On 4-manifolds which admit geometric decompositions or which are Seifert fibred

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ABSTRACT

We show that a closed 4-manifold which admits a finite decomposition into geometric pieces of finite volume is (essentially) either geometric or aspherical. We also consider the homotopy types of closed 4-manifolds which are Seifert fibred or which are the total spaces of bundles with base and fibre closed aspherical surfaces.

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Introduction

In this report we shall give several related but separate results on 4-dimensional geometries. The first result illustrates the limitations of geometry in dimension 4, by showing that a closed 4-manifold which admits a finite decomposition into geometric pieces of finite volume is (essentially) either geometric or aspherical. The geometric viewpoint is nevertheless of considerable interest in connection with complex surfaces. (See [Wl86]). In §2 we characterize algebraically the homotopy types of (orientable) Seifert fibred 4-manifolds. This class of 4-manifolds includes all but three of the 4-dimensional infrasolvmanifolds and all the manifolds with geometry $S^3 \times E^1$, $S^2 \times E^2$, $H^2 \times E^2$ or $\tilde{SL} \times E^1$, but also has infinitely many nongeometric members. We give examples of such manifolds which are not geometric but which have geometric decompositions. The final section considers bundles with base and fibre aspherical surfaces and their connections with complex surfaces and with the geometry $H^2 \times H^2$.

Notation. If π is a group π', $\zeta\pi$ and $\sqrt{\pi}$ are its commutator subgroup, centre and Hirsch-Plotkin radical, respectively. If $G \leq \pi$ is a subgroup then $N_\pi(G)$ and $C_\pi(G)$ are the normalizer and centralizer of G in π, respectively.

§1. Geometric decompositions

We shall say that an n-manifold M is *geometric* if it is homeomorphic to a complete geometric n-manifold of finite volume, and morely generally that it admits a *geometric decomposition* if it may be split along a finite collection of disjoint 2-sided hypersurfaces S such that each component of $M - \cup S$ is geometric. We shall call the hypersurfaces S *cusps* and the components of $M - \cup S$ *pieces* of M.

Every closed 1- or 2-manifold is geometric. It is conjectured that every connected-sum indecomposable closed 3-manifold admits a geometric decomposition. In dimension 4 there are 19 maximal geometries; one of these is in fact an infinite family of closely related geometries and one is not realized by any closed 4-manifold. However our first result suggests that the scope for the use of geometry in this dimension is limited.

Theorem 1. *If a closed 4-manifold M admits a geometric decomposition then it is either*

(i) geometric;

(ii) the components of $M - \cup S$ all have geometry $S^2 \times H^2$, and M has a 2-fold cover which is the total space of an S^2-bundle over a hyperbolic surface;

(iii) the components of $M - \cup S$ have geometry H^4, $H^3 \times E^1$, $H^2 \times E^2$ or $\tilde{SL} \times E^1$;

(iv) the components of $M - \cup S$ have geometry $H^2(C)$ or F^4;

(v) the components of $M - \cup S$ all have geometry $H^2 \times H^2$.

Manifolds in the last three classes are aspherical.

Proof. The proof consists in considering the possible ends (cusps) of complete geometric 4-manifolds of finite volume. The hypersurfaces bounding a component of $M - \cup S$ correspond to the ends of its interior. If the geometry is of solvable or compact type then there are no ends, since every lattice is then cocompact [Rg]. Thus we may concentrate on the eight geometries $S^2 \times H^2$, $H^2 \times E^2$, $H^2 \times H^2$, $\tilde{SL} \times E^1$, $H^3 \times E^1$, H^4, $H^2(C)$ and F^4. The ends of a geometry of constant negative curvature H^n are flat [Eb80]; since any lattice in a Lie group must meet the radical in a lattice it follows easily that the ends are also flat in the mixed euclidean cases $H^3 \times E^1$, $H^2 \times E^2$ and $\tilde{SL} \times E^1$. Similarly, the ends of $S^2 \times H^2$-manifolds are $S^2 \times E^1$-manifolds. Since the elements of $PSL(2, C)$ corresponding to the cusps of finite area hyperbolic surfaces are parabolic, the ends of F^4-manifolds are Nil^3-manifolds. The ends of $H^2(C)$-manifolds are also Nil^3-manifolds [Ep87], while the ends of $H^2 \times H^2$-manifolds are Sol^3-manifolds [Sh63]. Clearly if two pieces are contiguous their common cusps must be homeomorphic. In all cases the inclusion of a cusp into the closure of a piece induces a monomorphism on fundamental group.

