On the decomposition of 3-dimensional Poincaré duality complexes

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ABSTRACT

We show that if the fundamental group of an orientable PD_3-complex has infinitely many ends then it is either a proper free product or virtually free of finite rank. It follows that every PD_3-complex is finitely covered by one which is homotopy equivalent to a connected sum of aspherical PD_3-complexes and copies of $S^1 \times S^2$. Furthermore, any torsion element of the fundamental group of an orientable PD_3-complex has finite centraliser.

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§0. INTRODUCTION

Let P be a PD_3-complex, $\tilde{P}$ its universal covering space, and $\pi = \pi_1(P)$. It was known by Wall [6] that either π is finite and $\tilde{P} \simeq S^3$, or π has one end and $\tilde{P}$ is contractible (P aspherical), or π has two ends and P is homotopy equivalent to one of $P_3 \# P_3$, $S^1 \times P_2$, $S^1 \times S^2$ or $S^1 \tilde{\times} S^2$, or π has infinitely many ends. Turaev since showed that if π is a proper free product, then P is a nontrivial connected sum of PD_3-complexes [5], but it is still not known whether every orientable PD_3-complex that has a fundamental group with infinitely many ends is such a nontrivial connected sum, as is the case for orientable 3-manifolds. If this question, of [6], were to be answered in the affirmative, then it would mean that every orientable PD_3-complex P decomposes as the connected sum of PD_3-complexes P_i such that $\tilde{P}_i$ is either a homotopy 3-sphere, a homotopy 2-sphere, or contractible. Theorem 3.4 implies that this is at least true of some finite cover of P. Theorem 3.7 gives still further constraints on the occurrence of torsion in π. However, this is not enough to completely settle Wall's question. In particular, the question raised in [3] as to whether $\mathcal{S}_3 *_{C_2} \mathcal{S}_3$ may be the fundamental group of an orientable PD_3-complex remains unanswered. Here $\mathcal{S}_3$ denotes the symmetric group on three elements.

The approach here is motivated by ideas in Hillman's paper [3]. There the groups $H_*(C, \bar{H}^1(\pi, \mathbf{Z}\pi))$, for C a cyclic subgroup of π, are known by duality and a spectral sequence argument. In §1 we outline an interpretation for such homologies in the more general setting of groups acting on trees. The calculations in that context are independent of any duality properties. In §2 we make precise the application of this to PD_3-complexes, via the accessibility of π, and the Mayer-Vietoris sequence of [1] for graphs of groups. These observations lead naturally to the results in §3 where we also recover the main result of [3] as Corollary 3.8.

§1. TREES WITH ∞-VERTICES

In this section we introduce the notion of a module Π being presented by a tree X with ∞-vertices, and proceed to calculate the homology groups $H_*(C,\Pi)$ of a prime order cyclic group C acting by automorphisms on X. These turn out to be precisely determined by properties of the subtree of fixed points of X under the action of C.

We follow [2] in defining a graph X to be the disjoint union of a pair of sets EX and VX, called the edge and vertex sets respectively, together with a pair of functions $o, t : EX \to VX$ which specify for each edge ϵ an original vertex $o(\epsilon)$, and a terminal vertex $t(\epsilon)$. A nonempty graph X is called a tree if there exists a unique reduced path (called a geodesic) between any two distinct vertices, or equivalently if X is connected and contains no closed loops. A graph automorphism $g : X \to X$ is a bijection such that $g(EX) = EX$, $g(VX) = VX$, and, for $\epsilon \in EX$, $o(g(\epsilon)) = g(o(\epsilon))$ and $t(g(\epsilon)) = g(t(\epsilon))$. These compose (on the left) in the obvious way, and if a group G acts on the tree (or graph) X by automorphisms then we shall call X a G-tree (or G-graph respectively).

1.1 DEFINITION : A G-tree with ∞-vertices is a tree X with a distinguished G-invariant subset $V_fX \subset VX$ consisting of finite valence vertices (i.e: having only finitely many adjacent edges). Vertices which do not lie in V_fX are said to be ∞-vertices. (Note that an ∞-vertex need not have infinite valence).

Henceforth we shall assume every tree X to have this extra structure, and any graph automorphism of X to preserve the set V_fX. Note that an ∞-vertex structure is also inherited by any subtree of X simply by restriction of the set V_fX.

To any tree X with ∞-vertices we may associate a $\mathbf{Z}$-module $\Pi[X]$, which is said to be presented by the tree X, as follows. Let $\mathbf{Z}[V_fX]$ and $\mathbf{Z}[EX]$ denote the free $\mathbf{Z}$-modules with bases V_fX and EX respectively. Then $\Pi[X]$ is defined to be the cokernel of the map $\Delta : \mathbf{Z}[V_fX] \hookrightarrow \mathbf{Z}[EX]$ defined for each $\nu \in V_fX$ by the formula

$$\Delta(\nu) = \sum_{\{\epsilon | t(\epsilon)=\nu\}} \epsilon \;-\; \sum_{\{\epsilon | o(\epsilon)=\nu\}} \epsilon \,. \tag{1}$$

Furthermore, if X is a G-tree then $\Pi[X]$ naturally inherits a left $\mathbf{Z}G$-module structure.

