ON THE GEOMETRIES OF NON-ORIENTABLE FIBRE-BUNDLES WITH BASE AND FIBRE EUCLIDEAN SURFACES

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Abstract. We show that closed 4-manifolds which fibre over the torus or Klein bottle with fibre a torus or Klein bottle are geometric in the sense of Thurston.

Introduction. The purpose of this paper is to show that those 4-manifolds M that are total-spaces of F-bundles over S, where F and S are closed surfaces of genus 1, are geometric in the sense of Thurston. Ue [Ue1] has shown this to be true in the instances where M is orientable. Below we complete the above task by showing that those non-orientable bundle-spaces are also geometric. We provide a listing of all such manifolds M up to bundle-diffeomorphism type, and display an explicit embedding of the fundamental group $\Pi_1(M)$ into the isometry group of some standard model, endowing M with a geometric structure.

This is achieved by primarily noting [Oh] that $\Pi_1(M)$ largely determines the diffeomorphism type of such 4-manifolds. After generating a listing of group presentations of all possible $\Pi_1(M)$, the determination of bundle-diffeomorphism classes of M is reduced to that of group extension isomorphism classes of the associated short exact sequences of the given fibrations. Thenceforth we provide a faithful and discrete representation Φ of $\Pi_1(M)$ into the isometry group of one of the 4-dimensional geometries $X =: E^4,\ Nil^3 \times E,\ Sol^3 \times E$ or Nil^4; the quotient of X by the image of Φ yields a compact manifold diffeomorphic to M.

Nomenclature. Throughout, the symbols F and S denote closed surfaces of genus 1. Here we take the fundamental groups of such surfaces to be each generated by an ordered pair of loops $\Pi_1(F) = \langle \gamma, \delta \rangle$, $\Pi_1(S) = \langle \alpha, \beta \rangle$ together with the single relator:

$$\wr \alpha, \beta \wr_S = \begin{cases} [\alpha, \beta], & \text{if } S \text{ is the torus } T^2; \\ \alpha\beta\alpha^{-1}\beta, & \text{if } S \text{ is the Klein bottle } Kb. \end{cases}$$

Let M be the total space of an F-bundle over S defined by the projection $p\colon M \to S$. Since S is aspherical, the long exact homotopy sequence for a fibration provides the short exact sequence:

$$1 \longrightarrow \Pi_1(F) \xrightarrow{i_*} \Pi_1(M) \xrightarrow{p_*} \Pi_1(S) \longrightarrow 1$$

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where i_* is induced by the inclusion of the fibre, and p_* is induced by the bundle projection. Viewing this sequence as a group extension the action of $\Pi_1(S)$ on $\Pi_1(F)$ determines a group homomorphism $\vartheta\colon \Pi_1(S) \to \mathrm{Out}(\Pi_1(F))$. Below we obtain a presentation for $\Pi_1(M)$ in terms of the images γ, δ of the generators of $\Pi_1(F)$ under i_*, and a fixed chosen pair α, β that project onto the generators of $\Pi_1(S)$. Abusing notation the elements of this set $\{\alpha, \beta, \gamma, \delta\}$ are refered to as the *canonical generators* of $\Pi_1(M)$.

Definition. Two fibre-bundles $p\colon M \to S$ and $p'\colon M' \to S'$ are *bundle-diffeomorphic* when there is a pair of diffeomorphisms $\phi_1\colon M \to M'$ and $\phi_2\colon S \to S'$ such that $\phi_2 p = p'\phi_1$. A bundle-diffeomorphism is a *bundle-diffeomorphism over* S when $S = S'$ and the mapping $\phi_2 = \mathrm{id}_S$.

Definition. Two group extensions $N \overset{\iota}{\hookrightarrow} E \overset{\rho}{\twoheadrightarrow} G$ and $N' \overset{\iota'}{\hookrightarrow} E' \overset{\rho'}{\twoheadrightarrow} G'$ are *isomorphic* when there is a pair of group isomorphisms $\phi_1\colon E \to E'$ and $\phi_2\colon G \to G'$ such that $\phi_2\rho = \rho'\phi_1$. An extension isomorphism is an *extension-equivalence* when the pair $(N, G) = (N', G')$, the mapping $\phi_2 = \mathrm{id}_G$ and the restriction of ϕ_1 to the image of i is the identity.

The Classification of $\Pi_1(M)$

The projection p is locally trivial; hence we may choose a small closed disk $D \subset S$ so that the restriction of p to $p^{-1}(\mathrm{Int}(D))$ is just the product projection. Now denoting $M \setminus p^{-1}(\mathrm{Int}(D))$ by M_0 we may write M as the adjunction space $M_0 \cup_h (F \times D)$ where $h\colon \partial(F \times D) \to \partial(M_0)$ is a gluing homeomorphism of $S^1 \times F$, whose restriction to S^1 is the identity. In this way h defines a continuous loop of homeomorphisms of F and hence may be viewed as an element of $\Pi_1(\mathrm{Homeo}(F))$.

We now examine the F-bundle that is p restricted to M_0. The base space $S \setminus \mathrm{Int}(D)$ is homotopy equivalent to $S^1 \vee S^1$ and hence from the long homotopy sequence for a fibration we have:

$$1 \longrightarrow \Pi_1(F) \longrightarrow \Pi_1(M_0) \longrightarrow \Pi_1(S^1 \vee S^1) \longrightarrow 1.$$

Here $\Pi_1(S^1 \vee S^1)$ is the free product $\langle\alpha\rangle * \langle\beta\rangle$ so that this sequence splits and the group $\Pi_1(M_0)$ is isomorphic to the semi-direct product $\Pi_1(F) \rtimes_\theta (\langle\alpha\rangle * \langle\beta\rangle)$. The action $\theta\colon \langle\alpha\rangle * \langle\beta\rangle \to \mathrm{Out}(\Pi_1(F))$ is the group homomorphism defined by $\theta = \vartheta\varrho_*$, where ϱ_* is induced by the inclusion $S^1 \vee S^1 \overset{\varrho}{\hookrightarrow} S$. Note in particular that the word $\wr\alpha, \beta\wr_S \in (\langle\alpha\rangle * \langle\beta\rangle)$ is in the kernel of θ.

We now compute $\Pi_1(M)$ by applying Van Kampen's theorem to $M = M_0 \cup_{F \times S^1} (F \times D)$. Writing $\eta = \wr\alpha, \beta\wr_S$ we obtain a commutative diagram:

$$\begin{tikzcd}
\Pi_1(\partial(M_0)) \arrow[rr, hook] & & \Pi_1(F) \rtimes_\theta (\langle\alpha\rangle * \langle\beta\rangle) \arrow[drr, "\mathrm{id} \rtimes \varrho_*"] & & \\
\Pi_1(F) \times \langle\eta\rangle \arrow[u, "h_*", "\wr"'] \arrow[urr, hook, dashed, "\iota"] \arrow[drr, "\mathrm{id} \times 0"'] & & & & \Pi_1(M) \\
& & \Pi_1(F) \times \langle 1\rangle \arrow[urr, hook, "i_*"'] & &
\end{tikzcd}$$

Here the monomorphism i_* is the familiar map induced by the inclusion of the fibre; and the mapping ι is the composite of h_* and the canonical inclusion. Thus writing $j_* = (\mathrm{id} \rtimes \varrho_*) \circ \iota$ we have that

$$\Pi_1(M) \cong \left(\Pi_1(F) \rtimes_\theta (\langle\alpha\rangle * \langle\beta\rangle)\right) / \langle j_*((1,\eta))\rangle.$$

Here the composite map j_* is determined by h_*, the map induced by h. Moreover the map ι factors through the cartesian product:

$$\begin{tikzcd}
\Pi_1(F) \times \langle\eta\rangle \arrow[r, dashed, "f"] \arrow[dr, hook, "\iota"'] & \Pi_1(F) \times \langle\alpha\rangle * \langle\beta\rangle \arrow[d] \\
& \Pi_1(F) \rtimes_\theta (\langle\alpha\rangle * \langle\beta\rangle)
\end{tikzcd}$$

where the mapping f precomposed with the projection on to the second factor, has image an infinite cyclic group generated by $\wr\alpha,\beta\wr_S \in (\langle\alpha\rangle * \langle\beta\rangle)$. Furthermore the action of $\wr\alpha,\beta\wr_S$ on $\Pi_1(F)$ is trivial; and thus $j_*((1,\eta)) = \gamma^a\delta^b \wr \alpha,\beta\wr_S$. Therefore writing $\delta^{-b}\gamma^{-a} = \gamma^m\delta^n$ we obtain the presentation:

$$\Pi_1(M) = \left\langle \alpha,\beta,\gamma,\delta \;\middle|\; \begin{array}{ll} \wr\gamma,\delta\wr_F = 1, & \alpha(\gamma,\delta)\alpha^{-1} = \vartheta(\alpha)(\gamma,\delta), \\ \wr\alpha,\beta\wr_S = \gamma^m\delta^n, & \beta(\gamma,\delta)\beta^{-1} = \vartheta(\beta)(\gamma,\delta). \end{array} \right\rangle.$$

The pair $\binom{m}{n} \in \mathbb{Z}^2$ represent the obstruction to constructing a cross-section to p and is refered to as the *Euler number* of the above presentation. Henceforth this group presentation shall be abreviated by the triple $(\vartheta(\alpha), \vartheta(\beta); \binom{m}{n})$ known as the *characteristic triple* representing $\Pi_1(M)$ with respect to the given presentation.

The question of which particular Euler numbers correspond to the fundamental group of such a bundle-space with given action ϑ is answered by considering the first homeotopy group of the fibre: $\Pi_1(\mathrm{Homeo}(F))$.

In the case where $F = T^2$, we have that $\Pi_1(\mathrm{Homeo}(T^2)) \cong \mathbb{Z}^2$ [Ha1]; in such a way that every obstruction $\binom{m}{n} \in \mathbb{Z}^2$ is realized by some gluing homeomorphism h as above [FS]. Moreover every action ϑ can be realized as the monodromy of such a bundle-projection. Thus as the total-space M is aspherical, the set $\mathrm{Hom}\,(\Pi_1(S), \mathrm{Out}(\Pi_1(F)) \times \mathbb{Z}^2$ modulo transformations induced by changes in the base and and fibre groups characterizes the homotopy type of all T^2-bundle-spaces over S. However when the fibre is $F = Kb$, many such $\binom{m}{n} \in \mathbb{Z}^2$ do not define torsion-free groups. In particular $\Pi_1(\mathrm{Homeo}(Kb)) \cong \mathbb{Z}$ [Ha2]. Thus as above the homotopy type of Kb-bundles over S is characterized by the set $\mathrm{Hom}\,(\Pi_1(S), \mathrm{Out}(\Pi_1(F)) \times \mathbb{Z}$ modulo transformations induced by automorphisms of the base and fibre groups. In light of the consequences of Dehn-Nielsen Realization Theorem [ZVC] we shall be able to determine how $\mathbb{Z}$ embeds in $\mathbb{Z}^2$ and hence determine the possible Euler numbers associated with each given action ϑ that give rise to bona fide bundle-space groups.

Although we are interested in the bundle-diffeomorphism type of M it is interesting to know to what extent $\Pi_1(M)$ determines the diffeomorphism type of M.

THE DIFFEOMORPHISM TYPE OF M

By exploiting the S^1-bundle over S^1 structure of the base-space S, thereby viewing such 4-manifolds as bundle-spaces over S^1 with fibre some small Seifert manifold, Ohba [Oh] has shown for the following pairs (F, S): (T^2, T^2), (Kb, T^2) and (Kb, Kb) the homotopy type classes of M are in bijection with the diffeomorphism type classes of M. (The result for the case $(F, S) = (T^2, T^2)$ was originally proved by [F,S] through different means.) Thus in these cases two non-bundle-diffeomorphic total-spaces cannot have the same characteristic triple.

In the remaining case where the pair $(F, S) = (T^2, Kb)$, such a bundle ξ has a two-fold covering $\eth^*(\xi)$ that is a T^2-bundle over T^2 induced by the orientation double covering $\eth: T^2 \twoheadrightarrow Kb$. Therefore from above, the fundamental group of the total space $\widetilde{M}$ of the bundle $\eth^*(\xi)$ determines the diffeomorphism type of $\widetilde{M}$. Of course any one T^2-bundle over T^2 may cover many non-bundle-diffeomorphic (or even non-diffeomorphic) T^2-bundle-spaces over Kb. In any case the techniques to follow will guarantee that the characteristic triple provides a classification of bundle-diffeomorphism type, as this invariant not only describes the isomorphism class of $\Pi_1(M)$ but also the induced epimorphism $p_*: \Pi_1(M) \to \Pi_1(S)$.

THE BUNDLE-DIFFEOMORPHISM TYPE OF M

A bundle-diffeomorphism (ϕ_1, ϕ_2) induces a triple of group automorphisms defining an isomorphism of the corresponding homotopy sequences:

$$\begin{tikzcd}
\Pi_1(F) \arrow[r, hook] \arrow[d, "\phi_0"', "\wr"] & \Pi_1(M) \arrow[r, two heads, "p_*"] \arrow[d, "(\phi_1)_*"', "\wr"] & \Pi_1(S) \arrow[d, "(\phi_2)_*"', "\wr"] \\
\Pi_1(F') \arrow[r, hook] & \Pi_1(M') \arrow[r, two heads, "p'_*"] & \Pi_1(S').
\end{tikzcd}$$

In fact even more is true, as is outlined in the following three propositions:

Proposition A. *Let ξ be an F-bundle with projection $p: M \to S$. Then for any isomorphism $\psi_S: \Pi_1(S') \to \Pi_1(S)$ there is a bundle-diffeomorphism:*

$$\begin{tikzcd}
M' \arrow[r, two heads, "p'"] \arrow[d, "\hat\psi_M"', "\wr"] & S' \arrow[d, "\hat\psi_S"', "\wr"] \\
M \arrow[r, two heads, "p"] & S
\end{tikzcd}$$

for some F-bundle $p': M' \to S'$; where $\hat\psi_S$ is a diffeomorphism realizing ψ_S. If $(\vartheta(\alpha), \vartheta(\beta); \binom{m}{n})$ is the characteristic triple of p then $(\vartheta(\psi_S(\alpha)), \vartheta(\psi_S(\beta)); \binom{m'}{n'})$ is the characteristic triple of p', where the vector $\binom{m'}{n'}$ corresponds to the word $\psi_S(\wr\alpha, \beta\wr_S) = \gamma^{m'}\delta^{n'}$.

proof. By the Dehn-Nielsen Theorem there is a diffeomorphism $\hat\psi_S: S' \to S$ realizing ψ_S. Define $p': M' \to S'$ to be the bundle $(\hat\psi_S)^*(\xi)$, induced by $\hat\psi_S$.

□

Proposition B. *Let ξ be an F-bundle with projection $p: M \to S$. Then for any isomorphism $\psi_F: \Pi_1(F) \to \Pi_1(F')$ there is an bundle-diffeomorphism over S:*

$$\begin{tikzcd} M \arrow[dd, "\hat{\psi}_M"', "\wr"] \arrow[dr, "p"] & \\ & S \\ M' \arrow[ur, "p'"] & \end{tikzcd}$$

for some F-bundle $p': M' \to S$; where the restriction of $\hat{\psi}_M$ to the fibre over the basepoint of S is a diffeomorphism realizing ψ_F. If $(\vartheta(\alpha), \vartheta(\beta); \binom{m}{n})$ is the characteristic triple of p then $(\kappa\vartheta(\alpha)\kappa^{-1}, \kappa\vartheta(\beta)\kappa^{-1}; \binom{m'}{n'})$ is the characteristic triple of p', where κ represents the restriction of $(\hat{\psi}_M)_*$ to $\ker(p_*)$ in $\mathrm{Out}(\Pi_1(F))$; and $\binom{m'}{n'}$ is the vector defined by the word $\kappa(\gamma^m\delta^n) = \gamma^{m'}\delta^{n'}$.

proof. Let $\hat{\psi}_F: F \to F'$ be a diffeomorphism realizing ψ_F and consider the mapping $\phi: F \times \mathfrak{F}_S \overset{\hat{\phi}_F \times \mathrm{id}}{\longrightarrow} F' \times \mathfrak{F}_S$. This induces a bundle-diffeomorphism $\widetilde{\phi}: M(0) \to M(0)'$ where the bundle-space $M(0)'$ is represented by the triple $(\kappa\vartheta(\alpha)\kappa^{-1}, \kappa\vartheta(\beta)\kappa^{-1}; \binom{0}{0})$, with $\kappa \in \mathrm{Out}(\Pi_1(F))$ as above. Then writing $M = M(0) \setminus p_0^{-1}(\mathrm{Int}(D)) \cup_h D \times F$ we may construct M' as the adjunction space $M(0)' \setminus (p_0')^{-1}(\mathrm{Int}(D)) \cup_{h'} D \times F'$; where the gluing map $h': (p_0')^{-1}(\partial D) \to D \times F$ is defined by $h' = \widetilde{\phi} \circ h \circ \widetilde{\phi}^{-1}$. Whence a bundle-diffeomorphism is given by:

$$\hat{\psi}_M = \begin{cases} \widetilde{\phi}, & \text{on } M(0) \setminus p_0^{-1}(\mathrm{Int}(D)), \\ \mathrm{id} \times \hat{\psi}_F, & \text{on } D \times F. \end{cases}$$

□

Proposition C. *Let ξ be an F-bundle over S. Then for any extension equivalence $\psi: \Pi_1(M) \to \Pi_1(M')$ of the associated exact sequence for ξ, there is a bundle-diffeomorphism $\hat{\psi}: M \tilde{\to} M'$ over S realizing ψ, for some F-bundle $p: M' \to S$. If $(\vartheta(\alpha), \vartheta(\beta); \binom{m}{n})$ is the characteristic triple of p then $(\vartheta(\alpha), \vartheta(\beta); \binom{m'}{n'})$ is the characteristic triple of p' where the vector $\binom{m'}{n'}$ is defined by the word $\wr\upsilon(\alpha), \upsilon(\beta)\wr_S = \gamma^{m'}\delta^{n'}$ and υ is the set-theoretic cross-section to p determined by ψ and the canonical generators of $\Pi_1(M)$.*

proof. Let $D \subset S$ be a small disk that lifts to the product $D \times F$ under p. Then $S \setminus D$ retracts to $S^1 \vee S^1$ and therefore the restricted bundle projection $M \setminus p^{-1}(\mathrm{Int}(D)) \to S \setminus D$ admits a cross-section. The base-point $*$ of the fibre F determines a such a section $\sigma: S \setminus D \hookrightarrow M \setminus p^{-1}(\mathrm{Int}(D))$.

Extention equivalences correspond to a change $\varsigma \mapsto \varsigma'$ of set-theoretic cross-section $\hat{\varsigma}: \Pi_1(S) \to \Pi_1(M)$ to p_*; which in turn correspond to mappings υ of the free group on the generators of $\Pi_1(S)$ into $\Pi_1(M \setminus p^{-1}(\mathrm{Int}(D)))$. Any such a mapping υ is realized by a cross-section of the restricted bundle-projection as given above. By the homogeniety of F we may change its base-point via some diffeomorphism $(F,*)\xrightarrow{\cong}(F,*')$ that is isotopic to the identity. As for proposition B this self-diffeomorphism defines a diffeomorphism ϕ on $F\times\mathfrak{F}_S$ which induces a diffeomorphism $\hat{\phi}$ between M and $M(0)'\setminus(p_0')^{-1}(\mathrm{Int}(D))\cup_{h'}D\times F'$ where $h'=\widetilde{\phi}h\widetilde{\phi}^{-1}$, giving the desired bundle-diffeomorphism. □

In view of the above three propositions and the preceding comment we are lead to the crucial theorem inspiring our proposed classification.

Theorem. *Two F-bundles over S are bundle-diffeomorphic precisely when their associated exact sequences are isomorphic.*

Remark. This theorem was essentially proved in [Hi] chapter IV where it is shown that two surface-bundles over a surface with aspherical base and fibre are homotopy equivalent precisely when the associated group extensions of the bundles are isomorphic.

Proof. Let ξ' and ξ be two F-bundles over S with a given extension isomorphism (ϕ_0,ϕ_1,ϕ_2) between their associated homotopy sequences. By proposition A with $\phi_2\in\mathrm{Aut}(\Pi_1(S))$ there exists a bundle-diffeomorphism $\xi''\xrightarrow{\cong}\xi$ realizing the corresponding extension isomorphism $(\mathrm{id},\widetilde{\phi}_2,\phi_2)$ where $\widetilde{\phi}_2$ is the pullback along p_* of ϕ_2.

Now applying proposition B to the bundle ξ'' with $\phi_0\in\mathrm{Aut}(\Pi_1(F))$ we obtain a bundle ξ''' that is bundle-diffeomorphic to ξ'' with associated extension isomorphism $(\phi_0,\widetilde{\phi}_0,\mathrm{id})$. In which case the associated sequence of ξ''' is isomorphic to the associated sequence of ξ via the isomorphism $(\phi_0,\widetilde{\phi}_0\widetilde{\phi}_2,\phi_2)$. Therefore we have an extension equivalence between the associated exact sequences of ξ' and ξ''' given by the isomorphism $\phi_1^{-1}\widetilde{\phi}_0\widetilde{\phi}_2\colon\Pi_1(M''')\xrightarrow{\cong}\Pi_1(M')$.

Hence by proposition C the bundles ξ' and ξ''' are bundle-diffeomorphic. Thus by transitivity of the bundle-diffeomorphism relation it follows that ξ is bundle-diffeomorphic to ξ'. □

As an aid in attaining bundle-diffeomorphism classes of such fibre bundles we will make use of the following well known theorem about group extensions:

Theorem. *Let Ξ be the group extension:*

$$1\longrightarrow N\longrightarrow E\longrightarrow G\longrightarrow 1$$

with action $\vartheta\colon\to\mathrm{Out}(N)$. Then equivalence classes of Ξ are in bijection with the elements of $H^2_\kappa(G;\mathcal{Z}(N))$, where the center of N, $\mathcal{Z}(N)$ is regarded as a G-module via the action $\kappa\colon G\to\mathrm{Aut}(\mathcal{Z}(N))$ induced by ϑ.

In particular it is well known that extensions with distinct actions *vartheta* are non-equivalent. Furthermore the question of existence of an extension Ξ with given action ϑ is decided by examination of a correspondence between the homomorphism ϑ and an element of $H^3_\kappa(G;\mathcal{Z}(N))$. Such an extension Ξ is realised with respect to the action ϑ precisely when ϑ corresponds to the zero element of $H^3_\kappa(G;\mathcal{Z}(N))$. In our case the groups in question G all have cohomological dimension 2 and thus all possible actions ϑ are realized. Thus bundle-diffeomorphism classes *a priori* correspond to some quotient of $H^2_\kappa(\Pi_1(S);\mathcal{Z}(\Pi_1(F)))$; namely by those transformations on the base and fibre, as extension equivalences correspond to a change of set-theoretic cross-section to the group surjection p_*. We refer the reader unfamiliar with group cohomology to [Ro] and [Br].

We calculate the cardinality of extension equivalence classes as follows. By duality we have that

$$H^2_\kappa(\Pi_1(S);\mathcal{Z}(\Pi_1(F))) \cong H_0^{w_1(S)\otimes\kappa}(\Pi_1(S);\mathbb{Z}\otimes_{\mathbb{Z}}\mathcal{Z}(\Pi_1(F))).$$

Where the action of $\Pi_1(S)$ on the orientation module $\mathbb{Z}$ is via the orientation character $w_1(S)\colon \Pi_1(S) \to \{\pm 1\}$. Thus writing $\Pi_1(Kb) = \langle \gamma, \delta\rangle$ we have that $\mathcal{Z}(\Pi_1(Kb)) = \langle \gamma^2\rangle \cong \mathbb{Z}$ so that:

$$H^2_\kappa(\Pi_1(S);\mathcal{Z}(\Pi_1(F))) \cong \begin{cases} \mathbb{Z}^2/\langle \kappa(g) - w_1(S)(g)\rangle, & \text{if } F = T^2\text{; or} \\ \mathbb{Z}/\langle \kappa(g) - w_1(S)(g)\rangle, & \text{if } F = Kb. \end{cases}$$

where the elements g range over $\Pi_1(S)$.

