Embedding Seifert manifolds in the 4-sphere

J.S. Crisp

17 November, 1994

Abstract

The $\mathbf{Z}/2\mathbf{Z}$-signature theorem is used to obtain a relative bound on the Euler invariant $\varepsilon \in \mathbf{Q}$ of any Seifert manifold M, over a nonorientable closed surface, which embeds in 4-space. This bound is sharp whenever the singular fibres of M are all odd, and occur in pairs of opposite sign, and in the case of genuine circle bundles we obtain a complete classification of those which embed in 4-space. The torsion linking form is computed explicity for a Seifert manifold M over a nonorientable closed surface, and conditions are found for this to be hyperbolic, thereby giving extra necessary conditions for embedding M in S^4. These results allow us to show, for instance, that there is (up to orientation) exactly one manifold fibred over the orbifold $RP^2(2,2)$ which embeds in S^4.

1. Introduction

This paper is concerned with the general question of whether a given 3-manifold embeds in S^4. By an embedding we shall mean a locally flat topological embedding, unless otherwise stated. However, all of the embeddings constructed in Section 2 are in fact smooth, so that for instance Corollaries 1.2 and 1.3 hold equally well for smooth embeddings. Only orientable 3-manifolds embed in S^4, and here we consider specifically the class of orientable Seifert manifolds [S], focussing mainly on those which are fibred over a nonorientable closed surface. In what follows, we shall write P_k for the nonorientable closed surface of genus k; thus $P_1 = RP^2$ is the real projective plane, and $P_2 = Kl$ the kleinbottle.

We may describe an orientable *Seifert manifold* M by the un-normalised data $M(F;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ (following [NR] or [JN]), where M is fibred over the surface F with a finite collection of singular fibres γ_i, for $i = 1,..,n$, of type (α_i,β_i) where α_i,β_i are coprime, and $\alpha_i \geq 1$ is the multiplicity of γ_i. Note that [JN] also defined a *generalised Seifert (G.S.) manifold*, in which they allowed fibres of type $(0,1)$ as well. These are fibres fixed pointwise under the circle action as described in [O, §1.6]. We shall use this definition only as notational convenience in the proofs of Proposition 2.2 and Claim 4.5.

There is a fibre preserving homeomorphism which respects orientations between any two Seifert manifolds exactly when their Seifert data are equivalent via a finite sequence of the following modifications: (I) Add or delete any pair $(1,0)$; (II) Replace each pair (α_i,β_i) with $(\alpha_i,\beta_i+c_i\alpha_i)$ where the c_i are chosen such that $\Sigma_{i=1}^n c_i = 0$; (III) Permute the indices. Note that there are some manifolds which admit more than one Seifert structure. These consist of the Lens spaces, and the following two exceptional cases ([JN], [NR], [O]) :

$$M(RP^2;(\alpha,\beta)) \cong M(S^2;(2,1),(2,-1),(-\beta,\alpha))$$
$$\text{and} \qquad M(Kl;(1,0)) \cong M(S^2;(2,1),(2,1),(2,-1),(2,-1))\,.$$

For any Seifert manifold $M = M(F;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ define the *generalised Euler invariant* to be $\varepsilon(M) = \sum_{i=1}^{n} \beta_i/\alpha_i \in \mathbf{Q}$. Clearly $\varepsilon(M)$ is invariant under (I)–(III). Observe also that, by applications of (I)–(III), we may assume that a Seifert manifold M has a unique standard form $M(F;S,(1,\beta_{n+1}))$ where $S = \{\gamma_i(\alpha_i,\beta_i) \mid \alpha_i > \beta_i \geq 1,\ i = 1,2,..,n\}$ is the set of strictly singular fibres. In view of this, each Seifert manifold M is determined by its base surface F, the set S and its generalised Euler invariant $\varepsilon = \varepsilon(M) = \beta_{n+1} + \sum_{i=1}^{n} \beta_i/\alpha_i$. When $F = P_k$ is nonorientable we shall write $M_S(k;\varepsilon)$ for such a manifold. If we reverse the orientation of a Seifert manifold, then this may be reflected in its presentation by a change in the sign of each singular fibre, so that $-M(F;(\alpha_i,\beta_i)) \cong M(F;(\alpha_i,-\beta_i))$. This also simply changes the sign of $\varepsilon(M)$ so that we may write $-M_S(k;\varepsilon) \cong M_{-S}(k;-\varepsilon)$ where $-S$ has the obvious definition.

As a first criterion for embeddability we note that if a 3-manifold M embeds in S^4 then its torsion linking form must be hyperbolic, and in particular the torsion subgroup $\tau(M)$ of $H_1(M)$ must be a direct double $A \oplus A$ [KK]. In Section 3 we compute explicitly the torsion linking form for the Seifert manifolds $M_S(k;\varepsilon)$ over nonorientable closed surfaces. It follows that, for the linking to be hyperbolic, the following conditions must be satisfied. Let S_{even} denote the subset of S of fibres with even multiplicity. Then S_{even} consists of an even number, $2b$ say, of fibres (Remark 3.2), and there is a fixed t such that their multiplicities may all be written $\alpha_i = 2^t s_i$, with s_i odd (Proposition 3.6). In such cases we associate two constants t_S and b_S to the set S as follows. When S_{even} is empty (i.e: all fibres have odd multiplicity) write $t_S = b_S = 0$. Otherwise write $t_S = t$, and $b_S = b - 1$. In Section 4 we prove the following theorem.

1.1. Theorem : *For a given collection S of strictly singular fibres suppose that $M_1 = M_S(k_1;\varepsilon_1)$ and $M_2 = M_S(k_2;\varepsilon_2)$ both embed in S^4. Then*

$$\varepsilon_1 - \varepsilon_2 \in \{\frac{2K}{2^{t_S}}, \frac{2K-4}{2^{t_S}}, \frac{2K-8}{2^{t_S}}, \ldots, \frac{-2K+4}{2^{t_S}}, \frac{-2K}{2^{t_S}}\} \cap \mathbf{Z}\,,$$

where $K = 2^{t_S}(k_1 + k_2 + 4b_S) + 4b_S$.

In particular, if we suppose that every fibre of S has odd multiplicity (so that $t_S = b_S = 0$), fix k, and let ε_0 be the least rational value such that $M_S(k;\varepsilon_0)$ embeds in S^4, then $M_S(k;\varepsilon)$ embeds only if

$$\varepsilon \in \{\varepsilon_0, \varepsilon_0 + 4, \varepsilon_0 + 8, \ldots, \varepsilon_0 + 4k\}.$$

When moreover the strictly singular fibres appear in pairs of "opposite sign", that is $S = -S$, we can be more explicit.

1.2. Corollary : *Suppose that* $S = \{\gamma_i^+(\alpha_i, \beta_i), \gamma_i^-(\alpha_i, \alpha_i - \beta_i) \mid i = 1,..,n\}$, *where each* α_i *is odd. Then* $M_S(k;\varepsilon)$ *embeds in* S^4 *if and only if*

$$\varepsilon \in \{2k, 2k-4, \ldots, -2k\}.$$

Proof: Examples of embeddings for each value of ε in the given range are provided in the proof of Proposition 2.2. On the other hand, Theorem 1.1 implies that ε may take on at most $k+1$ different values, so that the set $\{2k, 2k-4, \ldots, -2k\}$ must exhaust the possibilities. •

Note that the simple fact that M embeds in S^4 if and only if $-M$ embeds is reflected in Corollary 1.2.

An orientable circle bundle over a closed surface F may be expressed as a Seifert manifold $M = M(F;(1,e))$ with no strictly singular fibres, where e is the Euler class classifying the bundle. The circle bundles (necessarily orientable) which embed in S^4 are classified as follows:

1.3. Corollary : *Let* M *be an* S^1*-bundle with Euler class* e*, over a closed surface* F*. Then* M *embeds in* S^4 *if and only if either*
(1) F *is orientable and* $e = 0$ *or* ± 1*; or,*
(2) $F = P_k$ *is nonorientable and* $e \in \{2k, 2k-4, \ldots, -2k\}$.

Proof: The case when F is nonorientable follows immediately from Corollary 1.2. In the case where F is orientable Proposition 2.4 states that M embeds (smoothly in fact) for $e = 0$ or ± 1. For the converse, we need only check the torsion subgroup $\tau(M)$ of $H_1(M)$, since

$$H_1(M) \cong \langle a_i, b_i (i = 1,..,g), h \mid eh = 0\rangle \cong \mathbf{Z}^{2g} \oplus \mathbf{Z}/e\mathbf{Z},$$

so that $\tau(M)$ is a direct double only if $e = 0$ or ± 1. •

Finally, as a further application of Theorem 1.1, when $S = \{(2,1),(2,1)\}$ ($t_S = 1$ and $b_S = 0$), and $F = RP^2$ we have $\varepsilon_1 - \varepsilon_2 \in \{4, 2, 0, -2, -4\}$, and since $S = -S$, $\varepsilon \in \{2, 0, -2\}$. Thus the only manifolds fibred over the orbifold $RP^2(2,2)$ which may possibly embed in S^4 are the manifolds $\pm M(RP^2; (2,3), (2,1))$ and $M(RP^2; (2,1), (2,-1))$. The last of these fails to have a hyperbolic torsion linking form, as we show in Claim 3.7, and hence does not embed. This is, in fact, the Hantzsche-Wendt flat manifold. However an embedding of $M(RP^2; (2,3), (2,1))$ may be obtained by considering 0-framed surgery on the 8^2_2 link (using Rolfsen's notation [R1]), as in Example 2.1.

The key ingredient in the proof of Theorem 1.1 is the use of the Atiyah-Singer-Hirzebruch signature theorem. When considering the embeddings of nonorientable closed surfaces in S^4, Massey [M] used the $\mathbf{Z}/2\mathbf{Z}$-signature theorem to determine the possible range of the Euler class of the normal bundle of any embedded surface. A modification of this argument allows the analagous result to be proved for embeddings of circle bundles over these surfaces (Corollary 1.3). This was shown to me by J.A. Hillman for the specific cases of bundles over RP^2 and Kl. Note that this is a strengthening of Massey's argument, since an embedded surface P_k with Euler class e implies the existence of an embedded S^1-bundle with the same Euler class (namely $\partial N(P_k)$), but not conversely.

Theorem 1.1 extends this constraint on embeddings to the whole class of Seifert manifolds over nonorientable closed surfaces. The proof takes on two phases. In the first place, given M_1 and M_2, we construct a 4-manifold with involution g such that, by the signature theorem, $\mathrm{sign}(g, \Sigma) = 2^{t_S - 1}(\varepsilon_1 - \varepsilon_2)$. In the second place we use some algebraic topology to obtain an inequality which restricts the possible values of this signature. This inequality appears to be of a similar nature to the inequality given by Gilmer and Livingston in [GL, 2.1] which puts a bound on the Casson-Gordon signature invariant of (a cyclic cover of) a rational homology sphere that embeds in S^4.

