**Tits alternatives for groups of small cohomological dimension**

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ABSTRACT

*We show that the first $L^2$-Betti number of an ascending HNN extension over a finitely generated base is 0, and apply this to some groups of small cohomological dimension.*

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A well-known theorem of Tits asserts that every finitely generated linear group is either virtually solvable or contains a nonabelian free subgroup [Ti72]. In this note we shall show that similar "Tits alternatives" hold for groups of cohomological dimension two, $PD_3$-groups and 2-knot groups. (Our arguments require some additional hypotheses).

We also give several equivalent characterizations of coherent elementary amenable 2-knot groups, and determine the deficiencies of such groups in most cases.

## §1. A vanishing theorem for $\beta_1^{(2)}(\pi)$

Let $X$ be a finite complex with fundamental group $\pi$. The von Neumann algebra $A(\pi)$ is the algebra of bounded operators on the Hilbert space completion $\ell^2(\pi)$ of $\mathbb{C}[\pi]$ which commute with the natural left $\pi$-action. The *$L^2$-Betti numbers* of $X$ are defined by $\beta_i^{(2)}(X) = dim_{A(\pi)}(\bar{H}_2^{(2)}(X))$ where the $L^2$-homology $\bar{H}_i^{(2)}(X) = \bar{H}_i(C_*^{(2)})$ is the homology of the Hilbert $A(\pi)$-complex $C_*^{(2)} = \ell^2 \otimes C_*(\tilde{X})$ of square summable chains on $\tilde{X}$. ([At76]. See also [Lü94]). They are multiplicative in finite covers, and for $i = 0$ or $1$ depend only on $\pi_1(X)$. (In particular, $\beta_0^{(2)}(\pi) = 0$ if $\pi$ is infinite). The alternating sum of the $L^2$-Betti numbers is the Euler characteristic $\chi(X)$ [At76]. The usual Betti numbers of a space or group (with coefficients $\mathbb{Q}$) shall be denoted by $\beta_i(X)$.

A group $\pi$ has *geometric dimension at most* 2, written $g.d.\pi \leq 2$, if it is the fundamental group of a finite aspherical 2-complex. Let $\pi'$ and $\sqrt{\pi}$ denote the commutator subgroup and Hirsch-Plotkin radical of $\pi$, respectively, and if $\pi$ is elementary amenable let $h(\pi)$ denote its Hirsch length. (See Chapter I of [H2]).

**Theorem 1.** *Let $\pi = H*_\phi$ be a finitely presentable group which is an ascending HNN extension with finitely generated base $H$. Then $\beta_1^{(2)}(\pi) = 0$.*

**Proof.** Let $t$ be the stable letter and let $H_n$ be the subgroup generated by $H$ and $t^n$, and suppose that $H$ is generated by $g$ elements. Then $[\pi : H_n] = n$, so $\beta_1^{(2)}(H_n) = n\beta_1^{(2)}(\pi)$. But each $H_n$ is also finitely presentable and generated by $g+1$ elements. Hence $\beta_1^{(2)}(H_n) \leq g+1$, and so $\beta_1^{(2)}(\pi) = 0$. $\square$

This result extends part of Theorem 2.1 of [Lü94], which considered extensions of $Z$ by finitely presentable normal subgroups.

**Corollary.** *Let $\pi$ be a finitely presentable group which is an ascending HNN extension with finitely generated base $H$. Then $def(\pi) \leq 1$, and $def(\pi) = 1$ if and only if $g.d.\pi \leq 2$ and $\beta_2(\pi) = \beta_1(\pi) - 1$.*

**Proof.** If $X$ is a finite 2-complex corresponding to a presentation $P$ for $\pi$ then $\chi(X) = 1 - def(P)$. Now $\chi(X) = \beta_2^{(2)}(\pi)$ by Theorem 1, Hence $def(P) \leq 1$, and if $def(P) = 1$ then $X$ is aspherical, by Theorem 1 of [Hi97]. In that case $g.d.\pi \leq 2$ and $\beta_2(\pi) = \beta_1(\pi) - 1$. Conversely, if $X$ is a finite aspherical 2-complex with $\pi_1(X) \cong \pi$ then $\chi(X) = 1 - \beta_1(\pi) + \beta_2(\pi) = 0$. After collapsing a maximal tree in $X$ we may assume it has a single 0-cell, and then the presentation read off the 1- and 2-cells has deficiency 1. $\square$

## §2. Tits alternatives

A group is *coherent* if every finitely generated subgroup is finitely presentable.