As we now possess the machinery to classify such bundle-spaces up to bundle-diffeomorphism, we proceed as proclaimed.

The Classification of Non-Orientable Total Spaces of T^2-Bundles over T^2 and their Geometries

The classification of T^2-bundles with orientable total spaces together with a given geometric structure was achieved by Ue [Ue]. Below in the spirit of [Ue] we complete the classification of T^2−bundles over T^2 to intially obtain a list of such bundle-spaces that are non-orientable. We then proceed to show that all of these 4-manifolds are geometric by explicitly giving faithful, discrete representations from the fundamental group into the appropriate isometry group. By writing $M = (A, B; \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right))$ we are refering to the bundle-space with fundamental group

$$\Pi_1(M) = \left\langle \alpha,\beta,l,h \,\middle|\, \begin{matrix} [l,h]=1, & \alpha(l,h)\alpha^{-1} = (l,h)A, \\ [\alpha,\beta] = l^a h^b, & \beta(l,h)\beta^{-1} = (l,h)B. \end{matrix} \right\rangle;$$

where A and B are matrices in $\mathrm{Aut}(\Pi_1(T^2)) = \mathrm{GL}(2,\mathbb{Z})$. The results are captured in the following:

Theorem. *Let M be the non-orientable total space of a T^2-bundle over T^2 with first Betti number $\beta_1(M)$. Then M is bundle diffeomorphic to one of the geometric 4-manifolds listed below:*

	Total Space	*Geometric Type*
$\beta_1(M) = 3$	$(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right),\ I; \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))$	E^4
	$(\left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ I; \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right))$	E^4

$\beta_1(M) = 2$	$\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, -I; \begin{pmatrix} 0 \\ (1) \end{pmatrix}\right)$	E^4
	$\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, -I; \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$	E^4
	$\left(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, -I; \begin{pmatrix} 0 \\ (1) \end{pmatrix}\right)$	E^4
	$\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ I; \begin{pmatrix} (1) \\ n \end{pmatrix}\right)_{n \in \mathbb{N} \setminus \{0\}}$	$Nil^3 \times E$
	$\left(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \ I; \begin{pmatrix} 0 \\ n \end{pmatrix}\right)_{n \in \mathbb{N} \setminus \{0\}}$	$Nil^3 \times E$
	$\left(A, \pm I; \begin{pmatrix} a \\ b \end{pmatrix}, \det(A) = -1, \operatorname{rank}(A - I) = 2\right)$	$Sol^3 \times E$

TABLE 1.

Remark. Each characteristic triple listed gives a distinct bundle-space with the exception of the infinite family of $Sol^3 \times E$-manifolds. In this case one may obtain further refinement of bundle-space representation by observing that there is a bijection between similarity classes of such a matrix A and the class number of the ring $\mathbb{Z}[\zeta]$; where ζ is an eigenvalue of A. In particular there are finitely similarity classes corresponding to each pair of eigenvalues ζ, ζ^{-1} of A. For discussion of such matters we refer the reader to [Ne].

For future reference we include a listing [Ue1] of all such orientable bundle-spaces.

Theorem. *Let M be the orientable total space of a T^2-bundle over T^2 with first Betti number $\beta_1(M)$. Then M is bundle diffeomorphic to one of the geometric 4-manifolds listed below:*

	Total Space	*Geometric Type*
$\beta_1(M) = 4$	$\left(I, \ I; \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$	E^4
$\beta_1(M) = 3$	$\left(I, \ I; \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$	E^4
$\beta_1(M) = 2$	$\left(\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \ I; \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$	E^4
	$\left(\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, -I; \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$	E^4
	$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ I; \begin{pmatrix} 0 \\ (1) \end{pmatrix}\right)$	E^4
	$\left(-I, \ I; \begin{pmatrix} (1) \\ 0 \end{pmatrix}\right)$	E^4
	$\left(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \ I; \begin{pmatrix} a \\ b \end{pmatrix}\right)_{\lambda, b \neq 0}$	Nil^4
	$\left(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, -I; \begin{pmatrix} (1) \\ (1) \end{pmatrix}\right)_{\lambda \neq 0}$	$Nil^3 \times E$
	$\left(\begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}, \ I; \begin{pmatrix} (1) \\ (1) \end{pmatrix}\right)_{\lambda \neq 0}$	$Nil^3 \times E$
	$\left(A, \pm I; \begin{pmatrix} a \\ b \end{pmatrix}, \det(A) = 1, \lvert\operatorname{trace}(A)\rvert > 2\right)$	$Sol^3 \times E$

TABLE 2.

Note. Some of the representations as they appear in Table 2. have been slightly altered from their original form in [Ue1] so as to conform to our conventions.

The following three classes of fibre preserving diffeomorphims correspond to the respective transformations of the algebraic representation $(A, B; \binom{a}{b})$ of the bundle space M:

(1) Change of framing of the fibre

$$(l, h) \mapsto (l, h)P, \quad \exists P \in \mathrm{GL}(2, \mathbb{Z}).$$
$$(A, B; \binom{a}{b}) \mapsto (P^{-1}AP, P^{-1}BP; P^{-1}\binom{a}{b}).$$

(2) Change of base of the base space

$$(\alpha, \beta) \mapsto (\alpha, \beta)P, \quad \exists P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}).$$
$$(A, B; \binom{a}{b}) \mapsto (A^pB^r, A^qB^s; \det(P)\binom{a}{b}).$$

(3) Change of cross-section

$$(\alpha, \beta) \mapsto (l^sh^t\alpha, l^ph^q\beta), \quad \text{where } t, s, p, q \in \mathbb{Z} \text{ are free.}$$
$$(A, B; \binom{a}{b}) \mapsto (A, B; \binom{a}{b} + (A^{-1} - I)\binom{p}{q} + (I - B^{-1})\binom{s}{t}).$$

A constructive realization of the above bundle isomorphisms from the algebraic transformations may be found in [FS].

In this case equivalence classes of triples modulo transformations of type (3) are in bijection with elements of $H^2((Z)^2; \mathbb{Z}^2) \cong \mathbb{Z}^{\not\models}/\langle A - I, B - I\rangle$.

The bundle-space M is orientable precisely when $\det(A) = 1 = \det(B)$. Below we shall only consider the case $\det(A) = -1 = -\det(B)$. For if $\det(A) = 1$ and $\det(B) = -1$ we may interchange A and B while if $\det(A) = -1 = \det(B)$ we may replace B by AB. On examination of the presentation for $H_1(M)$ we see that $\beta_1(M) = 4 - \operatorname{rank}(A - I, B - I, \binom{a}{b})$. Furthermore $\beta_1(M) = 4$ if and only if $M \cong T^4 = T^2 \times T^2$. We proceed by exhausting all possible cases for the conjugacy classes of representation $(A, B; \binom{a}{b})$ with respect to possible values of $\beta_1(M)$. In doing this we will make frequent implicit use of the following lemma. The symbol (1) is used to denote a matrix entry that is either 1 or 0.

Lemma. *The conjugacy class of a matrix A in $\mathrm{GL}(2, \mathbb{Z})$ is determined as follows:*

(1) *If* $\operatorname{rank}(A - I) = 0$ *then* $A = I$.

(2) *If* $\operatorname{rank}(A - I) = 1$ *then A is conjugate to either* $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}_{\lambda \neq 0}$ *if* $\det(A) = 1$ *or precisely one of the two matrices* $\begin{pmatrix} -1 & (1) \\ 0 & 1 \end{pmatrix}$ *if* $\det(A) = -1$.

(3) *Let* $\operatorname{rank}(A - I) = 2$ *and* $\det(A) = 1$.

  a) If $|\operatorname{trace}(A)| < 2$ *then A is conjugate to precisely one of the three matrices* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

b) If $|\mathrm{trace}(A)| = 2$ *then* $A = \begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}$.

c) If $|\mathrm{trace}(A)| > 2$ *then* A *has eigenvalues* $\zeta, 1/\zeta$ *for some* $\zeta \in \mathbb{R} \setminus \mathbb{Q}$.

(4) *If* $\mathrm{rank}(A - I) = 2$ *and* $\det(A) = -1$ *then* A *has eigenvalues* $\zeta, -1/\zeta$ *for some* $\zeta \in \mathbb{R} \setminus \mathbb{Q}$.

proof:
The first claim is clear. A proof of (2) may be found in [FS]. The proof of (3)a) and b) comes from examining the possible characteristic polynomials $\lambda^2 - \mathrm{trace}(A)\lambda + \det(A)$ of such matrices A and recognising that these polynomials are in fact class functions. The proofs of (3)c) and (4) follow quickly from inspection of the possible characteristic polynomials of such matrices. □

As a consequence of the following theorem due to Wall [Wa] our task: to prove that each characteristic triple appearing in table 1 represents a distinct bundle-diffeomorphism class; is reduced to the problem of distinguishing between only those representations admitting common geometries.

Theorem. *Two closed* 4*-manifolds admitting distinct geometries are not homotopy equivalent.*

Henceforth in initialy proving that Table 1. is a complete listing we assume the given characteristic triples admit the geometries as claimed.

Proof of the theorem:
From our convention that $\det(A) = -1 = -\det(B)$ we must have that $A \neq I$ so that $\mathrm{rank}(A - I)$ is either 1 or 2.

A) The case $\beta_1(M) = 3 \Longrightarrow \mathrm{rank}(A - I, B - I; \binom{a}{b}) = 1$**.**

Thus $\mathrm{rank}(A - I) = 1$ and so $A \sim \begin{pmatrix} -1 & (1) \\ 0 & 1 \end{pmatrix}$ and we must have $b = 0$. In this case $AB = BA$ and $\det(B) = 1$ together imply $B = \pm I$. But $\mathrm{rank}(B - I)$ cannot exceed 1 so we must have $B = I$. Under an appropriate change of cross-section we may transform $\binom{a}{0}$ to $\binom{0}{0}$ or $\binom{(1)}{0}$ depending upon whether the value of the entry (1) in A is respectively either 1 or 0. That is we obtain the three representations:

$$\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, I; \begin{pmatrix} (1) \\ 0 \end{pmatrix}\right), \qquad \left(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, I; \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

However there is a group isomorphism between $\left(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, I; \binom{0}{0}\right)$ and $\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, I; \binom{1}{0}\right)$ fixing the generators α and l and mapping β to h^{-1}. Hence following [Oh], these two representations represent two fibrations with diffeomorphic total spaces. But since $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is not conjugate to $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ any such diffeomorphism cannot be realized by a bundle diffeomorphism. Thus, as bundle spaces these representations are distinct. The remaining two groups are distinguished by their abelianizations and hence represent distinct manifolds.

B) The case $\beta_1(M) = 2 \Longrightarrow \mathrm{rank}(A - I, B - I; \binom{a}{b}) = 2$**.**

B_1) $\mathrm{rank}(A - I) = 1 \Longrightarrow A \sim \begin{pmatrix} -1 & (1) \\ 0 & 1 \end{pmatrix}$ and as before we must have $B = \pm I$.

If $B = I$ then $\mathrm{rank}(B - I) = 0$ and we must have $b \neq 0$. As before we may transform $\binom{a}{b}$ by an appropriate change of cross-section to either $\binom{0}{b}$ or $\binom{(1)}{b}$ depending upon whether the value of the entry (1) in A is respectively either 1 or 0. Furthermore conjugating A and B in the first instance by $-I$ and in the second instance by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we may assume that b is positive. That is we obtain the three infinite families of representations:

$$(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, I; \begin{pmatrix} (1) \\ n \end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}, \qquad (\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, I; \begin{pmatrix} 0 \\ n \end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}.$$

After fixing the two pairs (A, B) as above, the above families of triples correspond to group extension equivalence classes since

$$H^2(\mathbb{Z}^2;\mathbb{Z}^2) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}, & \text{for } (A, B = (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), I).; \\ \mathbb{Z}, & \text{for } (A, B = (\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}), I). \end{cases}$$

Abelianization of these representations shows they are mutualy non-isomorphic and hence represent distinct manifolds.
If $B = -I$ then under an appropriate change of cross-section we may assume $\binom{a}{b} = \binom{(1)}{(1)}$ or $\binom{0}{(1)}$ depending on whether the respective value of the entry (1) in A is 0 or 1. That is we obtain the following six representations:

$$(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, -I; \begin{pmatrix} (1) \\ (1) \end{pmatrix}), \quad (\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, -I; \begin{pmatrix} 0 \\ (1) \end{pmatrix}).$$

Note that

$$H^2(\mathbb{Z}^2;\mathbb{Z}^2) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}, & \text{for } (A, B = (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), -I); \\ \mathbb{Z}/2\mathbb{Z}, & \text{for } (A, B = (\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}), -I); \end{cases}$$

so that these triples are seen to capture all equivalence classes of the possible group extensions with the action (A, B) fixed as above. However we may transform $((\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), -I; \binom{0}{1})$ to $((\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), -I; \binom{1}{0})$ by replacing A by AB and then conjugating by $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$; yielding an extra extension isomorphism that is not an equivalence. The remaining five representations are distinct and may be distinguished as follows: Let

$$M_1 = ((\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), -I; \begin{pmatrix} 0 \\ 0 \end{pmatrix}),\ M_2 = ((\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), -I; \begin{pmatrix} 0 \\ 1 \end{pmatrix}),\ M_3 = ((\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), -I; \begin{pmatrix} 1 \\ 1 \end{pmatrix}),$$
$$M_4 = ((\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}), -I; \begin{pmatrix} 0 \\ 0 \end{pmatrix}),\ M_5 = ((\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}), -I; \begin{pmatrix} 0 \\ 1 \end{pmatrix}).$$

Then abelianizing we have:

$$\begin{aligned} H_1(M_1) &= \mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2, \\ H_1(M_2) &= \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} = H_1(M_3) = H_1(M_4), \\ H_1(M_5) &= \mathbb{Z}^2; \end{aligned}$$

distinguishing M_1 and M_5. Furthermore the commutator subgroups $\Pi_1(M_2)' \cong \mathbb{Z}^2 \cong \Pi_1(M_4)'$ and $\Pi_1(M_3)' \cong \mathbb{Z}$ are characteristic and hence distinguish M_3.

Finally the bundle-spaces M_2 and M_4 must be distinct as the centers of their fundamental groups are non-isomorphic. In particular $\mathcal{Z}(\Pi_1(M_2)) = \langle \beta^2 \rangle \cong \mathbb{Z}$, whilst $\mathcal{Z}(\Pi_1(M)_4) = \langle \alpha^2, \beta^2 \rangle \cong \mathbb{Z}^2$.

B$_{12}$) $\operatorname{rank}(B - I) = 1 \Longrightarrow B \sim \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}_{\lambda \neq 0}$. But as before, the relation $[A, B] = I$ gives $\lambda = 0$ and this case does not represent a bundle-space.

B$_2$) $\operatorname{rank}(A - I) = 2 \Longrightarrow A$ has eigenvalues $\zeta, -1/\zeta \in \mathbb{R} \setminus \mathbb{Q}$.

B$_{21}$) $\operatorname{rank}(B - I) = 0 \Longrightarrow B = I$ and we obtain the infinite family of manifolds:

$$(A, I; \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), \det(A) = -1, \operatorname{rank}(A - I) = 2)$$

B$_{22}$) $\operatorname{rank}(B - I) = 1 \Longrightarrow B \sim \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}_{\lambda \neq 0}$. Writing $B = I + N$ the relation $AB = BA$ implies that A commutes with N, contradicting $\det(A) = -1$. Thus this representation does not represent a bundle-space.

B$_{23}$) $\operatorname{rank}(B - I) = 2$. If $|\operatorname{trace}(B)| < 2$ then B is conjugate to one of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. An easy calculation together with the inequality $x^2 + y^2 \geq |xy|$, $\forall x, y \in \mathbb{Z}$ shows that the centralizer of any one of the above matricies in $\mathrm{GL}(2, \mathbb{Z})$ only contains matrices of positive determinant; contradicting $\det(A) = -1$. If $|\operatorname{trace}(B)| = 2$ then $B \sim \left(\begin{smallmatrix} -1 & \lambda \\ 0 & -1 \end{smallmatrix}\right)$. Writing $B = -I + N$ the relation $AB = BA$ yields a contradiction as in the case B$_{22}$) unless $\lambda = 0$ so that we obtain the infinite family of manifolds:

$$(A, -I; \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), \det(A) = -1, \operatorname{rank}(A - I) = 2)$$

Remark. Since $H^2(\mathbb{Z}^2; \mathbb{Z}^2)$ is some quotient of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ there are at most four such bundle-spaces correspondind to each choice of A.

If $|\operatorname{trace}(B)| > 2$ then by Lemma 3. [FS] the pair (A, B) is equivalent to a pair $(C, \pm I)$ where the eigenvalues of C are some q^{th} roots of the respective eigenvalues of A and hence are irrational. That is we obtain the infinite family of manifolds already covered in case B$_{21}$ and this case above.

And we have shown that Table 1. is a complete list of all non-orientable 4-manifolds that arise as total spaces of T^2-bundles over T^2 up to bundle diffeomorphism. Below we show that they admit the geometries as claimed above.

The isometry group $Isom(E^4)$ of flat 4-space is $\mathbb{R}^4 \rtimes O(4)$ where the action of $O(4)$ on $\mathbb{R}^4$ is the standard left matrix multiplication.

Lemma. *A representation $\Phi: \Pi_1(M) \to Isom(E^4)$ is faithful and discrete with cocompact image precisely when the restriction of Φ to some normal subgroup $\mathcal{A} \lhd \Pi_1(M)$ of finite index is faithful and discrete with cocompact image.*

Remark. If indeed M admits an E^4-geometry we are guaranteed of the existence of such a subgroup $\mathcal{A}$. For if $\Pi_1(M)$ embeds discretely into $\mathbb{R}^4 \rtimes O(4)$ via Φ with compact quotient, then the image $\mathcal{A}$ of $\Phi(\Pi_1(M))$ in the translation group $\mathbb{R}^4$ must be discrete and cocompact; and thus free abelian of rank 4. Moreover the quotient $\mathbb{R}^4/\mathcal{A} \simeq T^4$ is a compact manifold that finitely covers M; and therefore the index of $\mathcal{A}$ in $\Pi_1(M)$ is finite. In fact flat n-manifold groups (also known as *Bieberbach groups*) are characterized by being torsion free and virtually $\mathbb{Z}^n$. See for example [Ch].

proof:

It is clear that if $\Pi_1(M)$ embeds faithfully and discretely into $Isom(E^4)$ then so does $\mathcal{A}$. To see that cocompactness of $\Phi(\Pi_1(M))$ implies that $\Phi(\mathcal{A})$ is cocompact note we have an $n-$fold covering space projection $E^4/\Phi(\mathcal{A}) \xrightarrow{p} E^4/\Phi(\Pi_1(M))$; where n is the index of $\mathcal{A}$ in $\Pi_1(M)$. Take an open covering of the total space and view each non-regular set in this cover as a union of regular sets. Then since p is a local homeomorphism and hence an open map, any open covering of the total space projects onto an open covering of the base space. Whence we may pass to a regular finite open covering which may be lifted to obtain a finite open regular covering of $E^4/\Phi(\mathcal{A})$. Now since p is a local homeomorphism we may form appropriate unions of these regular sets to obtain a finite subcovering of the original covering.

Below we prove the reverse implication. Since M is the total space of a surface-bundle over a surface $S \neq \mathbb{RP}^2$ with fibre $F \neq \mathbb{RP}^2$, $\Pi_1(M)$ is torsion free. To see this suppose $g \in \Pi_1(M)$ has finite order. Then examining the exact homotopy sequence for a fibration:

$$1 \longrightarrow \Pi_1(F) \xrightarrow{i_*} \Pi_1(M) \xrightarrow{p_*} \Pi_1(S) \longrightarrow 1.$$

we must have that $g \in \ker(p_*)$ since $\Pi_1(S)$ is torsion free. Therefore $g \in \operatorname{im}(i_*)$ and hence $g = 1$ as $\Pi_1(F)$ is torsion free.

We show that Φ is faithful. Let $K = \mathrm{K}er(\Phi)$ so that we have a commutative diagram as shown below:

$$\begin{tikzcd}
\mathcal{A} \cap K \arrow[r, hook] \arrow[d, hook] & K \arrow[r, two heads] \arrow[d, hook] & K/(\mathcal{A} \cap K) \arrow[d, "\psi"] \\
\mathcal{A} \arrow[r, hook] \arrow[dr, "\Phi|\mathcal{A}"'] & \Pi_1(M) \arrow[r, two heads] \arrow[d, "\Phi"] & \Pi_1(M)/\mathcal{A} \\
 & Isom(E^4) &
\end{tikzcd}$$

where ψ is the induced map.

If $x \in \mathcal{A} \cap K$ is non-trivial then $\Phi(x) = 1$ since $x \in K$; but $\Phi(x) \neq 1$ since $x \in \mathcal{A}$ so we must have $\mathcal{A} \cap K = \{1\}$ and ψ is a monomorphism. However since $\Pi_1(M)$ is torsion free, K must be torsion free and ψ embeds K in the finite group $\Pi_1(M)/\mathcal{A}$. Thus K is trivial and the representation Φ is faithful. Furthermore it is clear that $\Phi(\mathcal{A})$ cocompact implies $\Phi(\Pi_1(M))$ is cocompact. And finally if $\Phi(\mathcal{A})$ is discrete and $\Phi(\Pi_1(M))$ is not discrete then there is a point $\vec{x} \in \mathbb{R}^4$ and an infinite squence (g_i) of elements of $\Phi(\Pi_1(M))$ such that $g_i(\vec{x})$ converges to $\vec{x}$. Now since the index of $\Phi(\mathcal{A})$ in $\Phi(\Pi_1(M))$ is finite at least one of the cosets of $\Phi(\mathcal{A})$ in $\Phi(\Pi_1(M))$ must contain infintely many elements of this sequence. Hence without loss of generality we may assume $g_i = a_i g$ where $a_i \in \Phi(\mathcal{A})$ and $g \in \Phi(\Pi_1(M))$ is fixed. Thus if

$d(-,-)$ is the euclidean metric on $\mathbb{R}^4$ then $d(a_n g(\vec{x}),\vec{x}) < \epsilon/2$ for all fixed $\epsilon > 0$ and all $n > N_0(\epsilon)$. Hence by the triangle inequality $d(a_n g(\vec{x}), a_m g(\vec{x})) \le \epsilon$ for all $n,m > N_0$. But since $\Phi(\mathcal{A})$ acts isometrically we have that $d(g(\vec{x}), a_n^{-1}a_m(g(\vec{x}))) \le \epsilon$, contradicting the assumption that $\Phi(\mathcal{A})$ is discrete. Thus $\Phi(\Pi_1(M))$ is discrete and the proof is complete. $\square$

The bundle-space $(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ I; \begin{pmatrix} 0 \\ 0 \end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha,\beta,l,h \,\middle|\, \begin{matrix} [l,h]=1, & \alpha(l,h)\alpha^{-1} = (l^{-1},h), \\ [\alpha,\beta]=1, & \beta(l,h)\beta^{-1} = (l,h). \end{matrix} \right\rangle .$$

We define a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as follows:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix},\ I),$$
$$h \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2,\beta,l,h\rangle$ of $\Pi_1(M)$ has index 2 and maps faithfully into the translation group $\mathbb{R}^4$ under Φ with discrete image $\langle \begin{pmatrix} 2\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. This image is indeed cocompact as $E^4/\Phi(\mathcal{A}) \simeq \mathbb{R}^4/\mathbb{Z}^4 = T^4$ a compact manifold. Thus this representation is both faithful and discrete with cocompact image so that $(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ I; \begin{pmatrix} 0 \\ 0 \end{pmatrix})$ admits an E^4 geometry.

The bundle-spaces $(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix},\ I; \begin{pmatrix} 0 \\ 0 \end{pmatrix})$ and $(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ I; \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ admit an E^4 geometry.

