STRATIFICATION THEORY FROM THE WEIGHTED POINT OF VIEW

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To the memory of Professor Nobuo Sasakura

Stratification theory is a fundamental tool in constructing topological trivialization for families of varieties or maps. The idea was first presented by R.Thom [21] (for further development see for instance [16] and [7], and also a good survey can be found in [6]). At that time the purpose was mainly to show that, in some suitable set up, topological stable maps are dense (successfully proved). Using the existence of some good tube systems for a regular stratification, they showed the existence of a vector field whose integration gives a topological trivialization.

This approach was good enough for the study of topological stability, but unfortunately, in our opinion, the expression was not explicit enough to show that some given family is topologically trivial.

In this paper, we investigate stratification theory in terms of the defining equations of strata and maps (without tube systems), offering a concrete approach for solving the above problem.

The key notion in stratification theory is the regularity between strata (nowadays many regularity conditions are known, see [1, 2, 22]). In §1, we present a criterion for Verdier's (w)-regularity conditions and Kuo's ratio test condition in terms of the defining equations of the strata. Next we give an explicit construction of a vector field for topological trivialization. The key step in our construction is the use of a new projection formula (Lemma 1.4) allowing us to treat at the same time the non-complete intersection case. In this approach, it is possible to consider a weighted version of (w)-regularity condition and Kuo's ratio test condition, and we do this in §2. We show that these conditions imply the integrability of the vector fields constructed by the method in §1. In §3, using the regularity conditions defined in §2, we prove the isotopy lemmas. Actually, using a partition of unity, we patch up together the vector fields constructed in §2.

With appropriate modifications one can use the same techniques in the complex analytic case to obtain analogous results. Because the modifications are standard, we concentrate ourselves only on the real case.

1. Regularity conditions

1.1. **A criterion for (w)-regularity.** Let $(x, y) = (x_1, ..., x_n, y_1, ..., y_m)$ denote a system of coordinates of $\mathbf{R}^{n+m}$ and X, Y submanifolds of $\mathbf{R}^{n+m}$. For notational convenience we also use $x_{n+s} = y_s$. We assume that

$$Y = \{(x, y) \in \mathbf{R} : x_1 = \cdots = x_n = 0\}.$$

Let π_P denote the orthogonal projection of $\mathbf{R}^{n+m}$ to the normal space of X at $P \in X$. Then, following [22], we say X is (w)-*regular* over Y at $0 \in Y$, if for any unit vector v tangent to Y $|\pi_P(v)| \lesssim |x|$ at $P \in X$ near 0. Here $A \lesssim B$ means there is some positive constant C with $A \leq CB$. Of course we may restrict v to the members of a basis of the tangent space of Y at 0.

We next assume that X is some open set in the regular locus of the variety defined as the zero locus of some C^2-functions $F_1(x,y)$, ..., $F_p(x,y)$ near 0, i.e., setting $F := (F_1, ..., F_p)$, the Jacobi matrix of F has rank k on X near 0, where $k \leq p$ is the codimension of X in $\mathbf{R}^{n+m}$. We note that the normal space of X is generated by the gradients of the functions F_j $(j = 1, ..., p)$ at each $P \in X$ near 0.

Let $j_1, ..., j_k$ be integers with $1 \leq j_1 < \cdots < j_k \leq p$. We set $J = \{j_1, ..., j_k\}$, $F_J = (F_{j_1}, ..., F_{j_k})$ and

$$dF_J = dF_{j_1} \wedge \cdots \wedge dF_{j_k}, \quad \text{where} \quad dF_j = \sum_{i=1}^{n+m} \frac{\partial F_j}{\partial x_i} dx_i,$$

$$d_xF_J = d_xF_{j_1} \wedge \cdots \wedge d_xF_{j_k}, \quad \text{where} \quad d_xF_j = \sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i} dx_i,$$

and we define d^xF_J by $dF_J = d_xF_J + d^xF_J$.

For $I \subset \{1, ..., n\}$, $S \subset \{1, ..., m\}$, $J \subset \{1, ..., p\}$ with $\#I + \#S = \#J =$ k, we set $\frac{\partial F_J}{\partial(x_I, y_S)}$ the Jacobian of (F_J) with respect to the variables x_i $(i \in I)$ and y_s $(s \in S)$. If $S = \emptyset$, we simply denote it by $\frac{\partial F_J}{\partial x_I}$. We then define $\|d_xF\|$, $\|d^xF\|$ by the following formulae:

$$\|d_xF\|^2 = \sum_J \|d_xF_J\|^2 \quad \text{where} \quad \|d_xF_J\|^2 = \sum_I \left|\frac{\partial F_J}{\partial x_I}\right|^2$$

$$\|d^xF\|^2 = \sum_J \|d^xF_J\|^2 \quad \text{where} \quad \|d^xF_J\|^2 = \sum_{I,S:S\neq\emptyset} \left|\frac{\partial F_J}{\partial(x_I, y_S)}\right|^2$$

Let $|x|$ denote the function defined by $|x|^2 = \sum_{i=1}^n |x_i|^2$. For a matrix M we have used $|M|$ as the absolute value of its determinant $\det(M)$ if M is a square matrix or 0 otherwise.

Theorem 1.1. *The following conditions are equivalent:*

(i) *X is (w)-regular over Y at 0.*

(ii) *$\|d^xF\| \lesssim |x| \; \|d_xF\|$ holds on X near 0.*

(iii) *For any C^1-functions φ_j $(j = 1, ..., p)$ near 0, and $s = 1, ..., m$,*

$$\left|\sum_{j=1}^p \varphi_j \frac{\partial F_j}{\partial y_s}\right| \lesssim |x| \;\sup \left\{\left|\sum_{j=1}^p \varphi_j \frac{\partial F_j}{\partial x_i}\right| : i = 1, ..., n\right\} \quad \textit{holds on } X \textit{ near } 0.$$

(iv) *For $J \subset \{1, ..., , p\}$, $I = \{i_1, ..., i_{k-1}\} \subset \{1, ..., n\}$ with $1 \leq i_1 < \cdots < i_{k-1} \leq n$, $s = 1, ..., m$,*

$$\left|\frac{\partial F_J}{\partial(x_I, y_s)}\right| \lesssim |x| \; \|d_xF\| \quad \textit{holds on } X \textit{ near } 0.$$

The condition (iii) is inspired by several conditions appeared in T.Gaffney's paper [8]. The first author thanks Leslie Wilson for informing him about the existence of that paper. It is also possible to obtain a similar result for Kuo's ratio test condition and we present it in §1.5

We next state some sufficient conditions for (w)-regularity. For $j = 1, ..., p$, we set

$$\|d_x F^{[j]}\|^2 = \sum_J \sum_{i_1<\cdots<i_{k-1}} \left| \frac{\partial F_J^{[j]}}{\partial x_{\{i_1,...,i_{k-1}\}}} \right|^2, \quad \text{and} \quad h_j = \frac{\|d_x F\|}{\|d_x F^{[j]}\|},$$

where

$$F_J^{[j]} = \begin{cases} (F_{j_1}, ..., \widehat{F_{j_a}}, ..., F_{j_k}) & \text{if } j \in J = \{j_1, ..., j_k\},\ j = j_a, \\ (0, ..., 0)\ ((k-1)\text{-tuple}) & \text{if } j \notin J. \end{cases}$$

Here, $\widehat{\ }$ is the notation indicating that we omit the letter (or the portion) to which $\widehat{\ }$ is attached.

Corollary 1.2. *X is (w)-regular over Y at 0, if for $j = 1, ..., p$ the following inequalities hold on X near 0,*

$$\left| \frac{\partial F_j}{\partial y_s} \right| \lesssim |x|\ h_j,\ \ s = 1, ..., m \quad \left(\text{or, equivalently, } \|d^x F_j\| \lesssim |x| \frac{\|d_x F\|}{\|d_x F^{[j]}\|} \right).$$

Corollary 1.3. *X is (w)-regular over Y at 0, if the following inequalities hold on X near 0,*

$$\|d^x F_j\| \lesssim |x|\ \|d_x F_j\|, \quad \|d_x F_j\| \lesssim h_j = \frac{\|d_x F\|}{\|d_x F^{[j]}\|}, \quad \text{for } j = 1, ..., p.$$

By (iii) in Theorem 1.1, the first inequality here is a necessary condition for (w)-regularity. Note that when $p = k = 1$, this is just Teissier's c-condition (see [19, 20]).

1.2. **Linear algebra.** We present here some lemmas in linear algebra which are needed later on. Let V denote a real vector space with dimension n, and V^* denote the dual space of V. Let $e_1, ..., e_n$ denote a basis of V, and $e_1^*, ..., e_n^*$ the dual basis of V^* defined by $e_i^*(e_j) = \delta_{ij}$. We consider a positive symmetric bilinear form of V:

$$\langle\ ,\ \rangle : V \times V \to \mathbf{R}.$$

For an element $v \in V$, $v^\vee$ denotes the linear functional of V defined by $v^\vee(w) = \langle w, v\rangle$. This induces an identification between V and V^* by $v \mapsto v^\vee$. Then, we have $e_i^\vee = \sum_{i=1}^n g_{ij} e_j^*$ where $g_{ij} = \langle e_i, e_j\rangle$. Thus, $e_i^* = \sum_{j=1}^n g^{ij} e_j^\vee$ where (g^{ij}) denotes the inverse matrix of (g_{ij}).

