Virtual surface bundle groups and Johnson's trichotomy

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ABSTRACT

We give a new exposition of Johnson's trichotomy for torsion free groups which are virtually extensions of one surface group by another, and show that all groups in his first two classes may be realized by aspherical closed smooth 4-manifolds.

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Johnson has established a trichotomy for Surface bundle groups, in order to study in how many ways such a group could be an extension of one Surface group by another [Jo94]. (Following Johnson, we shall use the term *Surface* to mean closed surface with negative Euler characteristic. Thus a Surface group is a PD_2-group with trivial centre). He has since extended this trichotomy to torsion free groups commensurable with such Surface bundle groups, and asks whether every such group is the fundamental group of some aspherical closed 4-manifold [Jo96]. We shall give an alternative exposition of this trichotomy and of his uniqueness of structure results in §1, and show in §2 that all groups in his first two classes may be thus realized. We shall make some remarks on the cases with $\chi = 0$ in §3.

§1. Johnson's trichotomy

A *virtual Surface bundle group* (or vSb group) is a torsion free group which has a subgroup of finite index which is an extension of one Surface group by another. If π is a group then $\zeta\pi$ and $\sqrt{\pi}$ shall denote its centre and Hirsch-Plotkin radical, respectively. If G is a subgroup of π then $C_\pi(G)$ shall denote the centralizer of G in π.

Lemma 1. *Let G_1 and G_2 be groups with no nontrivial abelian normal subgroup. If H is a normal subgroup of $G = G_1 \times G_2$ which contains no nontrivial direct product then either $H \leq G_1 \times \{1\}$ or $H \leq \{1\} \times G_2$.*

Proof. Let P_i be the projection of H onto G_i, for $i = 1, 2$. If $(h, h') \in H$, $g_1 \in G_1$ and $g_2 \in G_2$ then $([h, g_1], 1) = [(h, h'), (g_1, 1)]$ and $(1, [h', g_2])$ are in H. Hence $[P_1, P_1] \times [P_2, P_2] \leq H$. Therefore either P_1 or P_2 is abelian, and so is trivial, since P_i is normal in G_i, for $i = 1, 2$. $\square$

Theorem 2. *Let π be a group with a normal subgroup K such that K and π/K are PD_2-groups with trivial centres.*
(i) If $C_\pi(K) = 1$ and K_1 is a finitely generated normal subgroup of π then $C_\pi(K_1) = 1$ also.
(ii) The index $[\pi : KC_\pi(K)]$ is finite if and only if π is virtually a direct product of PD_2-groups.

Proof. (i) Let $z \in C_\pi(K_1)$. If $K_1 \leq K$ then $[K : K_1] < \infty$ and $\zeta K_1 = 1$. Let $M = [K : K_1]!$. Then $f(k) = k^{-1} z^M k z^{-M}$ is in K_1 for all k in K. Now $f(kk_1) = k_1^{-1} f(k) k_1$ and also $f(kk_1) = f(kk_1k^{-1}k) = f(k)$ (since K_1 is a normal subgroup centralized by z), for all k in K and k_1 in K_1. Hence $f(k)$ is central in K_1, and so $f(k) = 1$ for all k in K. Thus z^M centralizes K. Since π is torsion free we must have $z = 1$. Otherwise $p(K_1)$ is a nontrivial finitely generated normal subgroup of π/K, and so has trivial centralizer. Hence $p(z) = 1$. Now $[K, K_1] \leq K \cap K_1$ and so $K \cap K_1 \neq 1$, for otherwise $K_1 \leq C_\pi(K)$. Since z centralizes the nontrivial normal subgroup $K \cap K_1$ in K we must again have $z = 1$.

(ii) Since K has trivial centre $KC_\pi(K) \cong K \times C_\pi(K)$ and so the condition is necessary. Suppose that $f : G_1 \times G_2 \to \pi$ is an isomorphism onto a subgroup of finite index, where G_1 and G_2 are PD_2-groups. Let $L = K \cap f(G_1 \times G_2)$. Then $[K : L] < \infty$ and so L is also a PD_2-group, and is normal in $f(G_1 \times G_2)$. We may assume that $L \leq f(G_1)$, by Lemma 1. Then $f(G_1)/L$ is finite and is isomorphic to a subgroup of $f(G_1 \times G_2)/K \leq \pi/K$, so $L = f(G_1)$. Now $f(G_2)$ normalizes K and centralizes L, and $[K : L] < \infty$. Hence $f(G_2)$ has a subgroup of finite index which centralizes K, as in part (i). Hence $[\pi : KC_\pi(K)] < \infty$. $\square$