1.2 EXAMPLE : In §2 to follow we find that if G is an accessible group then the $\mathbf{Z}G$-module $H^1(G, \mathbf{Z}G)$ is presented by any terminal G-tree. This includes the case that G is finitely presentable ([2], Theorem VI.6.3) and, in particular, the case that G is the fundamental group of a PD_3-complex.

1.3 DEFINITION : For a tree X define the set of ends $\mathcal{E}X$ to be the set of classes of rays (i.e: infinite paths from a vertex in X) which differ as sets only within a bounded set. (Explicitly , rays γ_1 and γ_2 are equivalent if $\gamma_1 \cup \gamma_2 \setminus \gamma_1 \cap \gamma_2$ is a finite subset of X). Consequently we may think of there being a unique geodesic from any vertex in X to any end in $\mathcal{E}X$, or for that matter from any end to any other end. Let $e(X) = |\mathcal{E}X|$ denote the number of ends of X, and $\infty(X) = |VX \setminus V_f X|$ the number of ∞-vertices, each of which may be infinite. Finally write

$$\xi(X) = e(X) + \infty(X) - 1\,.$$

For convenience we refer to an element of the union $\mathcal{E}X \coprod VX \backslash V_f X$ as an end/∞-vertex. Thus $\xi(X)+1$ measures the number of end/∞-vertices of X.

1.4 LEMMA : *Let X be a tree with ∞-vertices. Then $\Pi[X]$ is a free $\mathbf{Z}$-module of (possibly infinite) rank $R = \max\{\xi(X), 0\}$.*

Proof : Suppose that $x = \sum_{\epsilon \in EX} m_\epsilon \epsilon$ represents a nontrivial element $[x] \in \Pi[X]$ such that $n[x] = 0$ for some $n \neq 0$. That is $n.x = \sum_{\nu \in V_f X} n_\nu \Delta(\nu)$, for some n_ν which are zero for all but finitely many vertices. We shall formally extend such an expression to a sum over all of VX by setting $n_\nu = 0$ whenever ν is an ∞-vertex. By a choice of the representative x, we may assume that $0 \leq n_\nu < n$ for each $\nu \in VX$. But then, since $n.m_\epsilon = n_{t(\epsilon)} - n_{o(\epsilon)}$, our choice forces $m_\epsilon = 0$ for each $\epsilon \in EX$. That is $x = 0$ contradicting the nontriviality of $[x]$. Thus $\Pi[X]$ has no torsion elements and so is free as a $\mathbf{Z}$-module.

Write $\mathrm{rk}[X]$ for the rank of $\Pi[X]$ as a (free) $\mathbf{Z}$-module. Given any edge ϵ_0 in X define the subtrees X_+ and X_- to be the components of $X \setminus \{\epsilon_0\}$ containing the vertices $t(\epsilon_0)$ and $o(\epsilon_0)$ respectively, with $V_f X_\pm$ defined by restriction of $V_f X$. Any edge ϵ_0 of a given tree X represents an element of $\Pi[X]$ which we shall write as $[\epsilon_0]_X$, suppressing the subscript whenever there is no ambiguity. We have $[\epsilon_0] = 0$ exactly when we may write $\epsilon_0 = \sum_{\nu \in VX} n_\nu \Delta(\nu)$ where n_ν is zero except for finitely many vertices $\nu \in V_f X$. By connectedness of trees this is equivalent to

$$n_\nu = \begin{cases} n_{o(\epsilon_0)} & \text{if } \nu \in X_- \\ n_{o(\epsilon_0)} + 1 & \text{if } \nu \in X_+\,, \end{cases}$$

since, for $\epsilon \in EX$, $n_{t(\epsilon)} - n_{o(\epsilon)} = 1$ if $\epsilon = \epsilon_0$ and is 0 otherwise. Thus

$$\begin{aligned} [\epsilon_0] = 0 &\iff \text{one of } X_+ \text{ or } X_- \text{ is finite with no } \infty\text{-vertices} \\ &\iff \text{one of } X_+ \text{ or } X_- \text{ has no ends and no } \infty\text{-vertices} \\ &\iff \text{either } \xi(X_+) = -1 \text{ or } \xi(X_-) = -1\,. \end{aligned}$$

Note that finite vertices have finite valence, and a finite valence graph with no ends must be finite.

If $\mathrm{rk}[X] > 0$ we may choose an edge ϵ_0 such that $[\epsilon_0] \neq 0$. Then each of X_+ and X_- has either an ∞-vertex or an end, and so $\xi(X) > 0$. Conversely, if $\xi(X) > 0$ there are at least two distinct end/∞-vertices, and choosing ϵ_0 to lie on the geodesic joining them we arrange that X_+ contains one end/∞-vertex and X_- the other. But then $[\epsilon_0] \neq 0$ and $\mathrm{rk}[X] > 0$. Thus we have shown that $\mathrm{rk}[X] = 0 \Leftrightarrow \xi(X) \leq 0$.