It is well known that this bilinear form $\langle\ ,\ \rangle$ induces bilinear forms on the exterior products $\bigwedge^k V$, $\bigwedge^k V^*$. This is defined in the following way. The set $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid i_1 < \cdots < i_k\}$ gives a basis of $\bigwedge^k V$, and the bilinear form is defined by

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k}\rangle = g_{i_1 j_1} \cdots g_{i_k j_k}.$$

Similarly, the set $\{e^*_{i_1} \wedge \cdots \wedge e^*_{i_k} \mid i_1 < \cdots < i_k\}$ gives a basis of $\bigwedge^k V^*$, and the bilinear form is defined by

$$\langle e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}, e^*_{j_1} \wedge \cdots \wedge e^*_{j_k}\rangle = g^{i_1 j_1} \cdots g^{i_k j_k}.$$

Let $a^j = \sum_{i=1}^n a^j_i e_i$ $(j = 1, ..., k)$ denote vectors in V, and $b^j = \sum_{i=1}^n b^j_i e^*_i$ $(j = 1, ..., k)$ covectors in V^*. Under the identification $(\bigwedge^k V)^* = \bigwedge^k V^*$, we have (by [17], theorem 9, p.78),

$$(b^1 \wedge \cdots \wedge b^k)(a^1 \wedge \cdots \wedge a^k) = \det(b^j(a^i))_{1 \le i,j \le k}.$$

Let $a^j = \sum_{i=1}^n a^j_i e_i$ $(j = 1, ..., p)$ denote vectors in V, and W the linear span of $a^1, ..., a^p$, assumed to be of dimension $k \le p$. We set $W^\perp = \{v \in V \mid \langle v, w\rangle = 0,\ \forall w \in W\}$. Let $\phi_{j_1,...,j_k}$ $(1 \le j_1 < \cdots < j_k \le p)$ denote some constants such that the denominator is not zero. For $v = (v_1, ..., v_n) \in V$, we set

$$\pi(v) = \sum_{i=1}^n \frac{\sum_{j_1<\cdots<j_k} \phi_{j_1,...,j_k}\big((a^{j_1})^\vee \wedge \cdots \wedge (a^{j_k})^\vee \wedge e^*_i\big)(a^{j_1} \wedge \cdots \wedge a^{j_k} \wedge v)}{\sum_{j_1<\cdots<j_k} \phi_{j_1,...,j_k}\big((a^{j_1})^\vee \wedge \cdots \wedge (a^{j_k})^\vee\big)(a^{j_1} \wedge \cdots \wedge a^{j_k})} e_i.$$

Lemma 1.4. *The image of π is $W^\perp$, and $\pi : V \to W^\perp$ is the orthogonal projection with respect to $\langle\ ,\ \rangle$.*

Since we could not find this lemma in literature, we present here a complete proof of it.

Proof. Obviously π is a linear map. Since $\pi(a^j) = 0$, we have $\pi(W) = 0$. Therefore it is enough to see that $\pi(v) = v$ for any $v \in W^\perp$. For $v \in W^\perp$ and $j = 1, ..., p$, we have $(a^j)^\vee(v) = \langle a^j, v\rangle = 0$. Thus we obtain

$$\begin{aligned}\sum_i \big((a^{j_1})^\vee \wedge \cdots \wedge (a^{j_k})^\vee \wedge e^*_i\big)(a^{j_1} \wedge \cdots \wedge a^{j_k} \wedge v)e_i &= \sum_i \det \begin{pmatrix} (a^{j_r})^\vee(a^{j_q}) & (a^{j_r})^\vee(v) \\ e^*_i(a^{j_q}) & e^*_i(v)\end{pmatrix}_{1 \le r,q \le k} e_i \\ &= \det\begin{pmatrix} (a^{j_r})^\vee(a^{j_q}) & 0 \\ a^{j_q} & v\end{pmatrix}_{1 \le r,q \le k} = \big((a^{j_1})^\vee \wedge \cdots \wedge (a^{j_k})^\vee\big)(a^{j_1} \wedge \cdots \wedge a^{j_k})v\end{aligned}$$

Then we obtain

$$\begin{aligned}\sum_{j_1<\cdots<j_k} \phi_{j_1,...,j_k} \sum_i \big((a^{j_1})^\vee \wedge \cdots \wedge (a^{j_k})^\vee \wedge e^*_i\big)(a^{j_1} \wedge \cdots \wedge a^{j_k} \wedge v)e_i \\ = \sum_{j_1<\cdots<j_k} \phi_{j_1,...,j_k}\big((a^{j_1})^\vee \wedge \cdots \wedge (a^{j_k})^\vee\big)(a^{j_1} \wedge \cdots \wedge a^{j_k})v.\end{aligned}$$

Thus we obtain $\pi(v) = v$ for $v \in W^\perp$. □

Note that

$$\begin{aligned}\left(\sum_{j_1<\cdots<j_k} \phi_{j_1,...,j_k}\big((a^{j_1})^\vee \wedge \cdots \wedge (a^{j_k})^\vee\big)(a^{j_1} \wedge \cdots \wedge a^{j_k})\right)\pi(v) \\ = \sum_{i=1}^n \sum_{j_1<\cdots<j_k} \phi_{j_1,...,j_k}\big((a^{j_1})^\vee \wedge \cdots \wedge (a^{j_k})^\vee \wedge e^*_i\big)(a^{j_1} \wedge \cdots \wedge a^{j_k} \wedge v)e_i\end{aligned}$$

for any constants $\phi_{j_1,...,j_k}$.

Since $v = \sum_{i=1}^n (e_i)^*(v)e_i$, if we set $\phi_{j_1,...,j_k} = 1$, we get the following formula

$$\pi(v) = \frac{\sum_{j_1<\cdots<j_k} \det \begin{pmatrix} \langle a^{j_r}, a^{j_q}\rangle & \langle a^{j_r}, v\rangle \\ a^{j_r} & v \end{pmatrix}}{\sum_{j_1<\cdots<j_k} \det(\langle a^{j_r}, a^{j_q}\rangle)_{1\le r,q\le k}}.$$

Remark 1.5. Let V be the tangent space T_xM of a manifold M at a point x. Then V^* is its cotangent space T_x^*M. Let $\langle\ ,\ \rangle$ be a Riemannian metric. Let $x_1, ..., x_n$ be a local coordinate system around x, $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ the usual basis of T_xM, $dx_1, ..., dx_n$ the usual (dual) basis of T_x^*M.

Let $F_1, ..., F_p$ be C^1-functions and assume the rank of the Jacobian of $(F_1, ..., F_p)$ is rank k on $X \subset \bigcap_{j=1}^p F_j^{-1}(0)$, and $\phi_{j_1,...,j_k}$ $(1 \le j_1 < \cdots < j_k \le p)$ positive functions on the zero set of $(F_1, ..., F_p)$. Applying the previous lemma with $a^j = \mathrm{grad} F_j$, where $\mathrm{grad} F_j$ is the gradient vector of F_j i.e. a vector with $dF_j = (\mathrm{grad} F_j)^\vee$, we obtain that the orthogonal projection of $v \in T_xM$ to the tangent space T_xX is expressed by the following form.

$$(1.1)\quad v \mapsto \sum_i \frac{\sum_{j_1<\cdots<j_k} \phi_{j_1,...,j_k}(dF_{j_1} \wedge \cdots \wedge dF_{j_k} \wedge dx_i)(\mathrm{grad} F_{j_1} \wedge \cdots \wedge \mathrm{grad} F_{j_k} \wedge v)}{\sum_{j_1<\cdots<j_k} \phi_{j_1,...,j_k} \|dF_{j_1} \wedge \cdots \wedge dF_{j_k}\|^2} \frac{\partial}{\partial x_i}.$$

1.3. **Proof of Theorem 1.1.**

Lemma 1.6. *X is (w)-regular over Y at 0 iff the following inequalities hold on X near 0 for each $J \subset \{1, ..., p\}, \#J = k$:*

(a) $\quad |\langle dF_J \wedge dx_i, dF_J \wedge dy_s\rangle| \lesssim |x|\ \|dF_J\|^2$ *for* $1 \le i \le n; 1 \le s \le m$

(b) $\quad |\langle dF_J \wedge dy_i, dF_J \wedge dy_s\rangle| \lesssim |x|\ \|dF_J\|^2$ *for* $1 \le i < s \le m$

(c) $\quad \|dF_J\|^2 - \|dF_J \wedge dy_s\|^2 \lesssim |x|\ \|dF_J\|^2$ *for* $1 \le s \le m$

Proof. This is a consequence of the definition of (w)-regularity and (1.1). We can see this taking particular functions $\phi_{j_1,...,j_k}$ in the expression (1.1). Namely, one can take $\phi_{j_1,...,j_k} = 1$, if $J = \{j_1, ..., j_k\}$; 0, otherwise. □

Proof of (ii) $\Longrightarrow$ (i). Since the left hand side of (c) does not contain the terms containing $\frac{\partial F_j}{\partial y_s}$ for $1 \le j \le p$, and $\|dF_J\|^2 = \|d_xF_J\|^2 + \|d^xF_J\|^2$, (c) is equivalent to the following inequality.

$$\|d^xF\|^2 \lesssim |x|\ \|d_xF\|^2.$$

By Cauchy-Schwarz, (b) comes from (c). Thus we have that X is (w)-regular over Y at 0 iff the inequalities (a) and (c) hold on X near 0.

We now assume (ii). We then have $\|d^xF\|^2 \lesssim |x|^2\ \|d_xF\|^2$ on X near 0. Since we may assume that $|x|$ is very small, (c) trivially holds. By Cauchy-Schwarz inequality, this implies (a). This completes the proof. □

Proof of (i) $\Longrightarrow$ (iii). We first remark the following equality:

$$\sum_{i\neq n+s}\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial x_i}\langle dF_J\wedge dx_i, dF_J\wedge dy_s\rangle = \left\langle dF_J\wedge\sum_{j=1}^{p}\varphi_j\left(dF_j-\frac{\partial F_j}{\partial y_s}dy_s\right), dF_J\wedge dy_s\right\rangle$$
$$= -\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial y_s}\|dF_J\wedge dy_s\|^2.$$

Then, by (a) and (b), we have

$$\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial y_s}\right|\|dF_J\wedge dy_s\|^2\leq\sum_{i\neq n+s}\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial x_i}\right||\langle dF_J\wedge dx_i, dF_J\wedge dy_s\rangle|$$
$$\lesssim\sum_{i\neq n+s}\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial x_i}\right|\ |x|\ \|dF\|^2.$$

Then by (c) we have

$$\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial y_s}\right|\|dF_J\|^2\leq\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial y_s}\right|\left(\|dF_J\wedge dy_s\|^2+\ |x|\ \|dF\|^2\right)$$
$$\lesssim\sum_{i=1}^{n+m}\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial x_i}\right|\ |x|\ \|dF\|^2.$$

Summing up for all J and s we get the desired inequality, which completes the proof. $\square$

In §2 we shall prove the generalized weighted version (Theorem 2.1) of the equivalence (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv), and this will complete the proof of our theorem.

