Injective maps between Artin groups

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Abstract. A sufficient condition is given for the injectivity of a homomorphism between Artin monoids which moreover ensures injectivity of the induced map between Artin groups in the case where both groups are of finite type. We list numerous examples of monoid homomorphisms satisfying this injectivity condition, all of which happen to be so-called LCM-homomorphisms. In the case of an LCM-homomorphism there is a natural way to realise the corresponding map on Artin groups geometrically as the map induced on fundamental groups by an inclusion of certain finite simplicial complexes. An interesting group homomorphism which is not realised in this way exhibits the Artin group of type B_n as a subgroup of finite index of the classical $(n+1)$-string braid group (type A_n). This subgroup is in fact the group of n-string braids over an annulus.

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Introduction

In this paper we consider maps between Artin groups which are canonically induced by maps defined on the corresponding Artin monoids. Thus our primary objects of study are a class of these monoid homomorphisms called LCM-homomorphism which we define in Section 2. By the main theorem of Section 1, Theorem 1.3, these LCM-homomorphisms are shown to be injective, and moreover this implies injectivity of the corresponding group homomorphism when both Artin groups are of finite type.

Furthermore, it can be shown (Proposition 2.3) that each LCM-homomorphism induces an injective map between the corresponding Coxeter groups as well, thus recovering some well-known relationships among reflection groups. For example, it is well known [1] that the generators of a finite Coxeter group W may be partitioned into two sets of mutually commuting elements whose products generate a dihedral subgroup of order $2h$, where h is called the Coxeter number for W. Corresponding to this, we show (see Table 1(i)) that every irreducible finite type Artin group has an Artin subgroup of dihedral type. This was already observed by Coxeter [6] in the case $W = S_n$, the symmetric group.

However, the strongest motivation for the definition of an LCM-homomorphism is the geometric interpretation which can be given to the corresponding group homomorphism. Each Artin group of finite type was shown by Brieskorn [2] to be the fundamental group of the space of regular orbits of a finite complex reflection group, which Deligne [7] subsequently showed to be a $K(\pi,1)$-space for the group. Analogous spaces have been defined for general Artin groups (corresponding also to the infinite Coxeter groups) which are conjecturally all $K(\pi,1)$ [4], and more recently Charney and Davis [5] have described certain finite simplicial complexes which are homotopy equivalent to these non-compact spaces. Using the simplicial complexes of [5] we construct for each LCM-homomorphism ϕ a corresponding simplicial map Φ. Theorem 3.4 then states that the Artin group homomorphism defined by ϕ is in fact the map induced on fundamental groups by the simplicial inclusion Φ.

In Section 4 we describe a mechanism for generating LCM-homomorphisms, and list various examples amongst the finite type and affine (or euclidean) type Artin monoids. Finally, in Section 5 we discuss a map between Artin groups which is of an altogether different nature, but which nevertheless provided the initial motivation for this work. I would like to thank Arun Ram for introducing me to the family of Artin groups.

1. Artin groups, lcm's and injectivity

A *Coxeter system* of rank n is a pair (W,S) consisting of a set S of n elements and a group W with presentation

$$\langle S \mid (st)^{m_{st}} = 1\,, \quad \text{for } s,t\in S \text{ such that } m_{st}\neq\infty\rangle\,,$$

where $m_{ss}=1$ and $m_{st}=m_{ts}\in\{2,3,4,...,\infty\}$ for $s\neq t$. Such a system is determined by its Coxeter graph Γ, which is the labelled graph with vertex set S and an edge labelled m_{st} between s and t whenever $m_{st}\neq 1$ or 2. (When drawing Γ, the labels equal to 3 are usually suppressed). We shall refer to Γ as the *type* of the Coxeter system, and say that (W,S), or just Γ, is of finite type if the group W is finite. Throughout this paper habitual use will be made of the notation (W,S) and (W',S') to refer to Coxeter systems of type Γ and Γ' respectively.

The *Artin group* A_Γ of type Γ is the group with presentation

$$\langle S \mid \langle st\rangle^{m_{st}} = \langle ts\rangle^{m_{st}}\,, \quad \text{for } s,t\in S \text{ such that } m_{st}\neq\infty\rangle\,, \qquad \textbf{(A)}$$

where $\langle st\rangle^{m_{st}}$ denotes the word $ststs...$ of length m_{st}. Note that by adding the relations $s^2=1$ for each $s\in S$ one obtains an equivalent form of the above presentation for W. Thus W is a quotient of A_Γ via the canonical homomorphism $A_\Gamma\to W$ such that $s\mapsto s$ for $s\in S$.

The *Artin monoid* A_Γ^+ of type Γ is the monoid (semigroup with 1) given by the presentation **(A)** considered as a monoid presentation. We shall use $\doteq$ to denote equality of elements in the Artin monoid, and $=$ to denote equality in the group. Clearly there is a canonical monoid homomorphism $A_\Gamma^+\to A_\Gamma$ which, by 5.5 and 5.6 of [3], is known to be injective for Γ of finite type.

Considered concretely, the monoid A_Γ^+ is the quotient of $F(n)^+$, the free monoid of positive words in the n letters of the alphabet S, obtained by imposing the *positive word equivalence* ($\doteq$) generated by the relations $\langle st\rangle^{m_{st}} \doteq \langle ts\rangle^{m_{st}}$ for $m_{st} \neq \infty$, as in **(A)**. Since these relations preserve the length of a word, there is a well-defined homomorphism $\ell : A_\Gamma^+ \to \mathbf{N}$, called the length, such that $\ell(U)$ is given by the length of any word in $F(n)^+$ representing U.

We now summarize some of the important properties of an Artin monoid which are proved in the paper of Brieskorn and Saito, [3]. The Artin monoid A_Γ^+ is cancellative ([3], 2.3). Namely, if $UA_1V \doteq UA_2V$ then $A_1 \doteq A_2$. For $U, V \in A_\Gamma^+$, we say U *divides* V, written $U \mid V$, if $V \doteq UV'$ for some $V' \in A_\Gamma^+$. Note that if $U \mid V$ then $\ell(U) \leq \ell(V)$. An element V is said to be a *common multiple* for a finite subset $\mathcal{U} = \{U_1, ..., U_r\}$ of A_Γ^+ if $U_i \mid V$ for each $i = 1, .., r$. It is shown in [3], 4.1, that if there exists a common multiple for $\mathcal{U}$ then there exists a (necessarily unique) *least* common multiple, that is a common multiple which divides all other common multiples, and which shall be denoted $[U_1, .., U_r]$. Observe that $\langle st\rangle^{m_{st}}$ is a common multiple for any pair of generators $s, t \in S$ with $m_{st} \neq \infty$, and is in fact the least common multiple, by the Reduction Lemma 2.1 of [3]. On the other hand, if $m_{st} = \infty$ then s and t have no common multiple at all.

Definition 1.1. We say that a homomorphism $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ (with $\phi(1) = 1$) between Artin monoids *respects lcm's* if
(i) $\phi(s) \neq 1$ for each generator $s \in S$, and
(ii) for each pair of generators $s, t \in S$, the pair $\phi(s), \phi(t)$ have a common multiple only if s and t do, in which case $[\phi(s), \phi(t)] \doteq \phi([s, t])$.

Given words U, V, denote by $\langle U, V\rangle^m$ the word obtained from $\langle uv\rangle^m$ by replacing each occurrence of the letter u (resp. v) with the word U (resp. V). Using this notation, condition (ii) of Definition 1.1 states precisely that $\phi(s)$ and $\phi(t)$ have least common multiple $[\phi(s), \phi(t)] \doteq \langle \phi(s), \phi(t)\rangle^{m_{st}}$ if $m_{st} \neq \infty$, and no common multiple at all if $m_{st} = \infty$.