It follows immediately that if π and K are as in the theorem whether (i) $C_\pi(K) \neq 1$ and $[\pi : KC_\pi(K)] = \infty$, (ii) $[\pi : KC_\pi(K)] < \infty$ or (iii)

$C_\pi(K)=1$ depends only on π and not on the subgroup K. In [Jo94] these cases are labeled as types I, II and III, respectively. (The action of π by conjugation on K determines a homorphism $\theta:\pi/K\to Out(K)$. In terms of this action: if Imθ is infinite and Ker$\theta\neq 1$ then π is of type I, if Imθ is finite then π is of type II, and if θ is injective then π is of type III). It also follows easily from the argument of part (i) that if ρ is a subgroup of finite index in π and $L=K\cap\rho$ then $C_\rho(L)=1$ if and only if $C_\pi(K)=1$. Since ρ is virtually a product of Surface groups if and only if π is, commensurate vSb groups are of the same type [Jo96].

Theorem 3. *Let π be a group with normal subgroups K and K_1 such that K, K_1, π/K and π/K_1 are PD_2-groups with trivial centres. Suppose that $C_\pi(K)\neq 1$. If $[\pi:KC_\pi(K)]=\infty$ then $K_1=K$ is unique. If $[\pi:KC_\pi(K)]<\infty$ then either $K_1=K$ or $K_1\cap K=1$; in the latter case K and K_1 are the only such normal subgroups which are PD_2-groups with torsion free quotients.*

Proof. Let $p:\pi\to\pi/K$ be the quotient epimorphism. Then $p(C_\pi(K))$ is a nontrivial normal subgroup of π/K, since $K\cap C_\pi(K)=\zeta K=1$. Suppose that $K_1\cap K\neq 1$. Now $\Sigma=K_1\cap(KC_\pi(K))$ contains $K_1\cap K$, and $K_1\cap K\cap C_\pi(K)=K_1\cap\zeta K=1$. Hence $\Sigma\not\leq C_\pi(K)$. Since Σ is normal in $KC_\pi(K)\cong K\times C_\pi(K)$ we must have $\Sigma\leq K_1$, by Lemma 1. Hence $\Sigma\leq K_1\cap K$. Hence $p(K_1)\cap p(C_\pi(K))=1$, and so $p(K_1)$ centralizes the nontrivial normal subgroup $p(C_\pi(K))$ in π/K. Therefore $K_1\leq K$ and so $[K:K_1]<\infty$. Since π/K_1 is torsion free we must have $K_1=K$.

If $K_1\cap K=1$ then $[K,K_1]=1$ so $K_1\leq C_\pi(K)$ and $[\pi:KC_\pi(K)]\leq[\pi/K:p(K_1)]<\infty$. Suppose K_2 is a normal subgroup of π which is a PD_2-group with $\zeta K_2=1$ and such that π/K_2 is torsion free and $K_2\cap K=1$. Then $H=K_2\cap(KK_1)$ is normal in $KK_1\cong K\times K_1$ and $[K_2:H]<\infty$, so H is a PD_2-group with $\zeta H=1$ and $H\cap K=1$. The projection of H to K_1 is nontrivial since $H\cap K=1$. Therefore $H\leq K_1$, by Lemma 1, and so $K_1\leq K_2$. Hence $K_1=K_2$. $\square$

Corollary. *If α and β are automorphisms of π such that $\alpha(K)\cap K=1$ then $\pi\cong K\times K$ and $\beta(K)=K$ or $\alpha(K)$. In particular, $Aut(K\times K)\cong Aut(K)^2\tilde{\times}(Z/2Z)$.* $\square$

Johnson shows also that each finitely generated normal subgroup K such that π/K is a PD_2-group with trivial centre has only finitely many distinct images $\phi(K)$ under automorphisms of π. (The key point is that a PD_2-group with trivial centre has only finitely many normal subgroups with a given number of generators). He conjectures that in all cases there are only finitely many such normal subgroups which are themselves PD_2-groups with trivial centre. The next theorem is a partial result in this direction.