Proof of Corollary 1.2. We define d_ℓ by the following formula: ${d_\ell}^2=\sum_J\sum_{I,S:\#S=\ell}\left|\frac{\partial F_J}{\partial(x_I,y_S)}\right|^2$.

By Cauchy-Schwarz inequality, the inequality in Corollary 1.2 implies $d_1\lesssim|x|d_0$, which implies (iv) in Theorem 1.1. $\square$

Proof of Corollary 1.3. Obvious. Use Corollary 1.2. $\square$

1.4. **Kuo's vector.** Let U be a neighborhood of 0, and $Y=\{x_1=\cdots=x_n=0\}$. We assume that X is the regular locus of the zero locus of C^2-functions F_j ($j=1,...,p$), and assume the codimension is k.

Let v be a tangent vector to Y. The purpose of this subsection is to give an explicit construction of a tangent vector ξ to X which is an extension of v and $dp(\xi)=v$ where p is the natural projection defined by $(x,y)\mapsto y$.

We first remark that any multiple of the orthogonal projection of $v=\sum_{s=1}^m c_s\frac{\partial}{\partial y_s}$ to the tangent space of X may not have this property (except the case $m=1$).

For $s=1,...,m$, we use the following notation:

$$d^{(s)}F_j=\sum_{i=1}^{n}\frac{\partial F_j}{\partial x_i}dx_i+\frac{\partial F_j}{\partial y_s}dy_s,\quad\text{and}\quad d^{(s)}F_J=d^{(s)}F_{j_1}\wedge\cdots\wedge d^{(s)}F_{j_k}.$$

Let us consider the orthogonal projection of $\frac{\partial}{\partial y_s}$ to the tangent space of X in the space defined by $y_1 = \cdots = y_{s-1} = y_{s+1} = \cdots = y_m = 0$.

This is expressed by

$$\sum_{i=1}^{n} \frac{\sum_J \phi_J \langle d^{(s)} F_J \wedge dx_i, d^{(s)} F_J \wedge dy_s \rangle}{\sum_J \phi_J \langle d^{(s)} F_J, d^{(s)} F_J \rangle} \frac{\partial}{\partial x_i} + \frac{\sum_J \phi_J \langle d^{(s)} F_J \wedge dy_s, d^{(s)} F_J \wedge dy_s \rangle}{\sum_J \phi_J \langle d^{(s)} F_J, d^{(s)} F_J \rangle} \frac{\partial}{\partial y_s},$$

where ϕ_J are any positive function on X. Its multiple whose $\frac{\partial}{\partial y_s}$-component is 1, is expressed by the following formula:

$$\sum_{i=1}^{n} \frac{\sum_J \phi_J (d^{(s)} F_J \wedge dx_i)(\mathrm{grad}_x F_J \wedge \frac{\partial}{\partial y_s})}{\sum_J \phi_J \|d_x F_J\|^2} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_s},$$

where $\mathrm{grad}_x F_J$ denotes the wedge product of the gradients in coordinate $x_1, ..., x_n$.

Let $v = \sum_j c_s \frac{\partial}{\partial y_s}$ be a unit vector tangent to Y. Then the above construction implies the following vector ξ is tangent to X:

$$\xi = \sum_{i=1}^{n} \frac{\sum_{j=1}^{m} c_s \sum_J \phi_J (d^{(s)} F_J \wedge dx_i)(\mathrm{grad}_x F_J \wedge \frac{\partial}{\partial y_s})}{\sum_J \phi_J \|d_x F_J\|^2} \frac{\partial}{\partial x_i} + v.$$

Since $\mathrm{grad}_x F_j$ has zero $\frac{\partial}{\partial y_s}$ components, we thus have

$$\xi = \sum_{i=1}^{n} \frac{\sum_J \phi_J (dF_J \wedge dx_i)(\mathrm{grad}_x F_J \wedge v)}{\sum_J \phi_J \|d_x F_J\|^2} \frac{\partial}{\partial x_i} + v.$$

We call this ξ a *Kuo's vector*, because T.-C.Kuo expressed this vector in such an explicit form in [15] when $m = p = 1$, (see also [11, 12, 14]). Here we remark that the $\frac{\partial}{\partial x_i}$-component of ξ (say ξ_i) is

$$\xi_i = \sum_{s=1}^{m} c_s \frac{\sum_{J,I:\#I=k-1} \phi_J \frac{\partial F_J}{\partial(x_I, x_i)} \frac{\partial F_J}{\partial(x_I, y_s)}}{\sum_{J,I} \phi_J |\frac{\partial F_J}{\partial x_I}|^2}.$$

We assume that X is (w)-regular over Y at 0. Then the inequality (ii) of Theorem 1.1 implies

$$\left| \frac{\sum_J \langle dF_J \wedge dx_i, d_x F_J \wedge v^\vee \rangle}{\sum_J \|d_x F_J\|^2} \right| \lesssim |x| \quad \text{for } i = 1, ..., n \text{ on } X \text{ near } 0.$$

This implies that ξ satisfies (stratified) Lipschitz condition on (X, Y) near 0.

1.5. **Ratio test conditions.** We use the same notation as in Theorem 1.1. We define $|(x,y)|$ and $|y|$ by

$$|(x,y)|^2 = \sum_{i=1}^{n} |x_i|^2 + \sum_{j=1}^{m} |y_j|^2, \quad \text{and} \quad |y|^2 = \sum_{s=1}^{m} |y_s|^2.$$

We say that X is (r)-*regular* over Y at 0, if for any unit vector v tangent to Y

$$|\pi_P(v)|\ |(x,y)| = o(|x|) \quad \text{when } P = (x,y) \to 0,\ P \in X.$$

Here "$A = o(B)$ when $P \to 0$" means $\lim_{P \to 0} |A/B| = 0$. This was defined in [12].

Theorem 1.7. *The following conditions are equivalent:*

(i) X *is* (r)*-regular over* Y *at* 0.

(ii) $\|d^x F\|\,|(x,y)| = o(|x|\,\|d_x F\|)$ *when* $(x,y) \to 0$ *on* X.

(iii) *For any* C^1*-function* φ_j $(j = 1, ..., p)$ *near* 0 *and* $s = 1, ..., m$,

$$\left|\sum_{j=1}^{p} \varphi_j \frac{\partial F_j}{\partial y_s}\right| \,|(x,y)| = o(|x|\,\|d_x F\|) \quad \textit{when } (x,y) \to 0 \textit{ on } X.$$

(iv) *For* $J \subset \{1, ..., , p\}$, $I = \{i_1, ..., i_{k-1}\} \subset \{1, ..., n\}$ *with* $1 \leq i_1 < \cdots < i_{k-1} \leq n$, $s = 1, ..., m$,

$$\left|\frac{\partial F_J}{\partial(x_I, y_s)}\right| |(x,y)| = o(|x|\,\|d_x F\|) \quad \textit{when } (x,y) \to 0 \textit{ on } X.$$

Proof. This is proved in exactly the same way as Theorem 1.1 (we omit the details). □

Proposition 1.8. *When* X *is* (r)*-regular over* Y *at* 0, *the flow of Kuo's vector field* ξ *is unique.*

Proof. We first remark that ξ does not depend on the choice of ϕ_J, and we set $\phi_J \equiv 1$. If X is (r)-regular over Y at 0, then the inequalities obtained by replacing $|(x,y)|$ by $|y|$ from the inequalities in (ii), (iii), (iv) of Theorem 1.7 hold. This shows that the Kuo's vector ξ satisfies Nagumo's criterion (Corollary 6.1 on page 32 of [9]) and we have uniqueness of the flow if $m = 1$. Assume that $m > 1$. Let C be the integral curve of v. Then the flow of ξ is in $\pi^{-1}(C)$ and we can reduce the problem to the case $m = 1$ and we are done. □

2. Weighted versions of (w)-regularity

In the previous section we present (w) regularity in terms of defining equations in some coordinate. This allow us to consider a weighted version of (w)-regularity condition. We present it in this section.

2.1. **A weighted version of (w)-regularity.** Let $(x,y) = (x_1, ..., x_n, y_1, ..., y_m)$ denote a system of coordinates of a neighborhood U of 0 in $\mathbf{R}^{n+m}$ and X, Y disjoint submanifolds of U. For notational convenience we put $x_{n+s} = y_s$. We assume that

$$Y = \{(x,y) \in U : x_1 = \cdots = x_n = 0\}. \tag{2.1}$$

We fix a weight $\mathbf{w} = (w_1, ..., w_n)$, and consider the function defined by

$$\|x\| = \|x\|_{\mathbf{w}} := \left(|x_1|^{\frac{2w}{w_1}} + \cdots + |x_n|^{\frac{2w}{w_n}}\right)^{\frac{1}{2w}}, \quad \text{where } w = w_1 w_2 \cdots w_n. \tag{2.2}$$

We next assume that X is some open set in the regular locus of the variety defined as the zero locus of C^2-functions $F_1(x,y)$, ..., $F_p(x,y)$ near 0, i.e., setting $F := (F_1, ..., F_p)$, the Jacobi matrix of F has rank k on X near 0, where k is the codimension of X in $\mathbf{R}^{n+m}$. We note that the normal space of X is generated by the gradients of functions F_j $(j = 1, ..., p)$ at each $P \in X$ near 0.