If M is a closed 4-manifold with a geometric decomposition of type (*ii*) the inclusions of the cusps into the closures of the pieces induce isomorphisms on π_2, and a Mayer-Vietoris argument in the universal covering space $\tilde{M}$ shows that $\tilde{M}$ is homotopy equivalent to S^2. Let κ be the subgroup of $\pi_1(M)$ which acts trivially on $\pi_2(M) \cong Z$. Then $[\pi_1(M) : \kappa] \leq 2$, and the cusps of the corresponding covering space M_κ are copies of $S^2 \times S^1$. Since the bundle projection to S^1 is unique up to isotopy and change of orientation of S^1 it follows that M_κ admits a natural codimension-2 foliation with leaves S^2. Hence M_κ is the total space of an S^2-bundle over a hyperbolic surface.

If at least one piece has an aspherical geometry then all do and M is aspherical. //

Is every closed 4-manifold with a geometric decomposition of type (*ii*) diffeomorphic to a $S^2 \times H^2$-manifold?

Closed 4-manifolds of type (*iii*) with decompositions into pieces of types $H^2 \times E^2$ or $\tilde{SL} \times E^1$ only need not be geometric. For instance, let $G = < u, v, x, y | [u, v] = [x, y] >$ be the fundamental group of $T \sharp T$, the closed orientable surface of genus 2, and let $\theta : G \to SL(2, Z)$ be the epimorphism determined by $\theta(u) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\theta(x) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ and $\theta(v) = \theta(y) = I$. Then the semidirect product $\pi = Z^2 \times_\theta G$ is the fundamental group of a torus bundle over $T\sharp T$ which has a geometric decomposition into two pieces of type $H^2 \times E^2$, but is not geometric, since $\sqrt{\pi} \cong Z^2$ but $[\pi : C_\pi(\sqrt{\pi})]$ is infinite.

Which flat 3-manifolds occur as cusps of pieces of type (*iii*)? If π is the fundamental group of an $\tilde{SL} \times E^1$-manifold then $(\mathrm{Ker} w_1(\pi))' \cap \sqrt{\pi} \cong Z$ is normal in π. Hence the fundamental group of a cusp of an $\tilde{SL} \times E^1$-manifold must have a chain of abelian normal subgroups $Z < Z^2 < Z^3$ and so the only orientable flat 3-manifolds that can arise are the 3-torus and the "half-turn" manifold (with fundamental groups Z^3 and $Z^2 \times_{-I} Z$, respectively). The Hantzsche-Wendt manifold is not a cusp of a $\tilde{SL} \times E^1$- or $H^2 \times E^2$-manifold, since its fundamental group (G_6) does not have a rank 2 abelian normal subgroup.

Does every complete but noncompact $\tilde{SL} \times E^1$-manifold also admit the geometry $H^2 \times E^2$?

If a closed 4-manifold admits a geometric decomposition, is that decomposition essentially unique? Any PD_n-group satisfying *max-c* (the maximal condition on centralizers) has an essentially unique minimal finite splitting along virtually poly-Z subgroups of Hirsch length $n-1$, by Theorem A2 of [Kr90]. Do all fundamental groups of manifolds with geometric decompositions have *max-c*? Scott has announced an analogue of the 3-dimensional Characteristic Variety Theorem of Jaco, Shalen and Johanssen for higher-dimensional manifolds with metrics of non-positive sectional curvature [Sc92]. Do all closed 4-manifolds of types (*iii*-*v*) admit such metrics?

§2. Seifert fibrations

A closed 4-manifold is Seifert fibred if it is the total space of an orbifold bundle with base a 2-orbifold and general fibre a torus or Klein bottle. (In [Zi85], [Ue90,91] it is required that the general fibre be a torus. This is always so if the manifold is orientable). The fundamental group π of such a 4-manifold has a rank two free abelian normal subgroup A such that π/A is virtually a surface group. Every Seifert fibred 4-manifold is finitely covered by the total space of a torus bundle over a closed surface. (This is clear if the base orbifold is good, and follows from Theorem A of [Ue91] in the remaining cases). The results of this section imply that (with two exceptions which are flat 4-manifolds) orientable quotients of torus bundles over aspherical surfaces by finite groups are at least homotopy equivalent to Seifert fibred manifolds.

