Embedding 3-manifolds with circle actions in 4-space

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ABSTRACT

We show that if M is an orientable 3-manifold which is Seifert fibred over an orientable base orbifold B and which embeds in $\mathbb{R}^4$ then the generalized Euler invariant of the fibration is determined up to sign by B unless $H_1(M;\mathbb{Z})$ is torsion free, in which case it can take at most one nonzero value (up to sign). In particular, no such manifold with 3 exceptional fibres and Euler invariant 0 embeds in $\mathbb{R}^4$.

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If a closed 3-manifold M embeds in $\mathbb{R}^4$ (as a TOP locally flat submanifold) it must be orientable, and the linking pairing on the torsion subgroup $T(M)$ of $H_1(M;\mathbb{Z})$ must be hyperbolic [KK]. In particular, $T(M) \cong U \oplus U$ for some finite abelian group U [Ha]. In §1 we shall observe that this imposes strong constraints on the generalized Euler invariants of orientable 3-manifolds which are Seifert fibred over orientable base orbifolds (i.e., admit effective S^1-actions) and which embed in $\mathbb{R}^4$. The present simple argument does not work when the base orbifold is nonorientable; in [CH] we use the $Z/2Z$-Index Theorem to constrain the Euler invariants in the latter case.

In §2 we use Seifert invariants to describe the linking pairing, and in §3 we

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consider the cases with at most 3 exceptional fibres. In particular, no such manifold with 3 exceptional fibres and Euler invariant 0 embeds in $\mathbb{R}^4$.

§1. The torsion subgroup

We shall assume henceforth that $M = M(g;S)$ is an orientable 3-manifold which is the total space of a Seifert bundle over the orbifold $B = T_g(\alpha_1, \dots \alpha_r)$ (where T_g is the orientable surface of genus g), with Seifert invariants $S = \{(\alpha_i, \beta_i) \mid 1 \leq i \leq r\}$, where $1 < \alpha_i$ and $(\alpha_i, \beta_i) = 1$, for all $i \leq r$. (We do not assume that $0 < \beta_i < \alpha_i$). If $r = 1$ we allow also the possibility $\alpha_1 = 1$. Let $\varepsilon = -\Sigma_{i=1}^{i=r}(\beta_i/\alpha_i)$ be the generalized Euler invariant of the Seifert bundle, and let $\Pi = \Pi_{i=1}^{i=r}\alpha_i$. As B is the connected sum of T_g and $S^2(\alpha_1, \dots \alpha_r)$ we have $M = M(0;S)\sharp_f M(g;\emptyset) = M(0;S)\sharp_f(T_g \times S^1)$, where $\sharp_f$ denotes fibre sum [CH].

Theorem. *Let $M = M(g;S)$ be an orientable 3-manifold which is Seifert fibred over an orientable base orbifold $B = T_g(\alpha_1, \dots \alpha_r)$. Then $H_1(M;\mathbb{Z}) \cong Z^{2g} \oplus (\bigoplus_{i \geq 0}(Z/\lambda_i Z))$, where λ_i is determined by $\{\alpha_1, \dots \alpha_r\}$ and is nonzero, for all $i > 0$, while $|\varepsilon|\Pi = \lambda_0 \Pi_{j \geq 1}\lambda_j$.*

Proof. The fundamental group $\pi_1(M)$ has a presentation $\langle a_1, \dots b_g, q_1, \dots q_r, h \mid (\Pi[a_i, b_i])(\Pi q_j) = 1, q_i^{\alpha_i}h^{\beta_i} = 1, h\ central\rangle$. Hence $H_1(M;\mathbb{Z}) \cong Z^{2g} \oplus Cok(A)$, where A is the matrix

$$A = \begin{pmatrix} 0 & 1 & \dots & 1 \\ \beta_1 & \alpha_1 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \beta_r & 0 & \dots & \alpha_r \end{pmatrix}.$$