Theorem 4. *Let π be a group with normal subgroups K and K_1 such that K, K_1 and π/K are PD_2-groups, π/K_1 is torsion free and $\chi(\pi/K) < 0$. Then either $K_1 = K$ or $K_1 \cap K = 1$ and $\pi \cong K \times K_1$ or $\chi(K_1) < \chi(\pi/K)$.*

Proof. Let $p : \pi \to \pi/K$ be the quotient epimorphism. If $K_1 \leq K$ then $K_1 = K$, as in Theorem 3. Otherwise $p(K_1)$ has finite index in π/K and so $\chi(K_1) \leq \chi(p(K_1)) \leq \chi(\pi/K)$. We may assume that $\chi(K_1) \geq \chi(\pi/K)$. Hence $\chi(K_1) = \chi(\pi/K)$ and so $p|_{K_1}$ is an epimorphism. Therefore K_1 and π/K have the same orientation type, by the nondegeneracy of Poincaré duality and the Wu relation, and so $K_1 \cong \pi/K$. Since PD_2-groups are hopfian $p|_{K_1}$ is an isomorphism. Hence $[K, K_1] \leq K \cap K_1 = 1$ and so $\pi = K.K_1 \cong K \times K_1$. $\square$

The next two corollaries follow by elementary arithmetic.

Corollary A. *If $\chi(K) = 0$ or $\chi(K) = -1$ and π/K_1 is a PD_2-group then either $K_1 = K$ or $\pi \cong K \times K_1$.* $\square$

Corollary B. *If K and π/K are PD_2-groups, $\chi(\pi/K) < 0$, and $\chi(K)^2 \leq \chi(\pi)$ then either K is the unique such subgroup or $\pi \cong K \times K$.* $\square$

Since there is a bijective correspondance between extensions of one Surface group by another and Surface bundles over Surfaces (see Theorem IV.1 of [Hi]), this has the following topological consequence.

Corollary C. *Let M be the total space of a bundle ξ with base B and fibre F aspherical closed surfaces such that $\chi(B) < \chi(F)$. If M' is the total space of another surface bundle ξ' with fibre F and $\pi_1(M') \cong \pi_1(M)$ then M' is diffeomorphic to M via a fibre-preserving diffeomorphism.* $\square$

§2. Realization

As remarked above every extension of one Surface group by another may be realized by some Surface bundle. We shall show that every vSb group of type I or II is the fundamental group of some aspherical closed smooth 4-manifold.

Theorem 5. *Let π be a torsion free group with normal subgroups $K < G < \pi$ such that K and G/K are PD_2-groups and $[\pi : G] < \infty$. Then π is the fundamental group of an aspherical closed smooth 4-manifold which is the total space of a bundle over a 2-dimensional orbifold.*

Proof. Let $p : \pi \to \pi/K$ be the quotient homomorphism. Since π is torsion free the preimage in π of any finite subgroup of π/K is a PD_2-group. As the finite subgroups of π/K have order at most $[\pi : G]$, we may assume that π/K has no nontrivial finite normal subgroup, and so is the orbifold fundamental group of some 2-dimensional orbifold B. Let F be the aspherical closed surface with $\pi_1(F) \cong K$. If π/K is torsion free then B is a closed aspherical surface, and the result follows from Theorem IV. 1 of [Hi]. In general, we may assume that B is the union of a punctured surface B_o with finitely many suborbifolds U_i which are quotients of D^2 by finite subgroups of $O(2)$ (i.e., either cone discs or segments of a disc with one or two reflector sides and possibly a corner point at the junction of two reflector sides).

Let H_i be the preimage in π of $\pi_1^{orb}(U_i)$. Then $[H_i : K] < \infty$ and H_i embeds as a discrete uniform subgroup of $Isom(\mathbb{X}^2)$, where $\mathbb{X}^2 = \mathbb{R}^2$ if $\chi(K) = 0$ and $\mathbb{X}^2 = \mathbb{H}^2$ otherwise. The obvious action of H_i on $\mathbb{X}^2 \times D^2$ determines a bundle with general fibre F over the orbifold U_i. The action of π/K on K determines an action of $\pi_1(B_o)$ on K and hence an F-bundle over B_o.