**Theorem 2.** *Let $\pi$ be a finitely generated group such that $c.d.\pi \leq 2$. Then $\pi$ is solvable if and only if it has no nonabelian free subgroups, is coherent and $\pi/\pi'$ is infinite.*

**Proof.** If $c.d.\pi < 2$ then $\pi$ is free and each condition implies that $\pi \cong Z$ or is trivial. So we may assume $c.d.\pi = 2$, and hence $H^2(\pi; Z[\pi]) \neq 0$. If $\pi$ is solvable then it is an ascending HNN extension with base $Z$ and presentation $\langle a,t \mid tat^{-1} = a^n\rangle$ for some $n \geq 1$ [Gi79]. Such groups are easily seen to be coherent. Hence the conditions are necessary. Conversely, if $\pi$ is coherent and $\pi/\pi'$ is infinite $\pi$ is an HNN extension with finitely presentable base $H$, by Theorem A of [BS78]. Hence $H^2(\pi; Z[\pi])$ is a quotient of $H^1(H; Z[\pi]) \cong H^1(H; Z[H]) \otimes Z[\pi/H]$, by Theorem 0.1 of [BG85], and so $H^1(H; Z[H]) \neq 0$. Since $H$ has no nonabelian free subgroups it must have two ends, so $H \cong Z$ and $\pi$ is solvable. $\square$

In fact we need only know that $H$ is $FP_2$ (*almost* finitely presentable). Does this remain true without any such coherence hypothesis?

**Corollary.** *Let $\pi$ be a finitely presentable group. Then the following are equivalent:*
*(i) $\pi$ is solvable and $c.d.\pi \leq 2$;*
*(ii) $\pi$ is solvable and $def(\pi) = 1$; and*
*(iii) $\pi$ has no nonabelian free subgroups, is coherent and $def(\pi) = 1$.*
**Proof.** Condition $(i)$ implies the others by [Gi79]. Conversely, $(ii)$ implies $(i)$ by the Corollary to Theorem 1, and $(iii)$ implies $(i)$ by the Corollary to Theorem 1 together with Theorem 2. $\square$

Are these conditions also equivalent to "$\pi$ has no nonabelian free subgroups, is coherent and $c.d.\pi \leq 2$"?

See [Wi96] for a more group-theoretic determination of the solvable groups of deficiency 1. Note also that if $def(\pi) > 1$ then $\pi$ contains nonabelian free subgroups [Ro77].

A $PD_n$-group is a Poincaré duality group of formal dimension $n$.

**Theorem 3.** *A $PD_3$-group $G$ is virtually poly-$Z$ if and only if it has no nonabelian free subgroups, has a subgroup of finite index with infinite abelianization and is coherent.*
**Proof.** The conditions are clearly necessary. Suppose that they hold, and let $G_1$ be a subgroup of finite index such that $\beta_1(G_1) > 0$. Then $G_1$ is again a $PD_3$-group, and is an ascending HNN extension $H*_\phi$ with finitely generated base. Since $G$ is coherent $H$ is finitely presentable, and since $H^3(G; Z[G]) \cong Z$ it follows from Lemma 3.4 of [BG85] that $H$ is normal in $G_1$ and $G_1/H \cong Z$.

Hence $H$ is a $PD_2$-group, by Theorem 1 of [Hi96]. Since a $PD_2$-group with no nonabelian free subgroup is virtually $Z^2$ the theorem follows. $\square$

Does this remain true without assuming $G$ coherent? If $\beta_1(G_1) \geq 2$ the hypothesis of coherence is redundant, for there is then an epimorphism $p : G_1 \to Z$ with finitely generated kernel, by [BNS87], and Theorem 1 of [Hi96] requires only that $H$ be finitely generated.

Every subgroup of the fundamental group of a virtually Haken 3-manifold either has a nonabelian free subgroup or is virtually poly-$Z$ [EJ73]. All nontrivial finitely generated subgroups of infinite index in 3-manifold groups have infinite abelianization. If a similar result is true for subgroups of coherent $PD_3$-groups we could recover this stronger result, for subgroups of $PD_3$-groups are either $PD_3$-groups or have cohomological dimension $\leq 2$ [St77].