Theorem 2. *Let M be a closed 4-manifold with fundamental group π. Then M is simple homotopy equivalent to the total space of a torus or Klein bottle bundle over an aspherical closed surface if and only if π has a normal subgroup $A \cong Z^2$ or $Z\tilde{\times}Z$ such that π/A is torsion free and has infinite abelianization and $\chi(M) = 0$.*

Proof. The conditions are clearly necessary. Suppose that they hold. Then M is aspherical, so π is a PD_4-group. It follows from the LHS spectral sequence that $H^2(\pi/A; R[\pi/A]) \cong H^4(\pi; R[\pi]) \cong R$ for $R = Z$ or a field. Since π/A is torsion free and has infinite abelianization it has cohomological dimension ≤ 3, by Corollary 2.3 of [Fa74], and so is a PD_2-group [Fa75]. Thus M is homotopy equivalent to the total space of a $K(A, 1)$-bundle over $K(\pi/A, 1)$. The homotopy equivalence is simple by Lemma V.2 of [H]. //

Every Seifert fibred 4-manifold with base an euclidean orbifold has Euler characteristic 0 and fundamental group solvable of Hirsch length 4, and so is homeomorphic to an infrasolvmanifold, by Theorem VI.2 of [H]. It is easily seen that no group of type Sol_0^4, Sol_1^4 or $Sol_{m,n}^4$ (with $m \neq n$) has a rank two free abelian normal subgroup. Hence the manifold must have one of the geometries E^4, $Nil \times E^1$, $Sol \times E^1$ or Nil^4. Conversely, excepting only three flat 4-manifolds, such manifolds are Seifert fibred. The fundamental group of a closed $Nil^3 \times E^1$- or Nil^4-manifold has a rank two free abelian normal subgroup, by Theorem VI.3 of [H]. If π is the fundamental group of a $Sol^3 \times E^1$-manifold then the Hirsch-Plotkin radical of the intersection of all index 4 subgroups is such a subgroup. (In the Nil^4 and $Sol^3 \times E^1$ cases there is an unique maximal such subgroup). Case-by-case inspection of the 74 flat 4-manifold groups shows that all but three have such subgroups [Hi93]. (Ue actually shows that in the orientable cases such Seifert fibred 4-manifolds are diffeomorphic to such infrasolvmanifolds, and the Seifert fibration is unique up to diffeomorphism, which may be assumed fibre-preserving in the Nil^4 and $Sol^3 \times E^1$ cases. There is a minor oversight in [Ue90]; in fact *neither* of the groups $G(\pm)$ is realised by a Seifert fibred 4-manifold).

Orientable Seifert fibred 4-manifolds with base a spherical or bad orbifold admit one of the geometries $S^2 \times E^2$ or $S^3 \times E^1$, and conversely every such geometric 4-manifold is Seifert fibred over such a base, although in general it may admit many distinct Seifert fibrations [Ue91]. (See also [Oh90] for some of the nonorientable cases). The homotopy type of a $S^2 \times E^2$-manifold is determined up to finite ambiguity by the fundamental group (which must be virtually Z^2), Euler characteristic (which must be 0) and Stiefel-Whitney classes. There are just nine possible fundamental groups. Six of these have infinite abelianization, and the above invariants determine the homotopy type in these cases. (See Chapter VII of [H]). The homotopy type of a $S^3 \times E^1$-manifold is determined by the fundamental group (which must be virtually Z), Euler characteristic (which must be 0), orientation character w_1 and first k-invariant in $H^4(\pi; \pi_3)$. (See Chapter VIII of [H]).

If $\pi < O(4) \times E(1)$ is the fundamental group of an $S^3 \times E^1$-manifold then $F = \pi \cap SO(4) \times R$ is the maximal finite normal subgroup of π and acts freely on S^3, and $\pi/F \cong Z$ or $D = (Z/2Z) * (Z/2Z)$. In the first case π is a semidirect product $F \times_\theta Z$, where θ is an automorphism of F induced by an isometry of S^3/F and in the second case $\pi \cong G *_F H$, where the projections of G and H to $O(4)$ are injective, with images acting freely on S^3, and $[G : F] = [H : F] = 2$. In particular, the double covers of the spherical space forms $G \backslash S^3$ and $H \backslash S^3$ corresponding to the subgroup F must be homeomorphic. Conversely, if this condition holds then $G *_F H$ is geometric (i.e., is the fundamental group of an $S^3 \times E^1$-manifold). The list of groups which act freely on S^3 is well known, but which automorphisms of such groups are realizable by isometries of a corresponding spherical space form has not yet been completely determined.