These pieces may be assembled to give a bundle with general fibre F over the orbifold B, whose total space is an aspherical closed smooth 4-manifold with fundamental group π. $\Box$

Corollary. *Virtual Surface bundle groups of type I are realizable.*

Proof. Let π be torsion free, and suppose that π has a subgroup σ of finite index with a normal subgroup ν such that ν and σ/ν are PD_2-groups with $\zeta\nu = \zeta(\sigma/\nu) = 1$. Let $G = \cap h\sigma h^{-1}$ and $K = \nu \cap G$. Then $[\pi : G] < \infty$, G is normal in π, and K and G/K are PD_2-groups. If G is of type I then K is characteristic in G, by Theorem 3, and so is normal in π. As π/K is virtually a PD_2-group, Theorem 5 applies. $\Box$

This settles the realization question for groups of type I (and also for many others - see the corollaries to Theorem 4). It would also handle type III groups if it could be shown that if such a group G has a normal subgroup K such that both K and G/K are Surface groups then K is characteristic in G.

Groups of type II may not have such a characteristic surface subgroup (although this is nearly true). Nevertheless, all such groups may be realized geometrically, as we shall show next. A closed $\mathbb{H}^2\times\mathbb{H}^2$-manifold M is *reducible* if it has a finite cover isometric to a product of closed surfaces.

Theorem 6. *A group π is the fundamental group of a reducible $\mathbb{H}^2\times\mathbb{H}^2$-manifold if and only if it is torsion free, $\sqrt{\pi}=1$ and π has a subgroup of finite index which is isomorphic to a product of PD_2-groups.*

Proof. The conditions are clearly necessary. Suppose that they hold. Then π is a PD_4-group and has a subgroup of finite index which is a direct product $\alpha\beta\cong\alpha\times\beta$, where α and β are PD_2-groups. Let N be the intersection of the conjugates of $\alpha.\beta$ in π. Then N is normal in π, so $\sqrt{N}=1$ also, and $[\pi:N]<\infty$. Let $K=\alpha\cap N$ and $L=\beta\cap N$. Then K and L are PD_2-groups with trivial centre, and $KL\cong K\times L$ is normal in N and has finite index in π. Moreover N/K and N/L are isomorphic to subgroups of finite index in β and α, respectively, and so are also PD_2-groups. Since any automorphism of N must either fix these subgroups or interchange them, by Theorem 3, KL is normal in π and $[\pi:N_\pi(K)]\leq 2$.

Let $\nu=N_\pi(K)$. Then $L\leq C_\pi(K)\leq\nu$ and $\nu=N_\pi(L)$ also. After enlarging K and L, if necessary, we may assume that $L=C_\pi(K)$ and $K=C_\pi(L)$. Hence ν/K and ν/L have no nontrivial finite normal subgroup. (For if K_1 is normal in ν and contains K as a subgroup of finite index then $K_1\cap L$ is finite, hence trivial, and so $K_1\leq C_\pi(L)$). The action of ν/L by conjugation on K has finite image in $Out(K)$, and so ν/L embeds as a discrete cocompact subgroup of $Isom(\mathbb{H}^2)$, by the Nielsen conjecture [Ke83]. Together with a similar embedding for ν/K we obtain a homomorphism from ν to a discrete cocompact subgroup of $Isom(\mathbb{H}^2\times\mathbb{H}^2)$.

If $[\pi:\nu]=2$ let t be an element of $\pi-\nu$, and let $j:\nu/K\to Isom(\mathbb{H}^2)$ be an embedding onto a discrete cocompact subgroup S. Then $tKt^{-1}\cap K=1$ and t induces an isomorphism $f:\nu/G\to\nu/(tKt^{-1})$. Together j and $j\circ f^{-1}$ determine an embedding $J:\nu\to Isom(\mathbb{H}^2\times\mathbb{H}^2)$ onto a discrete cocompact subgroup of finite index in $S\times S$. Now $t^2\in\nu$ and $J(t^2)=(s,s)$, where $s=j(t^2K)$. We may extend J to an embedding of π in $Isom(\mathbb{H}^2\times\mathbb{H}^2)$ by defining $J(t)$ to be the isometry sending (x,y) to $(y,s.x)$. Thus (in either case) π acts isometrically and properly discontinuously on $\mathbb{H}^2\times\mathbb{H}^2$. Since π is torsion free the action is free, and so $\pi=\pi_1(M)$, where $M=\pi\backslash(\mathbb{H}^2\times\mathbb{H}^2)$. $\square$