## §3. 2-Knot groups

A 2-knot is a locally flat embedding $K : S^2 \to S^4$. Let $M(K)$ be the closed orientable 4-manifold obtained by surgery on $K$. Then $\pi_1(M(K)) \cong \pi K = \pi_1(S^4 - K(S^2))$ and $\chi(M(K)) = 0$. (See [H1]). The knot group $\pi K$ is finitely presentable and has infinite cyclic abelianization. Hence it is an HNN extension with finitely generated base and associated subgroups, by Theorem A of [BS78]. The knot group has no nonabelian free subgroups (or is amenable) if and only if the HNN extension is ascending and the base has no nonabelian free subgroups (or is amenable, respectively).

The group $\Phi$ with presentation $\langle a, t \mid tat^{-1} = a^2 \rangle$ is an example of particular interest. (It is a 2-knot group, and is the only solvable higher dimensional knot group of cohomological dimension 2).

**Theorem 4.** *Let $K$ be a 2-knot whose group $\pi = \pi K$ is an ascending HNN extension over a finitely presentable base $H$ with finitely many ends. Then either $\pi'$ is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical.*

**Proof.** Let $\pi = H*_\phi$ where $H$ is finitely presentable and $\phi : H \to H$ is a monomorphism. If $H$ is finite then $\phi$ is an automorphism and so $\pi' = H$. If $H$ has one end then $H^s(\pi; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 0.1 of [BG85]. Since $\beta_1^{(2)}(\pi) = 0$, by Theorem 1, $M(K)$ is aspherical, by "Eckmann's Theorem" of [Hi97]. (The latter theorem is implicit in [Ec94]). If $H$ has two ends then $\pi$ is elementary amenable and $h(\pi) = 2$. Let $T$ be the maximal locally finite normal subgroup of $\pi'$. Since $h(\pi'/T) = 1$ and $\pi'/T$ has no nontrivial locally-finite normal subgroup $[\pi'/T : \sqrt{\pi'/T}] \leq 2$. Since the meridianal automorphism of $\pi'$ induces a meridianal automorphism on $(\pi'/T)/\sqrt{\pi'/T}$ we must have $\pi'/T = \sqrt{\pi'/T}$. Hence $\pi'/T$ is a torsion free rank 1 abelian group, and so $H$ must be an extension of $Z$ by a finite normal subgroup $F$. The monomorphism $\phi$ must map $F$ isomorphically to itself, and hence $T = F$ is finite. Hence $\pi \cong \Phi$, by the corollary to Theorem IV.6 of [H]. $\square$

If $H$ is assumed only to be finitely generated and one ended is $M(K)$ still aspherical?

**Corollary A.** *Let $K$ be a 2-knot whose group $\pi = \pi K \cong H*_\phi$ is an ascending HNN extension over a finitely presentable base and which has an infinite amenable normal subgroup $A$. Then either $\pi'$ is finite or $A \cap \pi' = 1$ or $\pi \cong \Phi$ or $M(K)$ is aspherical.*

**Proof.** If $H$ has finitely many ends this follows from the theorem. If $H$ has infinitely many ends then $A \cap H$ is finite, hence $A \cap \pi'$ is finite. Since we then see that $\pi$ is virtually $\pi' \times Z$ we must have $\pi' = H$ and $M(K)'$ is a $PD_3$-complex. In particular $A \cap \pi' = 1$. $\square$

The groups of torus knots have one end, $H^2(\pi; \mathbb{Z}[\pi]) \neq 0$, are ascending HNN extensions over finitely generated free bases and have centre $A = Z$ such that $A \cap \pi' = 1$.

**Corollary B.** *If $H$ is $FP_3$ and has one end then $\pi'$ is a $PD_3^+$-group.*

**Proof.** This follows from Lemma 3.4 of [BG85], as in Theorem 3. $\square$

Does this remain true if we assume only that $H$ is $FP_2$ and has one end?

We can now give a version of the Tits alternative for 2-knot groups.