Let X_+° and $X_+^\bullet$ denote the trees $X_+ \cup \{\epsilon_0, o(\epsilon_0)\}$ with $V_f X_+^\circ = V_f X_+$ and $V_f X_+^\bullet = V_f X_+ \cup \{o(\epsilon_0)\}$. Similarly define X_-° and $X_-^\bullet$. Observe that, for any choice of edge ϵ_0 in X, the free $\mathbf{Z}$-module $\Pi[X]$ may be expressed thus as the quotient of a sum of free $\mathbf{Z}$-modules by a single relation :

$$\Pi[X] \cong \frac{\Pi[X_+^\circ] \oplus \Pi[X_-^\circ]}{([\epsilon_0]_{X_+^\circ}, 0) \sim (0, [\epsilon_0]_{X_-^\circ})} \quad \text{, or } \langle \Pi[X_+^\circ], \Pi[X_-^\circ] \mid [\epsilon_0]_{X_+^\circ} = [\epsilon_0]_{X_-^\circ} \rangle \,.$$

If we may assume that either $[\epsilon_0]_{X_+^\circ} \neq 0$ or $[\epsilon_0]_{X_-^\circ} \neq 0$ then, the relation above being non-vacuous, it follows that

$$\mathrm{rk}[X] = \mathrm{rk}[X_+^\circ] + \mathrm{rk}[X_+^\circ] - 1 \,.$$

Suppose that $rk[X] > 0$, and choose ϵ_0 such that $[\epsilon_0]_X \neq 0$. Then the assumption above is valid and since, in fact, both $[\epsilon_0]_{X_+^\circ} \neq 0$ AND $[\epsilon_0]_{X_-^\circ} \neq 0$ we may equally well argue that

$$\begin{aligned} \mathrm{rk}[X_\pm^\bullet] &= \mathrm{rk}[X_\pm^\circ] + \mathrm{rk}[\circ \overset{\epsilon_0}{-} \bullet] - 1 \\ &= \mathrm{rk}[X_\pm^\circ] - 1 \,. \end{aligned}$$

Thus we have $\mathrm{rk}[X] = \mathrm{rk}[X_+^\bullet] + \mathrm{rk}[X_-^\bullet] + 1$. On the other hand, it is easily checked that $\xi(X) = \xi(X_+^\bullet) + \xi(X_-^\bullet) + 1$ where $\xi(X_\pm^\bullet) = \xi(X_\pm) \geq 0$, as $[\epsilon_0]_X$ is nontrivial. We may now argue by induction on $m = \min\{\mathrm{rk}[X], \xi(X)\}$ to show that $\mathrm{rk}[X] = \xi(X)$ whenever m is finite and non-negative. But this leaves only the case $\mathrm{rk}[X] = \xi(X) = \infty$ and we are done. •

1.5 COROLLARY : *If G is an infinite group and X a terminal G-tree (see §2) then*

$$\mathrm{ends}(G) = e(X) + \infty(X) \,.$$

Proof : This follows immediately from Lemma 2.1 and Lemma 1.4 above. •

Suppose now that X is a C-tree, where $C = \langle g \rangle$ denotes a finite cyclic group of prime order p, and write $\Pi = \Pi[X]$ for the $\mathbf{Z}C$-module presented by X. Note that the set X^C of fixed points of X under the action of C is also a tree with ∞-vertices, in fact a subtree of X, where we set $V_f X^C = V_f X \cap X^C$. To compute the homology $H_q(C, \Pi)$, for $q > 0$, we need now to define the norm map $\bar{N} : \Pi_C \to \Pi^C$ with respect to C, which is induced by the map $N : \Pi \to \Pi^C$ given by $N(x) = x + g(x) + ... + g^{p-1}(x)$ for $x \in \Pi$. The fixed subtree X^C alone shall determine precisely the kernel and cokernel of $\bar{N}$.

For $\nu \in VX^C$, let O_ν (resp. T_ν) denote the set of edges of X which originate (resp. terminate) at ν, and write $E_\nu = O_\nu \cup T_\nu$. For $\epsilon \in E_\nu$, write X_ϵ for the tree (with ∞-vertices) obtained from the connected component of $X \setminus \{\nu\}$ containing ϵ by replacing ν with an infinite vertex. Then it follows that

$$\Pi[X] \cong \Big(\bigoplus_{\epsilon \in E_\nu} \Pi[X_\epsilon] \Big) / \mathbf{Z}\langle r \rangle \qquad \text{as a } \mathbf{Z}\text{-module},$$

where the relator r is just $\sum_{\epsilon \in T_\nu} [\epsilon] - \sum_{\epsilon \in O_\nu} [\epsilon]$ when $\nu \in V_f X$ and trivial otherwise. Since g fixes ν it clearly acts on the sets O_ν and T_ν, and permutes the connected components of $X \setminus \{\nu\}$ hence inducing $\mathbf{Z}$-isomorphisms $\Pi[X_\epsilon] \xrightarrow{\cong} \Pi[X_{g(\epsilon)}]$ such that, in particular, $g([\epsilon]) = [g(\epsilon)]$. Write $x \in \Pi[X]$ as $x = [x_\epsilon]_{\epsilon \in E_\nu}$ whenever x is represented by $(x_\epsilon)_{\epsilon \in E_\nu} \in \bigoplus_{\epsilon \in E_\nu} \Pi[X_\epsilon]$.