Lemma 1.2. *Suppose that $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ is a homomorphism which respects lcm's. If $U, V \in A_\Gamma^+$ are such that $\phi(U) \mid \phi(V)$ in $A_{\Gamma'}^+$, then $U \mid V$.*

Proof. This will follow by induction on the length of V. Note that, since each generator is mapped nontrivially, $\ell(U) \leq \ell(\phi(U))$. If $\ell(V) = 0$ then $\ell(U) = 0$, since $\ell(\phi(U)) \leq \ell(\phi(V)) = 0$, and the lemma holds trivially in this case. Suppose then that U and V both have nonzero lengths and write $U \doteq sU'$ and $V \doteq tV'$ for generators s, t and elements U', V' in A_Γ^+. Now, since $\phi(U) \doteq \phi(s)\phi(U')$ divides $\phi(V)$, both $\phi(s)$ and $\phi(t) \mid \phi(V)$. Hence $[\phi(t), \phi(s)]$ exists and divides $\phi(V)$. Now, since ϕ respects lcm's, it follows that $m_{st} \neq \infty$ and $\langle \phi(t), \phi(s)\rangle^{m_{st}} \mid \phi(V)$. But $\phi(V) \doteq \phi(t)\phi(V')$ and so, by cancellation, $\langle \phi(s), \phi(t)\rangle^{m_{st}-1}$ divides $\phi(V')$. Now, by an induction hypothesis we may suppose that $\langle st\rangle^{m_{st}-1} \mid V'$, and hence $\langle ts\rangle^{m_{st}} \doteq [s, t] \mid V$. Thus we may write $V \doteq sV''$. But by cancellation it follows that $\phi(U') \mid \phi(V'')$ and by induction that $U' \mid V''$ and hence $U \mid V$. $\square$

Theorem 1.3. *If a homomorphism $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ between Artin monoids respects lcm's then it is injective.*

Proof. Suppose that $\phi(U) \doteq \phi(V)$ for some $U, V \in A_\Gamma^+$. Since $\phi(U) \mid \phi(V)$ it follows by Lemma 1.2 that $U \mid V$. Similarly $V \mid U$, and comparing lengths we must have $U \doteq V$. □

We now turn our attention to the question of injectivity of maps between Artin groups. Any homomorphism $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ of Artin monoids naturally induces a homomorphism $\phi_A : A_\Gamma \to A_{\Gamma'}$ between the corresponding Artin groups.

Theorem 1.4. *Suppose that both Γ and Γ' are graphs of finite type. Then, if ϕ is injective, it follows that ϕ_A is an injective map between Artin groups.*

Proof. Apply Proposition 5.5 and Theorem 5.6 of [3]. Namely, if Γ is of finite type then the canonical map $A_\Gamma^+ \to A_\Gamma$ is injective, and every $X \in A_\Gamma$ may be written $X = C^{-1}B$ for $C, B \in A_\Gamma^+$. Suppose then that $\phi_A(X) = 1$ for some $X = C^{-1}B \in A_\Gamma$. That is $\phi_A(B) = \phi_A(C)$ and so, by injectivity of $A_{\Gamma'}^+ \to A_{\Gamma'}$, we have $\phi(B) \doteq \phi(C)$. But then, since ϕ is injective, $B \doteq C$ and so $X = 1$ as required. □

Take any Coxeter system (W, S) with graph Γ. Any subset T of S generates a subgroup W_T of W called a parabolic subgroup, and it is well-known that (W_T, T) is a Coxeter system with Coxeter graph $\Gamma(T)$, the full subgraph of Γ spanned by the vertices of T. The subgroup (resp. submonoid) of A_Γ (resp. A_Γ^+) generated by T is called a special subgroup (resp. submonoid). It follows immediately from Theorem 1.3 that the inclusion of $\Gamma(T)$ into Γ induces an obvious embedding $A_{\Gamma(T)}^+ \hookrightarrow A_\Gamma^+$, with image the special submonoid generated by T. Furthermore, it follows from Theorem 1.4 that, when Γ is of finite type, the special subgroup generated by any subset T of the generators is an Artin group of type $\Gamma(T)$, as first shown by Deligne, [7]. This has also been shown for general Γ by van der Lek [9], which suggests the following question:

Question. Given an Artin monoid homomorphism ϕ which respects lcm's (or is perhaps an LCM-homomorphism – Definition 2.1), is it in general true that the corresponding map ϕ_A is an injective group homomorphism?

Note that it is not even know whether the canonical map $A_\Gamma^+ \to A_\Gamma$ is always injective, and we do not have a solution to the word problem for infinite type Artin groups as we do for the finite type groups.

2. LCM-homomorphisms

We continue with some further notation from §5 of [3]. Let (W, S) be a Coxeter system of type Γ. If there exists a common multiple in A_Γ^+ for some subset T of the generating set S we shall denote the least common multiple by Δ_T, also called the fundamental element for T. Note that the process of rewriting positive words in reverse order induces an anti-isomorphism $\mathrm{rev} : A_\Gamma^+ \to A_\Gamma^+$ which converts (left) divisibility properties into right divisibility properties. It is shown in [3], Lemma 5.1(ii), that (if it exists) $\Delta_T \doteq \mathrm{rev}\Delta_T$ for $T \subseteq S$. Hence the definition of Δ_T is independent of whether we choose to consider left or right divisibility. It is also shown in [3], Theorem 5.6, that the fundamental element Δ_T exists if and only if the parabolic subgroup W_T is finite. (In fact Δ_T corresponds to the longest element of W_T). As usual (W, S) and (W', S') denote Coxeter systems of type Γ and Γ' respectively.

Definition 2.1. A homomorphism $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ between Artin monoids is said to be an *LCM-homomorphism* if the following hold :
(i) ϕ respects lcm's, and
(ii) there exists a collection $\{T(s) \mid s \in S\}$ of mutually disjoint nonempty subsets of S' such that $\Delta_{T(s)}$ exists and $\phi(s) \doteq \Delta_{T(s)}$ for each $s \in S$.

Examples of LCM-homomorphisms may be found in Tables 1 and 2 (see Section 4 for more details), many of which induce already familiar inclusions $W \hookrightarrow W'$ of the corresponding Coxeter groups [1, 11]. We devote the remainder of this section to establishing that the LCM condition is sufficient to ensure that there does exist a corresponding embedding of Coxeter groups. This is also a key step towards the geometric realisation of the map ϕ_A on Artin groups described in Section 3.

An element of an Artin monoid A_Γ^+ is said to be square free if it is not represented by any word which contains a square (s^2 for $s \in S$) as a subword (see also [3], 3.4). Denote by QFA_Γ^+ the set of all square free elements. Note that, by [3] Lemma 5.4, any fundamental element Δ_T is known to be square free. We say that a homomorphism $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ is *QF-injective* if ϕ maps QFA_Γ^+ into $QFA_{\Gamma'}^+$ injectively.

Lemma 2.2. *Suppose that $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ is an LCM-homomorphism. Then ϕ is QF-injective.*

Proof. By Theorem 1.3, ϕ is injective. So it suffices to show by induction on the word length that the image of every square free element U is square free. This is trivially true if $\ell(U) = 0$, and follows from the hypotheses if $\ell(U) = 1$, since then $\phi(U) \doteq \Delta_{T(s)}$ for some $s \in S$.