Aside from the fact that the main theorem is in any case true for this more general class, the choice to consider embeddings to be TOP locally flat (as opposed to smooth) was motivated by recent work of Hillman [H] where it is shown that in some cases the property of TOP locally flat embeddability is common to a whole class of $\mathbf{Z}[\pi]$-homology equivalent 3-manifolds (e.g: homology spheres, which all embed topologically but not always smoothly).

Acknowledgement :

I would like to thank Dr J.A. Hillman, who has supervised me during this work, for his many helpful suggestions and encouragement throughout.

2. Constructing Embeddings

In this section we consider various ways to construct smooth embeddings of Seifert manifolds, and in particular show the existence of all the embeddings claimed in the introduction. The simplest example is the standard embedding of S^3 into S^4 where $S^4 \cong D^4 \cup_{S^3} D^4$. We may consider the 3-sphere as Seifert fibred in many ways. Namely $S^3 \cong L(1,0) \cong M(S^2; (a_1, b_1), (a_2, b_2))$, whenever $a_1 b_2 + a_2 b_1 = 1$. More generally, any homology sphere embeds topologically [FQ]. These include, for instance, the manifolds obtained by $(-1/n)$-surgery on the trefoil knot [R2], which are Seifert fibred over S^2. However, not all of these embed smoothly.

Smooth embeddings exist for manifolds which are obtained by 0-framed surgery on a link in S^3 which is the union of two sublinks L_1, L_2, each of which is slice. The embedding is obtained from the standardly embedded 3-sphere by attaching, on one side of S^3, an embedded 2-handle to each component of L_1, and similarly, on the other side, to each component of L_2. (See also [GL, §4]).

2.1. Example : The manifold $M(RP^2; (2,3), (2,1))$ may be obtained by 0-framed surgery on the 8^2_2 link (using Rolfsen's notation [R1]), and hence we may embed it in S^4. This manifold has a surgery description **(A)** as in the figure, which is shown via the Kirby-Rolfsen [R2] calculus to be equivalent to the one claimed **(B)**.

(A)

HAND-DRAWN DIAGRAM TO BE INSERTED

(B)

Given Seifert manifolds M_1 and M_2, we write $M_1 \S M_2$ for the Seifert manifold constructed as follows. Take regular fibres f_i of M_i, $(i = 1, 2)$, and let $M_i^{(f_i)} = M_i \setminus \mathrm{int}(N(f_i))$. Then $M_1 \S M_2 = M_1^{(f_1)} \cup_h M_2^{(f_2)}$, where $h : \partial N(f_2) \to \partial N(f_1)$ is an orientation reversing fibre preserving homeomorphism. This operation is well-defined and if $M_i = M(F^{(i)}; (\alpha_j^{(i)}, \beta_j^{(i)}); j = 1, ..., n^{(i)})$, then $M_1 \S M_2 = M(F^{(1)} \sharp F^{(2)}; (\alpha_j^{(1)}, \beta_j^{(1)}); j = 1, ..., n^{(1)}, (\alpha_l^{(2)}, \beta_l^{(2)}); l = 1, ..., n^{(2)})$. If the M_i have singular fibres γ_i of type (p, q) and $(p, -q)$ respectively, then we have a well-defined $h : \partial N(\gamma_2) \to \partial N(\gamma_1)$ as above, and we may extend the above construction to define $M_1 \S_{\gamma_1} M_2$. This has Seifert data as before, except that the fibres γ_1 and γ_2 are eliminated. This 'fibre-sum' operation is clearly commutative and, if restricted to regular fibres, associative as well.

If M embeds in S^4 then M has a two-sided product neighbourhood in S^4. Thus we may embed two or more parallel copies of M, fibred in exactly the same way. Take two such which are the boundary components of $M \times I$ in S^4. Given a fibre γ of M then cleary $\partial(M^{(\gamma)} \times I) \cong M \S_\gamma -M$ also embeds in S^4. In fact, if we begin with $r+1$ parallel copies of M and take fibres γ_i $(i = 1, .., r)$ in M such that $\gamma_i \neq \gamma_{i+1}$ $(i = 1, .., r-1)$, then we may construct in this way an embedding of

$$M \S_{\gamma_1} -M \S_{-\gamma_2} M \S_{\gamma_3} \ldots \S_{\pm\gamma_r} \pm M.$$

This construction clearly gives smooth embeddings if we begin with a smooth embedding of M. For example, given any coprime pair (α, β) we may choose $a, b \in \mathbf{Z}$ such that $M = M(0; (\alpha, \beta), (a, b))$ is a 3-sphere and hence embeds smoothly. Thus $M \S_{(a,b)} -M \cong M(S^2; (\alpha, \beta), (\alpha, -\beta))$ also embeds smoothly in S^4. Similarly, if we write $M = M(S^2; (k, 1), (1, 0))$ (again a sphere), then $M \S -M \S M \cong M(S^2; (k, 1), (k, -1), (k, 1))$ embeds smoothly in S^4. These last manifolds may also be embedded via a 0-framed surgery, since

$$M(S^2; (k,1), (k,-1), (k,1)) \cong \qquad\qquad \cong \qquad\qquad .$$

Finally, let $M \hookrightarrow S^4$ be the smooth embedding of Example 2.1. Then $M \S_{(2,3)} -M \cong M(Kl; (2, 1), (2, -1))$ also embeds smoothly.

In general the boundary of the regular disk-bundle neighbourhood of any surface F smoothly embedded in S^4 is a circle bundle, $M(F; (1, e))$, with Euler number e equal to that of the normal bundle to F in S^4. Massey [M] constructed embeddings of the nonorientable surfaces P_k for all $e \in \{2k, 2k-4, \ldots, -2k\}$, and thus we have smooth embeddings of $M(P_k; (1, e))$ for all these values of e. We now extend these examples within the class of Seifert manifolds.

2.2. Proposition : *The following Seifert manifolds embed smoothly in S^4 for any $e \in \{2k, 2k-4, \dots, -2k\}$:*

$$M(P_k; (\alpha_1, \beta_1), (\alpha_1, -\beta_1), \dots, (\alpha_n, \beta_n), (\alpha_n, -\beta_n), (1, e)),$$

where each α_i is ODD. Note that here $S = -S$, and $\varepsilon(M) = e$.

Proof : Consider the lens space $L(\alpha_i, \beta_i)$ which we may write as a G.S. fibred manifold [JN]

$$L(\alpha_i, \beta_i) \cong M(S^2; (\alpha_i, \beta_i), (0, 1)) = L_i \quad \text{say}.$$

When α_i is odd L_i is a branched cyclic cover of S^3, branched over a knot. In this case, by [Zee], $\operatorname{punc} L_i = L_i \setminus \operatorname{int}(D^3)$ embeds smoothly in S^4 as the fibre of a twist-spun 2-knot K. Now let $\operatorname{punc} L_i$ be constructed from the G.S. fibration above by removing an open neighbourhood of a point on the fibre of type $(0, 1)$. Thus

$$\operatorname{punc} L_i \cong L_i^* \bigcup_{f \times I} D^2 \times I,$$

where $L_i^* = M(D^2; (\alpha_i, \beta_i))$ is Seifert fibred, f is a regular fibre, $I \cong [0, 1]$, and the identification maps ∂D^2 to f. Now $\operatorname{punc} L_i$ has a product neighbourhood $\operatorname{punc} L_i \times [-\epsilon, \epsilon]$ in the exterior $X(K)$ of K in S^4, and this may be written

$$\operatorname{punc} L_i \times [-\epsilon, \epsilon] = (L_i^* \times [-\epsilon, \epsilon]) \bigcup (D^2 \times I \times [-\epsilon, \epsilon]).$$

Thus the 3-manifold

$$\begin{aligned} \partial(L_i^* \times [-\epsilon, \epsilon]) &\cong L_i \,\S_{(0,1)} {-L_i} \\ &\cong M(S^2; (\alpha_i, \beta_i), (\alpha_i, -\beta_i)) \\ &= M_i \quad \text{say}, \end{aligned}$$

clearly embeds smoothly in S^4. Furthermore, this embedding has the property that there exists a 4-ball $B_i = D^2 \times I \times [-\epsilon, \epsilon]$ embedded in S^4 such that $M_i \cap B_i = \partial D^2 \times I \times [-\epsilon, \epsilon]$, and each circle $\partial D^2 \times \{r, s\}$ is a regular fibre in M_i. Notice that the embeddings of $M(P_k; (1, e))$ (for $e \in \{2k, 2k-4, \dots, -2k\}$) due to Massey [Ma] also have this property, since in fact each one bounds a disk bundle.

Now given any two such manifolds M_1 and M_2 embedded with this property, we may construct an embedding of $M_1 \S M_2$, also with this property, by taking connected sums of the 4-spheres. Remove the ball $\frac{1}{2}B_i = D^2 \times I \times [\frac{-\epsilon}{2}, \frac{\epsilon}{2}]$ from each sphere, and construct the sum by taking the union of the closures of the remaining components in the obvious way. Thus, by induction, we obtain a smooth embedding of $M_1 \S M_2 \S \dots \S M_n \S M(P_k; (1, e))$, which is exactly the desired manifold, for any $e \in \{2k, 2k-4, \dots, -2k\}$. $\bullet$

Note incidentally that this construction gives embeddings of

$$M(F;(\alpha_i,\beta_i),(\alpha_i,-\beta_i);i=1,..,n)\,,$$

with each α_i odd, for $F=S^2$ immediately, and, by the following claim, for any closed orientable surface. Let $M(T_g)$ denote the product bundle $T_g\times S^1$, where T_g denotes the closed orientable surface of genus g.

2.3. Claim : *Given a (smooth) embedding of $M=M(F;(\alpha_i,\beta_i))$ then we may construct a (smooth) embedding of $M\S M(T_g)\cong M(F\sharp T_g;(\alpha_i,\beta_i))$.*

Proof: Let f be a regular fibre of M embedded in S^4, and write B for the 3-ball with "equatorial" circle $C\subset\partial B$. Then f has in S^4 a neighbourhood $B\times f$ consistently fibred with M, so that $N=M\cap(B\times f)$ is a neighbourhood of f in M with $\partial N=C\times f$. Write $S^*=S^4\setminus\mathrm{int}(B\times f)$ and define $M^{(f)}$ as before to be $M\setminus\mathrm{int}(N)$. Then we have an embedding $\phi_M:M^{(f)}\hookrightarrow S^*$ such that the restriction $\partial M^{(f)}\hookrightarrow\partial S^*=\partial B\times f$ is a fibre preserving map with image $C\times f$. On the other hand, write $T_g^*=T_g\setminus\mathrm{int}(D^2)$. Since there are embeddings $T_g^*\hookrightarrow B$ with $\mathrm{Im}(\partial T_g^*)=C$, there exists some fibre preserving embedding $\phi_T:T_g^*\times f\hookrightarrow B\times f$ with $\mathrm{Im}(\partial T_g^*\times f)=C\times f$. Together ϕ_M and ϕ_T give an embedding of $M\S M(T_g)$ into $S^4=S^*\cup(B\times f)$.