1.6 CLAIM : *Let X be a C-tree for $C = \langle g \rangle$ a cyclic group of prime order p, fix $\nu \in VX^C$ and define X_ϵ, for $\epsilon \in E_\nu$, as above. Suppose that $x = [x_\epsilon]_{\epsilon \in E_\nu}$ is an element of $\Pi[X]^C$. Then $g(x_\epsilon) = x_{g(\epsilon)}$ holds in $\Pi[X_{g(\epsilon)}]$ for each $\epsilon \in E_\nu$.*

Proof : The hypothesis is that $x - g(x) = 0$, or equivalently that $[x_\epsilon - g(x_{g^{-1}(\epsilon)})]_{\epsilon \in E_\nu} = 0$ in $\Pi[X]$. The result then follows trivially in the case ν is an ∞-vertex, and otherwise we must have $(x_\epsilon - g(x_{g^{-1}(\epsilon)}))_{\epsilon \in E_\nu} = (m_\epsilon[\epsilon])_{\epsilon \in E_\nu}$ where $m_\epsilon = m$ if $\epsilon \in T_\nu$ and $m_\epsilon = -m$ if $\epsilon \in O_\nu$, for $m \in \mathbf{Z}$ fixed. Thus for fixed $\epsilon \in T_\nu$ the following p equations hold in $\Pi[X_\epsilon]$,

$$g^{p-i}(x_{g^i(\epsilon)} - g(x_{g^{i-1}(\epsilon)})) = g^{p-i}(m[g^i(\epsilon)]) = m[\epsilon] \text{ , for } i = 1, .., p.$$

Adding these gives $p.m[\epsilon] = 0$ in $\Pi[X_\epsilon]$. But since $\Pi[X_\epsilon]$ is torsion free (by Lemma 1.4) it follows that $m[\epsilon] = 0$ for each $\epsilon \in T_\nu$. By a similar argument for $\epsilon \in O_\nu$ we are done in all cases. •

1.7 REMARK : It follows from Claim 1.6 that if $x \in \Pi[X]^C$ is represented as a sum of elements from only those $\Pi[X_\epsilon]$ such that $g(\epsilon) \neq \epsilon$, then $x = N(y)$ for some $y \in \Pi[X]$.

1.8 LEMMA : *Let X be a C-tree for $C = \langle g \rangle$ a cyclic group of prime order p, and let $\Pi = \Pi[X]$ be the $\mathbf{Z}C$-module presented by X. Then the norm map $\bar{N} : \Pi_C \to \Pi^C$ has cokernel $(\mathbf{Z}/p\mathbf{Z})^R$ with $R = \mathrm{rk}(\Pi[X^C]) = \max\{(\xi(X^C), 0\}$.*

Proof : Write $\hat{\Pi} = \Pi[X^C]$. We may define a quotient map $\phi : \Pi \to \hat{\Pi}$ with kernel $\hat{K} = \langle [\epsilon] : \epsilon \in EX \setminus X^C \rangle$. Since ϕ is a $\mathbf{Z}C$-map ($\hat{K}$ a $\mathbf{Z}C$-submodule) it induces $\phi^C : \Pi^C \to \hat{\Pi} = \hat{\Pi}^C$ with kernel $\hat{K}^C = \hat{K} \cap \Pi^C$, and a well-defined map $\hat{N} : \hat{\Pi} \to \hat{\Pi}$ such that $\hat{N} \circ \phi = \phi^C \circ N$. We shall show that $\hat{K}^C \subset \mathrm{Im}(N)$. It follows that $\mathrm{coker}(\bar{N}) = \mathrm{coker}(N) = \mathrm{coker}(\phi^C \circ N) = \mathrm{coker}(\hat{N})$, and, since $\mathrm{Im}(\hat{N}) = p\hat{\Pi}$, the Lemma is easily proved with the help of Lemma 1.4.

Take an arbitrary $x \in \hat{K}^C$, writing $x = \sum_{i=1}^{k} m_i \epsilon_i$ with each $\epsilon_i \in EX \setminus X^C$. Let T denote the minimal C-subtree of X such that $\epsilon_i \in T$ for each $i = 1, .., k$. Note that T is finite and $T \cap X^C$ is a subtree of X^C (nonempty since C, being finite, always has fixed points). If ν_∞ is an ∞-vertex in $T \cap X^C$, then the trees X_ϵ , for $\epsilon \in E_{\nu_\infty}$, are in fact subtrees of X. For $\epsilon \in E_{\nu_\infty}$, write $X_{C\epsilon}$ for the subtree which is the union of the trees $X_{g^i(\epsilon)}$, for $i = 1, .., p$. Then $\Pi[X]$ is a direct sum, over the orbits $C\epsilon$, of the $\mathbf{Z}C$-submodules $\Pi[X_{C\epsilon}]$, and each component of x, corresponding to this sum, is itself an element of $\hat{K}^C$. Treating such components individually, the proof that $x \in \mathrm{Im}(N)$ ultimately reduces to cases $(*)$ where $T \cap X^C$ either consists of a single ∞-vertex ν_0, or contains no ∞-vertices at all.

