If M is an $S^2 \times E^2$-manifold with $\pi_1(M) \cong Z *_Z D$ then $H_2(M; Z/2Z)$ has a basis represented by embedded copies of RP^2 and Kb, with self-intersection numbers 1 and 0, respectively. Thus the characteristic element for the intersection pairing is $[Kb]$. The Poincaré dual to $[Kb]$ is U^2, and we may take the Poincaré dual to $[RP^2]$ as V. Are these two manifolds homotopy equivalent? In general we do not know whether both classes of k-invariants in $H^3(Z *_Z D; Z^u)$ are actually realized by closed 4-manifolds (or even by PD_4-complexes) M with this fundamental group and $\pi_2(M) \cong Z$.

The example $M = RP^2 \times T$ shows that in general $k_1(M)$ need not equal $\beta^u(v_2(M))$.

§3. Whitehead group.

It is well known that if π is a surface group then $Wh(\pi) = 0$ [Wd78].

Theorem 3. *Let $\pi = \pi_1(M)$ where M is a closed 4-manifold such that $\pi_2(M) \cong Z$ and $\chi(M) \leq 0$. Then $Wh(\pi) = 0$.*

Proof. If π is torsion free then it is a surface group, and the result is known. If π has nontrivial torsion then by Theorem VII.1 of [H] π is a semidirect product $\rho\tilde{\times}(Z/2Z)$ where ρ is a surface group, and π is isomorphic to a discrete subgroup of $Isom(X)$ which acts properly discontinuously on X, where $X = E^2$ or H^2. In Lemma V.1 of [H] it is shown that $Wh(\pi) = 0$ if $\pi \cong \rho \times (Z/2Z)$. Henceforth we shall assume that π has torsion but is not such a direct product. The singularities of the corresponding 2-orbifold X/π are then either marked points of order 2 or reflector curves; there are no corner points and no marked points of higher order. Let $|X/\pi|$ be the surface obtained by forgetting the orbifold structure of X/π, and let m be the number of marked points. Then $\chi(|X/\pi|) - (m/2) = \chi_{orb}(X/\pi) \leq 0$, by the Riemann-Hurwitz formula [Sc83], so either $\chi(|X/\pi|) \leq 0$ or $\chi(|X/\pi|) = 1$ and $m \geq 2$ or $|X/\pi| \cong S^2$ and $m \geq 4$.

We may separate X/π along embedded circles (avoiding the singularities) into pieces which are either (i) discs with at least two marked points; (ii) annuli with one marked point; (iii) annuli with one boundary a reflector curve; or (iv) surfaces other than D^2 with nonempty boundary. In each case the inclusions of the separating circles induce monomorphisms on orbifold fundamental groups, and so π is a generalized free product with amalgamation over copies of Z of groups of the form (i) $*^m(Z/2Z)$ (with $m \geq 2$); (ii) $Z*(Z/2Z)$; (iii) $Z \oplus (Z/2Z)$; or (iv) $*^m Z$, by the Van Kampen theorem for orbifolds [Sc83].

On applying Waldhausen's Mayer-Vietoris theorem for algebraic K-theory [Wd78] we see that $Wh(\pi) = 0$ for all such groups. //

Note that not all such orbifold groups arise in this way. For instance, the orbifold fundamental group of a torus with one marked point is $< x, y | [x, y]^2 = 1 >$, which has torsion free abelianization, and so cannot be a semidirect product as above.

Acknowledgment

I am grateful to Professor K.Tsukiyama for providing me with copies of his work on self homotopy equivalences.

References

[CH96] Cobb, R.J. and Hillman, J.A. S^2-bundles over aspherical surfaces and 4-dimensional geometries,
preprint, University of Sydney (1996).

[HK88] Hambleton, I.R. and Kreck, M.A. On the classification of topological 4-manifolds with finite fundamental group,
Math. Ann. 280 (1988), 85-104.

[H] Hillman, J.A. *The Algebraic Characterization of Geometric 4-Manifolds*,
London Mathematical Society Lecture Note Series 198, Cambridge University Press, Cambridge - New York - Melbourne (1994).

[Sc83] Scott, G.P. The geometries of 3-manifolds,
Bull. London Math. Soc. 15 (1983), 401-487.

[Ts80] Tsukiyama, K. Self-homotopy-equivalences of a space with two nonvanishing homotopy groups,
Proc. Amer. Math. Soc. 79 (1980), 134-139.

[Wd78] Waldhausen, F. Algebraic K-theory of generalized free products,
Ann. Math. 108 (1978), 135-256.