We show the first representation is geometric.

$$\Pi_1(M) = \left\langle \alpha,\beta,l,h \,\middle|\, \begin{matrix} [l,h]=1, & \alpha(l,h)\alpha^{-1} = (l^{-1},lh), \\ [\alpha,\beta]=1, & \beta(l,h)\beta^{-1} = (l,h). \end{matrix} \right\rangle .$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix},\ I),$$
$$h \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&1\\0&0&0&1 \end{pmatrix}).$$

Now by a change of ordered basis of $\mathbb{R}^4 = \langle e_1,e_2,e_3,e_4\rangle$ to $\mathbb{R}^4 = \langle e_1,e_2,e_3,e_3+2e_4\rangle$ we have a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix},\ I),$$
$$h \mapsto (\begin{pmatrix} 0\\0\\-\frac{1}{2}\\\frac{1}{2} \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}).$$

As before the subgroup $\mathcal{A}\langle\alpha^2,\beta,l,h\rangle$ of $\Pi_1(M)$ has index 2 and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix}2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that so that $(\left(\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right),\ I;\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right))$ admits a geometry of type E^4. Moreover since $(\begin{pmatrix}-1&0\\0&1\end{pmatrix},\ I;\begin{pmatrix}1\\0\end{pmatrix})$ is abstractly group isomorphic to $(\left(\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right),\ I;\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right))$, the composition of such an isomorphism with Φ above shows that $(\left(\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right),\ I;\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$ admits a geometry of type E^4.

The bundle-space $(\begin{pmatrix}-1&0\\0&1\end{pmatrix}, -I;\begin{pmatrix}0\\0\end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M)=\left\langle \alpha,\beta,l,h \,\middle|\, \begin{matrix}[l,h]=1, & \alpha(l,h)\alpha^{-1}=(l^{-1},h),\\ [\alpha,\beta]=1, & \beta(l,h)\beta^{-1}=(l^{-1},h^{-1}).\end{matrix}\right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M)\to\mathbb{R}^4\rtimes O(4)$ as follows:

$$l\mapsto(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ I),\qquad \beta\mapsto(\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}),$$
$$h\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I),\qquad \alpha\mapsto(\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}).$$

The subgroup $\mathcal{A}\langle\alpha^2,\beta^2,l,h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix}2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right), -I;\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right))$ admits an E^4 geometry.

The bundle-space $(\begin{pmatrix}-1&0\\0&1\end{pmatrix}, -I;\begin{pmatrix}0\\1\end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M)=\left\langle \alpha,\beta,l,h \,\middle|\, \begin{matrix}[l,h]=1, & \alpha(l,h)\alpha^{-1}=(l^{-1},h),\\ [\alpha,\beta]=h, & \beta(l,h)\beta^{-1}=(l^{-1},h^{-1}).\end{matrix}\right\rangle.$$

We define a representation $\phi\colon \Pi_1(M)\to\mathbb{R}^4\rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$l\mapsto(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ I),\qquad \beta\mapsto(\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\-1&0&0&-1\end{pmatrix}),$$
$$h\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I),\qquad \alpha\mapsto(\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4=\langle e_1,e_2,e_3,e_4\rangle$ to $\mathbb{R}^4=\langle 2e_1-e_4,e_2,e_3,e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M)\to\mathbb{R}^4\rtimes O(4)$ as shown below:

$$l\mapsto(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ I),\qquad \beta\mapsto(\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}),$$
$$h\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I),\qquad \alpha\mapsto(\begin{pmatrix}\frac12\\0\\0\\\frac12\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}).$$

The subgroup $\mathcal{A}\langle\alpha^2,\beta^2,l,h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\rangle\cong\mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right),-I;\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right))$ admits an E^4 geometry.

The bundle-space $(\begin{pmatrix}-1&0\\0&1\end{pmatrix},-I;\begin{pmatrix}1\\1\end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M)=\left\langle\alpha,\beta,l,h\,\middle|\,\begin{matrix}[l,h]=1, & \alpha(l,h)\alpha^{-1}=(l^{-1},h),\\ [\alpha,\beta]=lh, & \beta(l,h)\beta^{-1}=(l^{-1},h^{-1}).\end{matrix}\right\rangle.$$

We define a representation $\phi\colon\Pi_1(M)\to\mathbb{R}^4\rtimes\mathrm{GL}(4,\mathbb{Z})$ as follows:

$$l\mapsto(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ I),\qquad \beta\mapsto(\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&-1&0\\-1&0&0&-1\end{pmatrix}),$$
$$h\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},/I),\qquad \alpha\mapsto(\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&-1&0\\0&0&0&1\end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4=\langle e_1,e_2,e_3,e_4\rangle$ to $\mathbb{R}^4=\langle 2e_1-e_4,2e_2+e_4,e_3,e_4\rangle$ to obtain a representation $\Phi\colon\Pi_1(M)\to\mathbb{R}^4\rtimes O(4)$ as shown below:

$$l\mapsto(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ I),\qquad \beta\mapsto(\begin{pmatrix}0\\\frac12\\-\frac12\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}),$$
$$h\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I),\qquad \alpha\mapsto(\begin{pmatrix}\frac12\\0\\0\\\frac12\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}).$$

The subgroup $\mathcal{A}\langle\alpha^2,\beta^2,l,h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\rangle\cong\mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right),-I;\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ admits an E^4 geometry.

The bundle-space $(\begin{pmatrix}-1&1\\0&1\end{pmatrix},-I;\begin{pmatrix}0\\0\end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M)=\left\langle\alpha,\beta,l,h\,\middle|\,\begin{matrix}[l,h]=1, & \alpha(l,h)\alpha^{-1}=(l^{-1},lh),\\ [\alpha,\beta]=1, & \beta(l,h)\beta^{-1}=(l^{-1},h^{-1}).\end{matrix}\right\rangle.$$

We define a representation $\phi\colon\Pi_1(M)\to\mathbb{R}^4\rtimes\mathrm{GL}(4,\mathbb{Z})$ as follows:

$$l\mapsto(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ I),\qquad \beta\mapsto(\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}),$$
$$h\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I),\qquad \alpha\mapsto(\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&1\\0&0&0&1\end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4\rangle$ to $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_3 + 2e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$
\begin{aligned}
l &\mapsto \left(\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, I\right), &\qquad \beta &\mapsto \left(\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}\right),\\
h &\mapsto \left(\begin{pmatrix} 0\\0\\-\frac12\\\frac12 \end{pmatrix}, I\right), &\qquad \alpha &\mapsto \left(\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}\right).
\end{aligned}
$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, l, h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\2\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\left(\begin{smallmatrix} -1&1\\0&1 \end{smallmatrix}\right), -I; \left(\begin{smallmatrix} 0\\0 \end{smallmatrix}\right))$ admits an E^4 geometry.

The bundle-space $(\begin{pmatrix} -1&1\\0&1 \end{pmatrix}, -I; \begin{pmatrix} 0\\1 \end{pmatrix})$ admits an E^4 geometry.

$$
\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l^{-1}, lh), \\ [\alpha,\beta] = h, & \beta(l,h)\beta^{-1} = (l^{-1}, h^{-1}). \end{matrix} \right\rangle .
$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{R})$ as follows:

$$
\begin{aligned}
l &\mapsto \left(\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, I\right), &\qquad \beta &\mapsto \left(\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\-\frac12&-\frac12&-1&0\\-1&0&0&-1 \end{pmatrix}\right),\\
h &\mapsto \left(\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, I\right), &\qquad \alpha &\mapsto \left(\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} -1&0&0&0\\0&1&0&0\\0&-\frac12&-1&1\\0&0&0&1 \end{pmatrix}\right).
\end{aligned}
$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4\rangle$ to $\mathbb{R}^4 = \langle -2e_1 + 2e_2 + e_4, -4e_2 + e_3, e_3, e_3 + 2e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$
\begin{aligned}
l &\mapsto \left(\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, I\right), &\qquad \beta &\mapsto \left(\begin{pmatrix} 0\\-\frac14\\\frac14\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}\right),\\
h &\mapsto \left(\begin{pmatrix} 0\\0\\-\frac12\\\frac12 \end{pmatrix}, I\right), &\qquad \alpha &\mapsto \left(\begin{pmatrix} \frac12\\-\frac14\\0\\\frac14 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}\right).
\end{aligned}
$$

Then the subgroup $\mathcal{A}\langle \alpha^2, \beta^2, l, h\rangle$ of $\Pi_1(M)$ is of finite index and embeds discretely into $\mathbb{R}^4$ under Φ as $\langle \begin{pmatrix} 1\\-\frac12\\0\\\frac12 \end{pmatrix}, \begin{pmatrix} 0\\-\frac12\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\-\frac12\\\frac12 \end{pmatrix}\rangle$. As before these four vectors are linearly independent whence $\Phi(\mathcal{A})$ is cocompact. and $(\left(\begin{smallmatrix} -1&1\\0&1 \end{smallmatrix}\right), -I; \left(\begin{smallmatrix} 0\\1 \end{smallmatrix}\right))$ admits an E^4 geometry.

Let

$$Nil^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\};$$

Then Nil^3 naturally has the structure of an $\mathbb{R}$−bundle over $\mathbb{R}^2$:

$$0 \longrightarrow \langle z \rangle \longrightarrow \langle x, y, z \rangle \longrightarrow \langle x, y \rangle \longrightarrow 1.$$

This extension doesn't split since $[\langle x \rangle, \langle y \rangle] = \langle z \rangle$. All isometries of Nil^3 preserve orientation and the line bundle structure over $\mathbb{R}^2$ so we have an induced exact sequence:

$$0 \longrightarrow Isom^+(E) \longrightarrow Isom(Nil^3) \longrightarrow Isom(E^2) \longrightarrow 1.$$

Here $Isom^+(E) = \mathbb{R}$ is the group of orientation preserving isometries of E. The identity component $Isom^0(Nil^3)$ of $Isom(Nil^3)$ is $Nil^3 \rtimes \mathrm{SO}(2)$ where Nil^3 acts by left multiplication and the action of $\theta \in \mathrm{SO}(2)$ is defined by:

$$(x, y, z) \stackrel{\rho_\theta}{\longmapsto} (cx + sy, -sx + cy, z + s(cy^2 - cx^2 - 2sxy)/2);$$

where $s = \sin(\theta)$, $c = \cos(\theta)$ and $\theta \in [0, 2\pi)$. We take $Isom(Nil^3)/Isom^0(Nil^3)$ to be generated by the automorphism $(x, y, z) \stackrel{\lambda}{\longmapsto} (x, -y, -z)$. Furthermore we have $Isom(Nil^3 \times E) = Isom(Nil^3) \times Isom(E)$ where $Isom(E) = \mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z}$. The action of $\mathbb{Z}/2\mathbb{Z} = \langle \mathfrak{r} \rangle$ on $\mathbb{R}$ is by reflection or multiplication by -1. Below we shall denote the isometry of left multiplication by $X \in Nil^3$ simply by X.

Lemma. *Let* $\wp\colon Isom(Nil^3 \times E) \twoheadrightarrow Isom(Nil^3) \twoheadrightarrow Isom(E^2)$ *be the composite of the natural projection followed by the induced map of the* $\mathbb{R}$*-bundle projection. A representation* $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ *is faithful and discrete with cocompact image precisely when the following two properties are satisfied:*

(1) *The subgroup* $K \trianglelefteq \Pi_1(M)$ *whose image* $\Phi(K) = \Phi(\Pi_1(M)) \cap \ker(\wp)$ *embeds faithfully, discretely and cocompactly in* $\mathbb{R} \times Isom(E)$.
(2) *The quotient group* $\Pi_1(M)/K$ *embeds faithfully and discretely in* $Isom(E^2)$ *with cocompact image under the induced map from* Φ.

proof:
If $\Phi(\Pi_1(M))$ is discrete in $Isom(Nil^3 \times E)$ then clearly $\Phi(K)$ is discrete in $Isom(Nil^3 \times E)$ and hence discrete in $Isom^+(E)$. Hence by way of the quotient topology, $\Phi(\Pi_1(M))/\Phi(K)$ must be discrete in $Isom(E^2)$. Similarly if $\Pi_1(M)$ maps faithfully into $Isom(Nil^3 \times E)$ then so too does K; in which case $\Pi_1(M)/K$ must map faithfully into $Isom(E^2)$ via the induced map. If $\Phi(\Pi_1(M))$ is cocompact then the fibration $(Nil^3 \times E)/\Phi(\Pi_1(M)) \stackrel{\overline{p}}{\longrightarrow} E^2/\overline{\Phi}(\Pi_1(M)/K)$ shows that $\overline{\Phi}(\Pi_1(M)/K)$ is cocompact. Moreover the fibres of $\overline{p}$ are closed in $(Nil^3 \times E)/\Phi(\Pi_1(M))$ and hence compact. Thus $E^2/\Phi(K)$ is compact. Below we prove the reverse implication. Let

$X = \mathrm{K}er(\Phi)$ so that we have a commutative diagram as shown below:

$$\begin{tikzcd}
K\cap X \arrow[r, hook] \arrow[d, hook] & X \arrow[r, two heads] \arrow[d, hook] & X/(K\cap X) \arrow[d, "\psi"] \\
K \arrow[r, hook] \arrow[d, hook, "\Phi|K"] & \Pi_1(M) \arrow[r, two heads] \arrow[d, "\Phi"] & \Pi_1(M)/K \arrow[d, hook, "\overline{\Phi}"] \\
\mathbb{R}\times Isom(E) \arrow[r, hook] & Isom(Nil^3\times E) \arrow[r, two heads] & Isom(E^2)
\end{tikzcd}$$

By chasing a nontrivial element in $K\cap X$ into $Isom(Nil^3\times E)$ we see that $K\cap X$ is trivial. In which case the map ψ is injective. Hence if $x\in X$ is non-trivial then $\overline{\Phi}\psi(x)$ is non-trivial, contradicting the fact that $x\in \mathrm{K}er(\Phi)$. Thus $X=\{1\}$ and Φ is faithful. If both $\Phi(K)\trianglelefteq Isom^+(E)\times Isom(E)$ and $\overline{\Phi}(\Pi_1(M)/K)\trianglelefteq Isom(E^2)$ are discrete. Then $\Phi(\Pi_1(M))\trianglelefteq Isom(Nil^3\times E)$ is discrete by way of the quotient topology. If $\Phi(K)$ is cocompact then we have a covering space projection $Isom(Nil^3\times E)/\Phi(K)\xrightarrow{p} Isom(Nil^3\times E)/\Phi(\Pi_1(M))$ showing that $\Phi(\Pi_1(M))$ is cocompact. $\square$

The family of bundle-spaces $(\begin{pmatrix}-1&0\\0&1\end{pmatrix}, I; \begin{pmatrix}0\\n\end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}$ **admit geometries of type** $Nil^3\times E$**.**

$$\Pi_1(M)=\left\langle \alpha,\beta,l,h \;\middle|\; \begin{matrix}[l,h]=1, & \alpha(l,h)\alpha^{-1}=(l^{-1},h),\\ [\alpha,\beta]=h^n, & \beta(l,h)\beta^{-1}=(l,h).\end{matrix}\right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M)\to Isom(Nil^3\times E)$ as follows:

$$\begin{aligned}
\Phi(l)(x,y,z,w) &= (x,y,z,w+1),\\
\text{i.e}\quad \Phi(l) &= (I,\ (1,1)).\\
\Phi(h)(x,y,z,w) &= (x,y,z+h_0,w)\\
\text{i.e}\quad \Phi(h) &= (\begin{pmatrix}1&0&h_0\\0&1&0\\0&0&1\end{pmatrix},\ (0,1)).\\
\Phi(\beta)(x,y,z,w) &= (x,y+1,z,w)\\
\text{i.e}\quad \Phi(\beta) &= (\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix},\ (0,1)).\\
\Phi(\alpha)(x,y,z,w) &= (x+1,y,z+y,-w)\\
\text{i.e}\quad \Phi(\alpha) &= (\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\ (0,\mathfrak{r})).
\end{aligned}$$

where h_0 is non-zero.

The relations $[l,h]=1$, $\beta(l,h)\beta^{-1}=(l,h)$ and $\alpha(l,h)\alpha^{-1}=(l^{-1},h)$ always hold. The relation $[\alpha,\beta]=h^n$ gives $h_0=1/n$ and we have the desired representation. To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1,y_1,z,w)\}\le Isom(Nil^3\times E)$ in the image of Φ for any fixed $x_1,y_1\in\mathbb{R}$ is the group $\Phi(K)=\langle\Phi(l),\Phi(h)\rangle\cong\mathbb{Z}^2$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient T^2. Similarly $\overline{\Phi}(\Pi_1(M)/K)=\langle (x,y)\overset{[\alpha]}{\mapsto}(x+1,y),(x,y)\overset{[\beta]}{\mapsto}(x,y+1)\rangle\cong\mathbb{Z}^2$ acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient T^2 so we have the desired representation.

The family of bundle-spaces $(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, I; \begin{pmatrix} 1 \\ n \end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}$ **admit geometries of type** $Nil^3 \times E$**.**

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix} [l,h]=1, & \alpha(l,h)\alpha^{-1} = (l^{-1},h), \\ [\alpha,\beta] = lh^n, & \beta(l,h)\beta^{-1} = (l,h). \end{matrix} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned} \Phi(l)(x,y,z,w) &= (x,y,z,w+1), \\ \text{i.e}\quad \Phi(l) &= (I,\ (1,1)). \\ \Phi(h)(x,y,z,w) &= (x,y,z+1,w) \\ \text{i.e}\quad \Phi(h) &= (\begin{pmatrix} 1&0&1\\0&1&0\\0&0&1 \end{pmatrix},\ (0,1)). \\ \Phi(\beta)(x,y,z,w) &= (x,y+1,z,w+k) \\ \text{i.e}\quad \Phi(\beta) &= (\begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix},\ (k,1)). \\ \Phi(\alpha)(x,y,z,w) &= (x+a,y,z+ay,-w) \\ \text{i.e}\quad \Phi(\alpha) &= (\begin{pmatrix} 1&a&0\\0&1&0\\0&0&1 \end{pmatrix},\ (0,\mathfrak{r})). \end{aligned}$$

The relations $[l,h]=1$, $\alpha(l,h)\alpha^{-1} = (l^{-1},h)$ and $\beta(l,h)\beta^{-1} = (l,h)$ are automatically satisfied. The relation $[\alpha,\beta] = lh^n$ gives $a = n$ and $k = -1/2$. To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1,y_1,z,w)\} \leq Isom(Nil^3 \times E)$ in the image of Φ for any fixed $x_1, y_1 \in \mathbb{R}$ is the group $\Phi(K) = \langle \Phi(l), \Phi(h)\rangle \cong \mathbb{Z}^2$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient T^2. Similarly $\overline{\Phi}(\Pi_1(M)/K) = \langle (x,y)\overset{[\alpha]}{\mapsto}(x+n,y), (x,y)\overset{[\beta]}{\mapsto}(x,y+1)\rangle \cong \mathbb{Z}^2$ acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient T^2 so we have the desired representation.

The family of bundle-spaces $(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, I; \begin{pmatrix} 0 \\ n \end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}$ **admit geometries of type** $Nil^3 \times E$**.**

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix} [l,h]=1, & \alpha(l,h)\alpha^{-1} = (l^{-1},lh), \\ [\alpha,\beta] = h^n, & \beta(l,h)\beta^{-1} = (l,h). \end{matrix} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned} \Phi(l)(x,y,z,w) &= (x,y,z,w+1), \\ \text{i.e}\quad \Phi(l) &= (I,\ (1,1)). \\ \Phi(h)(x,y,z,w) &= (x,y,z+1,w+m) \\ \text{i.e}\quad \Phi(h) &= (\begin{pmatrix} 1&0&1\\0&1&0\\0&0&1 \end{pmatrix},\ (m,1)). \\ \Phi(\beta)(x,y,z,w) &= (x,y+1,z,w+k) \\ \text{i.e}\quad \Phi(\beta) &= (\begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix},\ (k,1)). \\ \Phi(\alpha)(x,y,z,w) &= (x+a,y,z+ay,-w) \\ \text{i.e}\quad \Phi(\alpha) &= (\begin{pmatrix} 1&a&0\\0&1&0\\0&0&1 \end{pmatrix},\ (0,\mathfrak{r})). \end{aligned}$$

The relations $[l,h]=1$, $\beta(l,h)\beta^{-1}=(l,h)$ and $\alpha l\alpha^{-1}=l^{-1}$ always hold. The relation $\alpha h\alpha^{-1}=lh$ gives $m=-1/2$. The relation $[\alpha,\beta]=h^n$ gives $a=n$ and $k=n/4$. To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1,y_1,z,w)\}\leq Isom(Nil^3\times E)$ in the image of Φ for any fixed $x_1,y_1\in\mathbb{R}$ is the group $\Phi(K)=\langle\Phi(l),\Phi(h)\rangle\cong\mathbb{Z}^2$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient T^2. Similarly $\overline{\Phi}(\Pi_1(M)/K)=\langle(x,y)\overset{[\alpha]}{\mapsto}(x+n,y),(x,y)\overset{[\beta]}{\mapsto}(x,y+1)\rangle\cong\mathbb{Z}^2$ acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient T^2 so we have the desired representation.

The Lie group $Sol^3=\mathbb{R}^2\rtimes\mathbb{R}$ where the action of $t\in\mathbb{R}$ on $\mathbb{R}^2=\{\begin{pmatrix}x\\y\end{pmatrix}\}$ is left multiplication by the matrix $\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix}$. The surfaces defined by constant values of t foliate Sol^3 and are preserved by all isometries. The identity component $Isom^0(Sol^3)$ of the isometry group is again Sol^3. The quotient group $Isom(Sol^3)/Isom^0(Sol^3)$ is isomorphic to the dihedral group of order 8 generated by the automorphisms:

$$(x,y,t)\longmapsto(\pm x,\pm y,t),\quad\text{and}\quad(x,y,t)\longmapsto(\pm y,\pm x,-t).$$

As before we have $Isom(Sol^3\times E)=Isom(Sol^3)\times Isom(E)$ so that we have an induced exact sequence:

$$1\longrightarrow Isom(E^2)\longrightarrow Isom(Sol^3\times E)\longrightarrow Isom(E)\times Isom(E)\longrightarrow 1.$$

Where $Isom(E)\times Isom(E)$ acts on $\mathbb{R}^2$ as a subgroup of $Isom(E^2)$.