Clearly, this argument holds equally well for smooth embeddings. •

As an example, given the standard embedding of $S^3\cong M(S^2;(1,e))$, a bundle with Euler number $e=\pm1$, we may also smoothly embed $M(T_g;(1,e))$ with $e=\pm1$ for any orientable surface T_g.

2.4. Proposition : *The circle bundle $M(T_g;(1,e))$ embeds smoothly in S^4 for $e=0,\pm1$.*

Proof : Apart from the case just mentioned, we see that $M(T_g;(1,0))=T_g\times S^1$ embeds smoothly in the neighbourhood $D^3\times S^1$ of any circle in S^4. Alternatively, we have a different embedding of $T_g\times S^1$ as the boundary of the product neighbourhood of $T_g\hookrightarrow S^3\hookrightarrow S^4$. •

3. The Torsion Subgroup $\tau(M)$

If a 3-manifold M embeds in S^4 it must have a hyperbolic torsion linking form, and in particular the torsion subgroup $\tau(M)$ of $H_1(M)$ must be a direct double $A \oplus A$ [KK]. In this section we determine a direct sum decomposition of $\tau(M)$ for an arbitrary Seifert manifold over a nonorientable closed surface, and then proceed to compute the torsion linking explicitly.

3.1. Lemma : *Given $M = M(P_k; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$, where each α_i is written $2^{t_i} s_i$ for s_i odd, we may suppose that the α_i's are ordered such that $t_1 = t_2 = \ldots = t_r > t_{r+1} \geq \ldots \geq t_n \geq 0$. Then, if $t_1 \neq 0$, the torsion subgroup of $H_1(M)$ is given by*

$$\tau(M) = (\bigoplus_{i=3}^{n} \mathbf{Z}/\alpha_i \mathbf{Z}) \oplus \begin{cases} \mathbf{Z}/2\alpha_1\mathbf{Z} \oplus \mathbf{Z}/2\alpha_2\mathbf{Z}\,, & \text{if } r \text{ is even,} \\ \mathbf{Z}/4\alpha_1\mathbf{Z} \oplus \mathbf{Z}/\alpha_2\mathbf{Z}\,, & \text{if } r \text{ is odd,} \end{cases}$$

while, if $t_1 = 0$ (i.e: every α_i is odd), we have

$$\tau(M) = (\bigoplus_{i=1}^{n} \mathbf{Z}/\alpha_i \mathbf{Z}) \oplus \begin{cases} \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}\,, & \text{if } \Sigma\beta_i \text{ is even,} \\ \mathbf{Z}/4\mathbf{Z}\,, & \text{if } \Sigma\beta_i \text{ is odd.} \end{cases}$$

Note that, by inserting pairs $(1,0)$ if necesssary, we may assume that $n \geq 2$.

Proof: The group $H_1(M)$ has the following presentation [S], [O], [JN] :

$$\langle h, a_1, .., a_k, q_1, .., q_n \mid 2h = 0,\ \alpha_j q_j + \beta_j h = 0,\ 2\sum_{i=1}^{k} a_i + \sum_{i=1}^{n} q_j = 0 \rangle\,, \quad (\spadesuit)$$

where each generator has an explicit representative relating to the Seifert structure as described following Remark 3.2. Replacing a_k with $a = a_1 + a_2 + \ldots + a_k$ this becomes $\mathbf{Z}^{k-1} \oplus \tau$ where

$$\tau = \langle h, a, q_j (j = 1, .., n) \mid 2h = 0,\ \alpha_j q_j + \beta_j h = 0,\ 2a + \Sigma q_j = 0 \rangle\,, \quad (\clubsuit)$$

and as we shall see τ is in fact the torsion subgroup $\tau(M)$.

Suppose first of all that $t_1 > 0$. In this case, we may assume (by a Seifert equivalence if necessary) that every β_j is odd, and so rewrite the appropriate relations in the above presentation as $\alpha_j q_j = h$. Now, for $j \geq 2$, replace q_j with $\bar{q}_j = q_j + \frac{\alpha_1}{2^{t_j}} q_1$, and replace a with

$$\bar{a} = a - (\sum_{j=2}^{n} \frac{\alpha_1}{2^{t_j+1}}) q_1 - \sum_{j=3}^{n} \frac{(s_j - 1)}{2} \bar{q}_j + \begin{cases} \frac{1}{2} q_1\,, & \text{if } r \text{ is even,} \\ -\frac{(s_2 - 1)}{2} \bar{q}_2\,, & \text{if } r \text{ is odd.} \end{cases}$$

It is easily checked that this last equation contains an integral multiple of q_1 in each case. Now the presentation for τ reduces to one of the following,

$$\tau_{\text{even}} \cong \langle h, \bar{a}, q_1, \bar{q}_2, .., \bar{q}_n \mid 2h = 0,\ \alpha_1 q_1 = h,\ \alpha_j \bar{q}_j = 0,\ 2\bar{a} + \bar{q}_2 + \sum_{j=3}^{n} s_j \bar{q}_j = 0 \rangle \,,$$

$$\tau_{\text{odd}} \cong \langle h, \bar{a}, q_1, \bar{q}_2, .., \bar{q}_n \mid 2h = 0,\ \alpha_1 q_1 = h,\ \alpha_j \bar{q}_j = 0,\ 2\bar{a} + q_1 + \sum_{j=2}^{n} s_j \bar{q}_j = 0 \rangle \,,$$

as r is even or odd respectively.

If r is even, replace $\bar{q}_2$ with $\bar{\bar{q}}_2 = \bar{q}_2 + \sum_{j=3}^{n} s_j \bar{q}_j$. With this, the relation $\alpha_2 \bar{q}_2 = 0$ may be replaced with $\alpha_2 \bar{\bar{q}}_2 = 0$ (note that α_j divides $\alpha_2 s_j$ for each $j \geq 3$), and τ may be written

$$\begin{aligned}
\tau_{\text{even}} &\cong \langle h, \bar{a}, q_1, \bar{\bar{q}}_2, .., \bar{q}_n \mid 2h = 0,\ \alpha_1 q_1 = h,\ \alpha_2 \bar{\bar{q}}_2 = \alpha_j \bar{q}_j = 0,\ 2\bar{a} + \bar{\bar{q}}_2 = 0 \rangle \\
&\cong \langle \bar{a}, q_1, \bar{q}_3, .., \bar{q}_n \mid 2\alpha_1 q_1 = 0,\ 2\alpha_2 \bar{a} = 0,\ \alpha_j \bar{q}_j = 0\ (j \geq 3) \rangle \\
&\cong \mathbf{Z}/2\alpha_1\mathbf{Z} \oplus \mathbf{Z}/2\alpha_2\mathbf{Z} \oplus (\bigoplus_{i=3}^{n} \mathbf{Z}/\alpha_i\mathbf{Z}) \,.
\end{aligned}$$

If r is odd, replace q_1 with $\bar{q}_1 = q_1 + \sum_{j=2}^{n} s_j \bar{q}_j$. With this, the relation $\alpha_1 q_1 = h$ may be replaced with $\alpha_1 \bar{q}_1 = h$, and τ may be written

$$\begin{aligned}
\tau_{\text{odd}} &\cong \langle h, \bar{a}, \bar{q}_1, \bar{q}_2, .., \bar{q}_n \mid 2h = 0,\ \alpha_1 \bar{q}_1 = h,\ \alpha_j \bar{q}_j = 0\ (j \geq 2),\ 2\bar{a} + \bar{q}_1 = 0 \rangle \\
&\cong \langle \bar{a}, \bar{q}_1, \bar{q}_2, .., \bar{q}_n \mid 4\alpha_1 \bar{a} = 0,\ \alpha_j \bar{q}_j = 0\ (j \geq 2) \rangle \\
&\cong \mathbf{Z}/4\alpha_1\mathbf{Z} \oplus (\bigoplus_{i=2}^{n} \mathbf{Z}/\alpha_i\mathbf{Z}) \,.
\end{aligned}$$

Finally, suppose that $t_1 = 0$ and every α_j is odd. We may assume that $\alpha_j q_j = h$ for $j = 1, .., s$, and 0 for $j > s$, where $s \equiv \sum_{j=1}^{n} \beta_j \pmod 2$. Putting

$$\bar{q}_j = \begin{cases} q_j + \alpha_1 q_1 = q_j + h \,, & \text{for } j = 2, ..s, \\ q_j \,, & \text{for } j > s. \end{cases}$$

$$\bar{a} = a - (\sum_{j=2}^{s} \frac{\alpha_1}{2}) q_1 - \sum_{j=3}^{n} \frac{(s_j - 1)}{2} \bar{q}_j + \begin{cases} \frac{1}{2} q_1 \,, & \text{if } s \text{ is even,} \\ -\frac{(s_2 - 1)}{2} \bar{q}_2 \,, & \text{if } s \text{ is odd,} \end{cases}$$

the rest of the argument follows exactly as in the preceding cases. •

3.2. Remark : Since $\tau(M)$ must be a direct double for M to embed in S^4, it follows from the previous Lemma 3.1 that $M(P_k;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ will only embed if the prime power factors of the α_i's occur in pairs and, when each multiplicity is odd, only if $\sum_{i=1}^n \beta_i$ is even. By looking at the torsion linking we find that, in addition, the nonzero t_i's must all be equal (Proposition 3.6), and when each multiplicity is odd we have the stronger condition that $\sum_{i=1}^n \alpha_i\beta_i \equiv 2k \bmod 4$ (Lemma 3.5).

Let $<,>: \tau(M)\times\tau(M)\to \mathbf{Q}/\mathbf{Z}$ denote the torsion linking form on $\tau(M)$ of the Seifert manifold M discussed above. We now compute this form on the original generators of the presentation (♣) which is derived from the following description of M.

Take a basepoint $*$ in the surface P_k and let $\sigma_1,\ldots,\sigma_k$ be orientation reversing loops in P_k at $*$ (each corresponding to a factor RP^2 in the connected sum). Also choose loops $\xi_1,..,\xi_n$ at $*$ in P_k which bound disks $N_1,..,N_n$ respectively meeting only at $*$. Orient and label all these loops so that they appear as in Figure 3.3(a) below, and from now on consider them all as 1-cycles.