Take $U \in QFA_\Gamma^+$ with $\ell(U) \geq 2$. Then we may write $U \doteq \langle st\rangle^m V$ where s, t are distinct generators, $m \geq 2$, and we may suppose that neither s nor t divide V. Furthermore $m \leq m_{st}$, since otherwise U fails to be square free. Thus $\phi(\langle st\rangle^m)$

divides $\phi([s,t])$. Now, by the LCM-homomorphism conditions

$$\phi([s,t]) \doteq [\phi(s),\phi(t)] \doteq [\Delta_{T(s)},\Delta_{T(t)}] \doteq \Delta_{T(s)\cup T(t)}$$

which is square free, by [3] Lemma 5.4 again. Hence $\phi(\langle st\rangle^m)$ must be square free. By induction we may suppose also that $\phi(V)$ is square free since V is square free and $\ell(V) < \ell(U)$. It now follows that $\phi(U) \in QFA^+_{\Gamma'}$ is square free.

Suppose by way of contradiction that $\phi(U) \doteq \phi(\langle st\rangle^m)\phi(V)$ is not square free. Since $\phi(V) \in QFA^+_{\Gamma'}$, we may write $\phi(\langle st\rangle^m)$ as AaB ($a \in S'$ a generator) such that $B\phi(V)$ is square free, but $aB\phi(V)$ is not. Then, by Lemma 3.4 of [3], $a \mid B\phi(V)$. Now Lemma 3.1 and 3.2 of [3] together imply that either $a \mid B$ or $b \mid \phi(V)$ for some $b \in T(s)\cup T(t)$. Since $a \mid B$ implies that AaB is not square free, a contradiction, we must have the latter. Suppose then that $b \mid \phi(V)$ and $b \in T(s)$. Now, since $\Delta_{T(s)} \doteq \mathrm{rev}\Delta_{T(s)}$, it follows that b divides $\Delta_{T(s)}$ on the right as well as on the left. Thus $\phi(sV) \doteq \Delta_{T(s)}\phi(V)$ is not square free, and since $\ell(sV) < \ell(U)$ we may assume by induction that sV is not square free either. But V is square free, and so, by Lemma 3.4 of [3] again, $s \mid V$ — a contradiction. Finally, the case $b \in T(t)$ leads to a similar contradiction. ☐

Proposition 2.3. *Suppose that $\phi : A^+_\Gamma \to A^+_{\Gamma'}$ is an LCM-homomorphism. Then ϕ induces (via the canonical maps $A^+_\Gamma \to W$ and $A^+_{\Gamma'} \to W'$) an injective homomorphism $\phi_W : W \to W'$.*

Proof. Recall that one canonically obtains a presentation for W (resp. W') by simply adding the relations $s^2 = 1$, for each generator s, to the standard presentation for A_Γ (resp. $A_{\Gamma'}$). Thus ϕ induces a well defined homomorphism $\phi_W : W \to W'$ simply because, for each generator $s \in S$, $\phi(s^2) \doteq \Delta^2_{T(s)} \doteq \Delta_{T(s)}\mathrm{rev}\Delta_{T(s)}$ which collapses to 1 under the canonical map. By a theorem of Tits ([12] Théorème 3) it is known that the canonical maps $QFA^+_\Gamma \to W$ and $QFA^+_{\Gamma'} \to W'$ are bijections. Clearly then, since ϕ is QF-injective by Lemma 2.2, the induced map ϕ_W is also injective. ☐

3. A geometric realisation of ϕ_A

The following definitions may be found in [5]. Recall that the derived complex of a poset $\mathcal{P}$ is an abstract simplicial complex consisting of the set of finite chains in $\mathcal{P}$ partially ordered by inclusion. Given a Coxeter system (W,S) of type Γ, define the complexes K, Σ and $\tilde{\Sigma}$ to be the geometric realisations of the derived complexes of the following three posets respectively :

$$\begin{aligned}
\mathcal{S}_f &= \{T \subseteq S \mid W_T \text{ is finite}\} = \{T \subseteq S \mid \Delta_T \text{ exists}\} \\
W\mathcal{S}_f &= \{wW_T \mid w \in W\,,\ T \in \mathcal{S}_f\} \\
W &\times \mathcal{S}_f\,.
\end{aligned}$$

The partial orderings, denoted $<$, are defined on $\mathcal{S}_f$ and $W\mathcal{S}_f$ by set inclusion, and on the set $W \times \mathcal{S}_f$ as follows. For $(w_1, T_1), (w_2, T_2) \in W \times \mathcal{S}_f$, we say that $(w_1, T_1) < (w_2, T_2)$ if and only if
(1) $w_1 T_1 < w_2 T_2$ and,
(2) for all $t \in T_1$, $l(w_1^{-1} w_2) < l(t w_1^{-1} w_2)$.
where l denotes the reduced word length of a Coxeter group element. This is precisely the length ℓ of the corresponding element in QFA_Γ^+ under the canonical bijection.

In [5] Σ is called the modified Coxeter complex, and $\tilde{\Sigma}$ the Salvetti complex associated to (W, S). The Salvetti complex was originally introduced in [10]. Note that K may be considered a subcomplex of Σ via the simplicial map induced by the injective map $\mathcal{S}_f \to W\mathcal{S}_f$ such that $T \mapsto W_T$ for $T \in \mathcal{S}_f$, and as such K is a fundamental domain for the obvious (left) action of W on Σ. Also, the group W acts freely on $\tilde{\Sigma}$ (via left multiplication onto the first factor of $W \times \mathcal{S}_f$) with orbit space denoted $Z = \tilde{\Sigma}/W$. In [5], Charney and Davis showed that Z is homotopy equivalent to the W-orbit space of the complex hyperplane complement associated to any representation of W as a real linear reflection group (see also [4]). Thus $\pi_1(Z) \cong A_\Gamma$ and Z is in many cases a $K(A_\Gamma, 1)$ space, and is conjecturally so in general.

Consider now a second Coxeter system (W', S') of type Γ'. We may similarly define posets $\mathcal{S}'_f, W'\mathcal{S}'_f, W' \times \mathcal{S}'_f$ and complexes $K', \Sigma', \tilde{\Sigma}'$ and $Z' = \tilde{\Sigma}'/W'$. Suppose for the remainder of this section that $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ is an LCM-homomorphism. From Proposition 2.3 we know that ϕ canonically induces an injection $\phi_W : W \to W'$. It is reasonable to ask then what sort of maps are induced on the objects which we have just introduced. Recall that the LCM property requires that each element $s \in S$ is associated to a subset $T(s) \subset S'$ such that $\phi(s) = \Delta_{T(s)}$. Define functions

$$
\begin{aligned}
& p : \mathcal{S}_f \to \mathcal{S}'_f \\
& \quad \text{such that } p(T) = \bigcup_{s \in T} T(s) \text{ for each } T \subseteq S\,, \\
& \hat{p} : W\mathcal{S}_f \to W'\mathcal{S}'_f \\
& \quad \text{such that } \hat{p}(wW_T) = \phi_W(w) W'_{p(T)} \text{ for each } w \in W\,,\ T \in \mathcal{S}_f\,, \\
& \tilde{p} : W \times \mathcal{S}_f \to W' \times \mathcal{S}'_f \\
& \quad \text{such that } \tilde{p}(w, T) = (\phi_W(w), p(T)) \text{ for each } (w, T) \in W \times \mathcal{S}_f\,.
\end{aligned}
$$

One can easily check that p is well-defined as follows. Given $T \in \mathcal{S}_f$, observe that $\phi(\Delta_T)$ is a common multiple for the set $p(T)$. Hence $\Delta_{p(T)}$ exists, and so $p(T)$ is an element of $\mathcal{S}'_f$. In the following we denote $\phi_W(w)$ by $\bar{w}$, for $w \in W$.

Lemma 3.1. *For arbitrary $w \in W$ and $T \in \mathcal{S}_f$ we have the following:*
(i) $w \in W_T$ if and only if $\bar{w} \in W'_{p(T)}$, and
(ii) $l(w) < l(tw)$ for all $t \in T$ if and only if $l(\bar{w}) < l(\tau\bar{w})$ for all $\tau \in p(T)$.