We define $\|d_x F\|_{\mathbf{w}}$, $\|d^x F\|_{\mathbf{w}}$, $D_{\mathbf{w}}(\ell)$ by the following formula:

$$\|d_x F\|_{\mathbf{w}}^2 = \sum_J \|d_x F_J\|_{\mathbf{w}}^2 \quad \text{where} \quad \|d_x F_J\|_{\mathbf{w}}^2 = \sum_I \left(\|x\|_{\mathbf{w}}^{w_I} \left| \frac{\partial F_J}{\partial x_I} \right| \right)^2 \tag{2.3}$$

$$\|d^x F\|_{\mathbf{w}}^2 = \sum_J \|d^x F_J\|_{\mathbf{w}}^2 \quad \text{where} \quad \|d^x F_J\|_{\mathbf{w}}^2 = \sum_{I,S:S\neq\emptyset} \left(\|x\|_{\mathbf{w}}^{w_I} \left| \frac{\partial F_J}{\partial(x_I, y_S)} \right| \right)^2 \tag{2.4}$$

$$D_{\mathbf{w}}(\ell)^2 = \sum_J \sum_{I,S:\#S=\ell} \left(\|x\|_{\mathbf{w}}^{w_I} \left| \frac{\partial F_J}{\partial(x_I, y_S)} \right| \right)^2. \quad \text{Here} \quad w_I = \sum_{i\in I} w_i. \tag{2.5}$$

We consider the singular metric of $\mathbf{R}^{n+m}$ defined by

$$\left\langle \|x\|_{\mathbf{w}}^{w_i} \frac{\partial}{\partial x_i}, \|x\|_{\mathbf{w}}^{w_j} \frac{\partial}{\partial x_j} \right\rangle = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.6}$$

Here, we understand that $w_{n+1} = \cdots = w_{n+m} = 0$. We first remark that

$$\langle dx_{i_1} \wedge \cdots \wedge dx_{i_k}, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \rangle = \|x\|_{\mathbf{w}}^{2(w_{i_1}+\cdots+w_{i_k})}$$

and $\langle dF, dF \rangle = \|dF\|_{\mathbf{w}}^2$, $\langle d_x F, d_x F \rangle = \|d_x F\|_{\mathbf{w}}^2$, etc.

Theorem 2.1. *The following conditions are equivalent:*

(i) $D_{\mathbf{w}}(m) \lesssim D_{\mathbf{w}}(m-1) \lesssim \cdots \lesssim D_{\mathbf{w}}(1) \lesssim D_{\mathbf{w}}(0)$ *holds on* X *near* 0.

(ii) $\|d^x F\|_{\mathbf{w}} \lesssim \|d_x F\|_{\mathbf{w}}$ *holds on* X *near* 0.

(iii) *For any* C^1*-functions* φ_j *(*$j = 1, ..., p$*) near* 0*, and* $s = 1, ..., m$*,*

$$\left| \sum_{j=1}^p \varphi_j \frac{\partial F_j}{\partial y_s} \right| \lesssim \sum_{i=1}^n \|x\|_{\mathbf{w}}^{w_i} \left| \sum_{j=1}^p \varphi_j \frac{\partial F_j}{\partial x_i} \right| \quad \textit{holds on } X \textit{ near } 0.$$

(iv) *For* $J \subset \{1, \ldots, p\}$*,* $I = \{i_1, ..., i_{k-1}\} \subset \{1, \ldots, n\}$ *with* $1 \leq i_1 < \cdots < i_{k-1} \leq n$*,* $s = 1, ..., m$*,*

$$\|x\|_{\mathbf{w}}^{w_{i_1}+\cdots+w_{i_{k-1}}} \left| \frac{\partial F_J}{\partial(x_I, y_s)} \right| \lesssim \|d_x F\|_{\mathbf{w}} \quad \textit{holds on } X \textit{ near } 0.$$

(v) *For* $J \subset \{1, \ldots, p\}$*,* $i = 1, ..., n$*,* $s = 1, ..., m$*,*

$$|\langle dF_J \wedge dx_i, d_x F_J \wedge dy_s \rangle| \lesssim \|x\|_{\mathbf{w}}^{w_i} \|d_x F_J\|_{\mathbf{w}}^2 \quad \textit{holds on } X \textit{ near } 0.$$

(vi) *For some positive* C^1*-functions* ϕ_J *on* X *with* $J \subset \{1, \ldots, p\}$*,* $i = 1, ..., n$*,* $s = 1, ..., m$*,*

$$\left| \sum_J \phi_J \langle dF_J \wedge dx_i, d_x F_J \wedge dy_s \rangle \right| \lesssim \|x\|_{\mathbf{w}}^{w_i} \sum_J \phi_J \|d_x F_J\|_{\mathbf{w}}^2 \quad \textit{holds on } X \textit{ near } 0.$$

We say that X is *weighted* (w)*-regular* over Y at 0 with respect to $\mathbf{w}$ (or $\mathbf{w}$-(w)*-regular* for short), if one of the above equivalent conditions holds in some coordinate system (x, y) with (2.1). If $w_1 = \cdots = w_n = 1$, these coincide with the usual (w)-regular condition. However, it is not immediate to see that (ii) is the same condition as in the homogeneous case.

We also state the weighted version of Theorem 1.3.

Corollary 2.2. *X is* $\mathbf{w}$*-*(w)*-regular over Y at 0, if the following inequalities hold on X near 0:*

$$\|d^x F_j\|_{\mathbf{w}} \lesssim \|d_x F_j\|_{\mathbf{w}}, \quad \|d_x F_j\|_{\mathbf{w}} \lesssim \frac{\|d_x F\|_{\mathbf{w}}}{\|d_x F^{[j]}\|_{\mathbf{w}}}, \qquad \textit{for } j = 1, ..., p.$$

Proof. The proof is similar to that of Corollary 1.3, and we omit it. ☐

2.2. **Proof of Theorem 2.1.** Since implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv), (v) $\Longrightarrow$ (vi) are clear, it is enough to see (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (i), (iii) $\Longrightarrow$ (v), and (vi) $\Longrightarrow$ (iii).

Proof of (iii) $\Longrightarrow$ (i). We assume (iii). By the inequality in (iii), we have

$$\left|\frac{\partial F_J}{\partial(x_I, y_s)}\right| \lesssim \sum_{i=1}^n \|x\|_{\mathbf{w}}^{w_i} \left|\frac{\partial F_J}{\partial(x_I, x_i)}\right| \tag{2.7}$$

and $D_{\mathbf{w}}(1) \lesssim D_{\mathbf{w}}(0) = \|d_x F\|_{\mathbf{w}}$. Similarly we also have

$$\left|\frac{\partial F_J}{\partial(x_I, y_{s_1}, y_{s_2})}\right| \lesssim \sum_{i=1}^n \|x\|_{\mathbf{w}}^{w_i} \left|\frac{\partial F_J}{\partial(x_I, y_{s_1}, x_i)}\right|$$

and, using (2.7), we obtain $D_{\mathbf{w}}(2) \lesssim D_{\mathbf{w}}(1)$ on X near 0. In a similar way we obtain that $D_{\mathbf{w}}(\ell) \lesssim D_{\mathbf{w}}(\ell - 1)$ $(\ell = 2, ..., m)$, and this completes the proof. ☐

Proof of (iii) $\Longrightarrow$ (v). We assume (iii). Using (iii) in a similar way to the previous proof, we have

$$\begin{aligned}
|\langle dF_J \wedge dx_i, d_x F_J \wedge dy_s\rangle| &= \left|\sum_I \|x\|_{\mathbf{w}}^{2(w_I+w_i)} \frac{\partial F_J}{\partial(x_I, x_i)} \frac{\partial F_J}{\partial(x_I, y_s)}\right| \\
&\lesssim \|x\|_{\mathbf{w}}^{w_i} \sum_{\ell=1}^n \sum_I \|x\|_{\mathbf{w}}^{2w_I+w_\ell+w_i} \left|\frac{\partial F_J}{\partial(x_I, x_i)}\right| \left|\frac{\partial F_J}{\partial(x_I, x_\ell)}\right| \leq \|x\|_{\mathbf{w}}^{w_i} \|d_x F_J\|_{\mathbf{w}}^2
\end{aligned}$$

and we are done. ☐

Proof of (vi) $\Longrightarrow$ (iii). Since

$$\sum_{i=1}^n \frac{\partial F_j}{\partial x_i} \langle dF_J \wedge dx_i, d_x F_J \wedge dy_s\rangle \;=\; -\sum_{t=1}^m \frac{\partial F_j}{\partial y_t} \langle dF_J \wedge dy_t, d_x F_J \wedge dy_s\rangle \;=\; -\frac{\partial F_j}{\partial y_s} \|dF_J\|_{\mathbf{w}}^2,$$

we obtain

$$\sum_{j=1}^p \varphi_j \sum_{i=1}^n \frac{\partial F_j}{\partial x_i} \sum_J \phi_J \langle dF_J \wedge dx_i, d_x F_J \wedge dy_s\rangle = -\sum_{j=1}^p \varphi_j \frac{\partial F_j}{\partial y_s} \sum_J \phi_J \|dF_J\|_{\mathbf{w}}^2.$$

Thus, we obtain

$$\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial y_s}\right|\sum_J \phi_J\|d_xF_J\|^2_{\mathbf{w}} \le \sum_{i=1}^{n}\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial x_i}\right|\left|\sum_J \phi_J\left\langle d^{(s)}F_J\wedge dx_i, d^{(s)}F_J\wedge dy_s\right\rangle\right|$$
$$\lesssim \sum_{i=1}^{n}\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial x_i}\right|\|x\|^{w_i}_{\mathbf{w}}\ \sum_J\phi_J\|d_xF_J\|^2_{\mathbf{w}}.$$

Dividing by $\|d_xF_J\|^2_{\mathbf{w}}$, we obtain (iii). □

Proof of (iv) $\Longrightarrow$ (iii). (iv) implies $D_{\mathbf{w}}(1)\lesssim\|d_xF\|_{\mathbf{w}}$ on X near 0. Then, by Cauchy-Schwarz inequality, we have

$$|\langle d^{(s)}F_J\wedge dx_i, d^{(s)}F_J\wedge dy_s\rangle| \lesssim \|x\|^{w_i}_{\mathbf{w}}\|d_xF_J\|^2_{\mathbf{w}} \quad \text{on } X \text{ near } 0.$$