Given $\epsilon \in X^C$ such that $o(\epsilon)$ has at least two adjacent edges and neither $o(\epsilon)$ nor $t(\epsilon)$ is an ∞-vertex, define a new C-tree X' by removing ϵ from X and identifying $t(\epsilon)$ with $o(\epsilon)$. Clearly X' and X present isomorphic $\mathbf{Z}C$-modules, since this modification corresponds to simply eliminating the generator ϵ via the relation $\Delta(o(\epsilon)) = 0$. By a finite (or zero) number of such operations on the edges of $T \cap X^C$ we arrive, in all cases $(*)$, at a tree X'' with subtree T'' corresponding to T such that $T'' \cap X''^C$ consists of a single vertex ν_0. Note that via the isomorphism $\Pi[X] \cong \Pi[X'']$ the element x is now expressed in terms of edges $\epsilon''_1, .., \epsilon''_k \in T''$. Thus applying Claim 1.6 at the vertex ν_0 in X'', and the Remark 1.7, we have that $x \in \mathrm{Im}(N)$. •

1.9 LEMMA : *Let X be a C-tree for $C = \langle g \rangle$ a cyclic group of prime order p, and let $\Pi = \Pi[X]$ be the $\mathbf{Z}C$-module presented by X. Then the norm map $\bar{N} : \Pi_C \to \Pi^C$ has kernel $\mathbf{Z}/p\mathbf{Z}$ if $\xi(X^C) = -1$, and 0 otherwise.*

Proof : For convenience we shall write $M = \mathbf{Z}[EX]$, $K = \Delta(\mathbf{Z}[V_f X])$ and $\phi : M \to \Pi$ for the surjective map with kernel K. Now N is induced under ϕ by a similar map $\mathcal{N}$ on M, so that the following diagram commutes

$$\begin{tikzcd}
M \arrow[r, two heads, "\mathcal{N}"] \arrow[d, two heads, "p \circ \phi"'] & \mathcal{N}(M) \subset M^C \arrow[d, "\phi|_{\mathcal{N}(M)}"] \arrow[dl, dotted, "\phi'"'] \\
\Pi_C \arrow[r, "\bar{N}"] & \Pi^C
\end{tikzcd}$$

where p is the obvious projection of Π onto its C-coinvariant module. Write K' for $\ker(p \circ \phi)$, and note that $K + (g-1)M \subseteq K'$. Conversely, if $x \in K'$ then

$\phi(x) \in \ker p = (g-1)\Pi$ and so $\phi(x) = (g-1)\phi(y) = \phi((g-1)y)$ for some $y \in M$. Thus $x \in K+(g-1)M$, proving the equality $K' = K+(g-1)M$. Furthermore, M is a direct sum of copies of the augmentation module $\mathbf{Z}$ and a free $\langle g\rangle$-module, and so $H_2(\langle g\rangle, M) = 0$. That is $\ker \mathcal{N} = (g-1)M$. Consequently, there exists a map ϕ' which commutes with the diagram (as shown) and such that $\ker \phi' = \mathcal{N}(K)$.

Now, by the diagram, $\ker \bar{N} = \phi'(K^C \cap \mathcal{N}(M))$ which may be identified with the quotient $(K^C \cap \mathcal{N}(M))/\mathcal{N}(K)$. Note that K^C is freely generated as a $\mathbf{Z}$-module by the set $V_fX^C \cup \{\mathcal{N}(\Delta(\nu)) : \nu \in V_fX \setminus V_fX^C\}$, while $\mathcal{N}(K)$ is $\mathbf{Z}$-free on $\{\mathcal{N}(\Delta(\nu)) : \nu \in V_fX\}$. Thus $K^C/\mathcal{N}(K)$ is simply a $\mathbf{Z}/p\mathbf{Z}$-vector space with basis V_fX^C, and $\ker \bar{N}$ is just the subspace of those elements whose representatives lie also in $\mathcal{N}(M)$.

Take any element $\bar{a} \in \ker\bar{N}$ with representative $a \in K^C \cap \mathcal{N}(M)$ which we may assume to have the form $a = \sum_{\nu \in VX^C} n_\nu \Delta(\nu)$ where $0 \leq n_\nu < p$ and n_ν is zero except for finitely many $\nu \in V_fX^C$. Expressing a in terms of the free $\mathbf{Z}$-basis EX for M, the coefficient m_ϵ of $\epsilon \in EX$ will be given as $m_\epsilon = n_{t(\epsilon)} - n_{o(\epsilon)}$. Since $a \in \mathcal{N}(M)$ and $\mathcal{N}(M)$ is generated ($\mathbf{Z}$-freely) by the set

$$\{\mathcal{N}(\epsilon) : \epsilon \in EX\} = \{p\epsilon : \epsilon \in EX^C\} \cup \{(1+g+..+g^{p-1})\epsilon : \epsilon \in EX \setminus EX^C\},$$

we must have $m_\epsilon \equiv 0 \pmod p$ for all $\epsilon \in EX^C$, and hence $n_{o(\epsilon)} = n_{t(\epsilon)}$ (since $0 \leq n_\nu < p$). By the connectedness of X^C, the coefficients n_ν must then take a value n_0 constant over the whole set of fixed vertices. It now follows that $\bar{a}$ can be chosen to be nontrivial (that is $n_0 \neq 0$), if and only if X^C is finite and has no ∞-vertices, that is $\xi(X^C) = -1$. Furthermore, any such choice of $\bar{a}$ will have order p and generate the whole of $\ker \bar{N}$. •