If F, G and H act freely on S^3 and $[G : F] = [H : F] = 2$ then $G *_F H$ is the fundamental group of an $S^3\times E^1$-manifold, provided that either F is not cyclic or at least one of G or H is cyclic. Suppose that F is cyclic but that neither G nor H is cyclic. Then each of the groups G and H is the fundamental group of $M(o;(2,1),(2,-1),(\alpha,\beta))$ for some $\alpha\geq 2$ and relatively prime β. The group $\pi=G*_F H$ has a presentation of the form $<x,u,v|x^{4k}=1,uxu^{-1}=x^a,vxv^{-1}=x^b,u^2=x^c,v^2=x^d>$, where $a^2\equiv b^2\equiv 1$ modulo $(4k)$. The index 2 subgroup generated by x and $t=uv$ is $<x,t|x^{4k}=1,txt^{-1}=x^{ab}>$, and is realised by the mapping torus of a self homeomorphism of $L(4k,ab)$. If the double covers of the spherical space forms $G\backslash S^3$ and $H\backslash S^3$ are homotopy equivalent then $\pi=G*_F H$ is realised by the union of two twisted I-bundles via a homotopy equivalence, which is a finite (but possibly nonsimple?) PD_4-complex with $\chi=0$. If π is realised by a finite simple PD_4-complex with $\chi=0$ must it be geometric?

In particular, $G=Q(24)$ and $H=Q(8)\times Z/3Z$ correspond to the spherical space forms $M(o;(2,1),(2,-1),(6,1))$ and $M(o;(2,1),(2,-1),(2,3))$, and the 2-fold covering spaces associated to $F\cong Z/12Z$ are the lens spaces $L(12,1)$ and $L(12,5)$, respectively. These lens spaces are not homotopy equivalent. The groups $G=Q(40)$ and $H=Q(8)\times Z/5Z$ correspond to spherical space forms doubly covered by $L(20,1)$ and $L(20,9)$, respectively, which are homotopy equivalent but not homeomorphic. (These examples were found by J.Crisp).

A manifold with geometry $H^2\times E^2$ or $\tilde{SL}\times E^1$ is Seifert fibred with base a hyperbolic orbifold. However not all such Seifert fibred 4-manifolds are geometric. An orientable Seifert fibred 4-manifold over an orientable hyperbolic base is geometric if and only if it is an elliptic surface; the relevant geometries are then $H^2\times E^2$ and $\tilde{SL}\times E^1$ [Ue90,91].

Theorem 3. *Let M be a closed orientable 4-manifold with fundamental group π. Then M is homotopy equivalent to a manifold which is Seifert fibred over a hyperbolic orbifold if and only if $\sqrt{\pi}\cong Z^2$, $G=\pi/\sqrt{\pi}$ has a subgroup of finite index which is torsion free and has infinite abelianization and $\chi(M)=0$.*

Proof. The conditions are clearly necessary. Suppose that they hold. Then M is aspherical, so π is a PD_4-group. It follows as in Theorem 2 that $H^2(G;Z[G]) \cong Z$ and hence that G is virtually a PD_2-group. If F is a finite normal subgroup of G then the preimage of F in π is torsion free and virtually abelian. Since M is orientable this preimage must be abelian and so G has no nontrivial finite normal subgroup. Moreover G is not virtually abelian, for otherwise π would be virtually solvable and hence $h(\sqrt{\pi}) \geq 3$. Therefore G acts effectively and properly discontinuously on H^2, by the solution to the Nielsen realization problem for surfaces [Ke83]. Since π is torsion free it is the fundamental group of a closed aspherical 4-manifold $M(\pi)$ which is Seifert fibred over $G\backslash H^2$ [CR72]. //

In particular, if $\sqrt{\pi}$ is central ($\zeta\pi \cong Z^2$) then the corresponding Seifert fibered manifold $M(\pi)$ admits an effective torus action with finite isotropy subgroups. The LHS spectral sequence for homology gives an exact sequence $H_2(\pi/\zeta\pi;Q) \to H_1(\zeta\pi;Q) \to H_1(\pi;Q)$. Since $H_2(\pi/\zeta\pi;Q) \cong Q$ it follows that $\zeta\pi/\zeta\pi \cap \pi'$ has rank at least 1. Hence $M(\pi)$ fibres over S^1 with monodromy of finite order. In these cases we have $Wh(\pi) = 0$ [NS85]. Is M s-cobordant to $M(\pi)$?

Corollary. *Let M be a closed orientable 4-manifold with fundamental group π. Then M is homotopy equivalent to a $\tilde{SL} \times E^1$-manifold or a $H^2 \times E^2$-manifold if and only if $\sqrt{\pi} \cong Z^2$, $\pi/\sqrt{\pi}$ has a subgroup of finite index which is torsion free and has infinite abelianization, $[\pi : C_\pi(\sqrt{\pi})] < \infty$ and $\chi(M) = 0$.*

Proof. The necessity of the conditions is proven in Theorem VI.13 of [H]. Suppose that they hold. Then M is homotopy equivalent to a manifold $M(\pi)$ which is Seifert fibred over a hyperbolic base orbifold, and the image of π in $Aut(\sqrt{\pi}) \cong GL(2,Z)$ is a finite group. The result follows as in Theorem B of §5 of [Ue91]. //

The $H^2 \times E^2$ case may be distinguished as follows.