**Corollary C.** *A 2-knot group $\pi$ is torsion free and virtually poly-$Z$ of Hirsch length 4 if and only if it is an ascending HNN extension $\pi \cong H*_\phi$ where $H$ is $FP_3$, has no nonabelian free subgroup and has a subgroup of finite index which maps onto $Z^2$.*

**Proof.** The conditions are clearly necessary. Conversely, if they hold then $\pi' = H$ and is a $PD_3$-group, by Corollary B, and $H$ is virtually poly-$Z$ by Theorem 3 (together with the remark following it). $\square$

This corollary applies in particular if $K$ is a 2-knot with a minimal Seifert hypersurface and $\pi = \pi K$ is amenable.

**Corollary D.** *Let $\pi$ be a 2-knot group. Then the following are equivalent:*
*(i) $\pi'$ is virtually nilpotent;*
*(ii) $\pi'$ is locally virtually poly-$Z$;*
*(iii) $\pi$ is coherent and elementary amenable; and*
*(iv) $\pi'$ is finite or $\pi \cong \Phi$ or $\pi$ is torsion free virtually poly-$Z$ and $h(\pi) = 4$.*
**Proof.** In all cases, $\pi$ has no nonabelian free subgroup, and so it is an ascending HNN extension $\pi \cong H*_\phi$ with finitely generated base $H$. Moreover $H \leq \pi' = \cup t^n H t^{-n}$ and so $H$ is finitely presentable, in all cases. Thus Theorem 4 applies. Clearly $(i)$ implies $(ii)$. If $(ii)$ holds then $\pi$ is elementary amenable. Since elementary amenable groups of finite cohomological dimension are virtually solvable $(i)$, $(ii)$ and $(iii)$ imply $(iv)$. On the other hand $(iv)$ implies $(i)$, $(ii)$ and $(iii)$. $\square$

This class probably includes all elementary amenable 2-knot groups (and perhaps even all amenable 2-knot groups?).

**Theorem 5.** *Let $K$ be a 2-knot with a minimal Seifert hypersurface, and suppose that $\pi = \pi K$ has a nontrivial abelian normal subgroup $A$. Then $A \cap \pi'$ is finite cyclic or is torsion free.*
**Proof.** By assumption, $\pi = HNN(H; \phi : I \cong J)$ for some finitely presentable group $H$ and isomorphism of $\phi$ of subgroups $I$ and $J$, where $I$ and $J$ are 3-manifold groups. Let $t \in \pi$ be the stable letter. If $A$ has rank $> 1$ then $M(K)$ is aspherical, by Theorem 10 of [Hi97]. We may assume that $A$ has rank $\leq 1$. Since $A \cap I$ is an abelian normal subgroup of a 3-manifold group it is cyclic. Moreover either $A \cap H = A \cap I$ or $A \cap H = A \cap J$ (by Britton's Lemma). Hence $A \cap \pi' = \cup_{n \in Z} t^n (A \cap I) t^{-n}$ is a monotone union. If $A \cap I$ is finite cyclic then $A \cap I = t^n (A \cap I) t^{-n}$, for all $n$, and so $A \cap \pi' = A \cap I$. Otherwise $A \cap \pi'$ is a torsion free rank 1 abelian group. $\square$

This was first proven in [Yo92], under the additional assumption that $A$ be finitely generated. This assumption excludes Example 10 of Fox (with commutator subgroup $\Phi' \cong Z[\frac{1}{2}]$), which has a minimal Seifert hypersurface.

## §4. Deficiency of virtually solvable 2-knot groups

In this brief section we shall estimate the deficiencies of the 2-knot groups considered in Corollary D of Theorem 4. (See Chapters IV- VI of [H] for the determination of these 2-knot groups).

The groups $Z$ and $\Phi$ clearly have deficiency 1. If a 2-knot group has nontrivial finite commutator subgroup then it has deficiency $\leq 0$, by the corollary to Theorem 1. Many such 2-knot groups are the groups of twist spins of classical knots, and so have presentations of deficiency 0. If $\pi'$ is finite but $\pi$ is not the group of a twist spun knot then $\pi' \cong Q(8) \times (Z/qZ)$, for some odd $q > 1$, and $\pi$ has a 2-generator 2-relator presentation $\langle t, z \mid ztz^q t^{-1} z^{-1} = tz^{-q} t^{-1}, t^{-1} zt = tz^q t^{-1} z^{2q-1} \rangle$, and so again has deficiency 0.