We can summarise the previous two lemmas by the following:

1.10 THEOREM : *Suppose that the finite cyclic group C of prime order p acts on the tree with ∞-vertices X, and let $\Pi = \Pi[X]$ be the left $\mathbf{Z}C$-module presented by X. Then*

$$H_i(C, \Pi) = \begin{cases} (\mathbf{Z}/p\mathbf{Z})^{R_+} & \text{if } i \text{ odd,} \\ (\mathbf{Z}/p\mathbf{Z})^{R_-} & \text{if } i > 0 \text{ even,} \end{cases}$$

where $R_+ = \max\{\xi(X^C), 0\}$ and $R_- = \max\{-\xi(X^C), 0\}$.

Proof : This follows immediately from Lemmas 1.8 and 1.9 given that we may calculate the homology of C from the kernel and cokernel of the norm map. •

§2. ACCESSIBLE GROUPS

A G-tree X with ∞-vertices is said to be terminal (see [2]) if each edge stabiliser is finite and each vertex stabiliser has at most one end, and $V_f X$ is taken to be precisely the set of vertices with finite stabilisers. A group G is said to be accessible if there exists a terminal G-tree. For a group G we shall consider the group cohomology $H^1(G, \mathbf{Z}G)$ as a left $\mathbf{Z}G$-module with action defined in terms of the natural right action by $g.m = xg^{-1}$ for $g \in G$ and $x \in H^1(G, \mathbf{Z}G)$.

2.1 LEMMA : *Let G be an accessible group with terminal G-tree X. Then X presents $H^1(G, \mathbf{Z}G)$ as a left $\mathbf{Z}G$-module.*

Proof : Recall that a graph of groups $(\mathcal{G}, Y)$ consists of a graph Y together with a collection $\mathcal{G}$ of groups G_v, for each $v \in VY$, and subgroups $G_e \subset G_{o(e)}$, for each $e \in EY$, with injective homomorphisms $\phi_e : G_e \to G_{t(e)}$. Fixing a choice of maximal subtree T of Y, we define the fundamental group of $(\mathcal{G}, Y)$ as

$$\pi_1(\mathcal{G}, Y, T) = \langle t_e\,,\ G_v \mid \mathrm{rel}\, G_v\,,\ t_e a t_e^{-1} = \phi_e(a) \text{ for } a \in G_e\,,\ t_e = 1 \text{ for } e \in ET\, \rangle$$

noting that up to isomorphism this group is independent of the choice of T.

By the Bass-Serre structure theorem ([4], I.5.4 Theorem 13) G is the fundamental group of a finite graph of groups $(\mathcal{G}, Y)$ where $Y = G \backslash X$ and the edge and vertex groups of $(\mathcal{G}, Y)$ are isomorphic to the corresponding edge and vertex stabilisers of X respectively. Furthermore, X is isomorphic to the G-tree $\tilde{X}$ defined with vertex and edge sets

$$V\tilde{X} = \coprod_{v \in VY} G/G_v\,, \qquad E\tilde{X} = \coprod_{e \in EY} G/G_e\,,$$

such that $o(gG_e) = gG_{o(e)}$ and $t(gG_e) = gt_eG_{t(e)}$ for $g \in G$ and $e \in EY$, and with G acting by left multiplication on cosets. Since X is a terminal G-tree the vertex groups of Y have at most one end, so that the cohomology Mayer-Vietoris sequence of [1] gives rise to a short exact sequence of right $\mathbf{Z}G$-modules

$$0 \to \bigoplus_{v \in V_f Y} \mathbf{Z}[G_v \backslash G] \xrightarrow{\Delta_r} \bigoplus_{e \in EY} \mathbf{Z}[G_e \backslash G] \longrightarrow H^1(G, \mathbf{Z}G) \to 0\,,$$

where $V_f Y = \{v \in VY \mid G_v \text{ finite}\}$ or just the set $G \backslash V_f \tilde{X}$. If we choose to consider this as a sequence of left $\mathbf{Z}G$-modules and left $\mathbf{Z}G$-maps (via the anti-isomorphism $g \mapsto g^{-1}$ of G), then the first two modules are naturally isomorphic to $\mathbf{Z}[V_f \tilde{X}]$ and $\mathbf{Z}[E\tilde{X}]$ respectively (by taking the coset $G_v g$ to $g^{-1}G_v$ etc..) and one can check that Δ_r becomes exactly the map $\Delta : \mathbf{Z}[V_f \tilde{X}] \hookrightarrow \mathbf{Z}[E\tilde{X}]$ of Definition 1.1. Thus $\tilde{X}$, or equivalently X, presents $H^1(G, \mathbf{Z}G)$ as a left $\mathbf{Z}G$-module. •

§3. RESULTS ON PD_3-COMPLEXES

A connected finitely dominated CW-complex P with a homomorphism $w : \pi_1(P) \to \{\pm 1\}$ is called a connected Poincaré complex of formal dimension n, or PD_n-complex, if it exhibits the equivariant Poincaré duality of an n-dimensional manifold with orientation class w. If Π is a left (resp. right) $\mathbf{Z}\pi$-module, denote by $\bar{\Pi}$ the left $\mathbf{Z}\pi$-module with the same underlying abelian group, and action given by $g.x = w(g)gx$ (resp. $w(g)xg^{-1}$) for all $g \in \pi$, $x \in \Pi$.