HAND-DRAWN DIAGRAM TO BE INSERTED

(a) (b)

3.3. Figure

Removing the interiors of the neighbourhoods N_i write $B = P_k \setminus \bigcup_{i=1}^n \mathrm{int}N_i$. The result is contractible to a bouquet of circles and thus there is a unique orientable S^1-bundle $B\tilde{\times}S^1$ over B, with a section $B\hookrightarrow B\tilde{\times}S^1$. We may now think of the σ_i and ξ_j as 1-cycles in $B\tilde{\times}S^1$. Also let f denote the fibre over $*\in B$ also considered as a 1-cycle in $B\tilde{\times}S^1$. Now let $N(\gamma_i)$, for $i=1,..,n$, be solid tori with cores γ_i, and write

$$M = B\tilde{\times}S^1 \cup_{h_i} \left(\bigcup_{i=1}^n N(\gamma_i)\right),$$

where each attaching homeomorphism h_i identifies the boundary of a meridinal disk in $N(\gamma_i)$ with a curve representing the homology class $\alpha_i[\xi_i]+\beta_i[f]$. Then M is the Seifert manifold $M(P_k;(\alpha_i,\beta_i))$ with singular fibres γ_i, and $H_1(M)$

has the presentation ($\spadesuit$) where $h = [f]$, $a_i = [\sigma_i]$, and $q_j = [\xi_j]$, for $i = 1,..,k$ and $j = 1,..,n$. Define the 1-chain $\sigma = \sigma_1 + \sigma_2 + \ldots + \sigma_k$, so that, in ($\clubsuit$), $a = a_1 + a_2 + \ldots + a_k = [\sigma]$. The relations in ($\clubsuit$) correspond to the presence of the following 2-chains in M. The union of fibres over σ_1 may be considered as a 2-chain A (the image of an annulus) with $\partial A = 2f$. For each $i = 1,..,n$, we may assume that there is a 2-chain D_i in $N(\gamma_i)$ with $\partial D_i = \alpha_i \xi_i + \beta_i f$ in M (thus D_i is homologous to a meridian disk). Finally, we may consider B as a 2-chain in M with $\partial B = 2\sigma + \sum_{i=1}^{n} \xi_i$. Note that we have described each of ∂A, ∂B and the ∂D_i explicitly as 1-cycles. We may now compute the following linkings. Let M be as in Lemma 3.1, with generalised Euler invariant ε.

3.4. Lemma : *The torsion linking form* $<,>: \tau(M) \times \tau(M) \to \mathbf{Q}/\mathbf{Z}$, *with respect to the generators of* ($\clubsuit$), *is given as follows:*

$$< q_i, q_j >= \begin{cases} \beta_i/\alpha_i\,, & \text{if } i = j, \\ 0\,, & \text{if } i \neq j, \end{cases} \qquad \begin{aligned} < a, q_i > &= -\beta_i/2\alpha_i\,, \\ < a, a > &= (2k+\varepsilon)/4\,, \\ < q_i, h > &=< h, h >= 0\,, \\ \text{and } < a, h > &= 1/2\,. \end{aligned}$$

Proof : The 1-cycles corresponding to the generators are shown in Figure 3.3(a) with f directed out of the page, and we use the right hand rule to determine a positive intersection. We may perturb each ξ_i to a homologous 1-cycle ξ_i' just inside $N(\gamma_i)$, and perturb σ to σ' as illustrated in Figure 3.3(b). Then we have the following intersection numbers: $\xi_i'.A = \xi_i'.B = 0$ and

$$\xi_i'.D_j = \begin{cases} \beta_i\,, & \text{if } i = j, \\ 0\,, & \text{if } i \neq j, \end{cases}$$

while $\sigma'.D_i = -\beta_i$, and $\sigma'.A = -1$, and $\sigma'.B = k$ (by a choice of σ', since by a different choice we may get $\sigma'.B = -k$). Now, for $j = 1,..,n$, let $C_j = 2D_j - \beta_j A$, so that $\partial C_j = 2\alpha_j \xi_j$. Then $\xi_i'.C_j = 2\xi_i'.D_j = \delta_{ij} 2\beta_i$, and consequently $< q_i, q_j >= \delta_{ij} \frac{\beta_i}{\alpha_i}$. Also $\sigma'.C_i = -2\beta_i + \beta_i = -\beta_i$, so that $< a, q_i >= \frac{-\beta_i}{2\alpha_i}$. Now write $C = \sum_{i=1}^{n} \hat{\alpha}_i C_i$, where $\hat{\alpha}_i = \frac{1}{\alpha_i} \prod_{j=1}^{n} \alpha_j$, and let $R = 2(\prod_{j=1}^{n} \alpha_j)B - C$. Then $\partial R = 4(\prod_{j=1}^{n} \alpha_j)\sigma$. Also

$$\sigma'.R = 2(\prod_{j=1}^{n} \alpha_j)k + \sum_{i=1}^{n} \hat{\alpha}_i \beta_i = (\prod_{j=1}^{n} \alpha_j)(2k+\varepsilon)\,.$$

Thus $< a, a >= \frac{(2k+\varepsilon)}{4}$. (Note that we may also use R to check that $< q_i, a >= \frac{-\beta_i}{2\alpha_i} =< a, q_i >$ which follows also by the symmetry of the form). Finally,

$\xi_i'.A = 0$ implies that $< q_i, h >= 0$ for each $i = 1, .., n$, and $\sigma'.A = -1$ implies that $< a, h >= -1/2 = 1/2$, since $\partial A = 2f$. Also we may perturb f to f' so that $f'.A = 0$. Thus $< h, h >= 0$. •

3.5. Lemma : *When every multiplicity α_i is odd, then the torsion linking form for $M(P_k; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ is given by the linking matrix*

$$T = \begin{pmatrix} \beta_1/\alpha_1 & 0 & & 0 & 0 \\ 0 & \beta_2/\alpha_2 & & 0 & 0 \\ & & \cdot & & \\ & & & \cdot & \\ 0 & 0 & & 0 & 1/2 \\ 0 & 0 & & 1/2 & (2k+\eta)/4 \end{pmatrix},$$

where $\eta = \sum_{i=1}^n \alpha_i \beta_i$. Also, η is invariant (mod 4) under Seifert data equivalences, and if the linking T is hyperbolic we must have $\eta \equiv 2k$ (mod 4).

Proof : From Lemma 3.4, the form is described by the matrix

$$\begin{pmatrix} \beta_1/\alpha_1 & 0 & & 0 & -\beta_1/2\alpha_1 \\ 0 & \beta_2/\alpha_2 & & 0 & -\beta_2/2\alpha_2 \\ & & \cdot & & \\ & & & \cdot & \\ 0 & 0 & & 0 & 1/2 \\ -\beta_1/2\alpha_1 & -\beta_2/2\alpha_2 & & 1/2 & (2k+\varepsilon)/4 \end{pmatrix}.$$

Changing the generators by substituting $\hat{a} = a + \sum_{i=1}^n \frac{(\alpha_i+1)}{2} q_i$ (noting that each α_i is odd) results in the matrix

$$\begin{pmatrix} \beta_1/\alpha_1 & 0 & & 0 & \beta_1/2 \\ 0 & \beta_2/\alpha_2 & & 0 & \beta_2/2 \\ & & \cdot & & \\ & & & \cdot & \\ 0 & 0 & & 0 & 1/2 \\ \beta_1/2 & \beta_2/2 & & 1/2 & x \end{pmatrix}.$$

where we have $x = \frac{k}{2} + \frac{\varepsilon}{4} + 2\sum_{i=1}^n (\frac{(\alpha_i+1)}{2})(\frac{-\beta_i}{2\alpha_i}) + \sum_{i=1}^n (\frac{(\alpha_i+1)}{2})^2 (\frac{\beta_i}{\alpha_i}) = \frac{k}{2} + \frac{\varepsilon}{4} + \frac{1}{4}\sum_{i=1}^n (\alpha_i^2 - 1)(\frac{\beta_i}{\alpha_i}) = \frac{k}{2} + \frac{1}{4}\sum_{i=1}^n \alpha_i \beta_i = \frac{(2k+\eta)}{4}$. Now, a substitution $\hat{q}_i = q_i - \beta_i h$, for each $i = 1, .., n$, leaves the matrix T.

We can check that η is invariant under Seifert equivalence, since if $\beta_1' = \beta_1 + \alpha_1$, and $\beta_2' = \beta_2 - \alpha_2$, then $(\alpha_1'\beta_1' + \alpha_2'\beta_2') - (\alpha_1\beta_1 + \alpha_2\beta_2) = \alpha_1^2 - \alpha_2^2 =$

$(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2) \equiv 0 \pmod 4$. Also, it is easily checked that the block $\begin{pmatrix} 0 & 1/2 \\ 1/2 & (2k+\eta)/4 \end{pmatrix}$ is hyperbolic if and only if $2k + \eta \equiv 0 \pmod 4$. •

In the case where the strictly singular fibres occur in opposite pairs (i.e: $S = -S$), as in Corollary 1.2, we can easily check that $\eta \equiv \varepsilon \pmod 4$, and moreover, since $\begin{pmatrix} \beta_i/\alpha_i & 0 \\ 0 & -\beta_i/\alpha_i \end{pmatrix} \sim \begin{pmatrix} 0 & 1/\alpha_i \\ 1/\alpha_i & 0 \end{pmatrix}$ the linking is clearly hyperbolic in this case if and only if $\varepsilon \equiv 2k \pmod 4$.

When considering the form in the general case, one can show the following.

3.6. Proposition : *The manifold $M = M(P_k; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ where we write $\alpha_i = 2^{t_i} s_i$ for s_i odd, has a hyperbolic torsion linking form only if, for each $i = 1, .., n$, either $t_i = t_1$ or $t_i = 0$. That is, all the nonzero t_i must be equal.*

Proof: We change our set of generators to a larger one as follows. Let $c_i = 2^{t_i} q_i$ and $d_i = s_i q_i$ for each $i = 1, .., n$. Then the set $\{a, h, c_i, d_i \,:\, i = 1, .., n\}$ generates $\tau(M)$, and we have (mod $\mathbf{Z}$)

$$
\begin{aligned}
< c_i, c_j > &= 2^{t_i} 2^{t_j} < q_i, q_j >= \begin{cases} 2^{t_i} \beta_i / s_i\,, & \text{if } i = j, \\ 0\,, & \text{if } i \neq j. \end{cases} \\
< c_i, d_j > &= 2^{t_i} s_j < q_i, q_j >= 0 \\
< d_i, d_j > &= s_i s_j < q_i, q_j >= \begin{cases} s_i \beta_i / 2^{t_i}\,, & \text{if } i = j, \\ 0\,, & \text{if } i \neq j. \end{cases} \\
< c_i, a > &= 2^{t_i} (\frac{-\beta_i}{2\alpha_i}) = \frac{-\beta_i}{2 s_i} \\
< d_i, a > &= s_i (\frac{-\beta_i}{2\alpha_i}) = \frac{-\beta_i}{2^{t_i+1}}\,.
\end{aligned}
$$