Let $E_i(A)$ be the ideal generated by the $(r+1-i) \times (r+1-i)$ subdeterminants of A, and let Δ_i be the positive generator of $E_i(A)$. Then $\Delta_0 = |det(A)| = |\varepsilon|\Pi$, while as the elements of each row are relatively prime Δ_i is the highest common factor of the $(r-i-1)$-fold products of distinct α_js, if $0<i<r$, and $\Delta_i=1$ if $i\geq\max\{r-1,1\}$. (In particular, if $r>2$ then Δ_{r-2} is the highest common factor of $\{\alpha_1,\dots\alpha_r\}$). Thus Δ_i depends only on $\{\alpha_1,\dots\alpha_r\}$ and is nonzero, for all $i>0$. If we set $\lambda_i=\Delta_i/\Delta_{i+1}$, for $i\geq 0$, then $Cok(A)\cong\bigoplus_{i\geq 0}(Z/\lambda_i Z)$, by the Elementary Divisor Theorem. In particular, $|\varepsilon|\Pi=\lambda_0\Pi_{j\geq 1}\lambda_j$. $\square$

Note that $T(M)\cong T(M(0;S))$ and $h\in T(M)$ if and only if $\varepsilon\neq 0$.

The elementary divisors λ_i may be determined more explicitly by localization. Let p be a prime, and let $v_i\geq 0$ be the p-adic valuation of α_i. (Thus $\alpha_i=p^{v_i}q_i$, where p does not divide q_i). If the indexing is such that $v_i\geq v_{i+1}$ for all i then the p-adic valuation of λ_j is v_{j+2}, for all $j\geq 1$. In particular, λ_{i+1} divides λ_i, for all $i\geq 0$.

Corollary. *If $T(M)$ is a nontrivial direct double then $T(M)$ and $|\varepsilon|$ are determined by $\{\alpha_1,\dots\alpha_r\}$.*

Proof. If $\varepsilon=0$ then $T(M)\cong\bigoplus_{i\geq 1}(Z/\lambda_i Z)$. Hence if $T(M)$ is a direct double $\lambda_{2i-1}=\lambda_{2i}$, for all $i>0$. If $\varepsilon\neq 0$ and $T(M)$ is a direct double $\lambda_{2i}=\lambda_{2i+1}$, for all $i\geq 0$, which is possible for at most one nonzero value of ε (up to sign). Clearly these two systems of equations can only be both satisfied if $\lambda_i=1$ for all $i>0$ and $\lambda_0=0$ or 1, in which case $T(M)=0$. $\square$

When M is Seifert fibred over a nonorientable base orbifold $T(M)$ is again largely determined by the set $\{\alpha_1,\dots\alpha_r\}$, but ε is not constrained at all by the condition that $T(M)$ be a direct double (see [CH]).

Although the β_is only contribute to the structure of $T(M)$ via ε, they play a more substantial role in the linking pairing. (See below).

If $T(M)=0$ and $\varepsilon\neq 0$ then $M(0;S)$ is a $\mathbb{Z}$-homology sphere, and so embeds as a TOP locally flat submanifold of $\mathbb{R}^4$ [FQ]. Hence $M = M(0;S)\sharp_f(T_g \times S^1)$ embeds also. If $T(M) = 0$ and $\varepsilon = 0$ then $H_1(M(0;S);\mathbb{Z}) \cong Z$, and $M(0;S)$ again embeds in $\mathbb{R}^4$ if moreover it admits a degree 1 map to $S^2 \times S^1$ which induces isomorphisms on homology with local coefficients [Hi]. However, this is not a necessary condition, and it is not known in general which (Seifert fibred) homology $S^2 \times S^1$s embed.

If M embeds in $\mathbb{R}^4$ then so does $M\sharp_f(T_g \times S^1)$. Is there a converse? The analogous implication in the case of nonorientable base orbifolds is not reversible. (See [CH]).

§2. Bilinear pairings

A *linking pairing* on a finite abelian group N is a symmetric bilinear function $\ell : N \times N \to \mathbb{Q}/\mathbb{Z}$ which is nonsingular in the sense that : $n \mapsto \ell(-,n)$ defines an isomorphism from N to $Hom(N,\mathbb{Q}/\mathbb{Z})$. Such a pairing splits uniquely as the orthogonal sum (over primes p) of its restrictions to the p-primary subgroups of N. It is *metabolic* if there is a subgroup $P < N$ such that $P = P^\perp = \{t \in N \mid \ell_M(t,p) = 0\ \forall p \in P\}$, and is *hyperbolic* if N is the direct sum of two such subgroups.