If a 2-knot group $\pi$ is a $PD_4$-group then $def(\pi) \leq 0$, by Theorem 5 of [Hi97]. If $\pi$ is a semidirect product $\pi = Z^3 \times_C Z$ then $C$ is conjugate to a matrix of the form $C = \begin{pmatrix} 0 & 0 & 1 \\ m & p & r \\ n & q & s \end{pmatrix}$. (See Theorem A.3 of [AR84]). Hence $\pi$ has a deficiency 1 presentation $\langle t, x, y, z \mid xz = zx, yz = zy, txt^{-1} = y^m z^n, tyt^{-1} = y^p z^q, tzt^{-1} = xy^r z^s \rangle$. If $n = p = 0$ then $\pi$ has a 2-generator deficiency 0 presentation $\langle t, x \mid txt^{-1}x = xtxt^{-1}, t^3 xt^{-3} = xtx^r tx^s t^{-2} \rangle$. Conversely, if $\pi$ has a 2-generator 2-relator presentation then $\pi'/\pi''$ is a cyclic $\mathbb{Z}[\pi/\pi']$-module, and so $C$ is conjugate to such a matrix. We do not know whether the other metabelian 2-knot groups have deficiency 0.

There are two virtually abelian 2-knot groups $G(\pm)$. These have presentations $\langle x, y, t \mid xy^2 x^{-1} y^2 = 1, txt^{-1} = (xy)^{\mp 1}, tyt^{-1} = x^{\pm 1} \rangle$ and so have deficiency 0. (The group $G(+)$ is the group of the 3-twist spin of the figure eight knot). There are three 2-knot groups $\pi$ such that $\pi'$ is free nilpotent of class 2 on 2 generators. These have presentations $\langle x, y, t \mid [x, [[x, y]] = 1, txt^{-1} = xy, tyt^{-1} = w \rangle$, where $w = x^{-1}$, $xy^2$ or $x$ (respectively), and so have deficiency 0. These are easily seen to have 2-generator 2-relator presentations also. (The first of these is the group of the 6-twist spin of the trefoil knot). For each odd $q > 1$ there is also a 2-knot group with a deficiency -1 presentation $\langle t, x, y, z \mid xz = zx, xyx^{-1}y^{-1} = z^q, txt^{-1} = xy, tyt^{-1} = x, tzt^{-1} = z^{-1} \rangle$; we do not know whether these groups have deficiency 0. The other virtually poly-$Z$ 2-knot groups are the groups of 2-twist spins of Montesinos knots, and so have deficiency 0.

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## §4. Various

If $K$ is a slice knot then $M(K)=\partial(D^5-R)$, where $R$ is an open regular neighbourhood of a slice disc $\Delta$ in $D^5$. We shall say that $K$ is $\pi_1$-*slice* if $\Delta$ can be chosen so that the inclusion of $M=M(K)$ into $D^5-R$ induces an isomorphism on fundamental groups.

**Theorem 6.** *Let $K$ be a 2-knot with group $\pi=\pi K$. If $K$ is $\pi_1$-slice then the homomorphism from $H_3(M';\mathbb{Z})=H_3(M(K);\Lambda)$ to $H_3(\pi';\mathbb{Z})=H_3(\pi;\Lambda)$ induced by $c_M$ is trivial. If moreover $M'$ is a $PD_3$-complex and $\pi$ is torsion free then $\pi'$ is a free group.*

**Proof.** Let $\Delta$ and $R$ be chosen as above. Since $c_M$ factors through $D^5-R$ the first assertion follows from the exact sequence of homology (with coefficients $\Lambda$) for the pair $(D^5-R,M)$. If $M'$ is a $PD_3^+$-complex with torsion free fundamental group then it is a connected sum of aspherical $PD_3$-complexes with handles $S^2\times S^1$ [Tu90]. It is easily seen that if $H_3(c_M;\Lambda)=0$ there is no aspherical summand, and so $\pi'$ is free. □

**Lemma 6.** *Let $\nu$ be the fundamental group of a closed orientable $\mathbb{H}^2\times\mathbb{E}^1$-manifold. Then $\nu$ admits no meridianal automorphism.*