Finally $< c_i, h >=< d_i, h >=< h, h >= 0$ and $< a, h >= 1/2$ as before. Since s_i is odd, we may put each $< c_i, a >$ entry to zero as in Lemma 3.5. This operation otherwise only affects the $< a, a >$ entry, and gives a linking matrix which is a block sum of the diagonal matrix $(< c_i, c_j >)$ with

$$
\begin{pmatrix}
\frac{s_1 \beta_1}{2^{t_1}} & 0 & & 0 & \frac{-\beta_1}{2^{t_1+1}} \\
0 & \frac{s_2 \beta_2}{2^{t_2}} & & 0 & \frac{-\beta_2}{2^{t_2+1}} \\
 & & \cdot & & \\
 & & \cdot & & \\
0 & 0 & & 0 & 1/2 \\
\frac{-\beta_1}{2^{t_1+1}} & \frac{-\beta_2}{2^{t_2+1}} & & 1/2 & x
\end{pmatrix}.
$$

Now suppose that contrary to the conclusion of the theorem there is some $i \in \{2,..,n\}$ such that $0 < t_i < t_1 = t$. Then we wish to show that the above form is not hyperbolic. Note that in this case α_1, α_i are even, so that β_1, β_i are odd (as well as s_1, s_i being odd). Replace the generator d_i with $\bar{d}_i = d_i + 2^{t-t_i} s_i d_1$ (or just $\bar{d}_i = s_i \bar{q}_i$ for $\bar{q}_i$ as in the proof of Lemma 3.1). Then

$$
\begin{aligned}
< \bar{d}_i, \bar{d}_i > &=< d_i, d_i > +(2^{t-t_i} s_i)^2 < d_1, d_1 >= \frac{s_i\beta_i + 2^{t-t_i} s_i^2 s_1 \beta_1}{2^{t_i}} \\
&= \frac{(\text{odd})}{2^{t_i}} \qquad \text{since } t_1 < t \text{ and } s_i\beta_i \text{ is odd,} \\
< \bar{d}_i, d_1 > &= 2^{t-t_i} s_i (\frac{s_1\beta_1}{2^t}) = \frac{(\text{odd})}{2^{t_i}}, \\
< \bar{d}_i, d_j > &= 0 \qquad \text{for } j \neq 1, i, \\
< \bar{d}_i, a > &= \frac{-\beta_i}{2^{t_i+1}} - 2^{t-t_i} s_i (\frac{\beta_1}{2^{t+1}}) = \frac{-(\beta_i + s_i\beta_1)}{2^{t_i+1}} = \frac{\frac{1}{2}(\text{even})}{2^{t_i}}, \\
< \bar{d}_i, h > &= 0\,.
\end{aligned}
$$

The previous block now becomes

$$
\begin{pmatrix}
\frac{(\text{odd})}{2^t} & 0 & \frac{(\text{odd})}{2^{t_i}} & 0 & . & 0 & \frac{(\text{odd})}{2^{t+1}} \\
0 & . & 0 & & & & . \\
\frac{(\text{odd})}{2^{t_i}} & 0 & \frac{(\text{odd})}{2^{t_i}} & 0 & . & 0 & \frac{\frac{1}{2}(\text{even})}{2^{t_i}} \\
0 & & 0 & . & & & . \\
. & & . & & . & & . \\
0 & & 0 & & & 0 & 1/2 \\
\frac{(\text{odd})}{2^{t+1}} & . & \frac{\frac{1}{2}(\text{even})}{2^{t_i}} & & . & 1/2 & x
\end{pmatrix},
$$

which is equivalent (by conjugation) to the linking matrix

$$
\begin{pmatrix}
\frac{(\text{odd})}{2^t} & 0 & 0 & 0 & . & 0 & \frac{(\text{odd})}{2^{t+1}} \\
0 & . & 0 & & & & . \\
0 & 0 & \frac{(\text{odd})}{2^{t_i}} & 0 & . & 0 & 0 \\
0 & & 0 & . & & & . \\
. & & . & & . & & . \\
0 & & 0 & & & 0 & 1/2 \\
\frac{(\text{odd})}{2^{t+1}} & . & 0 & . & . & 1/2 & x'
\end{pmatrix}.
$$

But, by Theorem 0.1 of [KK], a hyperbolic linking on a 2-group can have no diagonal summand and hence the form above is not hyperbolic. Linkings on finite abelian groups form an abelian semigroup under block sum, and [KK, Theorem 0.1] gives a set of generators and relations for the semigroup of linkings on 2-groups. According to [KK] any hyperbolic linking on a 2-group may be written in terms of these generators as $\bigoplus_i E_0^{k_i}$ where $E_0^k = \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix}$. Also we have the relations $2E_0^k = 2E_1^k$ for $k \geq 2$, where $E_1^k = \begin{pmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{pmatrix}$. But the other relations of [KK, Theorem 0.1] allow an E_0^k or E_1^k to be exchanged for diagonal summands $A^k(n) = (\frac{n}{2^k})$ only if some other diagonal summand is already present, which is impossible if we start with a hyperbolic linking. Finally, the linking we are considering above may be written $A^{t_i}(\text{odd}) \oplus \Lambda$ for some linking Λ which may also be expressed in terms of the standard generators. Consequently, no hyperbolic form is equivalent to this one. $\bullet$

Note that the odd order torsion linking can be treated as follows. The $n \times n$ block $\text{diag}(\frac{2^{t_i}\beta_i}{s_i})$ is equivalent (as a linking form) to

$$B = \bigoplus_{p \text{ prime},\ r \geq 1} B(p^r),$$

such that each $B(p^r)$ is given by the matrix $\text{diag}(\frac{b_1(p^r)}{p^r}, \frac{b_2(p^r)}{p^r}, \ldots, \frac{b_n(p^r)}{p^r})$, where $b_i(p^r) = \frac{\alpha_i\beta_i}{p^r}$ if p^r divides s_i, and is 0 otherwise. Thus there is one diagonal entry for each distinct prime power factor of each s_i. The linking B is hyperbolic if and only if each $B(p^r)$ is hyperbolic. That is if and only if

$$B(p^r) \sim \bigoplus_{i=1}^{m} \begin{pmatrix} 0 & \frac{1}{p^r} \\ \frac{1}{p^r} & 0 \end{pmatrix} = \bigoplus_{i=1}^{m} \begin{pmatrix} \frac{1}{p^r} & 0 \\ 0 & \frac{-1}{p^r} \end{pmatrix}.$$

Note that since $(\frac{b}{p^r}) \sim (\frac{m^2 b}{p^r})$ we may assume that each $b_i(p^r)$ is either 1 or a quadratic non-residue. Then by Wall's presentation ([W], [KK]) of the semigroup of linkings on p-groups (for a fixed odd prime p) — with generators $(\frac{1}{p^r})$ and $(\frac{n(p)}{p^r})$ and relations $(\frac{1}{p^r}) \oplus (\frac{1}{p^r}) = (\frac{n(p)}{p^r}) \oplus (\frac{n(p)}{p^r})$ for each $n(p)$ a quadratic non-residue (mod p) — it is a routine matter to check whether or not the linking $B(p^r)$ is hyperbolic. Thus in the case of Lemma 3.5 it is possible to give necessary and sufficient conditions for hyperbolicity.

In view of Proposition 3.6, we may consider the case where $t = 1$, in which case we have the following partial result :

3.7. Claim : *With $S = \{\gamma_i(2,1) \mid i = 1,..,2m\}$, the manifold $M_S(k;\varepsilon)$ has a hyperbolic torsion linking form if and only if $\delta = 2k + \varepsilon \equiv 0$ (mod 4).*

Proof : We consider only $m = 1$. The general case follows by the same argument applied m times. Note that, for the given S, ε and hence δ must be an integer. By Lemma 3.4, the torsion linking form has the following matrix:

$$\begin{pmatrix} 1/2 & 0 & 0 & -1/4 \\ 0 & 1/2 & 0 & -1/4 \\ 0 & 0 & 0 & 1/2 \\ -1/4 & -1/4 & 1/2 & \delta/4 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{\delta}{4} \end{pmatrix}.$$

In what follows we use the notation $(i) : +r(j)$ to mean replace the i^th generator, g_i say, with $g_i + rg_j$. That is add $r \times$ row j to row i and $r \times$ col j to col i. When $\delta \equiv 0$ (mod 4) this linking is equivalent to

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix} \overset{(1):+(3)}{\sim} \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix} \overset{(2):-(1)}{\sim} \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $\delta \equiv 1$ (mod 4) then we have

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \overset{(2):-(3)}{\sim} \begin{pmatrix} \frac{1}{2} & \frac{-1}{4} & \frac{1}{4} \\ \frac{-1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \overset{(1):-(3)}{\sim} \begin{pmatrix} \frac{1}{4} & \frac{-1}{4} & 0 \\ \frac{-1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix},$$

which fails to be hyperbolic, and similarly for $\delta \equiv -1$ (mod 4). Finally when $\delta \equiv 2$ (mod 4) we have

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \overset{(1):-(2)}{\sim} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \overset{(1):+2(3)}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix},$$

which again is not hyperbolic. •

In particular, the manifolds fibred over the orbifold $RP^2(2,2)$, that is the $M(RP^2;(2,1),(2,-1),(1,e))$, have hyperbolic torsion linking if and only if $e \equiv 2$ (mod 4). Thus $M(RP^2;(2,1),(2,-1))$ does not embed in S^4.

4. Proof of Theorem 1.1.

In what follows let $M = M_S(k;\varepsilon)$, and suppose that M is embedded in the 4-sphere. Then M separates S^4, and we may write $S^4 = X \cup_M Y$ for some 4-manifolds X and Y. Note also that, by the Mayer-Vietoris sequence, $H_1(M) \cong H_1(X) \oplus H_1(Y)$. With the notation of Lemma 3.1, the set S consists of fibres of multiplicity $\alpha_i = 2^{t_i} s_i$ for s_i odd, $i = 1,..,n$, and we may suppose, by Proposition 3.6, that $t^1 = t^2 = .. = t^r = t_S$ while $t^{r+1} = .. = t^n = 0$. In what follows, we simply write $t_S = t_1 = t$.

4.1 Lemma : *Take M as above. Then for one of X or Y, say X without loss of generality, there exists a covering $\psi_X : \tilde{X} \to X$ (which restricts to a covering $\psi_M : \tilde{M} \to M$ on the boundary) together with a covering involution (of $\tilde{X}$) which restricts on $\tilde{M}$ to a rotation of the fibres through π radians. Here $\tilde{M}$ has the Seifert structure induced by ψ_M.*

Proof : Since $\tau(M)$ must be a direct double, we may assume by Lemma 3.1, that it has a direct summand $\mathbf{Z}/2\alpha_1\mathbf{Z}$ generated by q_1, as in the proof. In fact $\hat{q} = s_1 q_1$ generates a direct summand of $\tau(M)$ of order 2^{t+1}. Thus we may define a surjective homomorphism

$$\phi : H_1(M) \longrightarrow \langle \zeta \mid 2^{t+1}\zeta = 0\rangle \cong \mathbf{Z}/2^{t+1}\mathbf{Z}$$

such that, other than $\hat{q}$ every generator of $H_1(M)$ maps to 0, while $\hat{q} \mapsto \zeta$. This map is effectively a projection onto the summand of $\tau(M)$ generated by $\hat{q}$.