If M is a closed oriented 3-manifold Poincaré duality determines a linking pairing $\ell_M : T(M) \times T(M) \to \mathbb{Q}/\mathbb{Z}$, which may be described as follows. Let w, z be 1-cycles representing elements of $T(M)$ and suppose that $mz = \partial C$ for some 2-chain C which is transverse to z and some nonzero $m \in \mathbb{Z}$. Then $\ell_M(w,z) = (w.C)/m \in \mathbb{Q}/\mathbb{Z}$. It follows easily from the Mayer-Vietoris theorem and duality that if M embeds in $\mathbb{R}^4$ then ℓ_M is hyperbolic. (If X and Y are the closures of the components of $\mathbb{R}^4 - M$ and T_X and T_Y are the kernels of the induced homomorphisms from $T(M)$ to $H_1(X;\mathbb{Z})$ and $H_1(Y;\mathbb{Z})$ (respectively) then $T(M) \cong T_X \oplus T_Y$ and the restriction of ℓ_M to each of these summands is trivial [KK]).

There are analogous pairings on covering spaces of M. In particular, if $\phi : \pi_1(M) \to Z$ is an epimorphism the homology groups of the associated covering space M_ϕ are Λ-modules, where $\Lambda = \mathbb{Z}[Z] = \mathbb{Z}[t,t^{-1}]$. There is a *Blanchfield pairing* on the Λ-torsion submodule of $H_1(M_\phi;\mathbb{Z})$ with values in $\mathbb{Q}(t)/\Lambda$ which is nonsingular modulo the maximal finite submodule and hermitean with respect to the involution sending t to t^{-1}. If M embeds in $\mathbb{R}^4$ then this Blanchfield pairing is again hyperbolic [Ka].

Assume now that $M = M(g;S)$. Then $T(M)$ is a subgroup of the group generated by h and $q_1, \dots q_r$, and the Seifert structure gives natural 2-chains relating these 1-cycles. For let N_i be a torus neighborhood of the i^{th} singular fibre, with meridian D_i, and let B_o be a section of the restriction of the Seifert fibration to $M - \cup_{i \geq 1} N_i$. Then $\partial D_i = \alpha_i q_i + \beta_i h$ and $\partial B_o = \Sigma q_i$. Moreover $h.B_o = 1$, $h.D_i = 0 = q_j.B_o$, $q_i.D_i = \beta_i$ and $q_i.D_j = 0$ if $i \neq j$, for all $1 \leq i, j, \leq r$. As these intersection numbers are independent of g, it is clear that $\ell_M \cong \ell_{M(0;S)}$.

For small values of r we can compute ℓ_M directly (see below). Assume that $r \geq 3$. If $\varepsilon \neq 0$ then $h \in T(M)$ and $M(0;S)$ is a $\mathbb{Q}$-homology 3-sphere, and is a $\mathbb{N}il^3$- or $\widetilde{\mathbb{SL}}$-manifold. Let $\Delta = \varepsilon\Pi$ and $\delta = \text{h.c.f.}\{\alpha_1, \dots \alpha_r\}$. Then $\Delta h = \partial(\Pi B - \Sigma(\Pi/\alpha_i)D_i)$ and $\Delta \alpha_i q_i = \partial(\Delta D_i + \beta_i(\Sigma(\Pi/\alpha_j)D_j) - \beta_i \Pi B)$. Hence $\ell_M(h,h) = \Pi/\Delta$, $\ell_M(h,q_i) = -\beta_i\Pi/\Delta\alpha_i$, $\ell_M(q_i,q_i) = \beta_i(\alpha_i\Delta + \beta_i\Pi)/\Delta(\alpha_i)^2$ and $\ell_M(q_i,q_j) = \beta_i\beta_j\Pi/\Delta\alpha_i\alpha_j$, if $i \neq j$. Since $T(M)$ is generated by h and the q_is and ℓ_M is nonsingular $h = 0$ if and only if Δ divides Π and each of $\beta_i\Pi/\alpha_i$, i.e., $\Delta = \text{h.c.f.}\{\Pi/\alpha_1, \dots \Pi/\alpha_r\}$. In particular, $H_1(M;\mathbb{Z}) \cong (Z/\delta Z)^{r-1}$ if and only if $\Delta = \delta^{r-1}$. In this case $T(M) = H_1(M;\mathbb{Z})$ is generated by any $r-1$ elements of $\{q_1, \dots q_r\}$, and $h = 0$. In general it may not be easy to find a minimal generating set for $T(M)$, and we should first localize at a prime.