Proof. Part (i) of this lemma follows by the injectivity of ϕ_W, from Proposition 2.3, together with the claim that $\phi_W(W_T) = W'_{p(T)} \cap \phi_W(W)$. Clearly $\phi_W(W_T) \subseteq W'_{p(T)} \cap \phi_W(W)$. To show the reverse inclusion, suppose $u \in W \setminus W_T$. We shall use the canonical bijection between W and QFA^+_Γ. Thus u corresponds to an element U of QFA^+_Γ and there exists some $s \in S \setminus T$ such that s appears in every word representing U. Note that, by the nature of the Artin relations, if a particular letter appears in some positive word then it appears in any other word positive equivalent to that one. But each $\tau \in T(s)$, a set disjoint from $p(T)$, appears in at least one, and hence in every word representing $\phi(U)$. Consequently $\phi_W(u) \notin W'_{p(T)}$.

As ϕ_W is induced by the QF-injective map ϕ (Lemma 2.2), we may represent w and $\bar{w}$ by square free elements V and $\bar{V} \doteq \phi(V)$ respectively. It is clear that, for $t \in T$, $l(w) < l(tw)$ if and only if tV is square free, and similarly for $\bar{w}$. Thus it suffices to show, for each fixed $t \in T$, that tV is square free if and only if $\tau\bar{V}$ is square free for every $\tau \in T(t)$. Now, if tV is square free the QF-injectivity of ϕ implies that $\Delta_{T(t)}\bar{V}$ is square free and, since each $\tau \in T(t)$ right divides $\Delta_{T(t)}$, $\tau\bar{V}$ is also square free. On the other hand, if tV is not square free, then $t \mid V$ by Lemma 3.4 of [3], and so $\Delta_{T(t)} \mid \bar{V}$. But then $\tau\bar{V}$ cannot be square free for any $\tau \in T(t)$. □

Lemma 3.2. *Each of p, $\hat{p}$ and $\tilde{p}$ is an injective order preserving map.*

Proof. Note that a map $f : \mathcal{P} \to \mathcal{P}'$ between posets is injective and strictly order preserving if the following, somewhat stronger, condition holds :

P : for all $X_1, X_2 \in \mathcal{P}$, $X_1 \leq X_2$ if and only if $f(X_1) \leq f(X_2)$.

We show in fact that each of p, $\hat{p}$ and $\tilde{p}$ satisfy **P**. Indeed, this is evidently true for p when one considers that the subsets $T(s)$ are mutually disjoint.

Observe that in general $w_1 W_{T_1} \leq w_2 W_{T_2}$ if and only if $T_1 \leq T_2$ and $w_1^{-1} w_2 \in W_{T_2}$. That $\hat{p}$ satisfies **P** then follows by Lemma 3.1(i), with $w = w_1^{-1} w_2$, and the fact that p satisfies **P**. Finally, $\tilde{p}$ satisfies **P** by a similar application of Lemma 3.1(ii) together with the fact that $\hat{p}$ satisfies **P**. □

The consequence of Lemma 3.2 is that p, $\hat{p}$ and $\tilde{p}$ induce injective simplicial maps $P : K \to K'$, $\hat{P} : \Sigma \to \Sigma'$ and $\tilde{\Phi} : \tilde{\Sigma} \to \tilde{\Sigma}'$ respectively. Note that $\tilde{\Phi}$ is a W-equivariant map, and hence induces a well-defined simplicial map on the orbit spaces $\Phi : Z \to Z'$.

Proposition 3.3. *The simplicial map $\Phi : Z \to Z'$ induced by $\tilde{p}$ is injective.*

Proof. Since $\tilde{\Phi}$ is injective it suffices to show that, for $x, y \in \tilde{\Sigma}$ and $w' \in W'$, $\tilde{\Phi}(x) = w'.\tilde{\Phi}(y)$ only if $w' \in \phi_W(W)$. In fact, since $\tilde{\Phi}$ is simplicial and W' acts via simplicial automorphisms, if $\tilde{\Phi}(\tilde{\Sigma})$ and $w'.\tilde{\Phi}(\tilde{\Sigma})$ intersect at all then they share some 0-simplex, or vertex of $\tilde{\Sigma}'$. But then, since vertices of $\tilde{\Phi}(\tilde{\Sigma})$ correspond to elements of $\tilde{p}(W \times \mathcal{S}_f)$ we must have $\tilde{p}(W \times \mathcal{S}_f) \cap w'.\tilde{p}(W \times \mathcal{S}_f) \neq \emptyset$ which implies that $w' \in \phi_W(W)$ as required. □

We may identify the Artin group A_Γ with $\pi_1(Z)$ as follows. Take the point $x_0 \in \tilde{\Sigma}$ corresponding to $(1,\emptyset) \in W \times \mathcal{S}_f$ as a basepoint, and its orbit Wx_0 as a basepoint in Z. Each edge, or 1-simplex, in the simplicial decomposition of $\tilde{\Sigma}$ corresponds to a chain $(w_1,T_1) < (w_2,T_2)$ in the poset $W \times \mathcal{S}_f$, and a simplicial path in $\tilde{\Sigma}$ may be described by a string of such inequalities. Thus write $\{(1,\emptyset) < (1,\{s\}) > (s,\emptyset)\}$ to describe a path γ_s in $\tilde{\Sigma}$ from x_0 to $s.x_0$, for each generator $s \in S$. Let σ_s be the loop at Wx_0 in Z which lifts to γ_s. Now identify elements of A_Γ with elements of $\pi_1(Z)$ via the map ψ which takes each generator $s \in S$ to the class $\psi(s) = [\sigma_s] \in \pi_1(Z)$. That ψ gives explicitly an isomorphism $A_\Gamma \to \pi_1(Z)$ is implicit in the work of Charney and Davis ([4], [5]). One may also check this by an explicit construction of the universal cover of Z in which the union of all lifts of the loops $\{\sigma_s : s \in S\}$ forms the Cayley graph for A_Γ with respect to the standard presentation.

Theorem 3.4. *Let $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ be an LCM-homomorphism and ϕ_A the induced Artin group homomorphism, and let A_Γ and $A_{\Gamma'}$ be identified with $\pi_1(Z)$ and $\pi_1(Z')$ as above. Then there exists an injective simplicial map $\Phi : Z \to Z'$ such that*

$$\Phi_* = \phi_A \ ,$$

where Φ_* denotes the map on fundamental groups induced by Φ.

Proof. By Proposition 3.3 there exists an injective simplicial map Φ induced by $\tilde{p}$. It remains to check that $\Phi_* = \phi_A$.

Let $T \in \mathcal{S}'_f$, and observe that if $s \in T$ and $v \in W'_T$ such that $l(v) < l(vs)$ then $(1,\{s\}) < (v^{-1},T)$, and hence

$$\gamma_s \ = \ \{(1,\emptyset) < (1,\{s\}) > (s,\emptyset)\} \ \sim \ \{(1,\emptyset) < (v^{-1},T) > (s,\emptyset)\} \ ,$$

where $\sim$ denotes homotopy of paths relative to their endpoints. Under the action of W on $\tilde{\Sigma}$, this homotopy of paths is translated by v to a similar homotopy

$$v\gamma_s \ \sim \ \{(v,\emptyset) < (1,T) > (vs,\emptyset)\} \ .$$

Take any word $V = s_1 s_2 ... s_k$, with the s_i necessarily distinct) elements of T, which represents an element of $QFA_{\Gamma'}^+$. Consider elements $v_1 = 1$ and $v_{i+1} = v_i s_i$, $i \leq k$, of the group W'. Then $l(v_i) < l(v_i s_i)$ for each i and, as an Artin group element, V is represented in $\pi_1(Z)$ by a loop which lifts to the path $\gamma_V = \gamma_{s_1}.(v_2\gamma_{s_2})...(v_k\gamma_{s_k})$. But then $\gamma_V \ \sim \ \{(1,\emptyset) < (1,T) > (s_1 s_2 ... s_k,\emptyset)\}$.