Lemma. *Let $\wp\colon Isom(Sol^3\times E)\twoheadrightarrow Isom(E)\times Isom(E)$ be the above projection. A representation $\Phi\colon\Pi_1(M)\to Isom(Sol^3\times E)$ is faithful and discrete with cocompact image precisely when:*

1. *The subgroup $K\trianglelefteq\Pi_1(M)$ whose image $\Phi(K)=\Phi(\Pi_1(M))\cap\ker(\wp)$ embeds faithfully, discretely and cocompactly in $Isom(E^2)$.*
2. *The quotient group $\Pi_1(M)/K$ embeds faithfully and discretely in $Isom(E)\times Isom(E)$ with cocompact image under the induced map from Φ.*

proof:
If $\Phi(\Pi_1(M))$ maps discretely into $Isom(Sol^3\times E)$ then so too does $\Phi(K)$ and hence is discrete in $Isom(E^2)$. Hence by way of the quotient topology $\Phi(\Pi_1(M))/\Phi(K)$ is discrete in $Isom(E)\times Isom(E)$. Similarly if $\Phi(\Pi_1(M))$ maps faithfully into $Isom(Sol^3\times E)$ then so does $\Phi(K)$; in which case $\Pi_1(M)/K$ must map faithfully into $Isom(E)\times Isom(E)$ via the induced map. If $\Phi(\Pi_1(M))$ is cocompact then the fibration $(Sol^3\times E)/\Phi(\Pi_1(M))\xrightarrow{\overline{p}}E^2/\overline{\Phi}(\Pi_1(M)/K)$ shows that $\overline{\Phi}(\Pi_1(M)/K)$ is cocompact. Moreover the fibres of $\overline{p}$ are closed in $(Sol^3\times E)/\Phi(\Pi_1(M))$ and hence compact. Thus $E^2/\Phi(K)$ is compact. Below we prove the reverse implication. Let

$X = \mathrm{K}er(\Phi)$ so that we have a commutative diagram as shown below:

$$\begin{tikzcd}
K \cap X \arrow[r, hook] \arrow[d, hook] & X \arrow[r, two heads] \arrow[d, hook] & X/(K \cap X) \arrow[d, "\psi"] \\
K \arrow[r, hook] \arrow[d, hook, "\Phi|K"] & \Pi_1(M) \arrow[r, two heads] \arrow[d, "\Phi"] & \Pi_1(M)/K \arrow[d, hook, "\overline{\Phi}"] \\
Isom(E^2) \arrow[r, hook] & Isom(Sol^3 \times E) \arrow[r, two heads] & Isom(E) \times Isom(E)
\end{tikzcd}$$

By chasing a nontrivial element in $K \cap X$ into $Isom(Sol^3 \times E)$ we see that $K \cap X$ is trivial. In which case the map ψ is injective. Hence if $x \in X$ is non-trivial then $\overline{\Phi}\psi(x)$ is non-trivial contradicting the fact that $x \in \mathrm{K}er(\Phi)$. Thus $X = \{1\}$ and Φ is faithful. If both $\Phi(K) \trianglelefteq Isom(E^2)$ and $\overline{\Phi}(\Pi_1(M)/K) \trianglelefteq Isom(E) \times Isom(E)$ are discrete. Then $\Phi(\Pi_1(M)) \trianglelefteq Isom(Sol^3 \times E)$ is discrete by way of the quotient topology. If $\Phi(K)$ is cocompact then we have a covering space projection $Isom(Sol^3 \times E)/\Phi(K) \xrightarrow{p} Isom(Sol^3 \times E)/\Phi(\Pi_1(M))$ showing that $\Phi(\Pi_1(M))$ is cocompact. $\square$

The family of bundle-spaces $(A, \pm I; \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), \det(A) = -1, \mathrm{rank}(A-I) = 2)$ **admit geometries of type** $Sol^3 \times E$**.**

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \;\middle|\; \begin{array}{ll} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l,h)A, \\ [\alpha,\beta] = l^a h^b, & \beta(l,h)\beta^{-1} = (l^{\pm 1}, h^{\pm 1}). \end{array} \right\rangle.$$

We define a representation $\Phi: \Pi_1(M) \to Isom(Sol^3 \times E)$ as follows: We have that $PAP^{-1} = \begin{pmatrix} e^t_0 & 0 \\ 0 & -e^{-t_0} \end{pmatrix}$ for some $P = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{R})$. Hence we define a representation $\Phi: \Pi_1(M) \to Isom(Sol^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(l)(x,y,t,u) &= (x+p, y+r, t, u) \\
\text{ie} \quad \Phi(l) &= ((\begin{pmatrix} p \\ q \end{pmatrix}, 0),\ (0,1)). \\
\Phi(h)(x,y,t,u) &= (x+q, y+s, t, u) \\
\text{ie} \quad \Phi(h) &= ((\begin{pmatrix} r \\ s \end{pmatrix}, 0),\ (0,1)). \\
\Phi(\beta)(x,y,t,u) &= (\epsilon x + x_1, \epsilon y + y_1, t, u+1) \\
\text{ie} \quad \Phi(\beta) &= ((\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, 0),\ (1,1))\lambda. \\
\Phi(\alpha)(x,y,t,u) &= (e^{t_0}x + x_0, -e^{-t_0}y + y_0, t + t_0, u) \\
\text{ie} \quad \Phi(\alpha) &= ((\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, t_0),\ (0,1))\rho \quad t_0 \neq 0.
\end{aligned}$$

where $\epsilon = \pm 1$ respective as to whether $B = \pm I$ and hence $(x,y,t,u) \stackrel{\lambda}{\longmapsto} (\epsilon x, \epsilon y, t, u)$. Similarly ρ is the isometry $(x,y,t,u) \stackrel{\rho}{\longmapsto} (x, -y, t, u)$.

The relations $[l,h]=1$, $\alpha(l,h)\alpha^{-1}=(l,h)A$ and $\beta(l,h)\beta^{-1}=(l^{\pm 1},h^{\pm 1})$ are automatically satisfied. The relation $[\alpha,\beta]=l^a h^b$ gives the relation:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \left(I - PAP^{-1}\right)^{-1}\left((1-\epsilon)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - P\begin{pmatrix} a \\ b \end{pmatrix}\right).$$

Here $K=\langle \Phi(l),\Phi(h)\rangle$. Since $P\in \mathrm{GL}(2,\mathbb{R})$ the vectors $\left(\begin{smallmatrix} p \\ r \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} q \\ s \end{smallmatrix}\right)$ are linearly independent and hence K acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient T^2. Similarly the group $\overline{\Phi}(\Pi_1(M)/K)=\langle t\overset{[\alpha]}{\mapsto}t+t_0, u\overset{[\beta]}{\mapsto}t+1\rangle$ is isomorphic to $\mathbb{Z}^2$ and acts on $\mathbb{R}^2\langle t,u\rangle$ with qutient T^2. Thus Φ is indeed faithful and discrete with cocompact image; and we have the desired representation completing the proof. $\square$

The Classification of Non-Orientable Total Spaces of T^2-Bundles over Kb and their Geometries

Theorem. *Let M be the non-orientable total space of a T^2-bundle over Kb with first Betti number $\beta_1(M)$. Then M is bundle diffeomorphic to one of the geometric 4-manifolds listed below:*

	Total Space	*Geometric Type*
$\beta_1(M)=3$	$(I,\ I;\left(\begin{smallmatrix} (1) \\ 0 \end{smallmatrix}\right))$	E^4
	$(\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right),\ I;\left(\begin{smallmatrix} (1) \\ (1) \end{smallmatrix}\right))_{\lambda\in\mathbb{N}\setminus\{0\}}$	$Nil^3\times E$
$\beta_1(M)=2$	$(I,\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right);\left(\begin{smallmatrix} (1) \\ (1) \end{smallmatrix}\right))$	E^4
	$(I,\left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right);\left(\begin{smallmatrix} 0 \\ (1) \end{smallmatrix}\right))$	E^4
	$(-I,\ I;\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right))$	E^4
	$(-I,\ I;\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))$	$Nil^3\times E$
	$(-I,\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right);\left(\begin{smallmatrix} (1) \\ 0 \end{smallmatrix}\right))$	E^4
	$(-I,\left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right);\left(\begin{smallmatrix} 0 \\ n \end{smallmatrix}\right))_{n\in\mathbb{N}\setminus\{0\}}$	$Nil^3\times E$
	$(\left(\begin{smallmatrix} -1 & \lambda \\ 0 & -1 \end{smallmatrix}\right),\ I;\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right))_{\lambda,n\neq 0}$	Nil^4
$\beta_1(M)=1$	$(I,-I;\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right))$	E^4
	$(I,-I;\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))$	$Nil^3\times E$
	$(\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right),-I;\left(\begin{smallmatrix} (1) \\ (1) \end{smallmatrix}\right))_{\lambda\neq 0}$	$Nil^3\times E$
	$(A,\ \pm I;\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right),\det(A)=1,\lvert\mathrm{trace}(A)\rvert>2)$	$Sol^3\times E$
	$(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),\ B;\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right),\det(B)=1,\lvert\mathrm{trace}(B)\rvert>2)$	$Sol^3\times E$
	$(-I,\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right);\left(\begin{smallmatrix} (1) \\ n \end{smallmatrix}\right))_{n\in\mathbb{N}\setminus\{0\}}$	$Nil^3\times E$

TABLE 3.

Remark. As before the classes of $Sol^3 \times E$-manifolds listed above admit further refinement, see [Ne].

By writing $M = (A, B; \binom{a}{b})$ we are refering to the bundle-space with fundamental group

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{array}{ll} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l,h)A, \\ \alpha\beta\alpha^{-1}\beta = l^a h^b, & \beta(l,h)\beta^{-1} = (l,h)B. \end{array} \right\rangle .$$

The following three classes of fibre preserving diffeomorphims correspond to the respective transformations of the algebraic representation $(A, B; \binom{a}{b})$ of the bundle space M:

(1) Change of framing of the fibre

$$(l,h) \mapsto (l,h)P, \quad \exists P \in \mathrm{GL}(2,\mathbb{Z}).$$
$$(A, B; \binom{a}{b}) \mapsto (P^{-1}AP, P^{-1}BP; P^{-1}\binom{a}{b}).$$

(2) Change of base of the base space

(i)
$$(\alpha,\beta) \mapsto (\alpha^{-1},\beta)$$
$$(A, B; \binom{a}{b}) \mapsto (A^{-1}, B; B^{-1}A^{-1}\binom{a}{b}).$$

(ii)
$$(\alpha,\beta) \mapsto (\alpha,\beta^{-1})$$
$$(A, B; \binom{a}{b}) \mapsto (A, B^{-1}; -B\binom{a}{b}).$$

(iii)
$$(\alpha,\beta) \mapsto (\beta\alpha\beta^{-1},\beta)$$
$$(A, B; \binom{a}{b}) \mapsto (BAB^{-1}, B; B\binom{a}{b}).$$

(iv)
$$(\alpha,\beta) \mapsto (\alpha^{-1}\beta,\alpha^{-1}\beta\alpha)$$
$$(A, B; \binom{a}{b}) \mapsto (A^{-1}B, B^{-1}; A^{-2}\binom{a}{b}).$$

(3) Change of cross-section

$$(\alpha,\beta) \mapsto (\alpha l^p h^q, \beta l^s h^t), \quad \text{where } t,s,p,q \in \mathbb{Z} \text{ are free.}$$
$$(A, B; \binom{a}{b}) \mapsto (A, B; \binom{a}{b} + B^{-1}A(B^{-1} - I)\binom{p}{q} + (B^{-1}A + I)\binom{s}{t}).$$

Here equivalence classes of triples modulo transformations of type (3) are in bijection with the elements of $H^2(\mathbb{Z}\tilde{\times}\mathbb{Z}; \mathbb{Z}^2) \cong \mathbb{Z}^2/\langle -A - I, B - I\rangle$.

The bundle-space M is orientable precisely when $\det(A) = -1$ and $\det(B) = 1$. Below we shall only consider the two cases: $\det(A) = \det(B)$. For if $\det(A) = 1$ and $\det(B) = -1$ we may replace A by $A^{-1}B$. On examination of the presentation for $H_1(M)$ we see that $\beta_1(M) = 3 - \mathrm{rank}(A - I, B - I)$.

Assuming that the bundle-spaces listed in table 3. admit the geometries as claimed, then to prove that table 3. is a complete listing of all such distinct bundle-spaces up fibre preserving diffeomorphism it suffices to distinguish between representations admitting identical geometries with the same values of $\beta_1(M)$.

Lemma 2. *Let the matrix $A \in \mathrm{GL}(2,\mathbb{Z})$ be such that $A^2 = I$.*
(1) *If $\det(A) = 1$ then A is conjugate to either $\pm I$.*
(2) *If $\det(A) = -1$ then A is conjugate to one of $\begin{pmatrix} -1 & (1) \\ 0 & 1 \end{pmatrix}$.*

proof : The eigenvalues of A square to give 1 and therefore belong to the set $\{\pm 1\}$. Hence the result follows from Lemma 1. $\square$

A) The case $\beta_1(M) = 3 \Longrightarrow \mathrm{rank}(A - I, B - I) = 0$.
Here we must have $A = I = B$ and hence by an appropriate change of cross-section we may assume $\binom{a}{b} = \binom{(1)}{(1)}$; corresponding to $H^2(\mathbb{Z}\widetilde{\times}\mathbb{Z};\mathbb{Z}^2) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, being the four extension equivalence classes with trivial action. However by changing the framing of the fibre we see that there are at most two such bundle-spaces distinguished by whether $\binom{a}{b} = \binom{0}{0}$ or not:

$$(I,\ I; \begin{pmatrix} 0 \\ 0 \end{pmatrix}),\quad (I,\ I; \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

These two manifolds are indeed distinct as their first homology groups are $\mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}^3$ respectively. The first manifold is the product bundle $Kb \times T^2$.

B) The case $\beta_1(M) = 2 \Longrightarrow \mathrm{rank}(A - I, B - I) = 1$.
B_1) The case $\det(A) = 1 = \det(B)$

B_{11}) $\mathrm{rank}(A - I) = 0 \Longrightarrow A = I$.
We must have $\mathrm{rank}(B - I) = 1$ therefore $B \sim \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}_{\lambda \neq 0}$. However the relation $ABA^{-1}B = I$ implies $B^2 = I$ and consequently $\lambda = 0$ so we have no such bundle-spaces.

B_{12}) $\mathrm{rank}(A - I) = 1 \Longrightarrow A \sim \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}_{\lambda \neq 0}$
Furthermore by conjugating A by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ if nessesary, we may assume that λ is positive. If $\mathrm{rank}(B - I) = 0$ then $B = I$ and we may transform $\binom{a}{b}$ into $\binom{(1)}{(1)}$ if λ is even, or $\binom{0}{(1)}$ if λ is odd by some appropriate change of cross-section. This corresponds to the fact that

$$H^2(\mathbb{Z}\widetilde{\times}\mathbb{Z};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } \lambda \in 2\mathbb{N}; \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \lambda \notin 2\mathbb{N}. \end{cases}$$

That is for each value of $\lambda \in \mathbb{N}$ we obtain two triples when λ is odd, and four triples when λ is even:

$$(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},\ I; \begin{pmatrix} (1) \\ (1) \end{pmatrix})_{\lambda \in 2\mathbb{N}\setminus\{0\}},\quad (\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},\ I; \begin{pmatrix} 0 \\ (1) \end{pmatrix})_{\lambda \in \mathbb{N}\setminus 2\mathbb{N}}.$$

However when λ is even, a tranformation of the fibre via the matrix $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ equates the two representations with Euler numbers $\binom{1}{0}$ and $\binom{1}{1}$. For the sake of brevity we tabulate this redundancy as it is shown above. Abelianization of these groups yields: $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/\lambda\mathbb{Z}$ or $\mathbb{Z}^2 \oplus \mathbb{Z}/2\lambda\mathbb{Z}$ when $\binom{a}{b} = \binom{0}{0}$ or $\binom{0}{1}$ respectively. And the remaining bundle-space has $H_1(M) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/\lambda\mathbb{Z}$; showing that these five infinite families of manifolds are all mutually non-diffeomorphic.

If $\mathrm{rank}(B-I) = 1$ we must have $B \sim \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}_{\mu\neq 0}$. However the relation $ABA^{-1}B = I$ implies that $\mu = 0$ and we have no such bundle-spaces.

B$_2$) The case $\det(A) = -1 = \det(B) \Longrightarrow \mathrm{rank}(A-I) = 1 = \mathrm{rank}(B-I)$.
Here we may assume $A = \begin{pmatrix} -1 & (1) \\ 0 & 1 \end{pmatrix}$ in which case we must have $B = \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix}$. The relation $ABA^{-1}B = I$ gives $* = (1)$ so that after some changes of cross-section we obtain the six representations: $(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} (1) \\ (1) \end{pmatrix})$,and $(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 \\ (1) \end{pmatrix})$. In this case for cosmetic purposes we abandon our convention that $\det(A) = \det(B)$ and replace A by $A^{-1}B$ so as to write these representations as:

$$(I, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} (1) \\ (1) \end{pmatrix}), \quad (I, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 \\ (1) \end{pmatrix}).$$

These triples represent the equivalence classes of the associated extensions since

$$H^2(\mathbb{Z}\tilde{\times}\mathbb{Z}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } B = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right); \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } B = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right). \end{cases}$$

When $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ the groups $H_1(M)$ are: $\mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2$, $\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ depending on whether $\binom{a}{b} = \binom{0}{0}$, $\binom{1}{0}$, $\binom{0}{1}$ or $\binom{1}{1}$ respectively. These latter two representations with isomorphic first homology are seen to be distinct by observing the isomorphism type of the quotient of $\Pi_1(M)$ by its center $\mathcal{Z}(\Pi_1(M)) = \langle \alpha^2, h \rangle$. Since the center is characteristic in $\Pi_1(M)$ any isomorphism of $\Pi_1(M)$ will induce an isomorphism on this quotient. When $\binom{a}{b} = \binom{0}{1}$ this quotient abelianizes to give $(\mathbb{Z}/2\mathbb{Z})^3$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ in the other case. Thus these four bundle-spaces are distinct. When $B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ the group $H_1(M)$ is either $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}^2$ depending on whether $\binom{a}{b} = \binom{0}{0}$ or $\binom{0}{1}$ respectively. The former case may be distinguished from the other two bundle-spaces with the same first homology group as follows. The kernel $K \lhd \Pi_1(M)$ of the projection:

$$\Pi_1(M) \twoheadrightarrow H_1(M) \twoheadrightarrow H_1(M)/\mathrm{Torsion} \cong \mathbb{Z}/2\mathbb{Z}.$$

is characterstic and is isomorphic to the Klein bottle group generated by β and l. However the above subgroup $K = \langle l, \beta^2 h^{-1} \rangle$ of the bundle-spaces $(I, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} (1) \\ 1 \end{pmatrix})$ is free abelian.

C) The case $\beta_1(M) = 1 \Longrightarrow \mathrm{rank}(A-I, B-I) = 2$.

Here we shall utilize the fact that M is double covered by the total-space $\widetilde{M}$ of a T^2-bundle over T^2. The covering projection $\widetilde{M}: \twoheadrightarrow M$ is induced by the orientation double covering $T^2 \twoheadrightarrow Kb$ and hence is associated with the kernel of the composite projection:

$$\eth\colon \Pi_1(M) \twoheadrightarrow H_1(M) \twoheadrightarrow H_1(M)/\text{Torsion} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Thus $\widetilde{M}$ is the bundle-space with algebraic representation $(A^2, B; (A-B)\binom{a}{b})$. Since $\mathrm{Ker}(\eth)$ is characteristic in $\Pi_1(M)$ any isomorphism $\psi\colon \Pi_1(M) \to \Pi_1(M')$ induces an isomorphism $\widetilde{\psi}\colon \Pi_1(\widetilde{M}) \to \Pi_1(\widetilde{M'})$. However the same bundle-space $\widetilde{M}$ may cover many distinct T^2-bundle spaces over Kb. We proceed in cases according to the value of $\beta_1(\widetilde{M})$.

C$_1$) The case $\beta_1(\widetilde{M}) = 4 \Longrightarrow \widetilde{M} = T^4$

We must have $A^2 = I = B$, thus $\det(A) = 1$ and hence by Lemma 2. $A = \pm I$. However $A = I$ contradicts $\mathrm{rank}(A-I, B-I) = 2$ so we must have $A = -I$. Therefore the matrix $A - B$ is invertible over $\mathbb{R}$ and thus the Euler number is trivial. That is we obtain the sole representaion:

$$(-I, I; \begin{pmatrix} 0 \\ 0 \end{pmatrix}).$$

Here $H_1(M) \simeq \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$.

C$_2$) The case $\beta_1(\widetilde{M}) = 3 \Longrightarrow \mathrm{rank}(A^2 - I, B - I; (A-B)\binom{a}{b}) = 1$.

C$_{21}$) The case $\det(A) = 1 = \det(B)$.

Here we have $\det(A^2) = 1 = \det(B)$ so that $\widetilde{M}$ is orientable. From Table 2. we see that there is only one orientable T^2-bundle over T^2. Therefore $A^2 = I = B$ so that from Lemma 2 we must have $A = \pm I$. As in the previous case $A = I$ yields a contradiction so $A = -I$. Thus we obtain a bigraded infinite family of triples indexed by $\binom{a}{b}$, in correspondence with $H^2(\mathbb{Z}\widetilde{\times}\mathbb{Z}; \mathbb{Z}^2) \cong \mathbb{Z}^2$, representing extension equivalence classes with the given action. [note: algebraic transformations of type (3) for such an action act identically on $\mathbb{Z}^2$.] However as both $\pm I$ are invariant to a change of basis we obtain just two representations up to extension isomorphism distinguished by whether $\binom{a}{b}$ is the zero vector or not. As one of these representations was already discovered in the previous case, we just obtain the single representation:

$$(-I, I; \begin{pmatrix} 1 \\ 0 \end{pmatrix});$$

with $H_1(M) \simeq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

C$_{22}$) The case $\det(A) = -1 = \det(B)$.

Here $\det(A^2) = 1$ and $\mathrm{rank}(A^2 - I) \leq 1$, thus $A^2 \sim \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ for some $\lambda \in \mathbb{Z}$. Writing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and examining this relation with the condition that $\det(A) = -1$, we find the only posibility is that $\lambda = 0$. Hence by lemma 2 we may assume $A = \begin{pmatrix} -1 & (1) \\ 0 & 1 \end{pmatrix}$.

If (1)=0 then writing B as $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and testing the relation $ABA^{-1}B = I$ with the condition that $\det(B) = -1$ we see that the only possibilities for B

are $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. However the latter matrix contradicts the assumption that $\mathrm{rank}(A-I, B-I) = 2$ so we must have $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Now a change of cross-section permits us to reduce the Euler number to $\begin{pmatrix} a \\ (1) \end{pmatrix}$ corresponding to $H^2(\mathbb{Z}\tilde{\times}\mathbb{Z};\mathbb{Z}^2) \cong \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. However the condition that $\mathrm{rank}(A^2-I, B-I; (A-B)\binom{a}{b}) = 1$ implies that $a = 0$ so that we obtain the two representations: $(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \begin{pmatrix} 0 \\ (1) \end{pmatrix})$. We now choose to cast aside our convention and write these representations as:

$$(-I, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \end{pmatrix}), \quad (-I, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

These two representations have $H_1(M) = \mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})^3$ or $\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ depending on whether $\binom{a}{b} = \binom{0}{0}$ or $\binom{0}{1}$ respectively; and hence represent distinct manifolds.

If the entry (1) in A is non-trivial then following similar methods to the above we find that $B = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ and the set $\{\binom{a}{b}, \binom{1}{2}\}$ is $\mathbb{Z}$-linearly independent. Now conjugating A and B by $\pm\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$ and then replacing A by $A^{-1}B$ and applying a cross-section change we obtain the family of representations:

$$(-I, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 \\ n \end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}.$$

Being in correspondence with equivalence classes given by $H^2(\mathbb{Z}\tilde{\times}\mathbb{Z};\mathbb{Z}^2) \cong \mathbb{Z}$. Here $H_1(M) = \mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})^2$ distinguishing this family of bundle-spaces from the other $Nil^3\times E$-manifold with $\beta_1(\widetilde{M}) = 3$ [case C_{21})]. For fixed $n\in\mathbb{N}$ such a bundle-space is doubly covered by the T^2-bundle over T^2 with characteristic triple: $(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, I; \binom{0}{2n})$. The diffeomorphism type of such a bundle-space is determined by n, and hence the above family of triples represent an infinite mutually non-diffeomorphic family of T^2-bundle-spaces over Kb indexed by $n\in\mathbb{N}$.

C_3) The case $\beta_1(\widetilde{M}) = 2 \Longrightarrow \mathrm{rank}(A^2-I, B-I; (A-B)\binom{a}{b}) = 2$.

C_{31}) The case $\det(A) = 1 = \det(B)$.

If $\mathrm{rank}(B-I) = 0$ then $B = I$ and hence A must be one of the matrices: $I, \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}$ or else A has irrational eigenvalues. The first two matrices contradict $\mathrm{rank}(A-I, B-I) = 2$ so do not yield a representation. Therefore $A = \begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}$ and $H^2(\mathbb{Z}\tilde{\times}\mathbb{Z};\mathbb{Z}^2) \cong \mathbb{Z}\oplus\mathbb{Z}/\lambda\mathbb{Z}$ corresponding to the fact that an appropriate change of cross-section assures that b is positive and reduces a modulo λ. Furthermore the condition $\mathrm{rank}(A^2-I, B-I; (A-B)\binom{a}{b}) = 2$ shows that $b \neq 0$. That is we obtain the two families of representations:

$$(\begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}, I; \begin{pmatrix} m \\ n \end{pmatrix})^{0\leq m<\lambda}_{n\neq 0}, \quad (A, I; \begin{pmatrix} a \\ b \end{pmatrix}, \det(A) = 1, \ |\mathrm{trace}(A)| > 2).$$

A representative set of distinct bundle-spaces represented by the above triples is in bijection with the module $\mathbb{Z}^2/\langle -A-I, A^{-2}-I, P\rangle$ where the matrix $P \in$

$\mathrm{GL}(2,\mathbb{Z})$ ranges over the centralizer of A in $\mathrm{GL}(2,\mathbb{Z})$. If $\operatorname{rank}(B-I)=1$ then we may assume $B=\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}_{\lambda\neq 0}$. However an easy calculation shows that there are no matrices $A\in \mathrm{SL}(2,\mathbb{Z})$ that satisfy $AB=B^{-1}A$. Thus we get no manifolds under these conditions.