Theorem 4. *Let M be a closed orientable 4-manifold with fundamental group π. Then*

M is homotopy equivalent to a closed $H^2 \times E^2$-manifold if and only if π has a finitely generated subgroup ρ such that ρ/ρ' is infinite, $\sqrt{\rho} = 1$, $[\pi : N_\pi(\rho)] < \infty$, $\zeta C_\pi(\rho) \cong Z^2$ and $\chi(M) = 0$.

Proof. The necessity of the conditions follows from Theorem VI.13 of [H]. Suppose that they hold. Then M is aspherical and so π is a PD_4-group. Let $C = C_\pi(\rho)$. Then C is also normal in $\nu = N_\pi(\rho)$, and $C \cap \rho = 1$, since $\sqrt{\rho} = 1$. Hence $\rho \times C \cong \rho.C \leq \pi$. Now ρ is nontrivial. If ρ were free then an argument using the LHS spectral sequence for $H^t(\pi; Q[\pi])$ would imply that ρ has two ends, and hence that $\sqrt{\rho} = \rho \cong Z$. Hence $c.d.\rho \geq 2$. Since moreover $Z^2 \leq C$ we must have $c.d.\rho = c.d.C = 2$ and $[\pi : \rho.C] < \infty$. Hence ρ is a PD_2-group of hyperbolic type.

Since $\zeta\nu \cong Z^2$ it follows easily that $\sqrt{\pi} \cong Z^2$ and that $[\pi : C_\pi(\sqrt{\pi})] < \infty$. The image of ρ in $\pi/\sqrt{\pi}$ is torsion free and of finite index. Hence the conditions of the above Corollary hold. Since π is virtually a product it must be of type $H^2 \times E^2$. //

Is it possible to give a more self-contained argument for this case? It is not hard to see that $\pi/\sqrt{\pi}$ acts isometrically and properly discontinuously on H^2. However it is more difficult to construct a suitable homomorphism from π to $Isom(E^2)$.

The closed 4-manifold obtained by surgery on a twist spun torus knot in general admits the geometry $\tilde{SL} \times E^1$. (See Chapter X of [H]). Every orientable $H^2 \times E^2$-manifold is double covered by a Kähler surface [Wl86]. Since the unique double cover of a 2-knot manifold $M(K)$ has first Betti number 1, no such manifold can be an $H^2 \times E^2$-manifold. Can such a knot manifold be Seifert fibred but not geometric?

Examples. If M is a compact manifold with boundary whose interior is an F^4-manifold of finite volume then $\pi_1(M)$ is a semidirect product $Z^2 \times_\theta F$ where $\theta : F \to GL(2, Z)$ is a monomorphism with image of finite index. The double $DM = M \cup_\partial M$ is Seifert fibred over a hyperbolic base but is not geometric, since $\sqrt{\pi} \cong Z^2$ but $[\pi : C_\pi(\sqrt{\pi})]$ is infinite. The orientable surface of genus 2 can be represented as a double in two distinct ways; we shall give corresponding examples of nongeometric torus bundles which admit geometric decompositions of type F^4.

1. Let $F(2)$ be the free group of rank two and let $\gamma : F(2) \to SL(2,Z)$ have image the commutator subgroup $SL(2,Z)'$, which is freely generated by $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 3 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. The natural surjection from $SL(2,Z)$ to $PSL(2,Z)$ induces an isomorphism of commutator subgroups, and the parabolic subgroup $PSL(2,Z)' \cap Stab(0)$ is generated by the image of $AB^{-1}A^{-1}B = \left(\begin{smallmatrix} -1 & 0 \\ -6 & -1 \end{smallmatrix}\right)$. Hence $[Stab(0) : PSL(2,Z)' \cap Stab(0)] = 6 = [PSL(2,Z) : PSL(2,Z)']$, and so $PSL(2,Z)'$ has a single cusp at 0. The quotient space $PSL(2,Z)' \backslash H^2$ is the once-punctured torus. Let $N \subset PSL(2,Z)' \backslash H^2$ be the complement of an open horocyclic neighbourhood of the cusp. The double DN is the closed orientable surface of genus 2. The semidirect product $\Gamma = Z^2 \times_\gamma F(2)$ is a lattice in $Isom(F^4)$, and the double of the bounded manifold with interior $\Gamma \backslash F^4$ is a torus bundle over DN.