In particular, if we take $T = p(\{a\}) = T(a)$ and $V = \phi(a) = \Delta_{T(a)}$ for some $a \in S$, then $\gamma_V \ \sim \ \tilde{\Phi}(\gamma_a)$. Consequently, $\phi_A(a) = \Phi_*(a)$ for each generator $a \in S$. □

Corollary 3.5. *If Γ and Γ' are both of finite type and if $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ is an LCM-homomorphism (viz: Table 1), then the map $\Phi : Z \to Z'$ of Proposition 3.3 above is a π_1-injective inclusion of finite $K(\pi,1)$-complexes.*

Proof. Simply combine Theorem 1.4 with Theorem 3.4. The result of [5] ensures that Z and Z' are both $K(\pi,1)$-complexes in this case. □

4. Constructing LCM-homomorphisms

In this section we describe a mechanism for generating LCM-homomorphisms between Artin monoids via so-called folding operations on the Coxeter graphs. Examples obtained in this way are listed in Table 1 for graphs of finite type, and Table 2 for graphs of affine type. These tables should be read in conjunction with Table 3 which displays the complete list of irreducible* finite type, and affine (or euclidean) type Coxeter graphs (see also [1], Ch.VI, §4.1 and §4.2), together with the Coxeter number for each finite type. The irreducible finite type graphs A_n, B_n, $C_n = D_n$, E_6, E_7, E_8, F_4, $G_6 = I_2(6)$, H_3 ,H_4 and $I_2(h)$ are shown in solid line (ignoring the vertex s_0 and the dashed edge/s adjacent to s_0). Each irreducible affine type $\widetilde{\Gamma}$ may be obtained as the affine extension of a finite type Γ by including the extra vertex s_0 and the dashed edge/s as shown.

As usual let Γ and Γ' be Coxeter graphs with vertex sets S and S' respectively.

Definition 4.1. We define a *folding of* Γ' *onto* Γ to be any surjective simplicial map $f : \Gamma' \to \Gamma$ (where the graphs are considered as 1-dimensional simplicial complexes) which satisfies the following condition. For every edge $\varepsilon = s \overset{m}{\bullet\!\!-\!\!\bullet} t$ in Γ (by "edge" we really mean a subgraph of dihedral type $I_2(m)$, $m \geq 3$), the restriction f_ε of f to $f^{-1}(\varepsilon)$ is of one of the types (A)–(D) listed below.

(A) For any positive integer k define the Coxeter graph $k.\varepsilon$ to be the disjoint union of k copies of ε, each with the same label m. Then the *k-fold trivial* folding $f_\varepsilon : k.\varepsilon \to \varepsilon$ maps each copy identically onto ε.

(B) Take any irreducible finite type Coxeter graph K with Coxeter number $m > 3$. Choose a partition $K_s \cup K_t$ of the vertices of K such that each of K_s and K_t represents a set of mutually commuting generators in the Artin monoid, or more precisely, such that there is no edge between any pair of vertices in K_s (resp. in K_t). Then the *dihedral* folding $f_\varepsilon : K \to \varepsilon$ is the unique simplicial map such that $f(K_s) = s$ and $f(K_t) = t$. Note that one may always choose such a partition, and that it is unique up to a relabelling of the two sets.

(C) Fix $m = 3$ and consider the graph A_4 with vertices $\{s_1, s_2, s_3, s_4\}$ as in Table 3. Define the `I`-*type* folding $f_\varepsilon : A_4 \to \varepsilon$ with *special vertex* t such that $f^{-1}(s) = \{s_1, s_4\}$ and $f^{-1}(t)$ is the whole edge spanned by $\{s_2, s_3\}$.

* If Γ is a disjoint union $\Gamma_a \cup \Gamma_b$ of graphs, then W, A_Γ and A^+_Γ all decompose accordingly as direct products. We say that Γ is *irreducible* if there is no such direct product decomposition, or equivalently if Γ is a connected graph.

(D) Define a *composite folding* $f_\varepsilon = f'_\varepsilon \circ g$ where f'_ε is a k-fold trivial folding onto ε, and g is a folding onto $k.\varepsilon$ which restricts to types (A), (B) and (C) only.

Note: If we take $m = 3$ in (B) then it reduces to the trivial case (A), since A_2 is the only finite type with Coxeter number 3.

We say that f is a *locally trivial* folding if every restriction f_ε is of the trivial type (A). In this case f is a genuine covering of spaces. For example, the k-fold locally trivial foldings of $\widetilde{A}_{kn}$ onto $\widetilde{A}_n$ of Table 2(i) correspond topologically to the k-fold coverings of the circle by itself. Otherwise we may think of a folding as a "branched" covering with various kinds of singular behaviour of types (B), (C) and (D).

Suppose that one attempts to define an LCM-homomorphism $\phi : A_\Gamma^+ \to A_{\Gamma'}^+$ by first setting $\phi(s) = \Delta_{T(s)}$, where $\{T(s) : s \in S\}$ is a collection of nonempty disjoint subsets of S', as required by Definition 2.1. Then, in order to check that ϕ extends to a well-defined homomorphism $A_\Gamma^+ \to A_{\Gamma'}^+$ and furthermore to check that this homomorphism respects lcm's, it suffices to check that

$$\langle \Delta_{T(s)}, \Delta_{T(t)} \rangle^{m_{st}} \doteqdot \langle \Delta_{T(t)}, \Delta_{T(s)} \rangle^{m_{st}} \doteqdot [\Delta_{T(s)}, \Delta_{T(t)}] \qquad \mathbf{(4.2)}$$

for each pair $s, t \in S$ such that $m_{st} \neq \infty$, and that $\Delta_{T(s)}$ and $\Delta_{T(t)}$ have no common multiple whenever $m_{st} = \infty$. In other words, it suffices to check the restriction of ϕ to each rank two special submonoid. Moreover, note that in general one has $[\Delta_{T(s)}, \Delta_{T(t)}] \doteqdot \Delta_{T(s) \cup T(t)}$.

Proposition 4.3. *Every folding $f : \Gamma' \to \Gamma$ induces an LCM-homomorphism $\phi^f : A_\Gamma^+ \hookrightarrow A_{\Gamma'}^+$ such that $\phi^f(s) \doteqdot \Delta_{f^{-1}(s)}$ for each $s \in S$.* *

Proof. By the preceding remarks it suffices to check condition (4.2) above for every pair of vertices $s, t \in S$. If there is no edge between s and t (that is s and t commute in A_Γ^+) then, since f is simplicial, there is no edge between any $s' \in f^{-1}(s)$ and $t' \in f^{-1}(t)$ (that is s' and t' commute in $A_{\Gamma'}^+$). It follows that $\Delta_{f^{-1}(s)}$ commutes with $\Delta_{f^{-1}(t)}$ as required.

This leaves only the cases of restrictions f_ε of type (A)–(D). For type (A), the k-fold trivial folding, it is easily checked that, for $m \neq \infty$,

$$\begin{aligned} \langle s_1 s_2 .. s_k, t_1 t_2 .. t_k \rangle^m &\doteqdot \langle s_1, t_1 \rangle^m ... \langle s_k, t_k \rangle^m \\ &\doteqdot \langle t_1, s_1 \rangle^m ... \langle t_k, s_k \rangle^m \doteqdot \langle t_1 t_2 .. t_k, s_1 s_2 .. s_k \rangle^m \end{aligned}$$

where s_i, t_i denote the vertices of the i^{th} copy of ε in $k.\varepsilon$. That this is a least common multiple is also clear since the $\langle s_i, t_i \rangle^m$ are pairwise commuting elements each of which must divide $[s_1 s_2 .. s_k, t_1 t_2 .. t_k]$. When $m = \infty$ there is clearly no

* Strictly speaking $f^{-1}(s)$ is a full subgraph of Γ' with vertex set $T(s)$ say. By an abuse of notation we write $\Delta_{f^{-1}(s)}$ to mean $\Delta_{T(s)}$. Similarly Δ_{A_4} denotes $\Delta_{\mathrm{vert}\, A_4}$.