We next remark the following equality:

$$\sum_{i=1}^{n}\frac{\partial F_j}{\partial x_i}\langle d^{(s)}F_J\wedge dy_s, d^{(s)}F_J\wedge dx_i\rangle = \left\langle d^{(s)}F_J\wedge dy_s, d^{(s)}F_J\wedge\left(d^{(s)}F_j-\frac{\partial F_j}{\partial y_s}dy_s\right)\right\rangle$$
$$= -\frac{\partial F_j}{\partial y_s}\|d_xF_J\|^2_{\mathbf{w}}.$$

Thus, we obtain

$$\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial y_s}\right|\|d_xF_J\|^2_{\mathbf{w}} \le \sum_{i=1}^{n}\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial x_i}\right|\left|\left\langle d^{(s)}F_J\wedge dx_i, d^{(s)}F_J\wedge dy_s\right\rangle\right|$$
$$\lesssim \sum_{i=1}^{n}\left|\sum_{j=1}^{p}\varphi_j\frac{\partial F_j}{\partial x_i}\right|\|x\|^{w_i}_{\mathbf{w}}\ \|d_xF\|^2_{\mathbf{w}}.$$

Summing up the inequalities, for all J and dividing by $\|d_xF\|^2_{\mathbf{w}}$, we obtain (iii). □

2.3. **ODE problems.** To construct a topological trivialization using the weighted (w)-condi- tion (or Kuo's ratio test condition), we consider the following classical ODE problem

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \tag{2.8}$$

where $x=(x_1,...,x_n)$ If $f(t,x)$ is continuous, it is well known that the solution $x=x(t)$ exists (see [9], theorems 2.1, page 12). We fix a weight $\mathbf{w}=(w_1,...,w_n)$ and consider the function $\|x\|=\|x\|_{\mathbf{w}}$ defined by (2.1).

We say $f(t,x)$ satisfies a *weighted Lipschitz condition* with respect to the weight $\mathbf{w}$ (or, $\mathbf{w}$-*Lipschitz*, for short) if the following condition holds:

$$\|f(t,x)-f(t,\overline{x})\|_{\mathbf{w}} \lesssim \|x-\overline{x}\|_{\mathbf{w}} \text{ for any } x,\ \overline{x} \text{ near } x_0.$$

We say that $f(t,x)$ is *weighted Lipschitz* , if it satisfies a weighted Lipschitz condition for some weight $\mathbf{w}$.

Lemma 2.3. *If $f(t,x)$ is weighted Lipschitz near $(t,x)=(t_0,x_0)$, then* (2.8) *admits at most one solution near $t=t_0$.*

Proof. Let $x(t)$, and $\overline{x}(t)$ be two solutions of (3.1) with $x(t_0) = \overline{x}(t_0) = x_0$. Since

$$x_i(t) - x_i(t_0) = \int_{t_0}^{t} f_i(t, x(s))ds, \quad \overline{x}_i(t) - \overline{x}_i(t_0) = \int_{t_0}^{t} f_i(t, \overline{x}(s))ds,$$

we have

$$x_i(t) - \overline{x}_i(t) = \int_{t_0}^{t} \{f_i(s, x(s)) - f_i(s, \overline{x}(s))\}ds.$$

Thus

$$\begin{aligned}
\sum_{i=1}^{n} |x_i(t) - \overline{x}_i(t)|^{\frac{2w}{w_i}} &= \sum_{i=1}^{n} \left| \int_{t_0}^{t} \{f_i(s, x(s)) - f_i(s, \overline{x}(s))\}ds \right|^{\frac{2w}{w_i}} \\
&\leq \sum_{i=1}^{n} \left(\int_{t_0}^{t} |f_i(s, x(s)) - f_i(s, \overline{x}(s))|ds \right)^{\frac{2w}{w_i}} \\
&\leq \sum_{i=1}^{n} |t - t_0|^{\frac{2w}{w_i} - 1} \int_{t_0}^{t} |f_i(s, x(s)) - f_i(s, \overline{x}(s))|^{\frac{2w}{w_i}} ds \qquad \text{(by Hölder)} \\
&\leq \sum_{i=1}^{n} \int_{t_0}^{t} |f_i(s, x(s)) - f_i(s, \overline{x}(s))|^{\frac{2w}{w_i}} ds \qquad (\text{since } |t - t_0| < 1).
\end{aligned}$$

We remark that the $\mathbf{w}$-Lipschitz condition implies that there is $C > 0$ such that

$$\sum_{i=1}^{n} |f_i(s, x) - f_i(s, \overline{x})|^{\frac{2w}{w_i}} \leq C \sum_{i=1}^{n} |x_i - \overline{x}_i|^{\frac{2w}{w_i}}.$$

Setting $\varphi(t) = \sum_{i=1}^{n} |x_i(t) - \overline{x}_i(t)|^{\frac{2w}{w_i}}$ we thus have

$$\varphi(t) \leq \int_{t_0}^{t} \sum_{i=1}^{n} |f_i(s, x) - f_i(s, \overline{x})|^{\frac{2w}{w_i}} ds = C \int_{t_0}^{t} \varphi(s)ds$$

We here set $\Phi(t) = \int_{t_0}^{t} \varphi(s)ds$. Then we have $\frac{d\Phi}{dt} = \varphi(t) \leq C\Phi(t)$. Since

$$\frac{d}{dt}\left(e^{-C(t-t_0)}\Phi(t)\right) = e^{-C(t-t_0)} \left(\frac{d\Phi}{dt}(t) - C\Phi(t) \right) \leq 0,$$

and $\Phi(t) \geq 0$, $\Phi(t_0) = 0$, we obtain $\Phi(t) = 0$, and we are done. $\square$

We say $f(t, x)$ satisfies a *weighted Nagumo condition* with respect to the weight $\mathbf{w}$ (or, $\mathbf{w}$-*Nagumo*, for short) if the following condition holds:

$$\|f(t, x) - f(t, \overline{x})\|_{\mathbf{w}} \leq \left\| \frac{x - \overline{x}}{t - t_0} \right\|_{\mathbf{w}} \quad \text{for any } t \text{ near } t_0 \text{ and any } x, \overline{x} \text{ near } x_0.$$

We say that $f(t, x)$ is *weighted Nagumo* , if it satisfies a weighted Nagumo condition for some weight $\mathbf{w}$.

Lemma 2.4. *If $f(t, x)$ is weighted Nagumo near $(t, x) = (t_0, x_0)$, then* (2.8) *admits at most one solution near $t = t_0$.*

Proof. Let $x(t)$, $\overline{x}(t)$ be same as the proof of Lemma 2.3. We define $\varphi(t) = (t-t_0)\|(x-\overline{x})/(t-t_0)\|_{\mathbf{w}}^{2w}$ and as in the previous proof we get

$$\varphi(t) \leq \int_{t_0}^{t} \frac{\varphi(s)}{s-t_0} ds.$$

We set $\Phi(t) = \int_{t_0}^{t} \frac{\varphi(s)}{s-t_0} ds$. Then

$$\frac{d}{dt}\left(\frac{\Phi(t)}{t-t_0}\right) = \frac{\varphi(t)-\Phi(t)}{(t-t_0)^2} \leq 0.$$

Since $\Phi(t) \geq 0$, $\Phi(t_0) = 0$, we obtain $\Phi(t) = 0$, and we are done. $\square$

Note that weighted Lipschitz trivially implies weighted Nagumo..

2.4. **Kuo's vector.** We consider the Euclidean space $\mathbf{R}^{n+m}$ with the singular metric defined by (2.6). Let U be a neighborhood of 0, and $Y = \{x_1 = \cdots = x_n = 0\}$. In this section we assume that X is the regular locus of the zero locus of C^2-functions F_j $(j = 1, ..., p)$, and assume the codimension is k. By elementary calculation, we can express the gradient vector field of the function F_j with respect to this singular metric is expressed as follows:

$$\mathrm{grad} F_j = \mathrm{grad}_x F_j + \sum_{s=1}^{m} \frac{\partial F_j}{\partial y_s}\frac{\partial}{\partial y_s}, \quad \text{where} \quad \mathrm{grad}_x F_j = \sum_{i=1}^{n} \|x\|_{\mathbf{w}}^{2w_i} \frac{\partial F_j}{\partial x_i}\frac{\partial}{\partial x_i}.$$

Using the same construction as §1.4, the vector field

$$\xi = \sum_{i=1}^{n} \frac{\sum_J \phi_J (dF_J \wedge dx_i)(\mathrm{grad}_x F_J \wedge v)}{\sum_J \phi_J \|d_x F_J\|_{\mathbf{w}}^2} \frac{\partial}{\partial x_i} + v.$$

is tangent to X. Here we remark that the $\frac{\partial}{\partial x_i}$-component of ξ (say ξ_i) is

$$\xi_i = \sum_{s=1}^{m} c_s \frac{\sum_J \phi_J \langle dF_J \wedge dx_i, d_x F_J \wedge dy_s\rangle}{\sum_J \phi_J \|d_x F_J\|_{\mathbf{w}}^2}$$

We assume that X is $\mathbf{w}$-(w)-regular over Y at 0. Then the inequality in (vi) of Theorem 2.1 implies

$$|\xi_i| \lesssim \|x\|_{\mathbf{w}}^{w_i} \quad \text{on } X \text{ near } 0. \tag{2.9}$$

This implies the $\mathbf{w}$-Lipschitz condition and the uniqueness of the flow of ξ on $X \cup Y$ by Lemma 2.3. Thus, we obtain the following:

Theorem 2.5. *The conditions in Theorem 2.1 are equivalent to the following condition:*
(vii) *The Kuo's vector ξ satisfies* (2.9) *for any unit vector $v = \sum_{s=1}^{m} c_s \frac{\partial}{\partial y_s}$.*

Remark 2.6. Assume that the closure of X is $X \cup Y$. We consider an extension of ξ to the neighborhood U of 0 and denote it by $\tilde{\xi}$. If the denominator $\|d_x F\|^2$ is not zero on

$U - Y$, there are no problems to extend ξ on U. In this case, the integration of ξ gives a $\mathcal{R}$-trivialization (if it is $\mathbf{w}$-Lipschitz or $\mathbf{w}$-Nagumo). In general, if we set

$$\tilde{\xi} = \begin{cases} \sum_{i=1}^n \frac{\sum_J \phi_J (dF_J \wedge dx_i)(\mathrm{grad}_x F_J \wedge v)}{\sum_J \phi_J \|d_x F_J\|_{\mathbf{w}}^2 + \sum_{j=1}^p |F_j|^2} \frac{\partial}{\partial x_i} + v & \text{on } U - Y, \\ v & \text{on } Y, \end{cases}$$

this is the desired extension on U. We remark that $\tilde{\xi}$ satisfies $(\mathbf{w})$-Lipschitz (or $(\mathbf{w})$-Nagumo), if ξ does so.