3.1 LEMMA (Hillman, [3]) : *Let P be a PD_3-complex with infinite fundamental group π. If C is a finite cyclic subgroup of π then there are isomorphisms $H_s(C, \bar{H}^1(\pi, \mathbf{Z}\pi)) \cong H_{s+3}(C, \mathbf{Z})$ for all $s \geq 1$.*

Proof: This follows from the spectral sequence for the projection of $\tilde{P}$ onto $\tilde{P}/C$ with the information that $H_q(\tilde{P}; \mathbf{Z}) = \mathbf{Z}, 0, \bar{H}^1(\pi, \mathbf{Z}\pi), 0, ...$ from the duality isomorphisms and the fact that $\tilde{P}$ is simply connected. (Note that since P is finitely dominated we may assume, without loss, that P is a finite 3-dimensional CW-complex). •

The thrust of the earlier sections §1 and §2 was to be able to calculate these homology groups independently of Lemma 3.1 (in fact without using duality) as is seen from Theorem 1.10 and the following lemma together.

3.2 LEMMA : *Given a group π and homomorphism $w : \pi \to \{\pm 1\}$, let Π be a $\mathbf{Z}\pi$-module and $\bar{\Pi}$ as above. If $C = \langle g \rangle$ is a finite cyclic subgroup of π then*

$$H_i(C, \bar{\Pi}) \cong \begin{cases} H_i(C, \Pi) & \text{if } i > 0 \text{ and } w(g) = 1, \\ H_{i+1}(C, \Pi) & \text{if } i > 0 \text{ and } w(g) = -1. \end{cases}$$

Proof: The case for $w(g) = 1$ is easy since then $\bar{\Pi} \cong \Pi$ as $\mathbf{Z}C$-modules. In the case $w(g) = -1$, $\bar{\Pi}$ and Π are distinguished as $\mathbf{Z}C$-modules only by the action of g, whereby the homology, $H_*(C, \bar{\Pi})$, of the complex

$$\cdots \longrightarrow \bar{\Pi} \xrightarrow{1-g} \bar{\Pi} \xrightarrow{1+g} \bar{\Pi} \xrightarrow{1-g} \bar{\Pi} \longrightarrow \mathbf{0},$$

is just that of the complex

$$\cdots \longrightarrow \Pi \xrightarrow{1+g} \Pi \xrightarrow{1-g} \Pi \xrightarrow{1+g} \Pi \longrightarrow \mathbf{0},$$

The lemma now follows by comparing this with the complex for $H_*(C, \Pi)$. •

3.3 REMARK : Given a PD_3-complex P with infinite fundamental group π, then since π is finitely presentable, it is accessible by Theorem VI.6.3 of [2], and there exists a terminal π-tree X, as in Lemma 2.1, which presents $\Pi =$

$H^1(\pi, \mathbf{Z}\pi)$ as a left $\mathbf{Z}\pi$-module. It is now clear, by Lemmas 3.1, 3.2, and Theorem 1.10, that if g is an element of π of prime order then exactly one of the following two cases must hold. Either $w(g) = 1$ and $\xi(X^{\langle g \rangle}) = -1$, or $w(g) = -1$ and $\xi(X^{\langle g \rangle}) = 1$. We shall apply this to prove the following two theorems.

3.4 THEOREM : *Let P be an orientable PD_3-complex with $\pi = \pi_1(P)$. Then π either has one end, is a proper free product, or is virtually free of finite rank. That is to say that P is either an aspherical complex, a nontrivial connected sum (by Turaev [5]), or finitely covered by some $\hat{P} \simeq \#^k(S^1 \times S^2)$.*

Proof : Let X be the π-tree of Remark 3.3 and $(\mathcal{G}, Y)$ the associated graph of group, which may be assumed to be finite since π is finitely generated. Assume that π is not a proper free product, and hence that the edge groups of $(\mathcal{G}, Y)$ are all nontrivial. It now suffices to show that either the vertex groups of $(\mathcal{G}, Y)$ are all finite, for then π must be virtually free of finite rank (see [4], II.2.6), or π has one end.