Table 1. $\phi_A : A_\Gamma \hookrightarrow A_{\Gamma'}$ for finite types.

	Γ	Γ'	Folding description
(i)	$I_2(h)$	$\Gamma(h)$	Dihedral folding
(ii)	B_n	D_{n+1}, A_{2n-1}	$(A_3, \pm\varepsilon)$
(iii)	B_n	A_{2n}	$(\mathtt{I}, s_n)$
(iv)	F_4	E_6	(A_3, ε)
(v)	H_4	E_8	(A_4, ε)
	H_3	D_6	(A_4, ε)

common multiple between $s_1 s_2..s_k$ and $t_1 t_2..t_k$ since none exists between s_1 and t_1 for a start.

For type (B), $[\Delta_{K_s}, \Delta_{K_t}]$ is just the fundamental element $\Delta \doteq \Delta_{K_s \cup K_t}$ and, by Lemma 5.8 of [3], we have $\Delta \doteq \langle \Delta_{K_s}, \Delta_{K_t} \rangle^m \doteq \langle \Delta_{K_t}, \Delta_{K_s} \rangle^m$, which is exactly what is required.

When f_ε is of type (C) we have $\Delta_{f_\varepsilon^{-1}(s)} \doteq s_1 s_4$ while $\Delta_{f_\varepsilon^{-1}(t)} \doteq s_2 s_3 s_2 \doteq s_3 s_2 s_3$. One can easily check ($\leq \frac{1}{2}$-page calculation) that

$$\langle s_1 s_4, s_2 s_3 s_2 \rangle^4 \doteq \Delta_{A_4} \doteq \mathrm{rev}\Delta_{A_4} \doteq \langle s_2 s_3 s_2, s_4 s_1 \rangle^4 ,$$

as required. Note that A_4 has Coxeter number 5 and so $\Delta_{A_4} \doteq \langle s_1 s_3, s_2 s_4 \rangle^5$.

Finally, a restiction f_ε as in (D) satisfies the required condition by much the same argument as used for type (A), but with each s_i (resp. t_i) replaced with a $\Delta_{g^{-1}(s_i)}$ (resp. $\Delta_{g^{-1}(t_i)}$). □

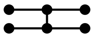
One may regard Definition 4.1 and the accompanying Proposition 4.3 purely as a mechanism for describing examples of LCM-homomorphisms. The most fundamental of these, Table 1(i), arise from the dihedral foldings of Definition 4.1(B). Let $\Gamma(h)$ denote a connected Coxeter graph of finite type with Coxeter number h. Then the dihedral folding of $\Gamma(h)$ onto $I_2(h)$ defines an embedding of the dihedral type Artin monoid, Artin group or Coxeter group into the corresponding object of type $\Gamma(h)$.

Notation (K, ε)**.** If a folding $f : \Gamma' \to \Gamma$ restricts over a given edge ε in Γ to a dihedral folding $f_\varepsilon : K \to \varepsilon$ and is trivial elswhere, then we shall call f a (K, ε)-*folding*. Note that the restriction f_ε depends on the choice of labelling the partition of K, or equivalently on the orientation of ε. When this choice leads to distinct foldings onto the same Γ, we distinguish them as $(K, +\varepsilon)$- and $(K, -\varepsilon)$-foldings.

For example, B_n has a single edge ε with label 4, and there are distinct (A_3, ε)- and $(A_3, -\varepsilon)$-foldings of D_{n+1} and A_{2n-1} respectively onto B_n. Furthermore, since the graph B_n is a tree, or simply-connected (as are all the irreducible finite and affine types other than $\widetilde{A}_n$) then the trivial parts of these foldings are uniquely determined. Thus these are the unique $(A_3, \pm\varepsilon)$-foldings onto B_n.

Due to the symmetry in F_4 there is only one (A_3, ε)-folding onto F_4, namely that of E_6, Table 1(iv). On the other hand it is the symmetric partition of A_4 which makes for unique (A_4, ε)-foldings of E_8 onto H_4 and D_6 onto H_3 (Table 1(v)). Since A_3 and A_4 are the only irreducible types, other than the dihedral types, with Coxeter numbers 4 and 5 respectively, it is clear that these are all the possible (K, ε)-foldings onto irreducible finite type graphs.

Notation $(\mathtt{I}, s_*)$**.** If a folding $f : \Gamma' \to \Gamma$ restricts over a given edge ε in Γ to an $\mathtt{I}$-type folding $f_\varepsilon : A_4 \to \varepsilon$ with special vertex s_*, and is trivial over every edge not adjacent to s_*, then we shall call f an $(\mathtt{I}, s_*)$*-folding*.

Note that since $f^{-1}(s_*) = \bullet\!\!-\!\!\bullet$ is "nontrivial" f must restrict to an $\mathtt{I}$-type folding over every edge adjacent to s_*. An example where there are two such edges is the $(\mathtt{I}, s_*)$-folding of $\widetilde{D}_5 =$  onto $\widetilde{C}_2 = \bullet\overset{4}{-\!\!-}\bullet\overset{4}{-\!\!-}\bullet$ given in Table 2(vi). Amongst the irreducible finite types the only possible $(\mathtt{I}, s_*)$-foldings are those of A_{2n} onto B_n, Table 1(iii).

Remark. At this stage it is easily checked that Table 1 lists all of the LCM-homomorphisms which may be obtained from foldings of irreducible finite type Coxeter graphs. In particular, these foldings all define injective maps between the corresponding Artin groups, by virtue of Theorem 1.4. Note also that all of the (K, ε)-foldings in Table 1 (which include the dihedral foldings) preserve the Coxeter number of the graphs.

We turn now to the LCM-homomorphisms of Table 2. Although we have not proved injectivity of the induced map on Artin groups in these cases, the geometric realisation of this map given by Theorem 3.5 suggests an alternative approach to this question. However, one may need to settle the conjecture that the simplicial complex Z defined in [5] is indeed a $K(A_\Gamma, 1)$-space when Γ is an affine type.

As has already been pointed out, Table 2(i) gives an example of the locally trivial (but "nontrivial") foldings that may arise when Γ is not simply-connected. Just as for B_n we have the three $(A_3, \pm\varepsilon)$- and $(\mathtt{I}, s_n)$-foldings onto $\widetilde{B}_n$ of Table 2(ii), and similarly for $\widetilde{C}_n$, Table 2(iii). Since $\widetilde{C}_n$ has two edges ε and ε' with label 4, it is possible for a folding f onto $\widetilde{C}_n$ to have both restrictions f_ε and $f_{\varepsilon'}$ nontrivial. In this way one gets one of the four further foldings of Table 2(iv) and (v), depending on whether f_ε and $f_{\varepsilon'}$ are dihedral or $\mathtt{I}$-type respectively. Table 2(vi)–(viii) give some further isolated examples. Note that, other than the dihedral type $I_2(6) = G_2$, there are exactly three types B_3, D_4 and A_5 with Coxeter number 6. Thus we get the six foldings onto $\widetilde{G}_2$ of Table 2(viii).