Thus, by theorem 2.1 of [9], page 94, the integration of $\tilde{\xi}$ gives a family of homeomorphisms which trivialize the family $(\mathbf{R}^n, X_y, y)$ $(y \in Y)$ near 0 where $X_y = \{x \in \mathbf{R}^n : (x, y) \in X\}$. Actually in this way we obtain a $\mathcal{K}$-trivialization of the family.

The notations $\mathcal{R}$, $\mathcal{K}$ we use here are the standard ones. For the definition and more about these equivalence relations, consult the excellent survey [24].

Theorem 2.7. *(Same notation as above) Suppose that X is $\mathbf{w}$-(w)-regular over Y at 0 and that the singular set of the variety defined by $F_1, ..., F_p$ is Y. Then the family $(\mathbf{R}^n, X_y, y)$ $(y \in Y)$ near 0 is topologically trivial.*

Theorem 2.8. *(Same notation as above) We assume that F_j $(j = 1, ..., p)$ are C^ω. We write for $j = 1, ..., p$ the weighted Taylor series of $F_j(x, y)$ by $H_{j,d_j}(x, y) + H_{j,d_j+1}(x, y) + \cdots$, where*

$$H_{j,\ell}(x, y) = \sum_{\alpha=(\alpha_1,...,\alpha_n)} c_{j,\ell,\alpha}(y) x_1^{\alpha_1} \cdots x_n^{\alpha_n}; \; w_1\alpha_1 + \cdots w_n\alpha_n = \ell$$

is a weighted homogeneous ℓ-form in variables x with respect to the weight $\mathbf{w} = (w_1, ..., w_n)$. If the zero locus of $(H_{1,d_1}(x, y), ..., H_{p,d_p}(x, y))$ has codimension k and its singular set is Y, then the family $(\mathbf{R}^n, X_y, y)$ $(y \in Y)$ is topologically trivial.

See [5], for similar topological triviality theorem.

Proof. It is enough to see the following condition which is (vi) in Theorem 2.1 with $\phi_J = \|d_x F_J\|_{\mathbf{w}}^{2d^{[J]}-2}$ where $d_J = \sum_{j \in J} d_j$, $d^{[J]} = \prod_{J' : \#J' = k, J' \neq J} d_{J'}$: There is $C > 0$ such that for $i = 1, ..., n$, $s = 1, ..., m$,

$$\left| \sum_J \|d_x F_J\|_{\mathbf{w}}^{2d^{[J]}-2} \langle dF_J \wedge dx_i, d_x F_J \wedge dy_s \rangle \right| \leq C \|x\|_{\mathbf{w}}^{w_i} \sum_J \|d_x F_J\|_{\mathbf{w}}^{2d^{[J]}} \tag{2.10}$$

holds on X near 0.

The weighted expansion of the right hand side of (2.10) with respect to the weight $\mathbf{w}$ is given by

$$H_d(x, y) + H_{d+1}(x, y) + \cdots ,$$

where $H_\ell(x, y)$ is a weighted homogeneous ℓ-form in variables x (not necessarily polynomial). By our supposition the zero set of $H_d(x, y)$ is in Y. We are also able to write the weighted expansion of the left hand side of (2.10) by

$$K_d(x, y) + K_{d+1}(x, y) + \cdots , \tag{2.11}$$

where $K_\ell(x,y)$ is a weighted homogeneous ℓ-form in variables x with respect to the weight $\mathbf{w}$. Using these expressions, we see that H_d is not zero outside the origin, and this implies our inequality. $\square$

Using the method above, we can show the following

Proposition 2.9. *Let V be a set defined by $F_1(x) = \cdots = F_a(x) = 0$ in $\mathbf{R}^n$. We consider a map $f : V \times \mathbf{R}^m \to \mathbf{R}^p$. We denote by $f_y : V \to \mathbf{R}^p$ the map defined by $f_y(x) = f(x,y)$ for $x \in V$. We also denote by the same letter f an extension of f to $\mathbf{R}^n \times \mathbf{R}^m$. Assume that the jacobi matrix $\frac{\partial(F,f)}{\partial x}$ is of constant rank (say k) on $(V - \{0\}) \times \mathbf{R}^m$. If the following inequalities hold on $V \times \mathbf{R}^m$ (resp. $\mathbf{R}^n \times \mathbf{R}^m$) for $s = 1,...,m$, $i_1,...,i_{k-1}$ with $1 \le i_1 < \cdots < i_{k-1} \le n$ then the family f_y, $y \in \mathbf{R}^m$, is C^0-$\mathcal{K}_{\mathcal{R}(V)}$-trivial (resp. C^0-$\mathcal{R}(V)$-trivial).*

$$\|x\|_{\mathbf{w}}^{w_{i_1}+\cdots+w_{i_{k-1}}} \left| \frac{\partial(F_J, f_{J'})}{\partial(x_{i_1},...,x_{i_{k-1}},y_s)} \right| \lesssim \|d_x(F,f)\|_{\mathbf{w}}.$$

For other equivalent conditions to this inequality, see Theorem 2.1.

This application was very much inspired by Maria Ruas's talk (related to J.N.Tomazella's Ph D. Thesis) at International Symposium "Topology of Singularities" held in Kochi, 1998. Here $\mathcal{K}_{\mathcal{R}(V)} = \mathcal{R}(V).\mathcal{C}$. We used the standard notation from [24]. For the definition and more about $\mathcal{R}(V)$, see [4].

2.5. **Weighted Nagumo regularity conditions.** Using Lemma 2.4, it is possible to give a similar treatment to §2.4. We replace $\|x\|_{\mathbf{w}}$ by $\|x/|y|\|_{\mathbf{w}}$ in (2.3), (2.4), (2.5), and denote them by $\|\widetilde{d}_x F\|_{\mathbf{w}}$, $\|\widetilde{d}^x F\|_{\mathbf{w}}$, $\widetilde{D}_{\mathbf{w}}(\ell)$, respectively. We define a new singular metric by replacing $\|x\|_{\mathbf{w}}$ by $\|x/|y|\|_{\mathbf{w}}$ in (2.6).

Theorem 2.10. *The following conditions are equivalent:*

(i) *For $\ell = 1,...,m$, $\widetilde{D}_{\mathbf{w}}(\ell) = o(\widetilde{D}_{\mathbf{w}}(\ell-1))$ when $(x,y) \to 0$, $(x,y) \in X$.*

(ii) *$\|\widetilde{d}^x F\|_{\mathbf{w}} = o(\|\widetilde{d}_x F\|_{\mathbf{w}})$, when $(x,y) \to 0$, $(x,y) \in X$.*

(iii) *For any C^1-functions φ_j $(j = 1,...,p)$ near 0, and $s = 1,...,m$,*

$$\left| \sum_{j=1}^{p} \varphi_j \frac{\partial F_j}{\partial y_s} \right| = o\left(\sum_{i=1}^{n} \left\| \frac{x}{|y|} \right\|_{\mathbf{w}}^{w_i} \left| \sum_{j=1}^{p} \varphi_j \frac{\partial F_j}{\partial x_i} \right| \right) \text{ when } (x,y) \to 0,\ (x,y) \in X.$$

(iv) *For $J \subset \{1,\dots,p\}$, $I = \{i_1,...,i_{k-1}\} \subset \{1,\dots,n\}$ with $1 \le i_1 < \cdots < i_{k-1} \le n$, $s = 1,...,m$,*

$$\left\| \frac{x}{|y|} \right\|_{\mathbf{w}}^{w_{i_1}+\cdots+w_{i_{k-1}}} \left| \frac{\partial F_J}{\partial(x_I, y_s)} \right| = o(\|\widetilde{d}_x F\|_{\mathbf{w}}) \text{ when } (x,y) \to 0,\ 0 \in X.$$

(v) *For $J \subset \{1,\dots,p\}$, $i = 1,...,n$, $s = 1,...,m$,*

$$|\langle dF_J \wedge dx_i, d_x F_J \wedge dy_s \rangle| = o\left(\left\| \frac{x}{|y|} \right\|_{\mathbf{w}}^{w_i} \|\widetilde{d}_x F_J\|_{\mathbf{w}}^2 \right) \quad \text{when } (x,y) \to 0,\ (x,y) \in X.$$

(vi) *For some positive C^1-functions ϕ_J on X with $J \subset \{1,\ldots,p\}$, $i=1,...,n$, $s=1,...,m$,*

$$\left|\sum_J \phi_J \langle dF_J \wedge dx_i, d_x F_J \wedge dy_s\rangle\right| = o\left(\left\|\frac{x}{|y|}\right\|_{\mathbf{w}}^{w_i} \sum_J \phi_J \|\widetilde{d}_x F_J\|_{\mathbf{w}}^2\right) \quad \textit{when } (x,y)\to 0,\ (x,y)\in X.$$

Proof. The proof is similar to that of Theorem 2.1, and we omit the details. □

We say that X is $\mathbf{w}$-*Nagumo regular* over Y at 0 if one (thus any) of the equivalent condition above holds. By the same way as the proof of Proposition 1.8, we obtain that the Kuo's vector ξ has a unique flow under $\mathbf{w}$-Nagumo regularity.