If $r \geq 3$ and $\varepsilon = 0$ then M is a flat 3-manifold or a $\mathbb{H}^2 \times \mathbb{E}^1$-manifold, and $h \notin T(M)$, so M is the mapping torus $F \times_\theta S^1$ of a periodic self homeomorphism θ of a closed orientable surface F. (If $g = 0$ the bundle projection to S^1 is essentially unique). Let $\phi : \pi_1(M) \to S^1$ be the epimorphism induced by the bundle projection. Then $M_\phi \cong F \times R$, and so $H_1(M_\phi; \mathbb{Z}) \cong H_1(F; \mathbb{Z})$ as an abelian group. Hence it is a torsion Λ-module (with no nontrivial finite submodule). The Blanchfield pairing is determined by the intersection form I_F on $H_1(F; \mathbb{Z})$ and the isometry $\theta_* = H_1(\theta)$ [Tr]. In particular, it is hyperbolic if and only if $H_1(F; \mathbb{Z}) = A \oplus B$ where $\theta_*(A) = A$, $\theta_*(B) = B$ and $I_F(A, A) = I_F(B, B) = 0$. (Note that if $M(0; S) = F \times_\theta S^1$ then $M(g; S) = (F \sharp T_g) \times_{\theta \sharp id} S^1$, and so we may again concentrate on the case $g = 0$).

§3. The cases with $r \leq 3$

If $r = 1$ then $\varepsilon \neq 0$, $H_1(M; \mathbb{Z}) = T(M) = (Z/\varepsilon\alpha Z)h$ and $\ell_M(h, h) = -\alpha/\beta$. If $r = 2$ let x, y be such that $x\alpha_1 - y\beta_1 = 1$, and let $k = yq_1 + xh$ and $k' = \alpha_1 q_1 + \beta_1 h = \partial D_1$. Then $q_1 = -\beta_1 k + xk'$ and $h = \alpha_1 k - yk'$, and $\varepsilon\alpha_1\alpha_2 k = \partial(\alpha_2 B - (x\alpha_2 + y\beta_2)D_1 - D_2)$. Hence $H_1(M; \mathbb{Z}) = (Z/\varepsilon\alpha_1\alpha_2 Z)k$. If $\varepsilon \neq 0$ then $T(M) = H_1(M; \mathbb{Z})$ and $\ell_M(k, k) = (x\alpha_2 + y\beta_2)/\varepsilon\alpha_1\alpha_2$, while if $\varepsilon = 0$ then $T(M) = 0$. Thus if $r \leq 2$ and $T(M)$ is a direct double $\varepsilon\Pi = 0$ or ± 1, and so $M(0; S) \cong S^3$ or $S^2 \times S^1$. Since these embed smoothly in $\mathbb{R}^4$ so does $M = M(0; S)\sharp_f (T_g \times S^1)$ [CH].

The simplest nontrivial case is when $r = 3$. If $r = 3$ then $\Delta_0 = |\varepsilon|\Pi$, $\Delta_1 = \delta = \text{h.c.f.}(\alpha_1, \alpha_2, \alpha_3)$ and $\Delta_j = 1$ for $j \geq 2$. We shall assume that $g = 0$ and $T(M)$ is a direct double.

If $\varepsilon = 0$ then $T(M) \cong Z/\delta Z$, and so $\delta = 1$. The equation $\varepsilon\Pi = 0$ implies that α_1 divides $\beta_1\alpha_2\alpha_3$ (etc.), and hence that $\alpha_1 = m_2 m_3$, $\alpha_2 = m_1 m_3$

and $\alpha_3 = m_1m_2$, where $m_i = \text{h.c.f.}(\alpha_j, \alpha_k)$ (for any permutation $\{i,j,k\}$ of $\{1,2,3\}$). Let $\mu = m_1m_2m_3$. We have $H_1(M;\mathbb{Z}) \cong Z$, and the abelianization homomorphism $\phi : \pi = \pi_1(M) \to Z$ is given by $\phi(h) = -\mu$, and $\phi(q_i) = m_i\beta_i$, for $1 \leq i \leq 3$. In this case $M \cong F \times_\theta S^1$, where $\chi(F) = m_1 + m_2 + m_3 - \mu$ and θ has order μ. (Note that $\chi(F) < 0$ unless $\{m_1, m_2, m_3\} = \{1,2,3\}$, in which case $M = M(0;(2,1),(3,-1),(6,-1))$ is the flat 3-manifold with holonomy $Z/6Z$).