Corollary. *Let M be a $\mathbb{H}^2 \times \mathbb{H}^2$-manifold. Then M is reducible if and only if it has a 2-fold covering space which is homotopy equivalent to the total space of a bundle over a hyperbolic 2-orbifold and with general fibre a hyperbolic surface.*

Proof. The isometry group $Isom(\mathbb{H}^2 \times \mathbb{H}^2)$ contains $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ as a normal subgroup of index 8. An irreducible lattice in $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ cannot have any nontrivial normal subgroups of infinite index, by Theorem IX.6.14 of [Ma]. Hence an $\mathbb{H}^2 \times \mathbb{H}^2$-manifold which is finitely covered by the total space of a surface bundle is reducible. The converse was proven in the theorem. $\square$

In general, we cannot assume that M is itself fibred over a 2-orbifold. Let G be a PD_2-group with $\zeta G = 1$ and let x be a nontrivial element of G. A uniform free action of G on $\mathbb{H}^2$ determines a uniform free action of $\pi = \langle G \times G, t \mid t(g_1, g_2)t^{-1} = (xg_2x^{-1}, g_1)\ \forall (g_1, g_2) \in G \times G, t^2 = (x, x)\rangle$ on $\mathbb{H}^2 \times \mathbb{H}^2$, by $(g_1, g_2).(h_1, h_2) = (g_1.h_1, g_2.h_2)$ and $t.(h_1, h_2) = (x.h_2, h_1)$, for all $(g_1, g_2) \in G \times G$ and $(h_1, h_2) \in \mathbb{H}^2 \times \mathbb{H}^2$. The group π has no normal subgroup which is a PD_2-group. (See Lemma 1).

§3. The cases with $\chi = 0$

If π is an extension of Z^2 by a normal Surface subgroup K (with $\zeta K = 1$) then $C_\pi(K) = \sqrt{\pi}$, and $[\pi : KC_\pi(K)] < \infty$ if and only if π is virtually $K \times Z^2$, so Johnson's trichotomy extends to such groups. The three types may be characterized by (I) $\sqrt{\pi} \cong Z$, (II) $\sqrt{\pi} \cong Z^2$, and (III) $\sqrt{\pi} = 1$. As these properties are shared by commensurate torsion free groups the trichotomy extends further to torsion free groups which are virtually such extensions. There is at present no uniqueness result corresponding to Theorem 3 above for such subgroups $K < \pi$, and (excepting for groups of type II) it is not known whether every such group is realized by some aspherical closed 4-manifold. (In fact, it also appears to be unknown in how many ways a 3-dimensional mapping torus may fibre over S^1).

On the other hand, if π is virtually an extension of a PD_2-group by a PD_2-group K with $\zeta K \neq 1$ then either $\sqrt{\pi} \cong Z^2$ (and so is a canonical normal PD_2-subgroup) or π is virtually poly-Z. The Johnson trichotomy is inappropriate here as $Out(Z^2) \cong GL(2,\mathbb{Z})$ is virtually free, and so the action θ is never injective. However all such groups may be realized by aspherical closed

4-manifolds. If π is virtually poly-Z then it is the fundamental group of an infrasolvmanifold. (See Chapter VI of [Hi]). Otherwise π is the fundamental group of an aspherical Seifert fibred 4-manifold, with general fibre a torus or Klein bottle, by Theorem 5.

References

[Hi] Hillman, J.A. *The Algebraic Characterization of Geometric 4-Manifolds*, London Mathematical Society Lecture Note Series 198, Cambridge University Press, Cambridge - New York - Melbourne (1994).

[Hi98] *On 4-manifolds which admit geometric decompositions*, J. Math. Soc. Japan 50 (1998),

[Jo93] Johnson, F.E.A. Surface fibrations and automorphisms of abelian extensions, Quart. J. Math. Oxford 44 (1993), 199-214.

[Jo94] Johnson, F.E.A. A group theoretic analogue of the Parshin-Arakelov rigidity theorem, Arch. Math. (Basel) 63 (1994), 354-361.

[Jo96] Johnson, F.E.A. Finite coverings and surface fibrations, J.Pure Appl. Math. 112 (1996), 41-52.

[Ke83] Kerkhoff, S.P. The Nielsen realization problem, Ann. Math. 117 (1983), 235-265.

[Ma] Margulis, G. *Discrete Subgroups of Semisimple Lie Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge Bd. 17, Springer-Verlag, Berlin - Heidelberg - New York (1991).