Suppose that $(\mathcal{G}, Y)$ has an infinite vertex group. Then either this is the only vertex group and π has one end, or it is one of the vertex groups $G_{o(e)}$ or $G_{t(e)}$ for some edge $e \in EY$. But in the latter case we show that both these groups are finite thus reaching a contradiction. Since G_e is nontrivial and finite, we may choose a nontrivial $g \in G_e \subset G_{o(e)}$ of prime order. By assumption, π is infinite and Remark 3.3 applies to show that, since P is orientable, $w(g) = 1$ and $\xi(X^{\langle g \rangle}) = -1$. That is $X^{\langle g \rangle}$ is finite with no ∞-vertices. But $G_{o(e)}$ is the stabiliser of some vertex of X which, since $g \in G_{o(e)}$, must lie in $X^{\langle g \rangle}$. Thus $G_{o(e)}$ must be a finite group. Finally take $g \in \phi_e(G_e) \subset G_{t(e)}$ to show similarly that $G_{t(e)}$ is finite. •

3.5 COROLLARY : *Let P be an orientable PD_3-complex. Then P is homotopy equivalent to a connected sum $V \# P_1 \# ... \# P_m$ where each P_i, for $i = 1, .., m$, is an aspherical PD_3-complex and V is a PD_3-complex with virtually free fundamental group.*

Proof : This follows from Theorem 3.4 by Turaev's theorem ([5], Theorem 1) and the fact that $\pi_1(P)$ is finitely presented, and by observing that $A * B$ is a virtually free group if both A and B are. •

3.6 COROLLARY : *Let P be an arbitrary PD_3-complex. Then $\pi_1(P)$ is virtually torsion free. Indeed, P is finitely covered by an orientable PD_3-complex which is homotopy equivalent to a (possibly empty) connected sum of aspherical PD_3-complexes and copies of $S^1 \times S^2$.*

Proof : This follows immediately by applying Corollary 3.5 to the orientation cover of P, and noting that aspherical PD_3-complexes are torsion free. •

3.7 THEOREM : *Let P be a PD_3-complex with $\pi = \pi_1(P)$. If $g \in \pi$ is a nontrivial element of prime order p such that $C_\pi(g)$ is infinite, then $p = 2$, $w(g) = -1$ and $C_\pi(g)$ has 2 ends.*

Proof : Again let X be the π-tree with finite edge stabilisers of Remark 3.3. Note that $C_\pi(g)$ acts on the subtree $X^{\langle g \rangle}$, also with finite edge stabilisers. (If $x \in C_\pi(g)$, $\epsilon \in X^{\langle g \rangle}$, then $g(x\epsilon) = gx\epsilon = xg\epsilon = x\epsilon$ implies that $x\epsilon \in X^{\langle g \rangle}$ also). Thus $C_\pi(g)$ being infinite implies that $X^{\langle g \rangle}$ has infinitely many edges, and so either an ∞-vertex or an end. That is $\xi(X^{\langle g \rangle}) \geq 0$. It follows from the Remark 3.3 that $w(g) = -1$, $p = 2$ (necessarily) and $\xi(X^{\langle g \rangle}) = 1$. Thus $X^{\langle g \rangle}$ has exactly two end/∞-vertices and contains a unique geodesic γ joining them. Since $C_\pi(g)$ respects the set of ∞-vertices and also acts on $\mathcal{E}X^{\langle g \rangle}$, it must preserve the geodesic γ. In fact, some subgroup H of index at worst 2 in $C_\pi(g)$ must fix each end/∞-vertex. If there is some ∞-vertex involved then the infinite group H stabilises every edge in γ which is a contradiction. Thus γ is an infinite geodesic joining two ends and H acts by translations. For any edge ϵ in γ the quotient $H/\mathrm{stab}_H(\epsilon)$ must be $\mathbf{Z}$. But any surjection $H \to \mathbf{Z}$ splits, and since $\mathrm{stab}_H(\epsilon)$ is finite, H must be virtually $\mathbf{Z}$ and have two ends. Hence $C_\pi(g)$ is also 2-ended as required. •

It follows that if P is an orientable PD_3-complex then any torsion element of $\pi_1(P)$ has finite centraliser. Thus, for example $\pi = G_1 *_A G_2$ is not the fundamental group of an orientable PD_3-complex if A is finite and a proper subgroup of both $N_{G_1}(A)$ and $N_{G_2}(A)$, since then $N_\pi(A)$ is infinite, and hence so is $C_\pi(a)$ for $a \in A$. As a corollary to this theorem we may also restate the main theorem of [3].

3.8 COROLLARY : *Let P be a PD_3-complex, with infinite fundamental group π. If π has a nontrivial finite normal subgroup N, then $P \simeq S^1 \times P_2$.*

Proof : In this case N contains a nontrivial element g of prime order p where $C_\pi(g)$ is a subgroup of finite index in π. Thus, applying Theorem 3.7, P is non-orientable, g has order 2, and π in fact has two ends. So, by Theorem 4.4 of [6], $P \simeq S^1 \times P_2$. •

Ideally we should like to improve Theorem 3.4 by reducing the case that π is virtually free. Beyond applying Theorem 3.7 as per the examples given, it appears that some other approach is needed to eliminate the cases where π is the fundamental group of a graph of groups with finite vertex groups and some nontrivial edge groups. Turaev ([5], Theorem 5) gives an explicit algebraic characterisation of the pair $(\pi_1(X), w(X))$ of a PD_3-complex X, but it is not clear how to apply this in general, let alone to decide whether the minimal example $\mathcal{S}_3 *_{C_2} \mathcal{S}_3$ proposed in [3] is or is not the fundamental group of an orientable PD_3-complex.

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