If $\operatorname{rank}(B-I)=2$ then we may assume exclusively either $B=\begin{pmatrix}-1&\lambda\\0&-1\end{pmatrix}$ or B has irrational eigenvalues. In the first case examining the relation $AB=B^{-1}A$ with the condition that $\det(A)=1$ shows that $\lambda=0$. That is $B=-I$ and the possiblities for A are: I, $\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}$, $\begin{pmatrix}-1&\lambda\\0&-1\end{pmatrix}$ or that A has irrational eigenvalues.

In the first instance, a change of cross-section reduces the Euler number to $\binom{(1)}{(1)}$ in correspondence with $H^2(\mathbb{Z}\tilde{\times}\mathbb{Z};\mathbb{Z}^2)\cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. However by changing the framing of the fibre we see that there are just two distinct representations depending on whether $\binom{a}{b}$ is trivial or not. That is we obtain the pair of triples:

$$(I,-I;\begin{pmatrix}0\\0\end{pmatrix}),\quad (I,-I;\begin{pmatrix}1\\0\end{pmatrix}).$$

These triples represent non-diffeomorphic manifolds which are distinguished by $H_1(M)$ being isomorphic to $\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})^3$ or $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ respectively.

By replacing A by $A^{-1}B$ we see that the following two possibilities for A give rise to the same family of representations [indexed by λ]. As before, modulo a change in the fibre we may assume that $\lambda\in\mathbb{N}$. Furthermore we are able transform the cross-section for this family so that $\binom{a}{b}=\binom{(1)}{(1)}$ if λ is even, or $\binom{0}{(1)}$ if λ is odd; in accordance with $H^2(\mathbb{Z}\tilde{\times}\mathbb{Z};\mathbb{Z}^2)$ having order four or two respectively. However when λ is even, the Euler number admits additional refinement via a change in the fibre equating $\binom{1}{1}$ and $\binom{1}{0}$. That is we obtain the six infinite families of representations:

$$(\begin{pmatrix}1&\lambda\\0&1\end{pmatrix},-I;\binom{a}{b})_{\lambda\in 2\mathbb{N}},\quad (\begin{pmatrix}1&\lambda\\0&1\end{pmatrix},-I;\begin{pmatrix}0\\(1)\end{pmatrix})_{\lambda\in\mathbb{N}\setminus 2\mathbb{N}};$$

where $\binom{a}{b}$ belongs to the set $\{\binom{0}{0},\binom{1}{0},\binom{0}{1}\}$. These triples are seen to represent distinct bundle-spaces as follows.

If λ is even, then the triple with trivial Euler number has $H_1(M)\cong\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})^2$ in contrast to the other two families with $H_1(M)\cong\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$. These remaining two bundle-spaces are distinguished by their double covers.

If λ is odd then $H_1(M)\cong\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})^2$ when the Euler number is trivial and is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ otherwise.

Finally when A has irrational eigenvalues we obtain the family of representations:

$$(A,-I;\binom{a}{b},\det(A)=1,|\operatorname{trace}(A)|>2).$$

Note, there are at most four bundle-spaces corresponding to each choice of A since $H^2(\mathbb{Z}\tilde{\times}\mathbb{Z};\mathbb{Z}^2)$ is some quotient of $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. Here the double covering of $(I,-I;\begin{pmatrix}1\\0\end{pmatrix})$ is $(I,-I;\begin{pmatrix}-2\\0\end{pmatrix})$ in contrast to the double coverings of the manifolds $(\begin{pmatrix}1&\lambda\\0&1\end{pmatrix},-I;\begin{pmatrix}(1)\\(1)\end{pmatrix})$ and $(\begin{pmatrix}-1&\lambda\\0&-1\end{pmatrix},-I;\begin{pmatrix}(1)\\(1)\end{pmatrix})$ which are of the form $(\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, -I; \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right))_{\mu\neq 0}$: a distinct family of bundle-spaces to the above $Nil^3 \times E$-manifold [from table 2.]. This latter family $(A, -I; \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), \det(A) = 1, |\mathrm{trace}(A)| > 2)$ of bundle-spaces differs from the previous family of $Sol^3 \times E$ bundle-spaces discovered; since all algebraic transformations preserve the conjugacy class of B. For the sake of brevity we tabulate these two families together. If B has irrational eigenvalues then there exists a matrix $P \in \mathrm{GL}(2,\mathbb{R})$ such that $P^{-1}BP = \begin{pmatrix} e^t & 0 \\ 0 & -e^t \end{pmatrix}_{t\neq 0}$. Writing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and examining the relation $P^{-1}ABA^{-1}BP = I$ we see that A is $\mathbb{R}$-conjugate to $\begin{pmatrix} 0 & -s \\ s^{-1} & 0 \end{pmatrix}$ for some real $s \neq 0$. Now conjugating this form of A by $\begin{pmatrix} s^{-1/2} & 0 \\ 0 & s^{1/2} \end{pmatrix}$ we see that A is $\mathbb{R}$-conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and hence $\mathbb{Z}$-conjugate to the same matrix. That is we obtain the family of representations:

$$(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B; \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), \ \det(B) = 1, \ |\mathrm{trace}(B)| > 2).$$

[Note: for the above family of triples there are at most two distinct bundle-spaces corresponding to each choice of B since $H^2(\mathbb{Z}\tilde{\times}\mathbb{Z};\mathbb{Z}^2)$ is some quotient of $\mathbb{Z}/2\mathbb{Z}$. Once again this family of $Sol^3 \times E$ bundle-spaces differs from the previous two discovered as any algebraic transformation of this representation preserves the conjugacy class of the matrix B.

C_{32}) The case $\det(A) = -1 = \det(B)$.
In this case $\mathrm{rank}(B - I) \geq 1$. We proceed acordingly. If $\mathrm{rank}(B - I) = 1$ then we may assume that $B = \begin{pmatrix} -1 & (1) \\ 0 & 1 \end{pmatrix}$. If (1)=0 the relation $AB = B^{-1}A$ implies that A is diagonal; hence $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the former matrix contradicts the assumtion that $\mathrm{rank}(A-I, B-I) = 2$. Thus $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the condition $\mathrm{rank}(A^2 - I, B - I; (A - B)\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)) = 2$ implies $b \neq 0$. Hence on replacing A by $A^{-1}B$, then conjugating and applying a change of cross-section we obtain the family of representations:

$$(-I, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} (1) \\ n \end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}.$$

This family of triples represents the equivalence classes of the corresponding extensions with the above given action as $H^2(\mathbb{Z}\tilde{\times}\mathbb{Z};\mathbb{Z}^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. For a fixed $n \in \mathbb{N}$ such bundle-spaces are doubly covered by the non-orientable T^2-bundle over T^2 represented by the triple: $(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, I; \left(\begin{smallmatrix} 0 \\ -2n \end{smallmatrix}\right))$, showing that n is an invariant of the diffeomorphism type. To see that triples with Euler number differing in the first component represent distinct bundle-spaces, we consider the characteristic subgroup $G \lhd \Pi_1(M)$ that is the intersection of the kernels of all homomorphsims $\varkappa\colon \Pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$. Here we have:

$$G = \begin{cases} \langle \alpha^2, \beta^2, l^2, h \rangle, & \text{if } (1) = 0; \\ \langle \alpha^2, \beta^2, l^2, h^2 \rangle, & \text{if } (1) = 1; \end{cases}$$

and hence in both cases the commutator subgroup of G,

$$[G,G] = \langle \alpha^2, \beta^2 \rangle = \langle h^{4n} \rangle.$$

Thus the isolator $\mathcal{I}([G,G]) = \langle h \rangle$ is characteristic in $\Pi_1(M)$; and the respective abelianized quotients of $\Pi_1(M)$ by $\langle h \rangle$ yield $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ or $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, distinguishing the triples. Here $H_1(M) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$ when n is even and $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ when n is odd when (1)=0. And $H_1(M) = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ when (1)=1.

If the entry (1)=1 in B then following similar methods to above we find that $A = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$ and $b \neq 0$. Hence replacing A by $A^{-1}B$ and changing the cross-section we may write this family of representations as: $(-I, \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right); \left(\begin{smallmatrix} 0 \\ n \end{smallmatrix}\right))_{n \in \mathbb{N} \setminus \{0\}}$; which were already discovered in case C_{22}. If $\operatorname{rank}(B - I) = 2$ then there exists a matrix $P \in \mathrm{GL}(2, \mathbb{R})$ such that $P^{-1}BP = \left(\begin{smallmatrix} e^t & 0 \\ 0 & -e^{-t} \end{smallmatrix}\right)_{t \neq 0}$. Writing $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and examining the relation $P^{-1}ABA^{-1}BP = I$ with the condition that $\det(A) = -1$ we see there are no such matrices $A \in \mathrm{GL}(2, \mathbb{R})$ that satisfy these properties. Thus this case gives no bundle-spaces.

We proceed by showing the bundle-spaces admit geometries as claimed. Obviously the product bundle $Kb \times T^2 = (I, I; \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right))$ is geometric of type E^4.

The bundle-space $(I,\ I; \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \;\middle|\; \begin{array}{ll} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l,h), \\ \alpha\beta\alpha^{-1}\beta = l, & \beta(l,h)\beta^{-1} = (l,h). \end{array} \right\rangle.$$

We define a representation $\phi \colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4, \mathbb{Z})$ as follows:

$$l \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ I),$$

$$h \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle$ to $\mathbb{R}^4 = \langle e_1, 2e_2 - e_3, e_3, e_4 \rangle$ to obtain a representation $\Phi \colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$l \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix},\ I),$$

$$h \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta, l, h \rangle$ of $\Pi_1(M)$ has index 2 and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(I,\ I; \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))$ admits an E^4 geometry.

The family of bundle-spaces $(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},\ I; \begin{pmatrix} (1) \\ (1) \end{pmatrix})_{\lambda\neq 0}$ admit geometries of type $Nil^3 \times E$.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix} [l,h]=1, & \alpha(l,h)\alpha^{-1} = (l, l^{\lambda}h), \\ \alpha\beta\alpha^{-1}\beta = l^{(1)}h^{(1)}, & \beta(l,h)\beta^{-1} = (l,h). \end{matrix} \right\rangle.$$

We define a representation $\Phi \colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(l)(x,y,z,w) &= (x, y, z+1, w), \\
\text{i.e} \quad \Phi(l) &= (\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ (0,1)). \\
\Phi(h)(x,y,z,w) &= (x+h_0, y, z+h_0 y, w) \\
\text{i.e} \quad \Phi(h) &= (\begin{pmatrix} 1 & h_0 & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ (0,1)). \\
\Phi(\beta)(x,y,z,w) &= (x+b_0, y, z+b_0 y + b_1, w+1) \\
\text{i.e} \quad \Phi(\beta) &= (\begin{pmatrix} 1 & b_0 & b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ (1,1)). \\
\Phi(\alpha)(x,y,z,w) &= (x, y+1, z, -w) \\
\text{i.e} \quad \Phi(\alpha) &= (\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ (0,\mathfrak{r}));
\end{aligned}$$

where h_0 is non-zero.

The relations $[l,h] = 1$, $\beta(l,h)\beta^{-1} = (l,h)$ and $\alpha l \alpha^{-1} = l$ are automatically satisfied. The relation $\alpha h \alpha^{-1} = l^{\lambda}h$ determines that $h_0 = -\lambda$; and the remaining relation $\alpha\beta\alpha^{-1}\beta = l^a h^b$ implies that $b_0 = -b\lambda/2$ and $b_1 = \frac{1}{4}(2a - \lambda b)$.

When $b = 0$ to see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1, y_1, z, w)\} \leq Isom(Nil^3 \times E)$ in the image of Φ for any fixed $x_1, y_1 \in \mathbb{R}$ is the group $\Phi(K) = \langle \Phi(l), \Phi(\beta) \rangle = \langle (z,w) \mapsto (z+1, w), (z,w) \mapsto z + a/2, w+1) \rangle \cong \mathbb{Z}^2$. Clearly K acts on $\mathbb{R}^2\langle z, w\rangle$ with quotient T^2. Similarly $\overline{\Phi}(\Pi_1(M)/K) = \langle (x,y) \overset{[\alpha]}{\mapsto} (x, y+1), (x,y) \overset{[h]}{\mapsto} (x - \lambda, y) \rangle \cong \mathbb{Z}^2$ acts on $\mathbb{R}^2\langle x, y\rangle$ with quotient T^2 so we have the desired representation when $b = 0$.

When $b = 1$ the group $\Phi(K) = \langle (z,w) \overset{[l]}{\mapsto} (z+1, w),\ (z,w) \overset{[\beta^2 h^{-1}]}{\longmapsto} (z, w+1) \rangle \cong \mathbb{Z}^2$; and acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient T^2. And the group $\overline{\Phi}(\Pi_1(M)/K) = \langle (x,y) \overset{[\alpha]}{\mapsto} (x, y+1), (x,y) \overset{[\beta^2 = h]}{\longmapsto} (x - \lambda, y) \rangle$ is similarly isomorphic to $\mathbb{Z}^2$ and acts on $\mathbb{R}^2\langle x, y\rangle$ with quotient T^2. Thus the family $(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},\ I; \begin{pmatrix} (1) \\ (1) \end{pmatrix})_{\lambda\neq 0}$ admit geometries of type $Nil^3 \times E$. In fact we have shown that this family of bundle-spaces also fibre over T^2 with fibre T^2.

The bundle-space $(I,\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \end{pmatrix})$ admits a geometry of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix} [l,h]=1, & \alpha(l,h)\alpha^{-1} = (l,h), \\ \alpha\beta\alpha^{-1}\beta = 1, & \beta(l,h)\beta^{-1} = (l^{-1},h). \end{matrix} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, I), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}),$$
$$h \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, l, h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\2\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(I, \begin{pmatrix} -1&0\\0&1 \end{pmatrix}; \begin{pmatrix} 0\\0 \end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(I, \begin{pmatrix} -1&0\\0&1 \end{pmatrix}; \begin{pmatrix} 1\\0 \end{pmatrix})$ admits a geometry of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \;\middle|\; \begin{matrix} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l,h), \\ \alpha\beta\alpha^{-1}\beta = l, & \beta(l,h)\beta^{-1} = (l^{-1},h). \end{matrix} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, I), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\-1&0&-1&0\\0&0&0&1 \end{pmatrix}),$$
$$h \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4\rangle$ to $\mathbb{R}^4 = \langle 2e_1 - e_3, e_2, e_3, e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, I), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}),$$
$$h \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} \frac{1}{2}\\0\\\frac{1}{2}\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, l, h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\2\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(I, \begin{pmatrix} -1&0\\0&1 \end{pmatrix}; \begin{pmatrix} 1\\0 \end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(I,\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ admits a geometry of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \;\middle|\; \begin{matrix} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l,h), \\ \alpha\beta\alpha^{-1}\beta = h, & \beta(l,h)\beta^{-1} = (l^{-1},h). \end{matrix} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$l \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}),$$

$$h \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle$ to $\mathbb{R}^4 = \langle e_1, 2e_2 - e_4, e_3, e_4 \rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$l \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}),$$

$$h \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, l, h \rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(I, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(I,\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ admits a geometry of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \;\middle|\; \begin{matrix} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l,h), \\ \alpha\beta\alpha^{-1}\beta = lh, & \beta(l,h)\beta^{-1} = (l^{-1},h). \end{matrix} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$l \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}),$$

$$h \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle$ to $\mathbb{R}^4 = \langle 2e_1 - e_3, 2e_2 - e_4, e_3, e_4 \rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$l \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}),$$

$$h \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, l, h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(I, \begin{pmatrix}-1&0\\0&1\end{pmatrix}; \begin{pmatrix}1\\1\end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(I,\ \begin{pmatrix}-1&1\\0&1\end{pmatrix}; \begin{pmatrix}0\\0\end{pmatrix})$ admits a geometry of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix}[l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l,h),\\ \alpha\beta\alpha^{-1}\beta = 1, & \beta(l,h)\beta^{-1} = (l^{-1}, lh).\end{matrix} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$l \mapsto (\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&1\\0&0&0&1\end{pmatrix}),$$
$$h \mapsto (\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4\rangle$ to $\mathbb{R}^4 = \langle e_1, e_2 - 2e_3, e_3, e_3 + 2e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$l \mapsto (\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}),$$
$$h \mapsto (\begin{pmatrix}0\\0\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, l, h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix}2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(I, \begin{pmatrix}1&-1\\0&1\end{pmatrix}; \begin{pmatrix}0\\0\end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(I,\ \begin{pmatrix}-1&1\\0&1\end{pmatrix}; \begin{pmatrix}0\\1\end{pmatrix})$ admits a geometry of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix}[l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l,h),\\ \alpha\beta\alpha^{-1}\beta = h, & \beta(l,h)\beta^{-1} = (l^{-1}, lh).\end{matrix} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$l \mapsto (\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\\frac{1}{2}&\frac{1}{2}&-1&1\\0&0&0&1\end{pmatrix}),$$
$$h \mapsto (\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&1&0&1\end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4\rangle$ to $\mathbb{R}^4 = \langle 4e_1 + e_3, 2e_2 - e_4, e_3, -2e_1 + e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0\end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} \frac14\\ \frac12\\ -\frac14\\ \frac12\end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}),$$
$$h \mapsto (\begin{pmatrix} \frac12\\0\\-\frac12\\1\end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} \frac14\\0\\-\frac14\\0\end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}).$$

The subgroup $\mathcal{A}\langle\alpha^2, \beta^2, l, h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} \frac12\\1\\0\\1\end{pmatrix}, \begin{pmatrix} \frac12\\0\\-\frac12\\0\end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0\end{pmatrix}, \begin{pmatrix} \frac12\\0\\-\frac12\\1\end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(I, \begin{pmatrix} 1&-1\\0&1\end{pmatrix}; \begin{pmatrix} 0\\1\end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(-I,\ I; \begin{pmatrix} 0\\0\end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l^{-1}, h^{-1}), \\ \alpha\beta\alpha^{-1}\beta = 1, & \beta(l,h)\beta^{-1} = (l,h). \end{matrix} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as follows:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0\end{pmatrix},\ I), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0\end{pmatrix},\ I),$$
$$h \mapsto (\begin{pmatrix} 0\\0\\0\\1\end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}).$$

The subgroup $\mathcal{A}\langle\alpha^2, \beta, l, h\rangle$ of $\Pi_1(M)$ has index 2 and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2\\0\\0\\0\end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0\end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(I, -I; \binom{0}{0})$ admits an E^4 geometry.

The bundle-space $(-I, I; \begin{pmatrix} 1\\0\end{pmatrix})$ admits a $Nil^3 \times E$ geometry.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l^{-1}, h^{-1}), \\ \alpha\beta\alpha^{-1}\beta = l, & \beta(l,h)\beta^{-1} = (l,h). \end{matrix} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(l)(x,y,z,w) &= (x,y,z-1,w),\\
\text{i.e}\quad \Phi(l) &= (\begin{pmatrix} 1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ (0,1)).\\
\Phi(h)(x,y,z,w) &= (x,y,z,w+1)\\
\text{i.e}\quad \Phi(h) &= (I,\ (1,1)).\\
\Phi(\beta)(x,y,z,w) &= (x,y+1,z,w)\\
\text{i.e}\quad \Phi(\beta) &= (\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix},\ (0,1)).\\
\Phi(\alpha)(x,y,z,w) &= (x+1,-y,-z-y,-w)\\
\text{i.e}\quad \Phi(\alpha) &= (\begin{pmatrix} 1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ (0,\mathfrak{r}))\rho.
\end{aligned}$$

Here ρ is the automorphism $(x,y,z,w)\overset{\rho}{\mapsto}(x,-y,-z,w)$.

All the relations are automatically satisfied. To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1,y_1,z,w)\} \le Isom(Nil^3 \times E)$ in the image of Φ for any fixed $x_1,y_1 \in \mathbb{R}$ is the group $\Phi(K) = \langle \Phi(l),\Phi(h)\rangle \cong \mathbb{Z}^2$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient T^2. Similarly $\overline{\Phi}(\Pi_1(M)/K) = \langle (x,y)\overset{[\alpha]}{\mapsto}(x+1,-y),(x,y)\overset{[\beta]}{\mapsto}(x,y+1)\rangle \cong \mathbb{Z}\tilde{\times}\mathbb{Z}$ acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient Kb so we have the desired representation.

The bundle-space $(-I,\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix};\begin{pmatrix} 0\\ 0\end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha,\beta,l,h \,\middle|\, \begin{array}{ll} [l,h]=1, & \alpha(l,h)\alpha^{-1} = (l^{-1},h^{-1}),\\ \alpha\beta\alpha^{-1}\beta = 1, & \beta(l,h)\beta^{-1} = (l^{-1},h). \end{array} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as follows:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0\end{pmatrix},\ I),\qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}),$$
$$h \mapsto (\begin{pmatrix} 0\\0\\0\\1\end{pmatrix},\ I),\qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2,\beta^2,l,h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2\\0\\0\\0\end{pmatrix},\begin{pmatrix} 0\\2\\0\\0\end{pmatrix},\begin{pmatrix} 0\\0\\1\\0\end{pmatrix},\begin{pmatrix} 0\\0\\0\\1\end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(-I,\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix};\begin{pmatrix} 0\\ 0\end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(-I,\ \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix};\begin{pmatrix} 1\\ 0\end{pmatrix})$ admits a geometry of type E^4.

$$\Pi_1(M) = \left\langle \alpha,\beta,l,h \,\middle|\, \begin{array}{ll} [l,h]=1, & \alpha(l,h)\alpha^{-1} = (l^{-1},h^{-1}),\\ \alpha\beta\alpha^{-1}\beta = l, & \beta(l,h)\beta^{-1} = (l^{-1},h). \end{array} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, I), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&1&-1&0\\0&0&0&1 \end{pmatrix}),$$
$$h \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4\rangle$ to $\mathbb{R}^4 = \langle e_1, 2e_2+e_3, e_3, e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$l \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, I), \qquad \beta \mapsto (\begin{pmatrix} 0\\\frac{1}{2}\\-\frac{1}{2}\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}),$$
$$h \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, l, h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\\frac{1}{2}\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(-I, \begin{pmatrix} -1&0\\0&1 \end{pmatrix}; \begin{pmatrix} 1\\0 \end{pmatrix})$ admits an E^4 geometry.

The family of bundle-spaces $(-I, \begin{pmatrix} -1&1\\0&1 \end{pmatrix}; \begin{pmatrix} 0\\n \end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}$ admit geometries of type $Nil^3 \times E$.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l^{-1}, h^{-1}), \\ \alpha\beta\alpha^{-1}\beta = h^n, & \beta(l,h)\beta^{-1} = (l^{-1}, lh). \end{matrix} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(l)(x,y,z,w) &= (x,y,z,w+1) \\
\text{i.e} \quad \Phi(h) &= (I,\ (1,1)). \\
\Phi(h)(x,y,z,w) &= (x,y,z+1,w+k), \\
\text{i.e} \quad \Phi(l) &= (\begin{pmatrix} 1&0&1\\0&1&0\\0&0&1 \end{pmatrix},\ (k,1)). \\
\Phi(\beta)(x,y,z,w) &= (x,y+1,z,-w) \\
\text{i.e} \quad \Phi(\beta) &= (\begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix},\ (0,\mathfrak{r})). \\
\Phi(\alpha)(x,y,z,w) &= (x+1,-y,-z-y+a_0,-w+a_1) \\
\text{i.e} \quad \Phi(\alpha) &= (\begin{pmatrix} 1&1&a_0\\0&1&0\\0&0&1 \end{pmatrix},\ (a_1,\mathfrak{r}))\rho.
\end{aligned}$$

Here ρ is the automorphism $(x,y,z,w) \overset{\rho}{\mapsto} (x,-y,-z,w)$.