3. Construction of vector fields for the isotopy lemmas

3.1. **A vector field for the first isotopy lemma.** Let $(x,y)=(x_1,...,x_n,y_1,...,y_m)$ denote a system of coordinates of $\mathbf{R}^{n+m}$. Let $p:\mathbf{R}^{n+m}\to\mathbf{R}^m$ denote the projection defined by $p(x,y)=y$. We set $X_0=0\times\mathbf{R}^m$. Let X_1, X_2, ..., X_s denote disjoint submanifolds of $\mathbf{R}^{n+m}$ satisfying the following conditions:

(i) The dimensions of any two (thus all) connected components of X_k are the same for $k=1,...,s$.
(ii) $m<\dim X_1<\dim X_2<\cdots<\dim X_s$.
(iii) X_j is weighted (w)-regular (or, $\mathbf{w}$-Nagumo regular) over X_i at each point of X_i, for $0\le i<j\le s$.
(iv) The restriction of p to $X_0\cup X_1\cup\cdots\cup X_k$ is proper for $k=1,...,s$.

Then, we show that $\{X_{i,y}\}_{0\le i\le s}$ $(y\in\mathbf{R}^m)$ is a topological trivial stratification (the first isotopy lemma), where $X_{i,y}=\{x\in\mathbf{R}^n:(x,y)\in X_i\}$ for $i=0,1,...,s$. We construct such a trivialization following [18] (see chapter I) under the following supposition:

(v) There is a C^2-map $F^{(k)}=(F_1^{(k)},...,F_{p_k}^{(k)}):\mathbf{R}^{n+m}\to\mathbf{R}^{p_k}$, so that the closure of X_k is the zero locus of $F^{(k)}$ and that X_k is its regular locus.

By the tubular neighborhood theorem, we have the following:

(1) There is a tubular neighborhood T_i of X_i and a submersion $\pi_i:T_i\to X_i$ for $i=0,1,...s$ so that $\pi_i|_{X_i}$ is the identity and $\pi_0=p|_{U_0}$.
(2) $\pi_i\circ\pi_j(x)=\pi_i(x)$ for $x\in T_i\cap T_j$ for $0\le i<j\le s$.

Locally we can use the construction in §2.4, understanding as a local system of coordinates some of components of $F^{(i)}$ (x-coordinates) and π_i (y-coordinates).

Without (iv), we claim there are C^1-vector fields ξ_i $(i=0,1,...,s)$ on X_i so that $\{\xi_i\}_{0\le i\le s}$ yields a continuous vector field on $X_0\cup X_1\cup\cdots\cup X_s$ so that

(3) $d(\pi_i|_{X_j})\xi_j=\xi_i$ on $X_j\cap T_i$ for $0\le i<j\le s$.
(4) $\{\xi_i,\xi_j\}$ on $\{X_i,X_j\cap T_i\}$ satisfies (stratified) weighted Lipschitz (or, Nagumo) condition for $0\le i<j\le s$.

We show this fact by induction on s. We first assume $s=1$. Because of partitions of unity, it is enough to see it locally and it has been already shown this fact in §2.4.

We assume that there are such ξ_i's for $i=0,1,...,s-1$, and C^1-vector fields ξ_{is} $(i=0,1,...,s-1)$ on X_s, such that for $i=0,1,...,s-1$,

(5) $d(\pi_i|_{U_i \cap X_s})\xi_{is} = \xi_i$ where U_i is a small neighborhood of X_i.
(6) $\{\xi_i, \xi_{is}\}$ on $\{X_i, X_s \cap U_i\}$ satisfies (stratified) weighted Lipschitz (or, Nagumo) condition.

Shrinking U_i's, if necessary, we have for $0 \le j \le i < s$

(7) $d(\pi_j|_{U_j \cap X_s})\xi_{is} = \xi_j$
(8) $\{\xi_j, \xi_{is}\}$ on $\{X_j, X_s \cap U_s\}$ satisfies (stratified) weighted Lipschitz (or, Nagumo) condition ((8) follows from (v)).

We remark $\{U_j \cap X_k\}_{0 \le j < k} \cup \{X_k\}$ is an open covering and we take its refinement $\{V_i\}_{0 \le i \le s}$ defined by

$$\begin{aligned} V_{s-1} &= U_{s-1} \cap X_s \\ V_{s-2} &= U_{s-2} \cap X_s - (\text{a small closed neighborhood of } X_{s-1}) \\ &\cdots \\ V_1 &= U_1 \cap X_s - (\text{a small closed neighborhood of } \bigcup_{i=2}^{s-1} X_i) \\ V_0 &= U_0 \cap X_s - (\text{a small closed neighborhood of } \bigcup_{i=1}^{s-1} X_i) \\ V_s &= X_s - (\text{a small closed neighborhood of } \bigcup_{i=0}^{s} X_i). \end{aligned}$$

Let $\{\psi_i\}_{0 \le i \le s}$ be a partition of unity on X_s subordinate to $\{V_i\}_{0 \le i \le s}$. Then $\xi_s = \sum_{i=0}^{s-1} \psi_i \xi_{is}$ is the desired vector field on X_s.

Using (iv), we show the integration of $\{\xi_i\}_{0 \le i \le s}$ yields a family of homeomorphisms which trivialize the family $(X_{i,y}, y)$ $(y \in \mathbf{R}^m)$.

3.2. **A lemma in linear algebra.** We use the notation in §1.2. Let $A = (a_i^j)_{i=1,\dots,n; j=1,\dots,p}$ be an n by p matrix and $b = (b^j)_{j=1,\dots,p}$ a column vector of dimension p. We set $\tilde{a}^j = (a_1^j, \dots, a_n^j, -b^j)$ for $j = 1, \dots, p$. Assume that $\operatorname{rank} A = k$. We set, for $i = 1, \dots, n$,

$$x_i = \frac{\sum_{j_1 < \cdots < j_k} \phi_{j_1,\dots,j_k} ((\tilde{a}^{j_1})^\vee \wedge \cdots \wedge (\tilde{a}^{j_k})^\vee \wedge e_i^*)(\tilde{a}^{j_1} \wedge \cdots \wedge \tilde{a}^{j_k} \wedge e_{n+1})}{\sum_{j_1 < \cdots < j_k} \phi_{j_1,\dots,j_k} \|a^{j_1} \wedge \cdots \wedge a^{j_k}\|^2}. \tag{3.1}$$

We assume that the denominator is not zero.

Lemma 3.1. *If* $\operatorname{rank}(A\ b) = k$*, then* $Ax = b$ *where* $x = (x_i)_{i=1,\dots,n}$.

Proof. It is enough to see that

$$\begin{pmatrix} A & -b \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = 0. \tag{3.2}$$

Consider the subspace W generated by $\tilde{a}^j$, $j = 1, \dots, p$. Multiplying the orthogonal projection of e_{n+1} to $W^\perp$ that the last component is 1, we get the lemma by Lemma 1.4. $\square$

Remark 3.2. When $n = p = k$, this reduces to G.Cramer's formula.

3.3. **A vector field for the second isotopy lemma.** To obtain an analogous treatment for the second isotopy lemma, we should construct a vector filed trivializing a family of maps under some reasonable conditions. We present here a construction of such a vector field in the case of two strata. Then using patching and inductuion (in a resonable setup) one can proceed in constructing a vector field for a second isotopy lemma type result (like §3.1).

We consider the Euclidean spaces $\mathbf{R}^{n+m}$ with coordinates $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_m)$, and $\mathbf{R}^{n'+m'}$ with coordinates $x' = (x'_1, ..., x'_{n'})$, $y' = (y'_1, ..., y'_{m'})$. Set

$$\begin{aligned} &Y = \{(x, y) \in \mathbf{R}^{n+m} : x = 0\}, \quad Y' = \{(x', y') \in \mathbf{R}^{n'+m'} : x' = 0\}, \\ &p : \mathbf{R}^{n+m} \to \mathbf{R}^m \quad \text{the projection defined by } p(x, y) = y, \text{ and} \\ &p' : \mathbf{R}^{n'+m'} \to \mathbf{R}^{m'} \quad \text{the projecton defined by } p'(x', y') = y'. \end{aligned}$$

Let X, X' be submanifolds of $\mathbf{R}^{n+m}$, $\mathbf{R}^{n'+m'}$, respectively. We assume

(i) X is the regular locus of the zero locus of C^2-functions F_j $(j = 1, ..., p)$. We assume the jacobi matrix of $(F_1, ..., F_p)$ has rank k on X.

We now consider a C^2-map $f : X \cup Y \to X' \cup Y'$ so that

(ii) $f(Y) \subset Y'$, and $f|_Y$ is a submersion,
(iii) $f(X) \subset X'$, and $f|_X$ is a submersion,
(iv) $p' \circ f = f \circ p$ on X near Y.

Suppose there are vector fields v', ξ' and v on Y', X' and Y respectively so that

(v) $d(p'|_{X'})\xi' = v'$,
(vi) $d(f|_Y)v = v'$.

We want to construct a vector field ξ on X such that

(1) $d(p|_X)\xi = v$,
(2) $d(f|_X)\xi = \xi'$.
(3) ξ and v define a C^0-vector field on $X \cup Y$.
(4) $\{v, \xi\}$ satisfies (stratified) weighted Lipschitz (or, Nagumo) condition on $\{Y, X\}$.

To describe such ξ, we write

$$\xi' = \sum_{j=1}^{n'} \xi'_j \frac{\partial}{\partial x'_j} + v' \ \text{ on } X', \quad \text{and} \quad \xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} + v \ \text{ on } X$$

where ξ'_j $(j = 1, ..., n')$ are C^1-functions on X', and ξ_i $(i = 1, ..., n)$ are C^1-functions on X. It is enough to find ξ_i's so that

$$\begin{pmatrix} \frac{\partial F_s}{\partial x_i} & \frac{\partial F_s}{\partial y_k} \\ \frac{\partial f_j}{\partial x_i} & \frac{\partial f_j}{\partial y_k} \\ \frac{\partial f'_\ell}{\partial x_i} & \frac{\partial f'_\ell}{\partial y_k} \end{pmatrix} \begin{pmatrix} \xi_i \\ v_k \end{pmatrix} = \begin{pmatrix} 0 \\ \xi'_j \\ v'_\ell \end{pmatrix} \tag{3.3}$$

where $f_j = x'_j \circ f$ $(j = 1, ..., n')$, $f'_\ell = y'_\ell \circ f$ $(\ell = 1, ..., m')$, and $v = \sum_{k=1}^m v_k \frac{\partial}{\partial y_k}$.