Now $\hat{q}$ may be identified with an element $(a, b) \in H_1(X) \oplus H_1(Y) \cong H_1(M)$, and one of $(a, 0)$ or $(0, b)$, without loss of generality $(a, 0)$ say, must have order 2^{t+1} not just in $H_1(M)$, but also under the quotient map ϕ as well. In this case the homomorphism

$$\alpha : H_1(X) \hookrightarrow H_1(M) \xrightarrow{\phi} \mathbf{Z}/2^{t+1}\mathbf{Z}\,,$$

such that $x \mapsto \phi((x, 0))$, must be surjective, since the image of a has order 2^{t+1} and so generates $\mathbf{Z}/2^{t+1}\mathbf{Z}$. Thus, taking

$$\alpha_\pi : \pi_1(X) \xrightarrow{\text{ab}} H_1(X) \xrightarrow{\alpha} \mathbf{Z}/2^{t+1}\mathbf{Z}\,,$$

there exists a (2^{t+1})-fold covering $\psi_X : \tilde{X} \to X$ with $\pi_1(\tilde{X}) = \ker \alpha_\pi$.

Let $h \in H_1(M)$ be the element of order 2 represented by a regular fibre f of M. As in the proof of Lemma 3.1, $h = \alpha_1 q_1 = 2^t \hat{q}$. Since the inclusion $M \subset X$ induces the projection $p_X : H_1(M) \to H_1(X)$, the fibre f of M represents in $H_1(X)$ the element $[f] = p_X(h) = p_X(2^t\hat{q}) = 2^t a$, and since $\alpha(a)$ has order 2^{t+1}, $\alpha([f])$ will have order 2 in $\mathbf{Z}/2^{t+1}\mathbf{Z}$. Thus the loop $f.f$ lifts to a loop in

$\tilde{M}$ ($[f]^2 \in \ker \alpha_\pi$), while f does not, which is to say that, for each regular fibre, $\psi_M^{-1}(f)$ is a collection of circles each of which covers f two-fold.

On the other hand, if we take the map

$$\mu : \pi_1(X) \xrightarrow{\alpha_\pi} \mathbf{Z}/2^{t+1}\mathbf{Z} \to \mathbf{Z}/2^t\mathbf{Z}\,,$$

such that $[f] \mapsto 0$, then we get an intermediate covering $\bar{\psi}_X : \bar{X} \to X$ where each fibre of M lifts to a fibre in $\bar{M}$. Thus $\tilde{X}$ is a 2-fold cover of $\bar{X}$ with covering automorphism as required. That is each regular fibre of $\tilde{M}$ covers a fibre of $\bar{M}$ 2-fold, and the same must necessarily be true for the singular fibres (of $\tilde{M}$ and $\bar{M}$) as well. •

4.2. A remark on 2-fold covers of Seifert manifolds : Observe that if $\tilde{M} \to M$ is a 2-fold covering of Seifert manifolds (a Seifert structure on M induces one on $\tilde{M}$) such that every fibre of M is twice covered by a single fibre of $\tilde{M}$ (that is, the covering involution acts on $\tilde{M}$ by rotation of the fibres through π radians), then a fibre $\tilde{\gamma}$ in $\tilde{M}$ of type $(\alpha, \tilde{\beta})$ must cover a fibre γ in M of type (α, β) where $\beta = 2\tilde{\beta} + c\alpha$ for some $c \in \mathbf{Z}$ [NR, 1.3]. Thus, for such a covering to exist, every fibre γ of M must have ODD multiplicity α. It is this fact that neccessitates the introduction of the intermediate 2^t-fold covering $\bar{M} \to M$.

It follows from this remark that, since $\tilde{M} \to \bar{M}$ is such a twofold covering, the singular fibres of $\bar{M}$ have odd multiplicities. Also recall that, by Lemma 3.6, we may assume that each multiplicity of a singular fibre in M is either odd already or $2^t s_i$ for s_i odd. Then in the covering $\bar{M} \to M$ each regular fibre of M lifts to 2^t distinct fibres, as do the singular fibres of odd multiplicity, while those fibres of even multiplicity $2^t s_i$ must each be 2^t-fold covered by a single fibre of multiplicity s_i. Considering the base spaces F and $\bar{F}$ as orbifolds, this covering induces a covering of orbifolds $\bar{F} \to F$, or a 2^t-fold branched covering of surfaces with a branch set consisting of the $2b$ points corresponding to the $2b$ singular fibres with even multiplicity. Thus if $F = P_k$, then $\tilde{F} = \bar{F}$ is also a nonorientable surface $P_{\tilde{k}}$ with genus $\tilde{k}$ calculated as follows. As for the statement of Theorem 1.1, define $b_S = 0$ if every fibre has odd multiplicity ($t = 0$) and otherwise

$$b_S = b - 1 = \frac{(\#\text{ of fibres in } M \text{ of EVEN multiplicity}) - 2}{2}.$$

Then, when $t \neq 0$, $\chi(\bar{F}) = 2^t(\chi(F) - 2b_S - 2) + 2b_S + 2$ which gives that $\tilde{k} = 2^t(k + 2b_S) - 2b_S$ (since $\chi(P_k) = 2 - k$). Note however that this formula also holds when $t = 0$ in which case $\tilde{k} = k$. Finally, if $\varepsilon, \bar{\varepsilon}, \tilde{\varepsilon}$ are the generalised Euler invariants for $M, \bar{M}, \tilde{M}$ respectively, then $\tilde{\varepsilon} = \frac{\bar{\varepsilon}}{2} = \frac{2^t\varepsilon}{2}$, as in [NR, 1.2] or [JN]. Now, with the number b_S defined above, we have the following lemma. (Here $\beta_i(X) = \dim H_i(X; \mathbf{Q}) = \dim H^i(X; \mathbf{Q})$ is the i^{th} Betti number.)

4.3. Lemma : *In Lemma 4.1, we have* $\beta_1(\tilde{X}) \leq 2^{t+1}(\beta_1(X) + b_S)$.

Proof : More generally consider a p^s-fold covering $\tilde{X} \to X$ for p prime, where, putting $K_0 = \pi_1(\tilde{X})$ and $G_0 = \pi_1(X)$, we have an exact sequence

$$1 \longrightarrow K_0 \longrightarrow G_0 \xrightarrow{\alpha_\pi} \mathbf{Z}/p^s\mathbf{Z} \longrightarrow 1\,.$$

We wish to estimate $\beta_1(\tilde{X}) = \text{rank } K_0^{\text{ab}}$ in terms of $\beta_1(X) = \text{rank } G_0^{\text{ab}}$. Now $\tau = \{x \in K_0 : x^n \in [K_0, K_0]\,,\ \text{for some } n \in \mathbf{Z}\}$ defines a subgroup of K_0 which is normal in G_0. (For $x \in \tau$, $g \in G_0$, we have $(gxg^{-1})^n = gx^ng^{-1} \in [K_0, K_0]$ for some n, since $x^n \in [K_0, K_0]$ and $[K_0, K_0] \lhd G_0$, which follows from $K_0 \lhd G_0$). Thus, writing $K = K_0/\tau = \mathbf{Z}^r$, where $r = \beta_1(\tilde{X})$, and $G = G_0/\tau$ we have an exact sequence $1 \longrightarrow K \longrightarrow G \longrightarrow \mathbf{Z}/p^s\mathbf{Z} \longrightarrow 1$. Here G^{ab} is a quotient of $G_0^{\text{ab}} = H_1(X)$, since G is a quotient of G_0, and in fact we may check that G^{ab} and G_0^{ab} have the same rank. Now $K \lhd G$ implies that $pK \lhd G$, and we may factor out by pK to get an exact sequence

$$1 \longrightarrow \bar{K} \longrightarrow \bar{G} \xrightarrow{\bar{\alpha}_\pi} \mathbf{Z}/p^s\mathbf{Z} \longrightarrow 1. \qquad (✠)$$

Here $\bar{K} = (\mathbf{Z}/p\mathbf{Z})^r$, and $\bar{G} \cong \langle \zeta, \bar{K} \mid \zeta^{p^s} \in \bar{K}, \zeta^{-1}x\zeta = Tx,\ \text{for } x \in \bar{K}\rangle$, where ζ acts on $\bar{K} = \mathbf{F}_p^r$ (considered now as an $\mathbf{F}_p$-vector space of dimension r) via some $T \in GL(r, \mathbf{F}_p)$. Let $\Gamma = \mathbf{F}_p[T]$ be the polynomial ring, which acts on $\bar{K}$ in the obvious way, so that we may also view $\bar{K}$ as a Γ-module. Now, since $\zeta^{p^s} \in \bar{K}$ and $\bar{K}$ is abelian, $T^{p^s}x = \zeta^{-p^s}x\zeta^{p^s} = x$. That is $\bar{K}$ is annihilated by $T^{p^s} - 1 = (T-1)^{p^s} \in \Gamma$ (this equality holds in Γ since the coefficients are mod p). Since Γ is a P.I.D, we have $\bar{K} = \bigoplus_{i=1}^{p^s}(\Gamma/(T-1)^i)^{e_i}$, for some $e_i \geq 0$. Each factor $\Gamma/(T-1)^i \cong \mathbf{F}_p^i$ as an $\mathbf{F}_p$-vector space, so that $\beta_1(\tilde{X}) = \dim\bar{K} = \Sigma_{i=1}^{p^s} ie_i \leq p^s\Sigma e_i$. On the other hand, $\bar{G}^{\text{ab}} \cong \langle \zeta, \bar{K}/(T-1)\bar{K} \mid \zeta^{p^s} \in \bar{K}/(T-1)\bar{K}\rangle$, and $\bar{K}/(T-1)\bar{K} = \bigoplus_{i=1}^{p^s}(\Gamma/(T-1))^{e_i} \cong \mathbf{F}_p^R$ where $R = \Sigma e_i$. Thus, so far, $\beta_1(\tilde{X}) \leq p^sR$, and we have a surjective homomorphism $\xi : G_0^{\text{ab}} \to G^{\text{ab}} \to \bar{G}^{\text{ab}}$, where $\bar{G}^{\text{ab}} = \langle \zeta, (\mathbf{Z}/p\mathbf{Z})^R \mid \zeta^{p^s} \in (\mathbf{Z}/p\mathbf{Z})^R\rangle$.