The relations $[l,h]=1$, $\alpha(l,h)\alpha^{-1}=(l^{-1},h^{-1})$ and $\beta l\beta^{-1}=l^{-1}$ are automatically satisfied. The relation $\beta h\beta^{-1}=lh$ implies that $k=-1/2$; and the relation $\alpha\beta\alpha^{-1}\beta=h^n$ gives the relations $a_0=(n+1)/2$ and $a_1=-n/4$. To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1,y_1,z,w)\}\leq Isom(Nil^3\times E)$ in the image of Φ for any fixed $x_1,y_1\in\mathbb{R}$ is the group $\Phi(K)=\langle\Phi(l),\Phi(h)\rangle\cong\mathbb{Z}^2$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient T^2. Similarly $\overline{\Phi}(\Pi_1(M)/K)=\langle(x,y)\overset{[\alpha]}{\mapsto}(x+1,-y),(x,y)\overset{[\beta]}{\mapsto}(x,y+1)\rangle\cong\mathbb{Z}\tilde{\times}\mathbb{Z}$ acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient Kb so we have the desired representation.

The Lie group Nil^4 is the semi-direct product of $\mathbb{R}^3=\{\begin{pmatrix}x\\y\\z\end{pmatrix}\}$ and $\mathbb{R}\langle t\rangle$ where the action of $t\in\mathbb{R}$ is left multiplication by the matrix $\begin{pmatrix}1&t&t^2/2\\0&1&t\\0&0&1\end{pmatrix}$. The identity component $Isom^0(Nil^4)$ of the isometry group is again Nil^4. The quotient group $Isom(Nil^4)/Isom^0(Nil^4)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ generated by the automorphisms:

$$(x,y,z,t)\mapsto(-x,-y,-z,t),\qquad (x,y,z,t)\mapsto(x,-y,z,-t).$$

The commutator subgroup of $Nil^4=Isom^0(Nil^4)$ is $\mathbb{R}^2\langle x,y\rangle$ so abelianization gives Nil^4 the structure of a plane-bundle over $\mathbb{R}^2\langle z,t\rangle$:

$$1\longrightarrow\mathbb{R}^2\langle x,y\rangle\longrightarrow Nil^4\longrightarrow\mathbb{R}^2\langle z,t\rangle\longrightarrow 1.$$

Passing to isometries we see that the above automorphisms preserve the bundle structure so we have an exact sequence:

$$1\longrightarrow Isom(\mathbb{R}^2)\longrightarrow Isom(Nil^4)\longrightarrow Isom(\mathbb{R}^2)\longrightarrow 1.$$

Lemma. *Let $\wp\colon Isom(Nil^4)\twoheadrightarrow Isom(E^2)$ be the induced map of the $\mathbb{R}^2$-bundle projection. A representation $\Phi\colon\Pi_1(M)\to Isom(Nil^4)$ is faithful and discrete with cocompact image precisely when the following two properties are satisfied:*

1. *The subgroup $K\trianglelefteq\Pi_1(M)$ whose image $\Phi(K)=\Phi(\Pi_1(M))\cap\ker(\wp)$ embeds faithfully, discretely and cocompactly in $Isom(E^2)$.*
2. *The quotient group $\Pi_1(M)/K$ embeds faithfully and discretely in $Isom(E^2)$ with cocompact image under the induced map from Φ.*

proof:
The proof is entirely analogous to that of the $Nil^3\times E$ or $Sol^3\times E$ case. ☐

The family of bundle-spaces $(\begin{pmatrix}-1&\lambda\\0&-1\end{pmatrix},\ I;\begin{pmatrix}m\\n\end{pmatrix})_{\lambda,n\neq0}$ admit geometries of type Nil^4.

$$\Pi_1(M)=\left\langle\alpha,\beta,l,h\ \middle|\ \begin{matrix}[l,h]=1, & \alpha(l,h)\alpha^{-1}=(l^{-1},l^{\lambda}h^{-1}),\\ \alpha\beta\alpha^{-1}\beta=l^mh^n, & \beta(l,h)\beta^{-1}=(l,h).\end{matrix}\right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to Isom(Nil^4)$ as follows:

$$\begin{aligned}
\Phi(l)(x,y,z,t) &= (x+l_0, y, z, t),\\
\text{i.e}\quad \Phi(l) &= (\begin{pmatrix} l_0\\ 0\\ 0\end{pmatrix}, 0).\\
\Phi(h)(x,y,z,t) &= (x, y+h_0, z, t)\\
\text{i.e}\quad \Phi(h) &= (\begin{pmatrix} 0\\ h_0\\ 0\end{pmatrix}, 0).\\
\Phi(\beta)(x,y,z,t) &= (x, y+b_1, z+1, t)\\
\text{i.e}\quad \Phi(\beta) &= (\begin{pmatrix} 0\\ b_1\\ 1\end{pmatrix}, 0).\\
\Phi(\alpha)(x,y,z,t) &= (-x-y-z/2, -y-z, -z, t+1)\\
\text{i.e}\quad \Phi(\alpha) &= (\begin{pmatrix} 0\\ 0\\ 0\end{pmatrix}, 1)\rho.
\end{aligned}$$

where h_0, l_0 are non-zero and ρ is the automorphism $(x,y,z,t) \overset{\rho}{\mapsto} (-x,-y,-z,t)$.

The relations $[l,h]=1$, $\alpha l \alpha^{-1} = l^{-1}$ and $\beta(l,h)\beta^{-1} = (l,h)$ are automatically satisfied. The relation $\alpha h \alpha^{-1} = l^{\lambda} h^{-1}$ implies that $l_0 = -h_0/\lambda$; and the relation $\alpha\beta\alpha^{-1}\beta = l^m h^n$ gives the relations $h_0 = -1/n$ and $b_1 = -ml_0 - 1/2$. Therefore $l_0 = 1/n\lambda$ and $b_1 = -m/n\lambda - 1/2$.

Here the group $\Phi(K) = \mathrm{Stab}_{\Phi(\Pi_1(M))}\langle z,t\rangle$ is $\langle \Phi(l), \Phi(h)\rangle = \langle (x,y) \mapsto (x + 1/n\lambda, y),\ (x,y) \mapsto (x, y-1/n)\rangle \cong \mathbb{Z}^2$ and acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient T^2. The group $\overline{\Phi}(\Pi_1(M)/K) = \langle (z,t) \overset{[\alpha]}{\mapsto} (z+1,t),\ (z,t) \overset{[\beta]}{\mapsto} (-z, t+1)$ is isomorphic to $\mathbb{Z}\tilde{\times}\mathbb{Z}$ and acts on $\mathbb{R}^2\langle z,t\rangle$ with quotient the Klein bottle. Thus this representation is faithful and discrete with cocompact image so that the family $(\begin{pmatrix} -1 & \lambda\\ 0 & -1\end{pmatrix},\ I; \binom{m}{n})_{\lambda, n\neq 0}$ admit geometries of type Nil^4.

The bundle-space $(I, -I; \binom{0}{0})$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{matrix} [l,h]=1, & \alpha(l,h)\alpha^{-1} = (l,h),\\ \alpha\beta\alpha^{-1}\beta = 1, & \beta(l,h)\beta^{-1} = (l^{-1}, h^{-1}). \end{matrix} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as follows:

$$\begin{aligned}
l &\mapsto (\begin{pmatrix} 0\\0\\1\\0\end{pmatrix},\ I), &\quad \beta &\mapsto (\begin{pmatrix} 0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}),\\
h &\mapsto (\begin{pmatrix} 0\\0\\0\\1\end{pmatrix},\ I), &\quad \alpha &\mapsto (\begin{pmatrix} 1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}).
\end{aligned}$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, l, h\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2\\0\\0\\0\end{pmatrix}, \begin{pmatrix} 0\\2\\0\\0\end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0\end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(I, -I; \binom{0}{0})$ admits an E^4 geometry.

The bundle-space $(I, -I; \binom{1}{0})$ admits a $Nil^3 \times E$ geometry.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{array}{ll} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l,h), \\ \alpha\beta\alpha^{-1}\beta = l, & \beta(l,h)\beta^{-1} = (l^{-1}, h^{-1}). \end{array} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(l)(x,y,z,w) &= (x, y, z+1, w), \\
\text{i.e} \quad \Phi(l) &= (\begin{pmatrix} 1&0&1\\0&1&0\\0&0&1 \end{pmatrix},\ (0,1)). \\
\Phi(h)(x,y,z,w) &= (x, y, z, w+1) \\
\text{i.e} \quad \Phi(h) &= (I,\ (1,1)). \\
\Phi(\beta)(x,y,z,w) &= (x, -y+1, -z, -w) \\
\text{i.e} \quad \Phi(\beta) &= (\begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix},\ (0,\mathfrak{r}))\rho. \\
\Phi(\alpha)(x,y,z,w) &= (x+1, y, z+y, w) \\
\text{i.e} \quad \Phi(\alpha) &= (\begin{pmatrix} 1&1&0\\0&1&0\\0&0&1 \end{pmatrix},\ (0,1)).
\end{aligned}$$

Here ρ is the automorphism $(x,y,z,w) \overset{\rho}{\mapsto} (x,-y,-z,w)$.

All the relations are automatically satisfied. To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1, y_1, z, w)\} \le Isom(Nil^3 \times E)$ in the image of Φ for any fixed $x_1, y_1 \in \mathbb{R}$ is the group $\Phi(K) = \langle \Phi(l), \Phi(h) \rangle \cong \mathbb{Z}^2$. Clearly K acts on $\mathbb{R}^2\langle z, w\rangle$ with quotient T^2. Similarly $\overline{\Phi}(\Pi_1(M)/K) = \langle (x,y) \overset{[\alpha]}{\mapsto} (x+1, y), (x,y) \overset{[\beta]}{\mapsto} (x, -y+1) \rangle \cong \mathbb{Z} \tilde{\times} \mathbb{Z}$ acts on $\mathbb{R}^2\langle x, y\rangle$ with quotient Kb so we have the desired representation.

The family of bundle-spaces $(\begin{pmatrix} 1&\lambda\\0&1 \end{pmatrix}, -I; \begin{pmatrix} (1)\\(1) \end{pmatrix})_{\lambda\neq 0}$ admit geometries of type $Nil^3 \times E$.

$$\Pi_1(M) = \left\langle \alpha, \beta, l, h \,\middle|\, \begin{array}{ll} [l,h] = 1, & \alpha(l,h)\alpha^{-1} = (l, l^{\lambda}h), \\ \alpha\beta\alpha^{-1}\beta = l^{(1)}h^{(1)}, & \beta(l,h)\beta^{-1} = (l^{-1}, h^{-1}). \end{array} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(l)(x,y,z,w) &= (x, y, z+1, w), \\
\text{i.e} \quad \Phi(l) &= (\begin{pmatrix} 1&0&1\\0&1&0\\0&0&1 \end{pmatrix},\ (0,1)). \\
\Phi(h)(x,y,z,w) &= (x+h_0, y, z+h_0 y, w) \\
\text{i.e} \quad \Phi(h) &= (\begin{pmatrix} 1&h_0&\\0&1&0\\0&0&1 \end{pmatrix},\ (0,1)). \\
\Phi(\beta)(x,y,z,w) &= (-x+b_0, y, -z+b_0 y, w+1) \\
\text{i.e} \quad \Phi(\beta) &= (\begin{pmatrix} 1&b_0&0\\0&1&0\\0&0&1 \end{pmatrix},\ (1,1))\rho. \\
\Phi(\alpha)(x,y,z,w) &= (x+a_0, y+1, z+a_0 y, -w) \\
\text{i.e} \quad \Phi(\alpha) &= (\begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix},\ (0,\mathfrak{r}));
\end{aligned}$$

where h_0 is non-zero and ρ is the automorphism $(x,y,z,w)\overset{\rho}{\mapsto}(-x,y,-z,w)$.

The relations $[l,h]=1$, $\beta(l,h)\beta^{-1}=(l^{-1},h^{-1})$ and $\alpha l\alpha^{-1}=l$ are automatically satisfied. The relation $\alpha h\alpha^{-1}=l^{\lambda}h$ determines that $h_0=-\lambda$; and the remaining relation $\alpha\beta\alpha^{-1}\beta=l^a h^b$ implies that $a_0=-b\lambda/2$, and $b_0=b\lambda/2-a$.

To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1,y_1,z,w)\}\leq Isom(Nil^3\times E)$ in the image of Φ for any fixed $x_1,y_1\in\mathbb{R}$ is the group $\Phi(K)=\langle\Phi(l),\Phi(\beta^2)\rangle=\langle(z,w)\mapsto(z+1,w),(z,w)\mapsto z,w+2)\rangle\cong\mathbb{Z}^2$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient T^2. Similarly the group $\overline{\Phi}(\Pi_1(M)/K)=\langle(x,y)\overset{[\alpha]}{\mapsto}(x-b\lambda/2,y+1),(x,y)\overset{[\beta]}{\mapsto}(-x+b\lambda/2-a,y),(x,y)\overset{[h]}{\mapsto}(x-\lambda,y)\rangle$ is isomorphic to the amalgamated free product $(\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z})*_{\mathbb{Z}}(\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z})$ when $b=0$, and to $\mathbb{Z}*_{\mathbb{Z}}(\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z})$ when $b=1$; and acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient the silvered annulus SAn or the silvered Möbius band SMb respectively. Thus we have the desired representation and the family $(\left(\begin{smallmatrix}1&\lambda\\0&1\end{smallmatrix}\right),-I;\left(\begin{smallmatrix}(1)\\(1)\end{smallmatrix}\right))_{\lambda\neq0}$ admit geometries of type $Nil^3\times E$. In fact we have shown that this family of bundle-spaces is not only Seifert fibered over Kb with fibre T^2 but is also Seifert fibred over either SAn or SMb with general fibre T^2.

The family of bundle-spaces $(A,\pm I;\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right),\det(A)=1,|\mathrm{trace}(A)|>2)$ **admit geometries of type** $Sol^3\times E$**.**

$$\Pi_1(M)=\left\langle\alpha,\beta,l,h\;\middle|\;\begin{matrix}[l,h]=1, & \alpha(l,h)\alpha^{-1}=(l,h)A,\\ \alpha\beta\alpha^{-1}\beta=l^ah^b, & \beta(l,h)\beta^{-1}=(l^{\pm1},h^{\pm1}).\end{matrix}\right\rangle.$$

We define a representation $\Phi\colon\Pi_1(M)\to Isom(Sol^3\times E)$ as follows: We have that $PAP^{-1}=\left(\begin{smallmatrix}e^{t}_0&0\\0&e^{-t_0}\end{smallmatrix}\right)$ for some $P=\left(\begin{smallmatrix}p&q\\r&s\end{smallmatrix}\right)\in\mathrm{GL}(2,\mathbb{R})$. Hence we define a representation $\Phi\colon\Pi_1(M)\to Isom(Sol^3\times E)$ as follows:

$$\Phi(l)(x,y,t,u)=(x+p,y+r,t,u)$$
$$\text{ie}\quad\Phi(l)=((\begin{pmatrix}p\\q\end{pmatrix},0),\ (0,1)).$$
$$\Phi(h)(x,y,t,u)=(x+q,y+s,t,u)$$
$$\text{ie}\quad\Phi(h)=((\begin{pmatrix}r\\s\end{pmatrix},0),\ (0,1)).$$
$$\Phi(\beta)(x,y,t,u)=(\epsilon x+x_1,\epsilon y+y_1,t,u+1)$$
$$\text{ie}\quad\Phi(\beta)=((\begin{pmatrix}x_1\\y_1\end{pmatrix},0),\ (1,1))\lambda.$$
$$\Phi(\alpha)(x,y,t,u)=(e^{t_0}x+x_0,e^{-t_0}y+y_0,t+t_0,-u)$$
$$\text{ie}\quad\Phi(\alpha)=((\begin{pmatrix}x_0\\y_0\end{pmatrix},t_0),\ (0,\mathfrak{r}))\quad t_0\neq0.$$

where $\epsilon=\pm1$ respective as to whether $B=\pm I$ and hence $(x,y,t,u)\overset{\lambda}{\longmapsto}(\epsilon x,\epsilon y,t,u)$.

The relations $[l,h]=1$, $\alpha(l,h)\alpha^{-1}=(l,h)A$ and $\beta(l,h)\beta^{-1}=(l^{\pm1},h^{\pm1})$ are automatically satisfied. The relation $[\alpha,\beta]=l^ah^b$ gives the relation:

$$\begin{pmatrix}x_1\\y_1\end{pmatrix}=\left(I+\epsilon PAP^{-1}\right)^{-1}\left((\epsilon-1)\begin{pmatrix}x_0\\y_0\end{pmatrix}+P\begin{pmatrix}a\\b\end{pmatrix}\right).$$

Here $K=\langle\Phi(l),\Phi(h)\rangle$. Since $P\in\mathrm{GL}(2,\mathbb{R})$ the vectors $\binom{p}{r}$ and $\binom{q}{s}$ are linearly independent and hence K acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient T^2.

The family of bundle-spaces $(\begin{pmatrix}0&-1\\1&0\end{pmatrix},B;\binom{a}{b},\det(B)=1,|\mathrm{trace}(B)|>2)$ **admit geometries of type** $Sol^3\times E$**.**

$$\Pi_1(M)=\left\langle \alpha,\beta,l,h\,\middle|\, \begin{matrix}[l,h]=1, & \alpha(l,h)\alpha^{-1}=(h,l^{-1}, \\ \alpha\beta\alpha^{-1}\beta=l^ah^b, & \beta(l,h)\beta^{-1}=(l,h)B.\end{matrix}\right\rangle.$$

We define a representation $\Phi\colon\Pi_1(M)\to Isom(Sol^3\times E)$ as follows: We have that $PBP^{-1}=\begin{pmatrix}e^t_1&0\\0&e^{-t_1}\end{pmatrix}$ for some $P=\begin{pmatrix}p&q\\r&s\end{pmatrix}\in\mathrm{GL}(2,\mathbb{R})$ that commutes with $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. Hence we define a representation $\Phi\colon\Pi_1(M)\to Isom(Sol^3\times E)$ as follows:

$$\begin{aligned}
\Phi(l)(x,y,t,u)&=(x+p,y+r,t,u)\\
\text{ie}\quad \Phi(l)&=((\begin{pmatrix}p\\q\end{pmatrix},0),\ (0,1)).\\
\Phi(h)(x,y,t,u)&=(x+q,y+s,t,u)\\
\text{ie}\quad \Phi(h)&=((\begin{pmatrix}r\\s\end{pmatrix},0),\ (0,1)).\\
\Phi(\beta)(x,y,t,u)&=(e^{t_1}x+x_1,e^{-t_1}y+y_1,t+t_0,u)\\
\text{ie}\quad \Phi(\beta)&=((\begin{pmatrix}x_1\\y_1\end{pmatrix},t_1),\ (0,1)).\\
\Phi(\alpha)(x,y,t,u)&=(y+x_0,-x+y_0,-t,u+1)\\
\text{ie}\quad \Phi(\alpha)&=((\begin{pmatrix}x_0\\y_0\end{pmatrix},t_0),\ (1,1))\rho\quad t_0\neq 0.
\end{aligned}$$

Here ρ is the automorphism $(x,y,t,u)\overset{\rho}{\longmapsto}(y,-x,-t,u)$.

The relations $[l,h]=1$, $\alpha(l,h)\alpha^{-1}=(l,h)A$ and $\beta(l,h)\beta^{-1}=(l^{\pm1},h^{\pm1})$ are automatically satisfied. The relation $[\alpha,\beta]=l^ah^b$ gives the relation:

$$\begin{pmatrix}x_0\\y_0\end{pmatrix}=PBP^{-1}\left(\left(PBP^{-1}-\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)\begin{pmatrix}x_1\\y_1\end{pmatrix}-P\begin{pmatrix}a\\b\end{pmatrix}\right).$$

Here $K=\langle\Phi(l),\Phi(h)\rangle$. Since $P\in\mathrm{GL}(2,\mathbb{R})$ the vectors $\binom{p}{r}$ and $\binom{q}{s}$ are linearly independent and hence K acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient T^2.

The families of bundle-spaces $(-I,\begin{pmatrix}-1&0\\0&1\end{pmatrix};\begin{pmatrix}(1)\\n\end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}$ **admit geometries of type** $Nil^3\times E$**.**

$$\Pi_1(M)=\left\langle \alpha,\beta,l,h\,\middle|\, \begin{matrix}[l,h]=1, & \alpha(l,h)\alpha^{-1}=(l^{-1},h^{-1}), \\ \alpha\beta\alpha^{-1}\beta=l^(1)h^n, & \beta(l,h)\beta^{-1}=(l^{-1},h).\end{matrix}\right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(l)(x,y,z,w) &= (x,y,z,w+1)\\
\text{i.e}\quad \Phi(h) &= (I,\ (1,1)).\\
\Phi(h)(x,y,z,w) &= (x,y,z+1,w),\\
\text{i.e}\quad \Phi(l) &= (\begin{pmatrix} 1&0&1\\ 0&1&0\\ 0&0&1\end{pmatrix},\ (0,1)).\\
\Phi(\beta)(x,y,z,w) &= (x,y+k,z,-w)\\
\text{i.e}\quad \Phi(\beta) &= (\begin{pmatrix} 1&0&0\\ 0&1&k\\ 0&0&1\end{pmatrix},\ (0,\mathfrak{r})).\\
\Phi(\alpha)(x,y,z,w) &= (x+1,-y,-z-y,-w+a)\\
\text{i.e}\quad \Phi(\alpha) &= (\begin{pmatrix} 1&1&0\\ 0&1&0\\ 0&0&1\end{pmatrix},\ (a,\mathfrak{r}))\rho.
\end{aligned}$$

Here ρ is the automorphism $(x,y,z,w)\overset{\rho}{\mapsto}(x,-y,-z,w)$.

The relations $[l,h]=1$, $\alpha(l,h)\alpha^{-1}=(l^{-1},h^{-1})$ and $\beta(l,h)\beta^{-1}=(l^{-1},h)$ are automatically satisfied. The relation $\alpha\beta\alpha^{-1}\beta = l^{(1)}h^n$ implies that $a=(1)/2$ and $k=-n$. To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1,y_1,z,w)\} \leq Isom(Nil^3 \times E)$ in the image of Φ for any fixed $x_1, y_1 \in \mathbb{R}$ is the group $\Phi(K) = \langle \Phi(l), \Phi(h)\rangle \cong \mathbb{Z}^2$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient T^2. Similarly $\overline{\Phi}(\Pi_1(M)/K) = \langle (x,y)\overset{[\alpha]}{\longmapsto}(x+1,-y), (x,y)\overset{[\beta]}{\longmapsto}(x,y-n)\rangle \cong \mathbb{Z}\tilde{\times}\mathbb{Z}$ acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient Kb so we have the desired representation.

The Classification of Total Spaces of Kb-Bundles over T^2 or Kb and their Geometries

Let M be the total space of a Kb-bundle over the surface S where S is either T^2 or Kb. If $S=T^2$ then the fundamental group of M is

$$\Pi_1(M) = \left\langle \alpha,\beta,\gamma,\delta \,\middle|\, \begin{array}{ll} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = \vartheta(\alpha)(\gamma,\delta),\\ [\alpha,\beta] = \gamma^m\delta^n, & \beta(\gamma,\delta)\beta^{-1} = \vartheta(\beta)(\gamma,\delta). \end{array}\right\rangle.$$

and extension equivalences are in bijection with:

$$H^2(\mathbb{Z}^2;\mathbb{Z}) \cong \mathbb{Z}/\langle \kappa(\alpha)-1, \kappa(\beta)-1\rangle$$

If $S=Kb$ then the fundamental group of M is

$$\Pi_1(M) = \left\langle \alpha,\beta,\gamma,\delta \,\middle|\, \begin{array}{ll} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = \vartheta(\alpha)(\gamma,\delta),\\ \alpha\beta\alpha^{-1}\beta = \gamma^m\delta^n, & \beta(\gamma,\delta)\beta^{-1} = \vartheta(\beta)(\gamma,\delta). \end{array}\right\rangle.$$

and extension equivalences are in bijection with:

$$H^2(\mathbb{Z}^2;\mathbb{Z}) \cong \mathbb{Z}/\langle -\kappa(\alpha)-1, \kappa(\beta)-1\rangle$$

where κ is the action on the center of $\mathbb{Z}) \cong \mathbb{Z}$ induced by ϑ.