2. Let $\delta : F(2) \to SL(2,Z)$ have image the subgroup which is freely generated by $U = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. Let $\bar{\delta} : F(2) \to PSL(2,Z)$ be the composed map. Then $\bar{\delta}$ is injective and $[PSL(2,Z) : \bar{\delta}(F(2))] = 6$. (Note that $\delta(F(2))$ and $-I$ together generate the level 2 congruence subgroup). Moreover $[Stab(0) : \bar{\delta}(F(2)) \cap Stab(0)] = 2$. Hence $\bar{\delta}(F(2))$ has three cusps, at 0, ∞ and 1, and $\bar{\delta}(F(2)) \backslash H^2$ is the thrice-punctured sphere. The corresponding parabolic subgroups are generated by U, V and VU^{-1}, respectively. Doubling the complement N of disjoint horocyclic neighbourhoods of the cusps in $\bar{\delta}(F(2)) \backslash H^2$ again gives a closed orientable surface of genus 2. (The presentation for $\pi_1(DN)$ derived from this construction is $< U, V, U_1, V_1, s, t | s^{-1}Us = U_1, t^{-1}Vt = V_1, VU^{-1} = V_1U_1^{-1} >$, which is equivalent to the usual presentation $< U, V, s, t | s^{-1}V^{-1}sV = t^{-1}U^{-1}tU >$). The semidirect product $\Delta = Z^2 \times_\delta F(2)$ is a lattice in $Isom(F^4)$, and the double of the bounded manifold with interior $\Delta \backslash F^4$ is again a torus bundle over DN.

3. If G is an orientable PD_2-group which is not virtually Z^2 and $\lambda : G \to SL(2,Z)$ is a homomorphism whose image is infinite and of infinite index then $\pi = Z^2 \times_\lambda G$ is the fundamental group of a closed orientable 4-manifold which is Seifert fibred over an orientable

hyperbolic surface but which neither is geometric nor has a geometric decomposition into pieces of type F^4. Are there such manifolds with no geometric decomposition at all? (The only relevant geometries are $H^2 \times E^2$, $\tilde{SL} \times E^1$ and F^4).

§3. Surface bundles

In the final section of Chapter X of [H] we gave alternative characterizations of certain minimal complex surfaces. Here we give another application of a similar argument.

Theorem 5. *Let S be a complex surface which is homotopy equivalent to the total space E of an F-bundle over B, where F and B are aspherical closed orientable surfaces and $\chi(B) < 0$. Then S is diffeomorphic to E.*

Proof. Since there are nonzero L^2 harmonic 1-forms on the upper half plane H the ℓ_2-cohomology group $H^1EL_2(\pi_1(B))$ is nontrivial and so there is a properly discontinuous holomorphic action of $\pi_1(B)$ on H and a $\pi_1(B)$-equivariant holomorphic map from the covering space S_F to H, with connected fibres, by Theorems 4.1 and 4.2 of [ABR92]. Let B_1 be the quotient of H by this action of $\pi_1(B)$. Then the quotient map from H to the complex curve B_1 is a covering projection, since $\pi_1(B)$ is torsion free.

The induced map $h : S \to B$ is holomorphic and is a submersion away from the preimage of a finite subset $D \subset B$. If D is empty then S is diffeomorphic to E, since such bundles are determined by their fundamental groups. Let F be the general fibre and F_d the fibre over $d \in D$. Then $\chi(S) = \chi(F)\chi(B) + \Sigma_{d\in D}(\chi(F_d) - \chi(F))$ and $\chi(F_d) \geq \chi(F)$, by Proposition III.11.4 of [BPV]. Since $\chi(S) = \chi(F)\chi(B)$ we must have $\chi(F_d) = \chi(F)$ for all $d \in D$. Moreover $\chi(F_d) > \chi(F)$ unless $\chi(F_d) = \chi(F) = 0$, by Remark III.11.5 of [BPV]. Therefore if D is nonempty S is elliptic, without singular fibres, and so is Seifert fibred. Since E is then also Seifert fibred the result in this case follows from Theorem C of [Ue90].//

If moreover h is a holomorphic fibre bundle then S is finitely covered by a cartesian product of complex curves. (See §V.6 of [BPV]). However there are such holomorphic submersions in which S has nonzero index and so which are not virtually products. (See §V.14 of [BPV]). What can be said if $\chi(B) = 0$? Any holomorphic submersion with base of genus at most 1 and fibre of genus at most 2 is a holomorphic fibre bundle [Ka68].