In our case of interest $p = 2$ and $s = t + 1$, and we take α, α_π as in the proof of Lemma 4.1. Also, since $\tau(M)$ is a direct double $\tau(X) \oplus \tau(Y)$ [KK, 6.1], we may assume that $\tau(X)$ has exactly $b_S + 1$ factors of even order. Furthermore, there exists some element of $\tau(X)$ of order 2^{t+1}, q say, whose image $\alpha(q)$ generates $\mathbf{Z}/2^{t+1}\mathbf{Z}$. By the construction of $\bar{G}$ we have a surjection $\hat{\xi}$ which induces ξ in the following commuting diagram,

$$\begin{tikzcd}
G_0 \arrow[r, "\text{ab}"] \arrow[d, "\hat{\xi}"'] & G_0^{\text{ab}} = H_1(X) \arrow[d, "\xi"'] \arrow[dr, "\alpha"] & \\
\bar{G} \arrow[r, "\text{ab}"] & \bar{G}^{\text{ab}} \arrow[r, "\bar{\alpha}"] & \mathbf{Z}/2^{t+1}\mathbf{Z}
\end{tikzcd}$$

where $\bar{\alpha}$ is induced by $\bar{\alpha}_\pi$ of the exact sequence (✠), and $\ker(\bar{\alpha}) = \bar{K}$. Clearly we can choose q such that $\alpha(q) = \bar{\alpha}(\zeta)$, still of order 2^{t+1}. Then $\xi(q) = \zeta + \kappa$ for some $\kappa \in \bar{K}$, and $\xi(2^{t+1}q) = 2^{t+1}\zeta + 2^{t+1}\kappa = 2^{t+1}\zeta$ since κ has order 2. But this must be 0 since q has order 2^{t+1}, and consequently

$$\bar{G}^{\mathrm{ab}} = \langle \zeta \mid 2^{t+1}\zeta = 0 \rangle \oplus (\mathbf{Z}/2\mathbf{Z})^R.$$

Then ξ taken "mod 2" gives a surjection $(\mathbf{Z}/2\mathbf{Z})^{\beta_1(X)+b_S+1} \to (\mathbf{Z}/2\mathbf{Z})^{R+1}$, and hence $R \leq \beta_1(X) + b_S$. But then $\beta_1(\tilde{X}) \leq 2^{t+1}(\beta_1(X) + b_S)$. •

4.4. Definition : By a *G-manifold* (G, Σ) we shall mean an oriented manifold Σ together with a finite group G which acts on Σ by orientation preserving homeomorphisms.

In this context we shall only consider Σ to be a connected 4-manifold, and suppose $G = \mathbf{Z}/2\mathbf{Z}$ acts by an involution $g : \Sigma \to \Sigma$. Furthermore we shall aim to construct $\mathbf{Z}/2\mathbf{Z}$-manifolds where the fixed point set of g is a closed connected surface F, smoothly embedded in Σ, such that g acts smoothly on a neighbourhood of F. That is F will have a smooth D^2-bundle neighbourhood $N(F)$ in Σ, on which g acts by rotation of the fibres. We shall call such a surface a *"good"* fixed point set. Given such F, denote by $e(F)$ the Euler class of the normal bundle to F in Σ, which is just the Euler class of $\partial N(F)$ considered as a circle bundle over F. Now the $\mathbf{Z}/2\mathbf{Z}$-signature theorem states that, for a closed $\mathbf{Z}/2\mathbf{Z}$-manifold Σ with "good" fixed point set F, we have

$$\mathrm{sign}(g, \Sigma) = e(F)\,.$$

Here $\mathrm{sign}(g, \Sigma)$ denotes the signature of the bilinear form $(\,,\,)_g$ on $H^2(\Sigma; \mathbf{Q})$ given by $(\sigma\,, \tau)_g = \sigma \cup g^*(\tau)$. We may look to various sources [JO], [G] for a statement of this theorem for Σ smooth, and g a diffeomorphism. However the smoothness is required (by the proof in [JO] for example) only in the neighbourhood of the fixed point set. Thus the signature theorem holds equally well in the present setting.

In proving Theorem 1.1 we shall also use the fact that $|\mathrm{sign}(g, \Sigma)| \leq \beta_2(\Sigma)$, and equals $\beta_2(\Sigma)$ mod 2, for Σ a closed 4-manifold.

Observe, for example, that the covering spaces $\tilde{X}$ constructed in Lemma 4.1 are $\mathbf{Z}/2\mathbf{Z}$-manifolds, where the action is by the covering involution ψ_X which rotates the fibres in the boundary by π radians. Any D^2-bundle over a surface F may be considered as a $\mathbf{Z}/p\mathbf{Z}$-manifold for any prime p. In this case, the action is by rotation of the disk fibre, and F becomes the "good" fixed point set. Consequently, if $\tilde{M}$ in Lemma 4.1 is a circle bundle with Euler number $\tilde{e}$, then the associated disk-bundle $D_{\tilde{M}}$ is a $\mathbf{Z}/2\mathbf{Z}$-manifold with boundary $(\mathbf{Z}/2\mathbf{Z}, \tilde{M})$.

Thus we may construct a closed $\mathbf{Z}/2\mathbf{Z}$-manifold $\Sigma_{\tilde{M}} = \tilde{X} \cup_{\tilde{M}} D_{\tilde{M}}$, which has a "good" fixed point set F with $e(F) = \tilde{e}$, and we have $\mathrm{sign}(g, \Sigma_{\tilde{M}}) = \tilde{e}$. Note that this is the case M an S^1-bundle with Euler class $e = 2\tilde{e}$.

More generally, if $\tilde{M}$ is Seifert fibred then the associated 'disk'-bundle, or more precisely the mapping cylinder

$$C_{\tilde{M}} = \tilde{M} \times [0,1]/((x_1, 0) \sim (x_2, 0) \text{ for } x_1, x_2 \text{ in the same fibre}),$$

is not in general a manifold. In particular, it ceases to be a manifold precisely at the isolated points $p_i = (x_i, 0)$ for x_i in the singular fibre γ_i of multiplicity > 1. However, we may modify $C_{\tilde{M}}$ by removing a neighbourhood which contains all such points. In this way we attempt to generalise the construction of the previous paragraph.

As in the statement of the theorem, given $S = \{\gamma_i(\alpha_i, \beta_i) : i = 1, .., n\}$, let $M_1 = M_S(k_1; \varepsilon_1)$ be fibred over F_1, and $M_2 = M_S(k_2; \varepsilon_2)$ over F_2, and suppose that both M_1 and M_2 embed in S^4. Then, we may write $S^4 = X_1 \cup_{M_1} Y_1$, where we may suppose there is a covering $\psi_{X_1} : \tilde{X}_1 \to X_1$ as described in Lemma 4.1 (and similarly for M_2).

Consider $C_{\tilde{M}_1}$. For each $i = 1, ..., n$, let $x_i \in \tilde{F}_1$ correspond to the fibre $\tilde{\gamma}_i$ of type $(s_i, \tilde{\beta}_i)$ in $\tilde{M}_1$. That is $x_i = \tilde{\pi}(\tilde{\gamma}_i)$ where $\tilde{\pi} : \tilde{M}_1 \to \tilde{F}_1$ is the fibre map. There exists a disk $\delta \in \tilde{F}_1$ such that $x_i \in \delta$ for each $i = 1, ..., n$. Let $B_1 = \tilde{\pi}^{-1}(\delta) \times [0, 1/2]/(\sim) \subset C_{\tilde{M}_1}$. This defines a neighbourhood which meets $\tilde{F}_1$ in δ (note that $C_{\tilde{M}_1}$ contains $\tilde{F}_1$ as $\{(x, 0) \in C_{\tilde{M}_1}\}$). Also B_1 is contractible to $\delta \sim *$, so that $\chi(B_1) = 1$.

4.5. Claim : *The boundary ∂B_1 is a rational homology sphere.* $(H_k(\partial B_1; \mathbf{Q}) = \mathbf{Q}, 0, 0, \mathbf{Q})$

Proof: We may write ∂B_1 as a G.S. fibred manifold [JN]

$$\partial B_1 \cong M(S^2; (0,1), (s_1, \tilde{\beta}_1), (s_2, \tilde{\beta}_2), \ldots, (s_n, \tilde{\beta}_n)),$$

with a fundamental group which abelianises to a group of finite order :

$$\begin{aligned} \pi_1(\partial B_1) &\cong \langle q_0, q_1, .., q_n, h \mid [q_i, h] = 1, q_i^{s_i} h^{\tilde{\beta}_i} = 1, h = 1, q_0 = \Pi_{i=1}^n q_i \rangle \\ &\cong \langle q_1, .., q_n \mid q_i^{s_i} = 1 \rangle . \end{aligned}$$

Thus $H_1(\partial B_1; \mathbf{Q}) = 0$, and the rest follows by Poincaré duality, given that ∂B_1 is closed, orientable and connected. Note that in fact ∂B_1 is a connected sum of lens spaces. •

Define the 'modified associated disk-bundle' to be $D_{\tilde{M}_1} = C_{\tilde{M}_1} \setminus \mathrm{int}(B_1)$, and let $\Sigma_1 = \tilde{X}_1 \cup_{\tilde{M}_1} D_{\tilde{M}_1}$ define a $\mathbf{Z}/2\mathbf{Z}$-manifold with boundary ∂B_1. Here g acts on Σ_1 via the covering automorphism of $\tilde{X}_1$, from Lemma 4.1, which extends across $D_{\tilde{M}_1}$ in the obvious way, with fixed point set $\tilde{F}_1 \setminus \mathrm{int}(\delta)$. Similarly construct the manifold Σ_2 with boundary ∂B_2. Note that $\tilde{M}_1$ and $\tilde{M}_2$ may be assumed, by a Seifert equivalence if necessary to have exactly the same collection of strictly singular fibre types. In fact, for $i = 1, 2$, $\tilde{M}_i = M(\tilde{F}_i; (s_1, 2\beta_1), .., (s_n, 2\beta_n), (1, e_i))$, and $\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 = e_1 - e_2$. Thus $B_1 \cong B_2$ ($\cong B$ say) in the strongest possible sense (that is by a homeomorphism which preserves the fibres, the set $\tilde{F}_i \cap B_i$, and the group action), and we may take the union

$$\Sigma = \Sigma_1 \bigcup_{\partial B} -\Sigma_2 \ .$$

to be a $\mathbf{Z}/2\mathbf{Z}$-manifold with "good" fixed point set $F = \tilde{F}_1 \sharp \tilde{F}_2$. Furthermore, F has Euler class $e(F) = \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 = \frac{2^t(\varepsilon_1 - \varepsilon_2)}{2}$. Note that $\partial(-\Sigma_2) \cong -\partial B$, while $\partial(\Sigma_1) \cong \partial B$. However, to consistently orient the union of oriented manifolds we require the attaching homeomorphism to be orientation reversing, and here it is taken to be the obvious one. Now, by the $\mathbf{Z}/2\mathbf{Z}$-signature theorem we may write

$$\mathrm{sign}(g, \Sigma) = \frac{2^t(\varepsilon_1 - \varepsilon_2)}{2}.$$

This signature is bounded by the value of $\beta_2(\Sigma)$, independently of the ε_i. It remains to express this constraint in terms of k_1 and k_2, and the constants t and b_S, which depend only on the set S.