Note that one may obtain the last four of these foldings by composing the $(B_3, \pm\varepsilon)$-foldings onto $\widetilde{B}_3$ and $\widetilde{F}_4$ respectively with $(A_3, \pm\varepsilon)$-foldings. Accordingly, these LCM-homomorphisms are composed from other LCM-homomorphisms as for example $A^{\pm}_{\widetilde{G}_2} \hookrightarrow A^{\pm}_{\widetilde{F}_4} \hookrightarrow A^{\pm}_{\widetilde{E}_6}$. In the same vein, the map of Table 2(v) may be ob-

Table 2. $\phi : A^+_\Gamma \hookrightarrow A^+_{\Gamma'}$ for affine types.

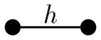
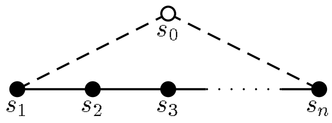
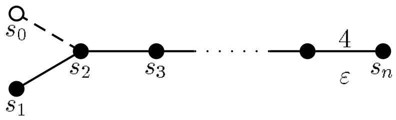
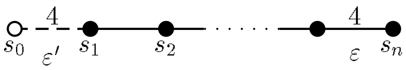
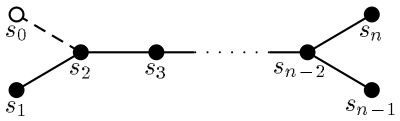
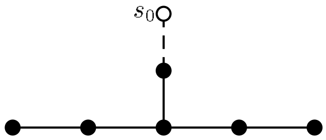
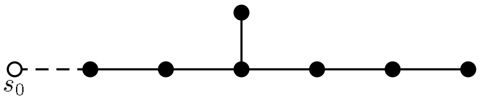
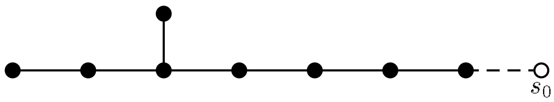
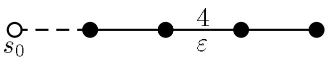
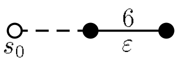
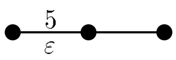
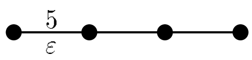
	Γ	Γ'	Folding description
(i)	$\widetilde{A}_n$	$\widetilde{A}_{kn}$	Locally trivial
(ii)	$\widetilde{B}_n$	$\widetilde{D}_{n+1}$, $\widetilde{D}_{2n}$	$(A_3, \pm\varepsilon)$
		$\widetilde{D}_{2n+1}$	$(\mathtt{I}, s_n)$
(iii)	$\widetilde{C}_n$	$\widetilde{B}_{n+1}$, $\widetilde{C}_{2n}$	$(A_3, \pm\varepsilon)$
		$\widetilde{C}_{2n+1}$	$(\mathtt{I}, s_n)$
(iv)	$\widetilde{C}_n$	$\widetilde{A}_{2n-1}$	(A_3, ε) & (A_3, ε')
		$\widetilde{A}_{2n}$	(A_3, ε) & $(\mathtt{I}, s_0)$
		$\widetilde{A}_{2n+1}$	$(\mathtt{I}, s_n)$ & $(\mathtt{I}, s_0)$
(v)	$\widetilde{C}_n$	$\widetilde{D}_{n+2}$	$(A_3, -\varepsilon)$ & $(A_3, -\varepsilon')$
(vi)	$\widetilde{C}_2$	$\widetilde{D}_5$	(T, s_1)
(vii)	$\widetilde{F}_4$	$\widetilde{E}_6$, $\widetilde{E}_7$	$(A_3, \pm\varepsilon)$
(viii)	$\widetilde{G}_2$	$\widetilde{B}_3$, $\widetilde{F}_4$	$(B_3, \pm\varepsilon)$
		$\widetilde{D}_4$, $\widetilde{E}_6$	$(D_4, \pm\varepsilon)$
		$\widetilde{D}_6$, $\widetilde{E}_7$	$(A_5, \pm\varepsilon)$

tained by composing $A^+_{\widetilde{C}_n} \hookrightarrow A^+_{\widetilde{B}_{n+1}} \hookrightarrow A^+_{\widetilde{D}_{n+2}}$. Further LCM-homomorphisms not already listed may also be obtained by composition of known ones, as for example $A^+_{\widetilde{C}_n} \hookrightarrow A^+_{\widetilde{C}_{2n+1}} \hookrightarrow A^+_{\widetilde{C}_{4n+2}}$. Including the restrictions of type (D) into Definition 4.1 ensures that examples such as this are also obtained via legitimate foldings, and hence are LCM-homomorphisms. In fact, the class of LCM-homomorphisms is closed under composition. This is a consequence of the following observation.

Proposition 4.4. *Let $\phi : A_\Gamma \to A_{\Gamma'}$ be an LCM-homomorphism, $T = \{t_1, t_2, .., t_k\}$ a subset of S, and $T' = T(t_1) \cup ... \cup T(t_k) \subseteq S'$ with the $T(t_i)$ as in Definition 2.1(ii). Then if Δ_T exists $\phi(\Delta_T) \doteq \Delta_{T'}$.*

Proof. Equivalently, $\phi([t_1, .., t_k]) \doteq [\Delta_{T(t_1)}, .., \Delta_{T(t_k)}]$. To see that this is true, note that the least common multiple of the t_i's may be computed using the algorithm of [3], section 3.3. Since this depends only on the relationships (namely (4.2)) already established amongst the $\Delta_{T(t_i)}$, the same formal procedure may be used (with $\Delta_{T(t_i)}$ replacing t_i) to compute the least common multiple of the $\Delta_{T(t_i)}$'s. □

Table 3. Coxeter graphs Γ of finite type and their affine extensions $\widetilde{\Gamma}$.

			Coxeter no, h
$I_2(h)$:		$(h \geq 3)$	
A_n ; $\widetilde{A}_n$:		$(n \geq 2)$	$n+1$
B_n ; $\widetilde{B}_n$:		$(n \geq 3)$	$2n$
C_n ; $\widetilde{C}_n$:		$(n \geq 2)$	$2n$
D_n ; $\widetilde{D}_n$:		$(n \geq 4)$	$2n-2$
E_6 ; $\widetilde{E}_6$:			12
E_7 ; $\widetilde{E}_7$:			18
E_8 ; $\widetilde{E}_8$:			30
F_4 ; $\widetilde{F}_4$:			12
G_2 ; $\widetilde{G}_2$:			6
H_3 : 	H_4 :		10 , 30

Finally note that we could widen the definition of a folding to allow for non-trivial foldings over edges with label ∞. This might give further examples, but none to add to the lists given here. However, with this widened definition, it may be reasonable to ask whether every LCM-homomorphism ϕ is essentially obtained from a folding of Coxeter graphs. By "essentially" we mean under the obvious assumption that the image of ϕ is not contained in any proper special subgroup of the target group.

5. The annular braid group, A_{B_n}

We conclude by giving an example of an injective Artin group homomorphism induced by a homomorphism between Artin monoids which does not respect lcm's and clearly is not an LCM-homomorphism. This example nevertheless has an interesting, but altogether quite different geometric interpretation. It is well known that A_{A_n} is canonically isomorphic to the $(n+1)$-string braid group $\mathcal{B}_{n+1}$. denote by $\mathcal{P}_{n+1}^{n+1}$ the subgroup of $\mathcal{B}_{n+1}$ consisting of braids which permute only the first n strings, also called $(n+1)$-pure braids. We shall prove that A_{B_n} is isomorphic to this finite index subgroup of $\mathcal{B}_{n+1}$. This curious fact has been noticed by others. Notably, Sofia Lambropoulou [8] has used it to derive invariants for links in a solid torus via a trace function on the union of all B_n-type Hecke algebras.