Now we consider the regularity condition for f, namely,

(vii) $(p, f) : X \to Y \times_{Y'} X'$ is a submersion.

See Lemma (I.2.4) in [7] and [10] for the relation between (vii) and Thom's (a_f)-condition.

If we assume (vii), (3.3) has a solution. So, by (v), it is enough to find ξ_i $(i = 1, ..., n)$ on X near Y so that

$$(3.4) \qquad \begin{pmatrix} \frac{\partial F_s}{\partial x_i} & \frac{\partial F_s}{\partial y_k} \\ \frac{\partial f_j}{\partial x_i} & \frac{\partial f_j}{\partial y_k} \end{pmatrix} \begin{pmatrix} \xi_i \\ v_k \end{pmatrix} = \begin{pmatrix} 0 \\ \xi'_j \end{pmatrix}.$$

Thus, it is enough to solve the following system on X near Y:

$$(3.5) \qquad A(\xi_i) = b \quad \text{where } A = \begin{pmatrix} \frac{\partial F_s}{\partial x_i} \\ \frac{\partial f_j}{\partial x_i} \end{pmatrix}, \ b = b' - b'', \ b' = \begin{pmatrix} 0 \\ \xi'_j \end{pmatrix}, \ b'' = \begin{pmatrix} \frac{\partial F_s}{\partial x_i} \\ \frac{\partial f_i}{\partial y_k} \end{pmatrix} (v_k)$$

Therefore, (vii) implies the existence of ξ_i satisfying (3.5) for any v_k and ξ'_j. In other words, (vii) implies

$$(3.6) \qquad \operatorname{rank} A = \operatorname{rank}(A\ b) = \ell \text{ (constant)} \quad \text{on } X \text{ near } 0.$$

By Lemma 3.1, ξ_i defined by (3.1) is a solution for (3.5).

Proposition 3.3. *We assume (i)-(vii). We consider the singular metric defined by* (2.6). $\xi = \sum_i \xi_i \frac{\partial}{\partial x_i} + v$ *is* $\mathbf{w}$*-Lipschitz, if the following inequalities hold for* $s = 1, ..., m$, $i = 1, ..., n$ *on* X *near* 0*:*

$$(3.7) \qquad \left| \sum_{J,J'} \phi_{J,J'} (d_x F_J \wedge d^{(s)} f_{J'} \wedge dx_i) \left(\operatorname{grad}_x F_J \wedge \operatorname{grad}_x f_{J'} \wedge \frac{\partial}{\partial y_s} \right) \right| \lesssim \|x\|_{\mathbf{w}}^{w_i} \left| \sum_{J,J'} \phi_{J,J'} \| d_x F_J \wedge d_x f_{J'} \|_{\mathbf{w}}^2 \right|$$

$$(3.8) \qquad \left| \sum_{J,J'} \phi_{J,J'} (d_x F_J \wedge \tilde{d} f_{J'} \wedge dx_i) \left(\operatorname{grad}_x F_J \wedge \operatorname{grad}_x f_{J'} \wedge \frac{\partial}{\partial t} \right) \right| \lesssim \|x\|_{\mathbf{w}}^{w_i} \left| \sum_{J,J'} \phi_{J,J'} \| d_x F_J \wedge d_x f_{J'} \|_{\mathbf{w}}^2 \right|$$

where $\tilde{d} f_{J'} = \bigwedge_{j \in J'} \tilde{d} f_j$, $\tilde{d} f_j = d_x f_j + \xi'_j dt$ *and* t *is a parameter,* $\#J + \#J' = \ell$, *and* $\phi_{J,J'}$ *are some* C^1*-functions so that the right hand side is nowhere zero on* X*.*

Proof. Obvious. □

Since the left hand side of (3.7) is

$$\left| \sum_{J,J',I:I \ni i} \phi_{J,J'} \|x\|_{\mathbf{w}}^{2w_I} \frac{\partial(F_J, f_{J'})}{\partial(x_{I-\{i\}}, y_s)} \frac{\partial(F_J, f_{J'})}{\partial x_I} \right|,$$

if $\phi_{J,J'}$ are non-negative, then (3.7) comes from

$$\phi_{J,J'}{}^{1/2}\|x\|_{\mathbf{w}}^{w_I-w_i}\left|\frac{\partial(F_J,f_{J'})}{\partial(x_{I-\{i\}},y_s)}\right| \lesssim \sum_{J,J';I}\phi_{J,J'}{}^{1/2}\|x\|^{w_I}\left|\frac{\partial(F_J,f_{J'})}{\partial x_I}\right| \quad \text{on } X \text{ near } 0. \tag{3.9}$$

Similarly, if $\phi_{J,J'}$ are non-negative, then (3.8) comes from

$$\phi_{J,J'}{}^{1/2}|\xi'_j|\|x\|_{\mathbf{w}}^{w_I-w_i}\left|\frac{\partial(F_J,f_{J'-\{j\}})}{\partial(x_{I-\{i\}})}\right| \lesssim \sum_{J,J';I}\phi_{J,J'}{}^{1/2}\|x\|_{\mathbf{w}}^{w_I}\left|\frac{\partial(F_J,f_{J'})}{\partial x_I}\right| \quad \text{on } X \text{ near } 0. \tag{3.10}$$

Proposition 3.4. *We assume (i)-(vii). If ξ' satisfies the $\mathbf{w}'$-Lipschitz condition, then* (3.8) *follows from the following inequality:*

$$\phi_{J,J'}\|f\|_{\mathbf{w}'}^{w'_j}\|x\|_{\mathbf{w}}^{w_I-w_i}\left|\frac{\partial(F_J,f_{J'-\{j\}})}{\partial(x_{I-\{i\}})}\right| \lesssim \sum_{J,J';I}\phi_{J,J'}\|x\|_{\mathbf{w}}^{w_I}\left|\frac{\partial(F_J,f_{J'})}{\partial x_I}\right| \quad \textit{on } X \textit{ near } 0, \tag{3.11}$$

for any I, J, J' with $J' \ni j$, $I \ni i$. Here, $\phi_{J,J'}$ are some non-negative C^1-functions so that the right hand side is nowhere zero on X.

Proof. It follows that $|\xi'_j| \lesssim \|f\|_{\mathbf{w}'}^{w'_j}$ on X near 0 and (3.10). □

In the propositions above we assume that $J \subset \{1,...,p\}$ and $J' \subset \{1,...,n'\}$.

Proposition 3.5. *We assume (i)-(iv). We also assume that F and f are real analytic. Consider the weighted expansion of F and $(f_1,...,f_{n'})$ with respect to the weight $\mathbf{w}$. We assume that the weighted initial form of $(f_1,...,f_{n'})$ has its degree a multiple of $\mathbf{w}'$. We assume that the jacobi matrix of weighted initial form of $(F_1,..,F_p,f_1,...,f_{n'})$ by the variables $(x_1,...,x_n)$ is of constant rank on X and that the closure of X is $X \cup Y$. Then for any $\mathbf{w}'$-Lipschitz vector field $\{\xi',v'\}$ on (X',Y') and any lift v on Y of v' there is a vector field ξ on X with (1)–(3) so that $\{\xi,v\}$ on (X,Y) is $\mathbf{w}$-Lipschitz.*

Proof. Same as that of Theorem 2.8, and we omit the details. □

Example 3.6 (Example 2.5.10 in [6]). *Let $F : \mathbf{R}^3 \to \mathbf{R}$ be the function defined by*

$$F(x,y,z) = y - x^{2k+1} - z^2x^{2\ell+1}.$$

We consider the restriction of the map $f : \mathbf{R}^3 \ni (x,y,z) \mapsto (y,z) \in \mathbf{R}^2$ to $F^{-1}(0)$. Set $Y = \{x = y = 0\}$ and $X = F^{-1}(0) - Y$. Note that $f|(X,Y)$ is always a Thom regular map. If $k \le \ell$, one can see we can apply Proposition 3.5. However if $k > \ell$, our assumption in Proposition 3.5 are not satisfied, and it is shown in [6] *that actually $\frac{\partial}{\partial z}$ does not admit a continuous lift by f on $F^{-1}(0)$.*

It is also possible to obtain a similar criterion for weighted Nagumo conditions as (3.7), (3.8), and we leave it to the reader.

We also remark that a pinch map does not satisfy Thom's (a_f)-condition, and so the use of (a_f)-condition is somehow restrictive. It is clear that there are maps with pinching so that there are ξ' and v' satisfying the following conditions:

(i) (3.6).

(ii) The inequalities that imply the weighted Lipschitz condition for ξ (defined by using (3.1)).

Example 3.7. *Consider the map* $f : \mathbf{R}^3 \to \mathbf{R}^3$ *defined by*

$$(x, y, t) \mapsto (x', y', t') = (x^2, x^4 y + x^3 t, t).$$

This map has pinching, i.e. the stratum $\{x = 0\}$ *is sent to* t'*-axis, and* f *is local diffeomorphism on the stratum* $\{x \neq 0\}$*. Following our method one can find that the vector field* $|x'|^{3/2}\frac{\partial}{\partial y'} + \frac{\partial}{\partial t'}$ *has a lift* $\frac{\partial}{\partial t}$ *by* f *(note that for example* $\frac{\partial}{\partial t'}$ *does not lift properly), and* $f_t : \mathbf{R}^2 \to \mathbf{R}^2$*,* $f_t(x, y) = (x^2, x^4 y + x^3 t)$ *is topologically right-left trivial.*

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