4.6. Claim : $\chi(\Sigma) = 2 - 2(\beta_1(\Sigma_1) + \beta_1(\Sigma_2)) + \beta_2(\Sigma)$.

Proof: By the Mayer-Vietoris sequence (with coefficients in $\mathbf{Q}$),

$$\begin{tikzcd} H_1(\partial B) \arrow[r] \arrow[d, equal] & H_1(\Sigma_1) \oplus H_1(\Sigma_2) \arrow[r, "\cong"] & H_1(\Sigma) \arrow[r, "0"] & H_0(\partial B) \arrow[d, equal] \\ \mathbf{0} & & & \mathbf{Q} \end{tikzcd}$$

we have $\beta_1(\Sigma) = \beta_1(\Sigma_1) + \beta_1(\Sigma_2)$, and, since Σ is a closed orientable 4-manifold, we may apply Poincaré duality to give

$$\begin{aligned} \chi(\Sigma) &= 2 - 2\beta_1(\Sigma) + \beta_2(\Sigma) \\ &= 2 - 2(\beta_1(\Sigma_1) + \beta_1(\Sigma_2)) + \beta_2(\Sigma) . \end{aligned}$$

•

4.7. Claim : $\chi(\Sigma) = 2^{t+1}\chi(X_1) + 2^{t+1}\chi(X_2) + \chi(\tilde{F}_1) + \chi(\tilde{F}_2) - 2.$

Proof: Since Σ is the union $\tilde{X}_1 \cup D_{\tilde{M}_1} \cup D_{\tilde{M}_2} \cup \tilde{X}_2$ along closed 3-manifolds,

$$\chi(\Sigma) = \chi(\tilde{X}_1) + \chi(\tilde{X}_2) + \chi(D_{\tilde{M}_1}) + \chi(D_{\tilde{M}_2}).$$

Also, for each $i = 1, 2$, the union $C_{\tilde{M}_i} = D_{\tilde{M}_i} \cup B_i$ gives $\chi(C_{\tilde{M}_i}) = \chi(D_{\tilde{M}_i}) + \chi(B_i)$. But that is $\chi(D_{\tilde{M}_i}) = \chi(\tilde{F}_i) - 1$, for each $i = 1, 2$, since $C_{\tilde{M}_i}$ is contractible to $\tilde{F}_i$, and $\chi(B_1) = \chi(B_2) = 1$. Thus,

$$\begin{aligned}\chi(\Sigma) &= \chi(\tilde{X}_1) + \chi(\tilde{X}_2) + \chi(\tilde{F}_1) + \chi(\tilde{F}_2) - 2 \\ &= 2^{t+1}\chi(X_1) + 2^{t+1}\chi(X_2) + \chi(F_1) + \chi(F_2) - 2.\end{aligned}$$

•

Combining **4.6** and **4.7** we have

$$\begin{aligned}\beta_2(\Sigma) &= 2(\beta_1(\Sigma_1) + \beta_1(\Sigma_2)) + 2^{t+1}(\chi(X_1) + \chi(X_2)) + \chi(\tilde{F}_1) + \chi(\tilde{F}_2) - 4 \\ &= \Xi_1 + \Xi_2 \ , \qquad \text{where} \qquad \Xi_i = 2\beta_i(\Sigma_i) + 2^{t+1}\chi(X_i) + \chi(\tilde{F}_i) - 2.\end{aligned}$$

4.8. Claim : *For* $i = 1, 2$, $\beta_1(\Sigma_i) \leq \beta_1(\tilde{X}_i) \leq 2^{t+1}(\beta_1(X_i) + b_S)$.

Proof: Write $\Sigma_i^0 = \Sigma_i \cup_{\partial B_i} B_i$, for $i = 1, 2$. That is $\Sigma_i^0 = \tilde{X}_i \cup_{\tilde{M}_i} C_{\tilde{M}_i}$, where $C_{\tilde{M}_i}$ is contractible to $\tilde{F}_i$. Then we have the Mayer-Vietoris sequence

$$H_1(\tilde{M}_i) \xrightarrow{\phi} H_1(\tilde{X}_i) \oplus H_1(\tilde{F}_i) \longrightarrow H_1(\Sigma_i^0) \longrightarrow 0$$

where $\phi = (\hat{\imath}_*, \hat{\jmath}_*)$ is induced by the inclusions $\hat{\imath} : \tilde{M}_i \hookrightarrow \tilde{X}_i$ and $\hat{\jmath} : \tilde{M}_i \hookrightarrow \tilde{C}_{\tilde{M}_i}$. But $\hat{\jmath} : H_1(\tilde{M}_i) \to H_1(\tilde{F}_i)$ is a surjection, so that $\dim(\mathrm{Im}\phi) \geq \beta_1(F_i)$. But then,

$$\begin{aligned}\beta_1(\Sigma_i^0) &= \beta_1(\tilde{X}_i) + \beta_1(\tilde{F}_i) - \dim(\mathrm{Im}\phi) \\ &\leq \beta_1(\tilde{X}_i).\end{aligned}$$

Thus, by Lemma 4.3, we have $\beta_1(\Sigma_i^0) \leq 2^{t+1}(\beta_1(X_i) + b_S)$.

On the other hand, the Mayer-Vietoris sequence for $\Sigma_i^0 = \Sigma_i \cup_{\partial B_i} B_i$,

$$\begin{array}{ccccccc} H_1(\partial B_i) & \longrightarrow & H_1(\Sigma_i) \oplus H_1(B_i) & \xrightarrow{\cong} & H_1(\Sigma_i^0) & \xrightarrow{0} & H_0(\partial B_i) \\ \| & & \| & & & & \| \\ \mathbf{0} & & \mathbf{0} & & & & \mathbf{Q} \end{array}$$

together with Claim 4.5 gives $\beta_1(\Sigma_i) = \beta_1(\Sigma_i^0)$, and we are done. •

Now, whenever $S^4 = X \cup_M Y$, the Mayer-Vietoris sequence (with coefficients in $\mathbf{Z}$) gives $H_i(M) \cong H_i(X) \oplus H_i(Y)$ if $i = 1, 2$, and otherwise

$$H_i(X) \cong H_i(Y) \cong \begin{cases} 0, & \text{if } i \geq 3\,, \\ \mathbf{Z}, & \text{if } i = 0\,. \end{cases}$$

Also, since $M \hookrightarrow X$ consequently induces epimorphisms $H_*(M) \to H_*(X)$, the relative exact sequence for (X, M) gives $\beta_1(M) = \beta_1(X) + \beta_2(X, M)$ which becomes $\beta_1(M) = \beta_1(X) + \beta_2(X)$ by Lefschetz duality.

So here we have that $\beta_1(X_i) + \beta_2(X_i) = \beta_1(M_i) = k_i - 1$, and $\beta_3(X_i) = \beta_4(X_i) = 0$. With **4.8** (for $i = 1, 2$), and these relations we now have

$$\begin{aligned} \Xi_i &\leq 2^{t+2}(\beta_1(X_i) + b_S) + 2^{t+1}\chi(X_i) + \chi(\tilde{F}_i) - 2 \\ &= 2^{t+1}(1 + \beta_1(X_i) + \beta_2(X_i) + 2b_S) + \chi(\tilde{F}_i) - 2 \\ &= 2^{t+1}(k_i + 2b_S) - \tilde{k}_i \quad = K_i \qquad \text{say.} \end{aligned}$$

Thus we have shown that $\beta_2(\Sigma) \leq K_1 + K_2 = K$. Also $\Xi_i \equiv \chi(\tilde{F}_i) \equiv \tilde{k}_i \equiv K_i$ mod 2, so that

$$\begin{aligned} &\operatorname{sign}(g, \Sigma) \equiv \beta_2(\Sigma) \equiv K \bmod 2\,, \\ \text{and} \qquad &|\operatorname{sign}(g, \Sigma)| \leq \beta_2(\Sigma) \leq K\,. \end{aligned}$$

It now follows that $\operatorname{sign}(g, \Sigma) \in \{K, K-2, K-4, \ldots, -K+2, -K\}$, and, since $\operatorname{sign}(g, \Sigma) = \frac{2^t(\varepsilon_1 - \varepsilon_2)}{2}$, where $\varepsilon_1 - \varepsilon_2$ must be an integer, that

$$\varepsilon_1 - \varepsilon_2 \in \{\frac{2K}{2^t}, \frac{2K-4}{2^t}, \frac{2K-8}{2^t}, \ldots, \frac{-2K+4}{2^t}, \frac{-2K}{2^t}\} \cap \mathbf{Z}\,.$$

Finally note that, since $\tilde{k}_i = 2^t(k_i + 2b_S) - 2b_S$ (following Remark 4.2), we have $K_i = 2^t(k_i + 2b_S) + 2b_S$, and so $K = 2^t(k_1 + k_2 + 4b_S) + 4b_S$, which completes the proof. •

References

[FQ] M.H. Freedman, and F. Quinn, *Topology of 4-Manifolds*, Princeton University Press, Princeton (1990).

[GL] P. Gilmer and C. Livingston, 'On embedding 3-manifolds in 4-space', Topology 22 (1983), 241-252.

[G] C.McA. Gordon, 'On the G-signature theorem in dimension four', in *A la Recherche de la Topologie Perdue*, ed. Guillou and Marin, Birkhäuser, 1986.

[H] J.A. Hillman, 'Embedding homology equivalent 3-manifolds in 4-space', Research Report 93-22, University of Sydney (1993), (revised 1994).

[JN] M. Jankins and W.D. Neumann, *Lectures on Seifert Manifolds*, Brandeis Lecture Notes 2, (1983).

[JO] K. Jänich and E. Ossa, 'On the signature of an involution', Topology 8 (1969), 27-30.

[KK] A. Kawauchi and S. Kojima, 'Algebraic classification of linking pairings on 3-manifolds', Math. Ann. 253 (1980), 29-42.

[M] W.S. Massey, 'Proof of a conjecture of Whitney', Pacific J. Math. 31 (1969), 143-156.

[NR] W.D. Neumann and F. Raymond, 'Seifert manifolds, plumbing, μ-invariant and orientation reversing maps', in Proceedings Sympos., Santa Barbara, CA 1977, Lecture Notes in Math. 664, Springer Verlag (1978) 163-196.

[O] P. Orlik, *Seifert Manifolds*, Lecture Notes in Math. 291, Springer Verlag (1972).

[R1] D. Rolfsen, *Knots and Links*, Publish or Perish, Berkeley (1976).

[R2] D. Rolfsen, 'Rational surgery calculus: Extension of Kirby's theorem', Pacific J. Math. 110 (1984), 377-386.

[S] H. Seifert, 'Topology of 3-dimensional fibred spaces', Acta Math. 60 (1933), 147-288.

[W] C.T.C. Wall, 'Quadratic forms on finite groups, and related topics', Topology 2 (1964), 281-298.

[Zee] E.C. Zeeman, 'Twisting spun knots', Trans. A.M.S. 115 (1965), 471-495.

To the memory of my father, C.F.C Crisp,
Sydney, November 17th, 1994.