Brieskorn [2] showed that the fundamental group of the regular orbit space of a finite real reflection group acting on complex space is just the corresponding Artin group of finite type. In particular A_{B_n} is the fundamental group of the orbit space of the following complex hyperplane complement

$$X_{B_n} \;=\; \{\, (x_1, x_2, .., x_n) \in \mathbf{C}^n \mid x_i \neq \pm x_j \text{ for } i \neq j\,;\; x_i \neq 0 \text{ for all } i \,\}\,,$$

under the action of $W_{B_n} = S_n \ltimes (\mathbf{Z}/2\mathbf{Z})^n$, acting as the group of signed permutations on the standard basis vectors for $\mathbf{C}^n$ (and their negatives). That is $A_{B_n} \cong \pi_1(X_{B_n}/W_{B_n})$. Also we have $A_{A_n} \cong \pi_1(X_{A_n}/W_{A_n})$ where

$$X_{A_n} \;=\; \{\, (\hat{x}_1, \hat{x}_2, .., \hat{x}_{n+1}) \in \mathbf{C}^{n+1} \mid \sum_{i=1}^{n+1} \hat{x}_i = 0\,;\; \hat{x}_i \neq \hat{x}_j \text{ for } i \neq j \,\}\,,$$

and $W_{A_n} = S_{n+1}$ acts by permutations of the standard basis for $\mathbf{C}^{n+1}$. Fixing a basepoint $*$ in X_{A_n}/W_{A_n}, loops at $*$ lift uniquely to paths $\hat{x}(t)$ from a fixed basepoint in X_{A_n} which determine braids in $\mathbf{C}\times I$ (with i^{th} string the path $(\hat{x}_i(t), t)$ in $\mathbf{C}\times I$). This describes an isomorphism $\pi_1(X_{A_n}/W_{A_n}) \cong \mathcal{B}_{n+1}$.

Proposition 5.1. *The group $A_{B_n} \cong \pi_1(X_{B_n}/W_{B_n})$ is isomorphic to the subgroup of index $n+1$ in $\pi_1(X_{A_n}/W_{A_n})$ corresponding to the subgroup $\mathcal{P}_{n+1}^{n+1}$ of $\mathcal{B}_{n+1}$.*

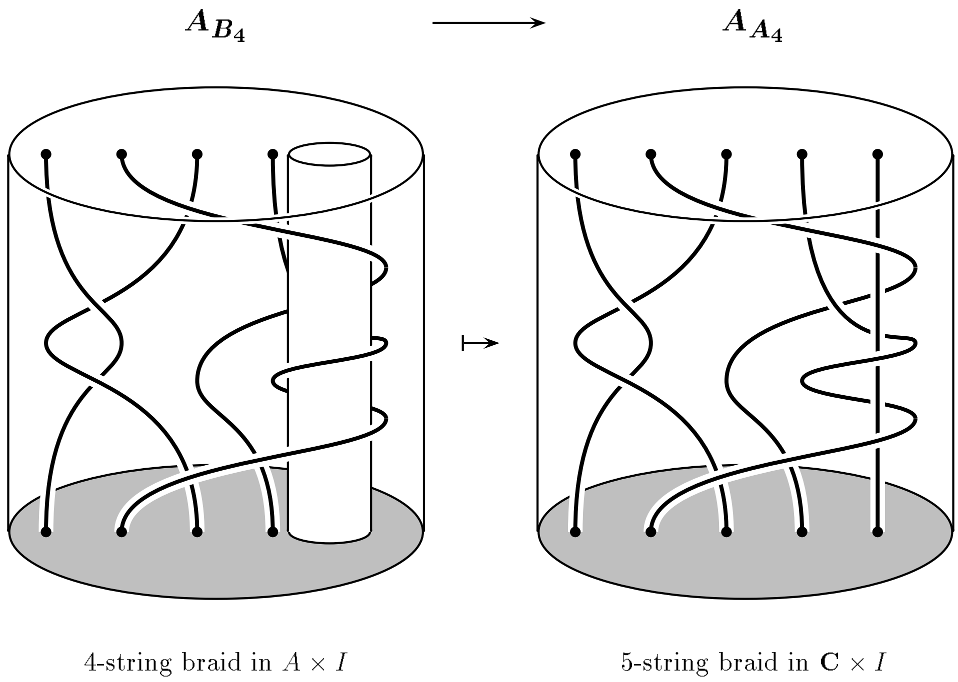


Figure 1. The 4-string annular braid group, A_{B_4}.

Proof. Since $(\mathbf{Z}/2\mathbf{Z})^n$ is a normal subgroup of W_{B_n}, the quotient group S_n is the full group of covering automorphisms for the intermediate covering space

$$\begin{aligned} Y_{B_n} &= X_{B_n}/(\mathbf{Z}/2\mathbf{Z})^n \\ &\cong \{\, (x_1, .., x_n) \in \mathbf{C}^n \mid x_i \neq x_j \text{ for } i \neq j \,;\, x_i \neq 0 \text{ for all } i \,\} \,, \end{aligned}$$

where S_n acts by permutations of the standard basis for $\mathbf{C}^n$. Thus $X_{B_n}/W_{B_n} \cong Y_{B_n}/S_n$. Now consider S_n acting on X_{A_n} as the subgroup of $W_{A_n} = S_{n+1}$ which stabilises the $(n+1)^{\mathrm{st}}$ basis vector (that is, by permutations on the first n basis vectors). With this convention we find that Y_{B_n} and X_{A_n} are S_n-equivariantly homeomorphic via the linear map $h : Y_{B_n} \to X_{A_n}$ such that $h(x_1, x_2, .., x_n) = (x_1 + \eta, x_2 + \eta, .., x_n + \eta, \eta)$ where $\eta = \frac{-1}{(n+1)} \sum\limits_{i+1}^{n} x_i$. The inverse map is given by $h^{-1}(\hat{x}_1, .., \hat{x}_n, \hat{x}_{n+1}) = (\hat{x}_1 - \hat{x}_{n+1}, .., \hat{x}_n - \hat{x}_{n+1})$. Consequently,

$$X_{B_n}/W_{B_n} \;\cong\; Y_{B_n}/S_n \;\cong\; X_{A_n}/S_n \;,$$

and A_{B_n} is isomorphic to the fundamental group of any of these spaces. Now, since X_{A_n}/S_n is an $(n+1)$-fold covering space of X_{A_n}/W_{A_n} (S_n having index $n+1$ in W_{A_n}) we have that $\pi_1(X_{A_n}/S_n)$ is a subgroup of index $n+1$ in $\pi_1(X_{A_n}/W_{A_n})$. Note that the only loops in X_{A_n}/W_{A_n} which lift to loops in X_{A_n}/S_n are precisely those which correspond to $(n+1)$-pure braids. Thus, this subgroup corresponds to the subgroup $\mathcal{P}_{n+1}^{n+1}$ of $\mathcal{B}_{n+1}$. □

It is indeed apparent from the description of Y_{B_n} that $A_{B_n} \cong \pi_1(Y_{B_n}/S_n)$ is the group of n-string braids in $\mathbf{C} \setminus \{0\} \times I$ (or $A \times I$ where A denotes an open annulus), and one sees that applying the linear isomorphism h corresponds to attaching a new string along $\{0\} \times I$ to obtain an $(n+1)$-string ($(n+1)$-pure) braid, as suggested by Figure 1.

Finally, one may check that the injective homomorphism $\phi_A : A_{B_n} \to A_{A_n}$ implicit in the above argument is exactly the map such that $\phi_A(s_i) = a_i$ for $i = 1, .., n-1$, and $\phi_A(s_n) = a_n^2$, where $\{s_1, .., s_n\}$ and $\{a_1, .., a_n\}$ are standard sets of generators for A_{B_n} and A_{A_n} respectively, and hence the associated monoid homomorphism ϕ is obviously not LCM, and fails to respect lcm's since $[a_{n-1}, a_n^2] \doteq a_n^2 a_{n-1} a_n$.

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