Theorem 6. *A closed 4-manifold M is finitely covered by a manifold homotopy equivalent to the total space E of an F-bundle over B, where F and B are closed aspherical surfaces and $\chi(F) < 0$ if and only if $\pi_1(M)$ has a normal subgroup G which is a PD_2-group with trivial centre and $\pi_2(M) = 0$.*

Proof. The conditions are clearly necessary. Conversely, if they hold then π has one end and so M is aspherical, since $\pi_2(M) = 0$. Hence $\pi = \pi_1(M)$ is a PD_4-group. Conjugation in π determines an embedding of $\pi/G.C_\pi(G)$ into $Out(G)$, which is virtually of finite cohomological dimension. Since $G \cap C_\pi(G) = \zeta G = 1$ the quotient π/G is an extension of $\pi/G.C_\pi(G)$ by $C_\pi(G)$, and so $v.c.d.\pi/G \leq c.d.C_\pi(G) + v.c.d.\pi/G.C_\pi(G) < \infty$. Hence π/G has a subgroup ρ of finite index such that $c.d.\rho < \infty$. It now follows from Theorem 9.11 of [B] that ρ is a PD_2-group. Therefore the cover of M corresponding to the preimage of ρ in π is homotopy equivalent to the total space of an F-bundle over B, where B and F are aspherical surfaces with $\pi_1(B) \cong \rho$ and $\pi_1(F) \cong G$. //

Note that $G.C_\pi(G) \cong G \times C_\pi(G)$ and so either π is virtually a product or $C_\pi(G)$ is a free group. If $C_\pi(G) \cong Z$ then $G.C_\pi(G)$ is a PD_3-group and so $\pi/G.C_\pi(G)$ is virtually Z. Hence π is virtually $H \times Z$, where H is an extension of Z by G.

Theorem 7. *Let M be a closed 4-manifold with fundamental group π. If π is a PD_4-group with a normal subgroup G which is a PD_2-group such that $C_\pi(G)$ is not a free group and if $\chi(M) = \chi(\pi) \neq 0$ then M is homotopy equivalent to a closed $H^2 \times H^2$-manifold which is finitely covered by the cartesian product of two H^2-manifolds.*

Proof. The intersection $G \cap C_\pi(G) = \zeta G$ is trivial, for otherwise $\sqrt{\pi} \neq 1$ and so $\chi(\pi) = 0$. Hence $G \times C_\pi(G) \cong \rho = G.C_\pi(G) \leq \pi$, and so $c.d.C_\pi(G) \leq 2$. Since $C_\pi(G)$ is not free

$c.d.G \times C_\pi(G) = 4$ and so ρ has finite index in π. (In particular, $[C_\pi(C_\pi(G)) : G]$ is finite). Hence ρ is a PD_4-group and $C_\pi(G)$ is a PD_2-group. Since $\chi(C_\pi(G))$ divides $\chi(\pi)$, which is nonzero, $C_\pi(G)$ is the fundamental group of an H^2-manifold.

The action of $\pi/C_\pi(G)$ by conjugation on G has finite image in $Out(G)$. Therefore $\pi/C_\pi(G)$ embeds as a discrete cocompact subgroup of $Isom(H^2)$, by the Nielsen conjecture [Ke83]. Similarly, $\pi/C_\pi(C_\pi(G))$ embeds as a discrete cocompact subgroup of $Isom(H^2)$. Now $[C_\pi(C_\pi(G)) : G]$ is finite and so the induced map from π/G to $Isom(H^2)$ has finite kernel. Together these homomorphisms give a homomorphism with finite kernel from π to a discrete uniform subgroup of $Isom(H^2 \times H^2)$. Thus π acts isometrically and properly discontinuously on $H^2 \times H^2$. Since π is torsion free the action is free, and the quotient manifold $\pi\backslash(H^2 \times H^2)$ is a $K(\pi, 1)$-space. Since $H^2(\rho; Z/2Z) \neq 0$ and $H^2(\rho; Q) \neq 0$ we may apply Theorem II.7 of [H] to see that the covering space of M with fundamental group ρ is aspherical. Hence M is itself aspherical and so the classifying map $c_M : M \to \pi\backslash(H^2 \times H^2)$ is a homotopy equivalence. //

Are these algebraic conditions necessary? The hypotheses that π be PD_4 and $\chi(M) = \chi(\pi) \neq 0$ could be replaced by $\pi_2(M) = 0$ and $\chi(M) > 0$. If an $H^2 \times H^2$ manifold fibres over a surface is it virtually a cartesian product? Conversely, is every such virtual product geometric?

Let $P = PSL(2, R)$ be the group of orientation preserving isometries of H^2. Then $Isom(H^2 \times H^2)$ contains $P \times P$ as a normal subgroup of index 8.

Theorem 8. *Let M be a closed $H^2 \times H^2$-manifold with fundamental group π. Then the following are equivalent:*

(i) *π has a subgroup of finite index which is a nontrivial direct product;*

(ii) *π has a nontrivial element with nonabelian centralizer;*

(iii) $\pi \cap (\{1\} \times P) \neq 1$;

(iv) $\pi \cap (P \times \{1\}) \neq 1$.