**Proof**. If $\nu$ is a lattice in $Isom^+(H^2\times E^1)$ then $\sqrt{\nu}\cong Z$ and either $\sqrt{\nu}=\zeta\nu$ and is not contained in $\nu'$ or $C_\nu(\sqrt{\nu})$ is a characteristic subgroup of index 2 in $\nu$. In either case $\nu$ admits no meridianal automorphism. □

Thus the only geometries realized by 2-knot manifolds $M(K)$ are $\mathbb{E}^4$, $\mathbb{N}il^3\times\mathbb{E}^1$, $\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$, certain of the $\mathbb{S}ol_{m,n}^4$- geometries, $\mathbb{S}^3\times\mathbb{E}^1$, $\widetilde{SL}\times\mathbb{E}^1$ and $\mathbb{H}^3\times\mathbb{E}^1$. Excepting the latter two geometries, the corresponding 2-knots are well understood.

**Corollary.** *Let $M$ be a closed $\mathbb{S}ol_1^4$-manifold with fundamental group $\pi$. Then $\beta_1(M)\leq 1$ and $M$ is orientable if and only if $\beta_1(M)=1$.*

**Proof.** The first assertion is clear if $\pi$ is a semidirect product $\Gamma_q\times_\theta Z$, and then follows in general. Hence if there is an epimorphism $p:\pi\to Z$ with kernel $K$ then $K$ must be virtually nilpotent of class 2 and the result follows from the theorem. □

Let $\pi$ be a torsion free virtually nilpotent group of Hirsch length 4 which is not virtually abelian. Then $\sqrt{\pi}$ has a characteristic subgroup $C\cong Z^2$. Since

$C$ is characteristic in $\sqrt{\pi}$ it is normal in $\pi$. Let $\theta : \pi \to Aut(C) \cong GL(2,\mathbb{Z})$ be the homomorphism induced by conjugation in $\pi$. After passing to a subgroup of index 2, if necessary, we may assume that $\pi$ acts via orientation preserving automorphisms of $C$, and so $\theta(\pi)$ is cyclic of order $\leq 6$. Let $p : \pi \to H = \pi/C$ be the canonical epimorphism. The preimage in $\pi$ of the maximal finite normal subgroup of $H$ is torsion free and so is isomorphic to $Z^2$ or $Z \times_{-1} Z$. Since the latter possibility is excluded, by the definition of $\pi^+$, $H$ has no nontrivial finite normal subgroup and so is a 2-dimensional crystallographic group. After again passing to subgroups of index 2, if necessary, we may assume that $H$ acts on $\sqrt{H} \cong Z^2$ via orientation preserving automorphisms also. If $H$ is abelian then $C_\pi(C)$ is a central extension of an abelian group, and so is nilpotent. Hence $[\pi : \sqrt{\pi}] \leq 6$ in this case. If $H$ is nonabelian then $[H : H'] \leq 9$ and $C_\pi(C) \cap p^{-1}(\sqrt{H})$ is nilpotent and $p^{-1}(H') \leq C_\pi(C) \cap p^{-1}(\sqrt{H})$, so $[\pi : \sqrt{\pi}] \leq [H : H']$. Taking into account the earlier passages to orientation preserving subgroups we see that in general $[\pi : \sqrt{\pi}] \leq 36$. Can this estimate be improved?

Let $N$ be the subgroup generated by all the normal subgroups which have no nonabelian free subgroup. Then $N$ is the maximal such subgroup, and clearly it contains the maximal elementary amenable subgroup. Is $N$ the maximal elementary amenable normal subgroup? If $N$ is nontrivial then either $N \cong Z$, $c.d.N = 2$ or $N = G$. If $N$ is a rank 1 abelian group then $N \cong Z$. (For otherwise $G' \leq C_G(N)$ and $[G' : G' \cap N] < \infty$, and $G'$ contains a subgroup isomorphic to $G' \cap N \times Z$, so $c.d.N = 1$ and again $N \cong Z$). If $c.d.N = 2$ then $N$ cannot be $FP_2$, for otherwise it would be a surface group and $G$ would be virtually the group of a surface bundle [Hi 87]. Since $N$ has no nonabelian free subgroup this would imply that $N$ and hence $G$ are virtually poly-$Z$, and so $N = G$.

Nontrivial classical knot groups have geometric dimension 2 and deficiency 1, but always have nonabelian free subgroups.

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