Here the mapping class group of the fibre Kb is the group $\mathrm{Out}(\Pi_1(Kb)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ which we take to be the four automorphisms:

$$(\gamma,\delta) \xmapsto{e} (\gamma,\delta), \qquad (\gamma,\delta) \xmapsto{\varphi_2} (\gamma^{-1},\delta),$$
$$(\gamma,\delta) \xmapsto{\varphi_1} (\gamma\delta,\delta), \qquad (\gamma,\delta) \xmapsto{\varphi_3} (\gamma^{-1}\delta,\delta).$$

These representatives genuinely commute as automorphisms of $\Pi_1(Kb)$, however whilst φ_2^2 is the identity automorphism φ_1 and φ_3 square to give non-trivial inner automorphisms. Note that e and φ_1 act identically on $\mathcal{Z}(\Pi_1(F)) = \langle\gamma^2\rangle$ whilst the maps φ_2 and φ_3 act on the center by inversion.

When the base-space is T^2, the action given by $\vartheta([\alpha,\beta])$ is the identity inner automorphism, and hence is well-defined up to multiplication by central elements of $\Pi_1(F)$. Therefore the Euler number must be central so that $\binom{m}{n} \in \langle\binom{2}{0}\rangle \cong \mathcal{Z}(\Pi_1(Kb))$.

When the base space is Kb, the action given by $\vartheta(\alpha\beta\alpha^{-1}\beta) = \vartheta(\beta^2)$ is the identity inner automorphism when $\vartheta(\beta) = e$ or φ_2; and in this case, as above the Euler number must represent a central element of $\Pi_1(F)$. However when $\vartheta(\beta) = \varphi_1$ or φ_3, the action $\vartheta(\alpha\beta\alpha^{-1}\beta)$ is by conjugation by δ^{-1}; and therefore the Euler number will lie in the coset of $\mathcal{Z}(\Pi_1(F))$ represented by δ^{-1}. That is the Euler number must be of the form $\binom{m}{n} \in \{\binom{2k}{1}\}$ where k is free in $\mathbb{Z}$.

Note that the orientation character $w_1(M)\colon \Pi_1(M) \to \mathrm{Out}(\Pi_1(F))$ always restricts nontrivially to $\langle\gamma\rangle$ so that these bundle-spaces are all non-orientable. Such Kb-bundle-spaces are classified by the following two theorems.

Theorem. *Let M be the total space of a Kb-bundle over T^2 with first Betti number $\beta_1(M)$. Then M is bundle diffeomorphic to precisely one of the geometric 4-manifolds listed below:*

	Total Space	*Geometric Type*
$\beta_1(M) = 3$	$(e, e; \binom{0}{0})$	E^4
	$(\varphi_1, e; \binom{0}{0})$	E^4
$\beta_1(M) = 2$	$(e, e; \binom{2n}{0})_{n\in\mathbb{N}\setminus\{0\}}$	$Nil^3 \times E$
	$(\varphi_1, e; \binom{2n}{0})_{n\in\mathbb{N}\setminus\{0\}}$	$Nil^3 \times E$
	$(\varphi_2, e; \binom{(2)}{0})$	E^4
	$(\varphi_3, e; \binom{(2)}{0})$	E^4
	$(\varphi_1, \varphi_2; \binom{(2)}{0})$	E^4

TABLE 4.

Above and henceforth the symbol (2) is used to denote a fixed integer equal to 0 or 2.

Theorem. *Let M be the total space of a Kb-bundle over Kb with first Betti number $\beta_1(M)$. Then M is bundle diffeomorphic to precisely one of the geometric 4-manifolds listed below:*

	Total Space	*Geometric Type*
$\beta_1(M)=2$	$(e,e;(\begin{smallmatrix}(2)\\0\end{smallmatrix}))$	E^4
	$(\varphi_1,e;(\begin{smallmatrix}(2)\\0\end{smallmatrix}))$	E^4
	$(e,\varphi_1;(\begin{smallmatrix}(2)\\-1\end{smallmatrix}))$	$Nil^3\times E$
$\beta_1(M)=1$	$(\varphi_2,e;(\begin{smallmatrix}0\\0\end{smallmatrix}))$	E^4
	$(\varphi_2,e;(\begin{smallmatrix}2n\\0\end{smallmatrix}))_{n\in\mathbb{N}\setminus\{0\}}$	$Nil^3\times E$
	$(\varphi_3,e;(\begin{smallmatrix}0\\0\end{smallmatrix}))$	E^4
	$(\varphi_3,e;(\begin{smallmatrix}2n\\0\end{smallmatrix}))_{n\in\mathbb{N}\setminus\{0\}}$	$Nil^3\times E$
	$(e,\varphi_2;(\begin{smallmatrix}(2)\\0\end{smallmatrix}))$	E^4
	$(e,\varphi_3;\left(\begin{smallmatrix}(2)\\-1\end{smallmatrix}\right))$	E^4
	$(\varphi_2,\varphi_1;(\begin{smallmatrix}0\\-1\end{smallmatrix}))$	E^4
	$(\varphi_2,\varphi_1;(\begin{smallmatrix}2n\\-1\end{smallmatrix}))_{n\in\mathbb{N}\setminus\{0\}}$	$Nil^3\times E$
	$(\varphi_1,\varphi_2;(\begin{smallmatrix}(2)\\0\end{smallmatrix}))$	E^4
	$(\varphi_1,\varphi_3;\left(\begin{smallmatrix}(2)\\-1\end{smallmatrix}\right))$	E^4

TABLE 5.

proof.

Abelianizing the presentation for $\Pi_1(M)$ and employing the function $r\colon \mathrm{Out}(\Pi_1(Kb))\to \mathrm{GL}(2,\mathbb{Z})$ defined by:

$$e\mapsto I,\ \varphi_1\mapsto \begin{pmatrix}1&0\\1&1\end{pmatrix},\ \varphi_2\mapsto\begin{pmatrix}-1&0\\0&1\end{pmatrix},\ \varphi_3\mapsto\begin{pmatrix}-1&0\\1&1\end{pmatrix},$$

we see that

$$\beta_1(M)=\begin{cases}4-\mathrm{rank}(\begin{pmatrix}0&m\\2&n\end{pmatrix},r\vartheta(\alpha)-I,r\vartheta(\beta)-I), & \text{if } S=T^2,\\ 3-\mathrm{rank}(\begin{pmatrix}0\\2\end{pmatrix},r\vartheta(\alpha)-I,r\vartheta(\beta)-I), & \text{if } S=Kb.\end{cases}$$

To obtain a listing of all such bundle-spaces our approach is to 'build up' the 4-manifold M as the bundle space of a 3-manifold bundle over the circle. This is done by firstly utilizing the S^1-bundle over S^1 structure of the base space S so as to induce a Kb over the fibre of this bundle with total space B a 3-manifold. Then M fibres over S^1 with fibre B. Known information about $\Pi_1(B)$ shall allow us to classify $\Pi_1(M)$ whilst preserving the Kb-bundle structure of M over S. This is roughly summarized in the diagram below:

$$\begin{tikzcd}
Kb \arrow[r, hook] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, hook] & S^1 \arrow[d, hook] \\
Kb \arrow[r, hook] & M \arrow[r, two heads] \arrow[d] & S \arrow[d] \\
& S^1 \arrow[r, equal] & S^1
\end{tikzcd}$$

We proceed in greater detail below.

From the long exact homotopy sequence for a fibration we have:

$$\begin{tikzcd}
1 \arrow[r] & \Pi_1(Kb) \arrow[r] & \Pi_1(M) \arrow[r, "p"] & \Pi_1(S) \arrow[r] & 1 \\[-20pt]
& \langle\gamma,\delta\rangle \arrow[r, mapsto] & \langle\alpha,\beta,\gamma,\delta\rangle \arrow[r, mapsto] & \langle\alpha,\beta\rangle.
\end{tikzcd}$$

Moreover there is a projection $q\colon \Pi_1(S) \twoheadrightarrow \mathbb{Z}\langle\alpha\rangle$ so that we have an epimorphism $qp\colon \Pi_1(M) \twoheadrightarrow \mathbb{Z}$ whose kernel $\ker(qp) = \langle\beta,\gamma,\delta\rangle$ is the fundamental group of the total space of a Kb-bundle over S^1 with homotopy sequence:

$$\begin{tikzcd}
1 \arrow[r] & \Pi_1(Kb) \arrow[r] & \ker(qp) \arrow[r, "\overline{p}"] & \Pi_1(S^1) \arrow[r] & 1 \\[-20pt]
& \langle\gamma,\delta\rangle \arrow[r, mapsto] & \langle\beta,\gamma,\delta\rangle \arrow[r, mapsto] & \langle\beta\rangle.
\end{tikzcd}$$

The image of β under $\overline{p}$ is a generator for $\mathbb{Z}$ and hence determines a right inverse s to $\overline{p}$ so that this sequence splits and $\ker(qp) \cong \Pi_1(Kb) \rtimes_\theta \mathbb{Z}\langle\beta\rangle$ where $\theta = \mathrm{im}(\vartheta|_{\langle\beta\rangle})$ is in the mapping class group of the fibre $\mathrm{Out}(\Pi_1(Kb))$ and is determined by the action of conjugation of β on γ and δ. That is $\ker(pq)$ is one of the four non-orientable flat 3-manifold groups: $B_1 - B_4$, where $B_{i+1} \cong \Pi_1(Kb) \rtimes_{\varphi_i} \mathbb{Z}$ and $\varphi_0 = e$; coresponding to the elements of $\mathrm{Out}(\Pi_1(Kb)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; (see [Hi1] (and also [Wo])).

Hence M fibres over S^1 with fibre some non-orientable flat 3-manifold represented by the split exact sequence:

$$\begin{tikzcd}
1 \arrow[r] & \ker(qp) \arrow[r] & \Pi_1(M) \arrow[r, "qp"] & \Pi_1(S^1) \arrow[r] & 1 \\[-20pt]
& \langle\beta,\gamma,\delta\rangle \arrow[r, mapsto] & \langle\alpha,\beta,\gamma,\delta\rangle \arrow[r, mapsto] & \langle\alpha\rangle.
\end{tikzcd}$$

Therefore $\Pi_1(M) \cong \ker(qp) \rtimes_\psi \mathbb{Z}$ where $\psi \in \mathrm{Out}(\ker(qp))$ is determined by the action of conjugation of α on β, γ and δ. Note that $\psi|_{\langle\gamma,\delta\rangle} = \mathrm{im}(\vartheta|_{\langle\alpha\rangle})$.

Kb-bundles over T^2.

We proceed in three cases according to the image of $\vartheta\colon \mathbb{Z}^2 \to (\mathbb{Z}/2\mathbb{Z})^2$.

The action of $\Pi_1(T^2)$ *on* $\Pi_1(Kb)$ *is trivial.*

Here we have an infinite family of bundle equivalences with respect to this action since $H^2(\mathbb{Z}^2;\mathbb{Z}) \cong \mathbb{Z}$.

In this case ϑ acts by inner automorphisms. In particular θ is trivial so that $\ker(qp)$ is the direct product $\Pi_1(Kb) \times \mathbb{Z} = B_1$. Therefore $\Pi_1(M) = B_1 \rtimes_\psi \mathbb{Z}$ where

$\psi \in \mathrm{Out}(B_1)$ and restricts trivially to $\langle\gamma,\delta\rangle < B_1$. Now $\mathrm{Out}(B_1) \cong \Gamma_2 \times \mathbb{Z}/2\mathbb{Z}$ where Γ_2 is the subgroup of $\mathrm{GL}(2,\mathbb{Z})$ consisting of matrices congruent mod(2) to upper triangular matrices. Such automorphisms are of the form:

$$(\beta,\gamma,\delta) \longmapsto (\beta^a\gamma^b,\ \beta^c\gamma^d\delta^{(1)},\ \delta);$$

where $\left(\begin{smallmatrix} a & c \\ b & d\end{smallmatrix}\right) \in \Gamma_2$ and $(1) \in \mathbb{Z}/2\mathbb{Z}$. Since ψ acts trivially on γ we have that $c = 0,\ d = 1$ and $(1) = 0$. Furthermore β and γ commute in B_1 so that the action of ψ on β corresponds to the relation $\alpha\beta\alpha^{-1} = \gamma^b\beta^a$. Thus $a = 1$ and b is free in $2\mathbb{Z}$. And we have an infinite family of representations with trivial monodromy over the base and Euler number $\left(\begin{smallmatrix} n \\ 0\end{smallmatrix}\right)$ where n is even. However changing the base of the fibre by inverting γ, we may assume that n is positive. That is we obtain the family of characteristic triples:

$$(e, e; \left(\begin{smallmatrix} n \\ 0\end{smallmatrix}\right))_{n\in 2\mathbb{N}}.$$

Each member of this family indeed represents a distinct bundle-space as each group abelianizes as: $H_1(M) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}$.

The action of $\Pi_1(T^2)$ *on* $\Pi_1(Kb)$ *is via a non-trival proper subgroup of* $\mathrm{Out}(\Pi_1(Kb))$.

The only non-trivial proper subgroup of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ up to group isomorphism is $\mathbb{Z}/2\mathbb{Z}$. Therefore by performing a basis change on $\Pi_1(T^2)$ we may assume that $\beta \in \ker(\vartheta)$ so that the image of α under ϑ is $\mathbb{Z}/2\mathbb{Z}$. Then as before θ is trivial so that $\ker(qp) = B_1$ and $\Pi_1(M) \cong B_1 \rtimes_\psi \mathbb{Z}$ where the restriction of $\psi \in \mathrm{Out}(B_1)$ to $\langle\gamma,\delta\rangle$ is non-trivial. Thus $a = 1$ and the Euler number is $\left(\begin{smallmatrix} n \\ 0\end{smallmatrix}\right)_{n\in 2\mathbb{Z}}$. Also since $\psi|_{\langle\alpha\rangle} \in \mathrm{Out}(\Pi_1(Kb))$ we must have $c = 0$; and all three non-trivial outer automorphisms $\varphi_1, \varphi_2, \varphi_3$ of $\Pi_1(Kb)$ may be realized as $\mathrm{im}(\vartheta|_{\langle\alpha\rangle})$ by choosing d and (1) appropriately. As with the infinite family of triples above we may assume that $n \geq 0$. That is we obtain the three infinite families: $(\varphi_1, e; \left(\begin{smallmatrix} n \\ 0\end{smallmatrix}\right))_{n\in 2\mathbb{N}}$, $(\varphi_2, e; \left(\begin{smallmatrix} n \\ 0\end{smallmatrix}\right))_{n\in 2\mathbb{N}}$, $(\varphi_3, e; \left(\begin{smallmatrix} n \\ 0\end{smallmatrix}\right))_{n\in 2\mathbb{N}}$. These representations admit further refinement as equivalence classes of the corresponding group extensions are counted by elements of $H^2(\mathbb{Z}^2;\mathbb{Z})$; being isomorphic to $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ respectively.

In the case where $\mathrm{im}(\vartheta|_{\langle\alpha\rangle}) = \varphi_1$ the infinite family of characteristic triples represents an infinite family of distinct bundle-spaces. This is easily seen by calculating the first homology group of the bundle-space to be: $H_1(M) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2n\mathbb{Z}$.

However in the case where $\mathrm{im}(\vartheta|_{\langle\alpha\rangle}) = \varphi_2$ we may apply a change of cross-section of the form $(\alpha,\beta) \mapsto (\alpha, \beta\gamma^{2k})$ for some appropriate $k \in \mathbb{Z}$ reducing the Euler number to one of the two possibilities: $\begin{pmatrix} (2) \\ 0\end{pmatrix}$. These two characteristic triples represent distinct bundle-spaces as the corresponding fundamental groups have distinct centers being $\langle\beta\rangle \cong \mathbb{Z}$ or trivial respective of whether $(2) = 0$ or 2.

Similarly in the case where $\mathrm{im}(\vartheta|_{\langle\alpha\rangle}) = \varphi_3$ isomorphisms as above permit us to transform the Euler number from $\begin{pmatrix} 2n' \\ 0\end{pmatrix}$ to $\begin{pmatrix} (2) \\ 0\end{pmatrix}$. The resulting two groups are distinguished by the center; being $\langle\beta\rangle \cong \mathbb{Z}$ or trivial respective of whether $(2) = 0$ or 2. Thus this pair of triples indeed represent distinct bundle-spaces. That is we obtain one infinite family and two pairs of characteristic triples:

$$(\varphi_1, e; \left(\begin{smallmatrix} n \\ 0\end{smallmatrix}\right))_{n\in 2\mathbb{Z}},\ (\varphi_2, e; \begin{pmatrix} (2) \\ 0\end{pmatrix}),\ (\varphi_3, e; \begin{pmatrix} (2) \\ 0\end{pmatrix}).$$

The action of $\Pi_1(T^2)$ *on* $\Pi_1(Kb)$ *is via the whole group* $\mathrm{Out}(\Pi_1(Kb))$.

Up to a change of basis of $\mathbb{Z}^2$ there is only one group epimorphism $\mathbb{Z}^2 \overset{\vartheta}{\twoheadrightarrow} (\mathbb{Z}/2\mathbb{Z})^2$. Hence applying a basis change to $\Pi_1(T^2)$ we may assume that the image of $\vartheta|_{\langle\alpha\rangle}$ is $\langle\varphi_1\rangle$ and hence the image of $\vartheta|_{\langle\beta\rangle}$ is $\langle\varphi_2\rangle$. Therefore $\ker(qp) \cong \Pi_1(Kb) \rtimes_{\varphi_2} \mathbb{Z} = B_3$ and $\Pi_(M) \cong B_3 \rtimes_\psi \mathbb{Z}$ where $\psi \in \mathrm{Out}(B_3)$ coincides with φ_1 when restricted to $\langle\gamma,\delta\rangle$. Here $\mathrm{Out}(B_3) \cong (\mathbb{Z}/2\mathbb{Z})^3$ and is generated by the automorphisms:

$$(\beta,\gamma,\delta) \overset{\mu_1}{\longmapsto} (\beta^{-1},\gamma,\delta),\ (\beta,\gamma,\delta) \overset{\mu_2}{\longmapsto} (\beta,\gamma\delta,\delta),\ (\beta,\gamma,\delta) \overset{\mu_3}{\longmapsto} (\beta\gamma^2,\gamma,\delta).$$

Note. There is misprint in [H1] where an automorphism corresponding to μ_2 appears as an inner automorpism of B_3.

Hence we see the only possiblities for ψ are μ_2 or $\mu_2\mu_3$. Thus noting that $\beta\gamma^2 = \gamma^{-2}\beta$ we obtain the two possible Euler numbers: $\binom{(-2)}{0}$. However we may change the sign of the Euler number by applying a transformation $(\gamma,\delta) \mapsto (\gamma^{-1},\delta)$ to the fibre. Thus we obtain the two representations:

$$(\varphi_1,\varphi_2;\tbinom{0}{0}), \qquad (\varphi_1,\varphi_2;\tbinom{2}{0}).$$

These extensions come from distinct equivalence classes as $H^2(\mathbb{Z}^2;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Infact these two triples represent distinct bundle-spaces since the center of $\Pi_1(M)$ is alternately $\langle\beta^2\rangle \cong \mathbb{Z}$ or trivial depending on whether the Euler number is respectively trivial or not.

Thus Table 4 contains a complete non-redundant listing of all Kb-bundles over T^2. We now show these bundle-spaces admit geometries as claimed.

The bundle-space $(\varphi_1, e; \binom{0}{0})$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha,\beta,\gamma,\delta \;\middle|\; \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma\delta,\delta), \\ [\alpha,\beta] = 1, & \beta(\gamma,\delta)\beta^{-1} = (\gamma,\delta). \end{matrix} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$\gamma \mapsto (\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&-1\end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ I),$$
$$\delta \mapsto (\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ I).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1,e_2,e_3,e_4\rangle$ to $\mathbb{R}^4 = \langle 2e_1+e_4,e_2,e_3,e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto (\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ I),$$
$$\delta \mapsto (\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix}\frac{1}{2}\\0\\0\\-\frac{1}{2}\end{pmatrix},\ I).$$

The subgroup $\mathcal{A}\langle \alpha, \beta, \gamma^2, \delta\rangle$ of $\Pi_1(M)$ has index 2 and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_1, e; \begin{pmatrix} 0 \\ 0 \end{pmatrix})$ admits an E^4 geometry.

The family of bundle-spaces $(e, e; \begin{pmatrix} 2n \\ 0 \end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}$ **admit geometries of type** $Nil^3 \times E$**.**

Remark. It is clear that when $n = 0$ the bundle-space $M \simeq T^2 \times Kb$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha, \beta, \gamma, \delta \,\middle|\, \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1,\ \alpha(\gamma,\delta)\alpha^{-1} = (\gamma,\delta), \\ [\alpha,\beta] = \gamma^{2n},\ \beta(\gamma,\delta)\beta^{-1} = (\gamma,\delta). \end{matrix} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(\gamma)(x,y,z,w) &= (x, y, z+1, -w), \\
\text{i.e}\quad \Phi(\gamma) &= (\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ (0, \mathfrak{r})). \\
\Phi(\delta)(x,y,z,w) &= (x, y, z, w+1) \\
\text{i.e}\quad \Phi(\delta) &= (I,\ (1,1)). \\
\Phi(\beta)(x,y,z,w) &= (x, y+b, z, w) \\
\text{i.e}\quad \Phi(\beta) &= (\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},\ (0,1)). \\
\Phi(\alpha)(x,y,z,w) &= (x+1, y, z+y, w) \\
\text{i.e}\quad \Phi(\alpha) &= (\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ (0,1));
\end{aligned}$$

where b is non-zero.

Then the relation $\gamma\delta\gamma^{-1}\delta = 1$ and the commutativity of α and β with γ and δ are preserved by Φ. The remaining relation $[\alpha,\beta] = \gamma^{2n}$ implies that $b = 2n$.

To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1, y_1, z, w)\} \leq Isom(Nil^3 \times E)$ in the image of Φ for any fixed $x_1, y_1 \in \mathbb{R}$ is the group $\Phi(K) = \langle \Phi(\gamma), \Phi(\delta)\rangle = \langle (z,w) \mapsto (z+1, -w), (z,w) \mapsto z, w+1)\rangle \cong \mathbb{Z}\tilde{\times}\mathbb{Z}$. Clearly K acts on $\mathbb{R}^2\langle z, w\rangle$ with quotient Kb. Similarly $\overline{\Phi}(\Pi_1(M)/K) = \langle (x,y) \overset{[\alpha]}{\mapsto} (x+1, y), (x,y) \overset{[\beta]}{\mapsto} (x, y+2n)\rangle \cong \mathbb{Z}^2$ acts on $\mathbb{R}^2\langle x, y\rangle$ with quotient T^2 so we have the desired representation.

The family of bundle-spaces $(\varphi_1, e; \begin{pmatrix} 2n \\ 0 \end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}$ **admit geometries of type** $Nil^3 \times E$**.**

$$\Pi_1(M) = \left\langle \alpha, \beta, \gamma, \delta \,\middle|\, \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1,\ \alpha(\gamma,\delta)\alpha^{-1} = (\gamma\delta,\delta), \\ [\alpha,\beta] = \gamma^{2n},\ \beta(\gamma,\delta)\beta^{-1} = (\gamma,\delta). \end{matrix} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(\gamma)(x,y,z,w) &= (x,y,z+1,-w),\\
\text{i.e}\quad \Phi(\gamma) &= (\begin{pmatrix} 1&0&1\\ 0&1&0\\ 0&0&1 \end{pmatrix},\ (0,\mathfrak{r})).\\
\Phi(\delta)(x,y,z,w) &= (x,y,z,w+1)\\
\text{i.e}\quad \Phi(\delta) &= (I,\ (1,1)).\\
\Phi(\beta)(x,y,z,w) &= (x,y+b,z,w)\\
\text{i.e}\quad \Phi(\beta) &= (\begin{pmatrix} 1&0&0\\ 0&1&b\\ 0&0&1 \end{pmatrix},\ (0,1)).\\
\Phi(\alpha)(x,y,z,w) &= (x+1,y,z+y,w+a)\\
\text{i.e}\quad \Phi(\alpha) &= (\begin{pmatrix} 1&1&0\\ 0&1&0\\ 0&0&1 \end{pmatrix},\ (a,1));
\end{aligned}$$

where b is non-zero.

Then the relations $\gamma\delta\gamma^{-1}\delta = 1,\ \alpha\delta\alpha^{-1} = \delta$ and $\beta(\gamma,\delta)\beta^{-1} = (\gamma,\delta)$ are preserved by Φ. The relation $\alpha\gamma\alpha^{-1} = \gamma\delta$ determines that $a = 1/2$; and the remaining relation $[\alpha,\beta] = \gamma^{2n}$ implies that $b = 2n$.