(v) M is finitely covered by the cartesian product of two hyperbolic surfaces;

Proof. Since π is torsion free each of the above conditions is invariant under passage to subgroups of finite index, and so we may assume without loss of generality that $\pi \leq P \times P$. Suppose that σ is a subgroup of finite index in π which is a nontrivial direct product. Since $\chi(\sigma) \neq 0$ neither factor can be infinite cyclic, and so the factors must be PD_2-groups. In particular, the centraliser of any element of either direct factor is nonabelian. Thus (i) implies (ii). Suppose that $g = (g_1, g_2) \in \pi \leq P \times P$ is nontrivial and that $C_\pi(< g >)$ is nonabelian. Since $C_{P\times P}(< g >) = C_P(< g_1 >) \times C_P(< g_2 >)$ and centralizers of nontrivial elements of infinite order in P are cyclic it follows that either g_1 or g_2 has finite order. Thus (ii) implies (iii) or (iv).

Let $K_1 = \pi \cap (\{1\} \times P)$ and $K_2 = \pi \cap (P \times \{1\})$. Then K_i is normal in π, and there are exact sequences $1 \to K_i \to \pi \to L_i \to 1$, where $L_i = pr_i(\pi)$ is the image of π under projection to the i^{th} factor of $P \times P$, for $i = 1$ and 2. Moreover K_i is normalised by L_{3-i}, for $i = 1$ and 2. Suppose that $K_1 \neq 1$. Then K_1 is non abelian, since it is normal in π and $\chi(\pi) \neq 0$. If L_2 were not discrete then elements of L_2 sufficiently close to the identity would centralize K_1. As centralizers of nonidentity elements of P are abelian, this would imply that K_1 is abelian. Hence L_2 is discrete. Now $L_2\backslash H^2$ is a quotient of $\pi\backslash H \times H$ and so is compact. Therefore L_2 is virtually a PD_2-group. Now $c.d.K_2 + v.c.d.L_2 \geq c.d.\pi = 4$, so $c.d.K_2 \geq 2$. In particular, $K_2 \neq 1$ and so a similar argument now shows that $c.d.K_1 \geq 2$. Hence $c.d.K_1 \times K_2 \geq 4$. Since $K_1 \times K_2 \cong K_1.K_2 \leq \pi$ it follows that π is virtually a product, and M is finitely covered by $(K_1\backslash H^2) \times (K_2\backslash H^2)$. Thus (iii) and (iv) are equivalent, and imply (v).

Clearly (v) implies (i). //

The final theorem strengthens a remark made in §VI.7 of [H].

Theorem 9. *A closed 4-manifold M with fundamental group π is s-cobordant to an X-manifold where $X = H^2 \times H^2$, H^4 or $H^2(C)$ if and only if π is torsion free and is* *isomorphic to a discrete uniform subgroup of $Isom(X)$ and $\chi(M) = \chi(\pi\backslash X)$.*

Proof. The conditions are clearly necessary. Suppose that they hold. Then $\chi(M) > 0$ and π is infinite and residually finite. Hence there is a subgroup σ of finite index such that the associated covering spaces M_σ and $\sigma\backslash X$ are orientable and $\chi(M) = \chi(\sigma\backslash X) > 2$. In particular, $H^2(M;Z)$ has elements of infinite order. Since the classifying map c_{M_σ} : $M_\sigma \to \sigma\backslash X$ is 2-connected it induces an isomorphism on H^2 and hence is a degree-1 map, by Poincaré duality. Therefore it is a homotopy equivalence, by Theorem II.3 of [H], and so M is aspherical. Hence $c_M : M \to \pi\backslash X$ is a homotopy equivalence.

It now follows from the work of Farrell and Jones [FJ90] that $Wh(\pi) = Wh(\pi \times Z) = 0$ and that $M \times S^1$ is homeomorphic to $(\pi\backslash X) \times S^1$. Hence M and $\pi\backslash X$ are s-cobordant.//

The Miyaoka-Yau characterization of quotients of the unit ball in C^2 (i.e., complex surfaces with geometry $H^2(C)$) suggests the following question. Let M be a closed oriented smooth 4-manifold such that $w_2(M)$ is the reduction of an integral class h with $h^2 \cap [M] = 3I(M) + 2\chi(M)$ and $\chi(M) = 3I(M) > 0$, where $I(M)$ is the index of M. Then M has an almost complex structure with first Chern class h, and for which the Miyaoka-Yau equation holds. (See Proposition 7.3 of [BPV]). Is M aspherical?

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