The fundamental group $\pi_1(M)$ has a presentation $\langle q_1, q_2, q_3, h \mid q_1q_2q_3 = 1, q_i^{\alpha_i}h^{\beta_i} = 1, h\ central\rangle$. Since $\text{h.c.f.}(\mu, m_1\beta_1) = m_1$ and $\text{h.c.f.}(m_1, m_2\beta_2) = 1$ there are integers e, f_1 and f_2 such that $e\mu + f_1m_1\beta_1 + f_2m_2\beta_2 = 1$. Let $t = h^{-e}q_1^{f_1}q_2^{f_2}$, $r_i = q_it^{-m_i\beta_i}$ and $s = ht^\mu$. Then $\phi(r_i) = \phi(s) = 0$ and $\phi(t) = 1$, and π has an equivalent presentation

$$\langle r_1, r_2, r_3, s, t \mid t^{e\mu-1}(r_1t^{m_1\beta_1})^{f_1}(r_2t^{m_2\beta_2})^{f_2} = s^e, (r_it^{m_i\beta_i})^{\alpha_i}(st^{-\mu})^{\beta_i} = 1,$$
$$r_1t^{m_1\beta_1}r_2t^{m_2\beta_2}r_3t^{m_3\beta_3} = 1, t^\mu r_is = sr_it^\mu, st = ts\rangle.$$

We may use the free differential calculus to obtain a presentation matrix for the Λ-module $H_1(M_\phi;\mathbb{Z})$, with 4 generators and 9 relations. Although this can be simplified by Tietze moves to one with 2 generators and 6 relations, the structure of the module is not clear. However if we tensor over Λ with the field $\mathbb{Q}[\zeta_\mu] = \mathbb{Q}[t,t^{-1}]/(\phi_\mu(t))$ of μ^{th} roots of unity (via the homomorphism sending $t \in \Lambda$ to ζ_μ) we find that the resulting $\mathbb{Q}[\zeta_\mu]$-vector space has dimension 1, and so is not a direct double. Hence the Blanchfield pairing cannot be hyperbolic and no such manifold can embed in $\mathbb{R}^4$.

The situation is less clear when $\varepsilon \neq 0$. We then have $\varepsilon\Pi = \eta\delta^2$, where $\eta = \pm 1$. On writing $\gamma_i = \alpha_i/\delta$ we obtain the equation $\gamma_1\gamma_2\beta_3 + \gamma_1\beta_2\gamma_3 + \beta_1\gamma_2\gamma_3 = -\eta$. Hence the γ_is must be pairwise coprime. (That is, δ is also the h.c.f. of any two of the α_is). Since $T(M) \cong (Z/\delta Z)^2$, it is easily seen that ℓ_M

is metabolic if and only if $(\delta)^2(\ell_M(q_1,q_2)^2 - \ell_M(q_1,q_1)\ell_M(q_2,q_2)) = \eta\beta_1\beta_2\beta_3$ is a square modulo (δ). If moreover δ is odd metabolic forms on $(Z/\delta Z)^2$ are hyperbolic, so ℓ_M is hyperbolic if and only if $\eta\beta_1\beta_2\beta_3$ is a square modulo (δ).

If $\varepsilon \neq 0$ and $\delta = 1$ then $M = M(0;(\gamma_1,\beta_1),(\gamma_2,\beta_2),(\gamma_3,\beta_3))$ is an homology sphere, and so always embeds in $\mathbb{R}^4$. The question of which such homology spheres embed smoothly is still open. (Compare Problem 4.2 of [Ki]). In particular, the Poincaré homology 3-sphere $M(0;(2,1),(3,-1),(5,-1))$ does not embed smoothly. On the other hand, the rational homology spheres $M(0;(2,1),(2,1),(2,-1))$ and $M(0;(3,1),(3,1),(3,-1))$ do embed smoothly. These three manifolds are of particular interest for the strategy of [Hi], as their fundamental groups are virtually solvable. (The only other 3-manifolds which are Seifert fibred over S^2, have virtually solvable fundamental group and embed in $\mathbb{R}^4$ are S^3, $S^2 \times S^1$ and $M(0;(2,1),(2,1),(2,-1),(2,-1))$ [CH]).

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