To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1,y_1,z,w)\} \leq Isom(Nil^3 \times E)$ in the image of Φ for any fixed $x_1, y_1 \in \mathbb{R}$ is the group $\Phi(K) = \langle \Phi(\gamma), \Phi(\delta)\rangle = \langle (z,w) \mapsto (z+1,-w), (z,w) \mapsto z, w+1)\rangle \cong \mathbb{Z}\widetilde{\times}\mathbb{Z}$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient Kb. Similarly $\overline{\Phi}(\Pi_1(M)/K) = \langle (x,y)\overset{[\alpha]}{\mapsto}(x+1,y), (x,y)\overset{[\beta]}{\mapsto}(x,y+2n)\rangle \cong \mathbb{Z}^2$ acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient T^2 so we have the desired representation.

The family of bundle-spaces $(\varphi_2, e; \begin{pmatrix} (2)\\ 0 \end{pmatrix})$ admit geometries of type E^4.

$$\Pi_1(M) = \left\langle \alpha,\beta,\gamma,\delta \;\middle|\; \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1,\ \alpha(\gamma,\delta)\alpha^{-1} = (\gamma^{-1},\delta),\\ [\alpha,\beta] = \gamma^{(2)},\ \beta(\gamma,\delta)\beta^{-1} = (\gamma,\delta). \end{matrix} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$\gamma \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix},\ I),$$

$$\delta \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&(2)&-1&0\\ 0&0&0&1 \end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4\rangle$ to $\mathbb{R}^4 = \langle e_1, e_2 + (1)e_3, e_3, e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\-(1)\\0 \end{pmatrix},\ I),$$

$$\delta \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix},\ I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle\alpha^2,\beta,\gamma^2,\delta\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle\begin{pmatrix}2\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\-(1)\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\rangle\cong\mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_2,e;\begin{pmatrix}(2)\\0\end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(\varphi_3,e;\begin{pmatrix}(2)\\0\end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M)=\left\langle\alpha,\beta,\gamma,\delta\,\middle|\,\begin{matrix}\gamma\delta\gamma^{-1}\delta=1,\ \alpha(\gamma,\delta)\alpha^{-1}=(\gamma^{-1}\delta,\delta),\\ [\alpha,\beta]=\gamma^{(2)},\ \beta(\gamma,\delta)\beta^{-1}=(\gamma,\delta).\end{matrix}\right\rangle.$$

We define a representation $\phi\colon\Pi_1(M)\to\mathbb{R}^4\rtimes\mathrm{GL}(4,\mathbb{Z})$ as follows:

$$\gamma\mapsto(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&-1\end{pmatrix}),\qquad\beta\mapsto(\begin{pmatrix}0\\1\\0\\0\end{pmatrix},I),$$
$$\delta\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},I),\qquad\alpha\mapsto(\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&(2)&-1&0\\0&0&0&1\end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4=\langle e_1,e_2,e_3,e_4\rangle$ to $\mathbb{R}^4=\langle 2e_1+e_4,e_2+(1)e_3,e_3,e_4\rangle$ to obtain a representation $\Phi\colon\Pi_1(M)\to\mathbb{R}^4\rtimes O(4)$ as shown below:

$$\gamma\mapsto(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}),\qquad\beta\mapsto(\begin{pmatrix}0\\1\\-(1)\\0\end{pmatrix},I),$$
$$\delta\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},I),\qquad\alpha\mapsto(\begin{pmatrix}\frac12\\0\\0\\-\frac12\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}).$$

The subgroup $\mathcal{A}\langle\alpha^2,\beta,\gamma^2,\delta\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\-(1)\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\rangle\cong\mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_3,e;\begin{pmatrix}(2)\\0\end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(\varphi_1,\varphi_2;\begin{pmatrix}0\\0\end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M)=\left\langle\alpha,\beta,\gamma,\delta\,\middle|\,\begin{matrix}\gamma\delta\gamma^{-1}\delta=1,\ \alpha(\gamma,\delta)\alpha^{-1}=(\gamma\delta,\delta),\\ [\alpha,\beta]=1,\quad\beta(\gamma,\delta)\beta^{-1}=(\gamma^{-1},\delta).\end{matrix}\right\rangle.$$

Modifying the geometric representation of $(\varphi_1,e;\begin{pmatrix}0\\0\end{pmatrix})$ we obtain the representation $\Phi\colon\Pi_1(M)\to\mathbb{R}^4\rtimes O(4)$ as shown below:

$$\gamma\mapsto(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}),\qquad\beta\mapsto(\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}),$$
$$\delta\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},I),\qquad\alpha\mapsto(\begin{pmatrix}\frac12\\0\\0\\-\frac12\end{pmatrix},I).$$

The subgroup $\mathcal{A}\langle\alpha,\beta^2,\gamma^2,\delta\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} \frac12 \\ 0 \\ 0 \\ -\frac12 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_1,\varphi_2;\begin{pmatrix} 0 \\ 0 \end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(\varphi_1,\varphi_2;\begin{pmatrix} 2 \\ 0 \end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha,\beta,\gamma,\delta \;\middle|\; \begin{array}{l} \gamma\delta\gamma^{-1}\delta = 1,\ \alpha(\gamma,\delta)\alpha^{-1} = (\gamma\delta,\delta), \\ [\alpha,\beta] = \gamma^2,\ \ \beta(\gamma,\delta)\beta^{-1} = (\gamma^{-1},\delta). \end{array} \right\rangle .$$

Modifying the geometric representation of $(\varphi_1,\varphi_2;\begin{pmatrix} 0 \\ 0 \end{pmatrix})$ we obtain the representation $\Phi\colon \Pi_1(M)\to\mathbb{R}^4\rtimes \mathrm{GL}(4,\mathbb{Z})$ as shown below:

$$\gamma\mapsto(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&-1 \end{pmatrix}), \qquad \beta\mapsto(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ -4&0&-1&0 \\ 0&0&0&1 \end{pmatrix}),$$
$$\delta\mapsto(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, I), \qquad \alpha\mapsto(\begin{pmatrix} \frac12 \\ 0 \\ 0 \\ -\frac12 \end{pmatrix}, I).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1,e_2,e_3,e_4\rangle$ to $\mathbb{R}^4 = \langle e_1-2e_3,e_2,e_3,e_4\rangle$ to obtain a representation $\Phi\colon\Pi_1(M)\to\mathbb{R}^4\rtimes O(4)$ as shown below:

$$\gamma\mapsto(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&-1 \end{pmatrix}), \qquad \beta\mapsto(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&1 \end{pmatrix}),$$
$$\delta\mapsto(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, I), \qquad \alpha\mapsto(\begin{pmatrix} \frac12 \\ 0 \\ 1 \\ -\frac12 \end{pmatrix}, I).$$

The subgroup $\mathcal{A}\langle\alpha,\beta^2,\gamma^2,\delta\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} \frac12 \\ 0 \\ 1 \\ -\frac12 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_1,\varphi_2;\begin{pmatrix} 2 \\ 0 \end{pmatrix})$ admits an E^4 geometry.

***Kb*-bundles over *Kb*.**

Here we fix the projection q to be:

$$q\colon \Pi_1(Kb)\twoheadrightarrow H_1(Kb)\twoheadrightarrow H_1(Kb)/\mathrm{Torsion}\cong\mathbb{Z}\langle\alpha\rangle.$$

As before writing $\Pi_1(M) = \ker(qp)\rtimes_\psi\mathbb{Z}$, we proceed in cases according to the image of $\vartheta\colon \mathbb{Z}\widetilde{\times}\mathbb{Z}\to(\mathbb{Z}/2\mathbb{Z})^2$.

When $\beta_1(M) = 1$, $\ker(qp) = H$ is characteristic and we may use the following lemma to avoid redundancies in our listing of such bundle-spaces.

Lemma. *Let $\Pi = H \rtimes_\theta \mathbb{Z}$ where the subgroup $H \lhd \Pi$ is characteristic; then $\phi\colon \Pi \to \Pi'$ is an isomorphism precisely when $\Pi' = \mathrm{im}(\phi|_H) \rtimes_{\theta'} \mathbb{Z}$, where the images of θ and θ' in the outer automorphism group of H are congugate via $\phi|_H$ up to inversion.*

Proof.

Let Π and Π' be groups as above with actions $\theta\colon \mathbb{Z} \to \mathrm{Aut}(H)$ and $\theta'\colon \mathbb{Z} \to \mathrm{Aut}(\mathrm{im}(\phi|_H))$ defined by respective cross-sections $s\colon \mathbb{Z} \to \Pi$ and $s'\colon \mathbb{Z} \to \Pi'$. And let $\epsilon \in \{\pm 1\} = \mathrm{Aut}(\mathbb{Z})$ be the induced isomorphism on the quotient of Π by H as shown below:

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] \arrow[d, "\phi|_H"'] & \Pi \arrow[r, two heads, "p"] \arrow[d, "\phi"'] & \mathbb{Z} \arrow[r] \arrow[d, "\epsilon"'] \arrow[l, bend left, "s"] & 1 \\
1 \arrow[r] & \phi(H) \arrow[r] & \Pi' \arrow[r, two heads, "p'"] & \mathbb{Z} \arrow[r] \arrow[l, bend left, "s'"] & 1
\end{tikzcd}$$

Now $\phi^{-1}s_2\epsilon$ is also a cross-section to p; thus writing $*$ to indicate multiplication in Π we have $s^{-1}(n) * \phi^{-1}s'\epsilon(n) \in \ker(p) = H$ for all $n \in \mathbb{Z}$. That is $\phi s(n) = s'\epsilon(n) * h_0$ for some $h_0 \in H$. Hence for any $h \in H$ applying ϕ to the element $\theta(n)(h) := s(n) * h * s^{-1}(h)$ yields the equation

$$\begin{aligned}
\phi|_H\theta(n)(h) &= \phi s(n) * \phi(h) * (\phi s(n))^{-1} \\
&= s'\epsilon(n) * h_0 * \phi|_H(h) * h_0^{-1} * \epsilon^{-1}{s'}^{-1}(n) \\
&= \theta'(\epsilon(n))(h_0 * \phi|_H(h) * h_0^{-1}) \\
&= \theta'(\epsilon(n))(C_{h_0}(\phi|_H(h)) \\
&= (\theta')^\epsilon C_{h_0}(\phi|_H(h));
\end{aligned}$$

where $C_{h_0}\colon H \to H$ denote conjugation by h_0. Thus passing to outer automorphisms we have that $[\theta(n)] = (\phi|_H)^{-1}[(\theta')^\epsilon(n)]\phi|_H$ completing the proof. $\square$

Remark. When $\mathrm{Out}(H) \cong (\mathbb{Z}/2\mathbb{Z})^k$ for some $k \in \mathbb{N}$, the groups Π and Π' are isomorphic precisely when $\theta = \theta'$

The action of $\Pi_1(Kb)$ *on* $\Pi_1(Kb)$ *is trivial.*

In this case there only two extension equivalence classes since $H^2(\mathbb{Z}\widetilde{\times}\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Using the same argument as above when $S = T^2$ with the exception that $a = -1$, this case yields the family of representations: $(e, e; \binom{n}{0})_{n\in 2\mathbb{Z}}$. However by applying an appropriate change of cross-section of the form $(\alpha, \beta) \mapsto (\alpha, \beta)\gamma^{\pm 2k})$ for some $k \in \mathbb{Z}$ we see that this family of bundle-spaces represents just two bundle-diffeomorphism classes given by the triples:

$$(e, e; \left(\begin{smallmatrix}(2)\\ 0\end{smallmatrix}\right)).$$

These two triple represent distinct manifolds. This may be seen by calculating the characteristic subgroup $\mathcal{I}(\Pi') = \{g \in \Pi_1(M) |\ g^n \in [\Pi_1(M), \Pi_1(M)],\ \exists n \in$

$\mathbb{N}\setminus\{0\}.\}$ of $\Pi_1(M)$ known as the *isolator* of the commutator subgroup. Here we have $\mathcal{I}(\Pi') = \langle \beta^{-1}\gamma^{(1)}, \delta\rangle$ which is isomorphic to $\mathbb{Z}^2$ or the Klein bottle group depending on whether $(1) = 0$ or 1 respectively.

The action of $\Pi_1(Kb)$ *on* $\Pi_1(Kb)$ *is via a non-trival proper subgroup of* $\mathrm{Out}(\Pi_1(Kb))$.

In this case $\mathrm{im}(\vartheta) = \langle\varphi_i\rangle$ for some fixed $i \in \{1,2,3\}$, and we may view this action as an epimorphism $\vartheta\colon \mathbb{Z}\tilde{\times}\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. If both generators α and β of $\mathbb{Z}\tilde{\times}\mathbb{Z}$ have non-trivial image under ϑ then we may apply an automorphism to $\mathbb{Z}\tilde{\times}\mathbb{Z}$ fixing β and mapping α to $\alpha\beta$. Therefore up to change of base of the base-space we may assume that either α or β is in the kernel of ϑ.

If $\beta \in \ker(\vartheta)$ then $\theta = \mathrm{im}(\vartheta|_{\langle\beta\rangle})$ is trivial and $\ker(qp) \cong B_1$. Thus $\Pi_1(M) \cong B_1 \rtimes_\psi \mathbb{Z}$ where $\psi|_{\langle\gamma,\delta\rangle} = \varphi_i = \mathrm{im}(\vartheta)$. Examining the outer automorphisms of B_1 we see that by taking $c = 0$, $a = -1$ and b an arbitrary even integer we are able to realize ψ for all values of i. That is we obtain the three families of representations: $(\varphi_1, e; \binom{n}{0})_{n\in 2\mathbb{Z}}$, $(\varphi_2, e; \binom{n}{0})_{n\in 2\mathbb{Z}}$, $(\varphi_3, e; \binom{n}{0})_{n\in 2\mathbb{Z}}$. Counting equivalence classes, the respective values of $H^2(\mathbb{Z}\tilde{\times}\mathbb{Z};\mathbb{Z})$ for the above three families of triples are $\mathbb{Z}/2\mathbb{Z}$,$\mathbb{Z}$ or $\mathbb{Z}$. In all of these cases by replacing γ by γ^{-1} if necessary, we may assume that $n \in \mathbb{N}$. However in the case where $\mathrm{im}(\vartheta|_{\langle\alpha\rangle}) = \varphi_1$; applying a change of cross-section of the form $(\alpha,\beta) \mapsto (\alpha, \beta\gamma^2 k)$ for an appropriate $k \in \mathbb{Z}$ permits us to assume that $n = (2)$. That is we obtain a pair and two infinite families of characteristic triples:

$$(\varphi_1, e; \left(\begin{smallmatrix}(2)\\0\end{smallmatrix}\right)),\ (\varphi_2, e; \left(\begin{smallmatrix}n\\0\end{smallmatrix}\right))_{n\in 2\mathbb{N}},\ (\varphi_3, e; \left(\begin{smallmatrix}n\\0\end{smallmatrix}\right))_{n\in 2\mathbb{N}}.$$

Here the two groups represented by $(\varphi_1, e; \left(\begin{smallmatrix}(2)\\0\end{smallmatrix}\right))$ are non-isomorphic and therefore represent distinct manifolds. This may be seen by calculating the isolator or the commutator subgroup. We have $\mathcal{I}(\Pi') = \langle\beta^{-1}\gamma^{(1)}, \delta\rangle$ being isomorphic to $\mathbb{Z}^2$ or $\mathbb{Z}\tilde{\times}\mathbb{Z}$ depending on whether $(1) = 0$ or 1 respectively. The family of triples $(\varphi_2, e; \binom{n}{0})_{n\in 2\mathbb{N}}$ may be seen to represent a corresponding family of distinct bundle-spaces as follows. The kernel $K \lhd \Pi_1(M)$ of the composite of qp followed the canonical projection:

$$\Pi_1(M) \twoheadrightarrow H_1(M) \twoheadrightarrow H_1(M)/\mathrm{Torsion} \cong \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$$

Here $K \lhd \Pi_1(M)$ is characteristic as is $\langle\delta\rangle \lhd \Pi_1(M)$, since

$$\langle\delta\rangle = \mathcal{I}([\mathcal{I}([\Pi_1(M), \Pi_1(M)]), \mathcal{I}([\Pi_1(M), \Pi_1(M)])).$$

Thus $\langle\delta\rangle$ is characteristic in K and any automorphism of $\Pi_1(M)$ induces an automorphism of $K/\langle\delta\rangle$. This quotient group $K/\langle\delta\rangle$ abelianizes to $\mathbb{Z}/4n\mathbb{Z}$ showing that n is indeed an invariant of this family of bundle-space representations. The same argument applied to the family $(\varphi_3, e; \binom{n}{0})_{n\in 2\mathbb{N}}$ yields the abelianization of the group $K/\langle\delta\rangle$ as $\mathbb{Z}/n\mathbb{Z}$ showing that this family of triples represent distinct bundle-spaces and is distinct from the previous family.

If $\alpha \in \ker(\vartheta)$ then $\theta = \varphi_i$ is non-trivial so that $\ker(qp) \cong B_{i+1}$ and $\Pi_1(M) \cong B_{i+1} \rtimes_\psi \mathbb{Z}$ where ψ acts trivially on $\langle\gamma,\delta\rangle < B_{i+1}$. Thus the monodromies are determined and we need only calculate the possible Euler numbers.

(1) $(i = 1).H^2(\mathbb{Z}\tilde{\times}\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$
In this case $\ker(qp) \cong B_2$ and the outer automorphism group $\mathrm{Out}(B_2)$ is

isomorphic to the subgroup $\Gamma_2 < \mathrm{GL}(2, \mathbb{Z}$ consisting of matrices congruent modulo 2 to lower triangular matrices.

More specifically the commutator subgroup $[B_2, B_2] = \langle\delta\rangle$ of B_2 is characteristic. Thus any automorphism ϕ of B_2 restricts to an automorphism of $\langle\delta\rangle \cong \mathbb{Z}$. Moreover ϕ induces an automorphism on the quotient $B_2/\langle\delta\rangle \cong \mathbb{Z}^2$; which may be represented by $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$, in the sense that

$$(\beta, \gamma, \delta) \stackrel{\phi}{\longmapsto} (\beta^a \gamma^b \delta^r, \beta^c \gamma^d \delta^s, \delta^\epsilon),$$

for some $r, s \in \mathbb{Z}$ and $\epsilon = \pm 1$. Now requiring that $\phi(\beta)$ commute with $\phi(\delta)$ shows that $A \in \Gamma_2$.

Now automorphisms that act identically on $B_2/\langle\delta\rangle$ (i.e when $a = 1 = d$ and $b = 0 = c$) are precisely the inner automorphisms of B_2. Thus $\mathrm{Inn}(B_2)$ is the kernel of the map on automorphisms induced by abelianization of B_2:

$$\mathrm{ab}^*\colon \mathrm{Aut}(B_2) \to \mathrm{Aut}(B_2/\langle\delta\rangle).$$

Therefore $\mathrm{Out}(B_2) \cong \Gamma_2$ is isomorphic to the image of ab^*; and may be generated by the following family of representative automorphisms:

$$(\beta, \gamma, \delta) \longmapsto (\beta^a \gamma^b \delta^{\frac{a-1}{2}}, \beta^c \gamma^d, \delta).$$

Thus the only automorphisms (unique up to inner automorphisms) that fix $\langle\gamma, \delta\rangle$ and invert β have $a = -1$, $d = 1$ and $c = 0$. Hence the Euler number is of the form $\left(\begin{smallmatrix} n \\ -1 \end{smallmatrix}\right)$, where n is even. A change of cross-section of the form $(\alpha, \beta) \mapsto (\alpha, \beta\gamma^2)$ transforms n into (2), and we obtain the pair of representations:

$$(e, \varphi_1; \begin{pmatrix} 0 \\ -1 \end{pmatrix}),\ (e, \varphi_1; \begin{pmatrix} 2 \\ -1 \end{pmatrix}).$$

these two triples indeed represent distinct bundle-spaces as $H_1(M)$ is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ when $(2) = 0$ and $\mathbb{Z}^2$ otherwise.

(2) $(i = 2).H^2(\mathbb{Z} \tilde{\times} \mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$
Here we see that the only outer automorphisms of B_3 that fix each element of $\langle\gamma, \delta\rangle$ and invert β are μ_1 and $\mu_1\mu_3$. Thus noting that $\beta\gamma^2 = \gamma^{-2}\beta$ we may apply an automorphism to the fibre that inverts γ so as to obtain the two representations:

$$(e, \varphi_2; \begin{pmatrix} 0 \\ 0 \end{pmatrix}),\ (e, \varphi_2; \begin{pmatrix} 2 \\ 0 \end{pmatrix}).$$

These bundle-spaces have $\Pi_1(M) \cong B_3 \rtimes \mathbb{Z}$ where $\mathrm{Out}(B_3) \cong (\mathbb{Z}/2\mathbb{Z})^3$. Since μ_1 and $\mu_1\mu_2$ are distinct elements of $\mathrm{Out}(B_3)$, by the above lemma these triples represent distinct bundle-spaces.

(3) $(i = 3).H^2(\mathbb{Z} \tilde{\times} \mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$
Here the group $\mathrm{Out}(B_4) \cong (\mathbb{Z}/2\mathbb{Z})^3$ and is generated by the automorphisms:

$$(\beta, \gamma, \delta) \stackrel{\nu_1}{\longmapsto} (\beta^{-1}\delta^{-1}, \gamma, \delta),\ (\beta, \gamma, \delta) \stackrel{\nu_2}{\longmapsto} (\beta, \gamma^{-1}, \delta),\ (\beta, \gamma, \delta) \stackrel{\nu_3}{\longmapsto} (\beta\gamma^2, \gamma, \delta).$$

Remark The outer automorphism ν_1 is incorrectly stated in [H1]
Of these, ν_1 and $\nu_1\nu_3$ are the only automorphisms of B_4 that act identically on the subgroup $\langle\gamma,\delta\rangle$ and invert β. Thus we obtain the two representations:

$$(e,\varphi_3;\begin{pmatrix}0\\-1\end{pmatrix}),\ (e,\varphi_3;\begin{pmatrix}2\\-1\end{pmatrix}).$$

These two triples are shown to represent distinct bundle-spaces by abelianizing to obtain: $H_1(M)\cong\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus(\mathbb{Z}/4\mathbb{Z})^2$ or $H_1(M)\cong\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})^2\oplus\mathbb{Z}/4\mathbb{Z}$ respectively.

The action of $\Pi_1(Kb)$ *on* $\Pi_1(Kb)$ *is via the whole group* $\mathrm{Out}(\Pi_1(Kb))$.

In this case, up to equivalence there are three distinct epimorphisms $\vartheta\colon\mathbb{Z}\widetilde{\times}\mathbb{Z}\twoheadrightarrow(\mathbb{Z}/2\mathbb{Z})^2$ distinguished by the image of β. This is because the subgroup $\langle\beta\rangle$ is characteristic in $\mathbb{Z}\widetilde{\times}\mathbb{Z}$ and the image of α under ϑ may be altered by an automorphism mapping α to $\alpha\beta$. Therefore a change of base of the base-space defines an equivalence of such epimorphisms, partioned into classes characterized by the image of β. We choose representatives for these monodromies $(\vartheta(\alpha),\vartheta(\beta))$ to be:

$$(\varphi_2,\varphi_1),\ (\varphi_1,\varphi_2),\ (\varphi_1,\varphi_3),$$

and proceed according to these three respective cases to determine the possible Euler numbers. Note that the respective extension equivalence classes corresponding to these actions have cardinalities in bijection with $H^2(\mathbb{Z}\widetilde{\times}\mathbb{Z};\mathbb{Z})\cong\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ respectively.

(1) In this case $\ker(qp)\cong B_2$ and therfore $\psi\in\mathrm{Out}(B_2)$ must be of the form:

$$(\beta,\gamma,\delta)\overset{\psi}{\longmapsto}(\gamma^b\delta^{-1}\beta^{-1},\gamma^{-1},\delta),$$

where b is even. Hence by inverting, γ if necessary ,we may assume that the Euler numbers are of the form $\begin{pmatrix}n\\-1\end{pmatrix}$ where $n\in\mathbb{N}$ is even. That is we obtain the family:

$$(\varphi_2,\varphi_1;\begin{pmatrix}n\\-1\end{pmatrix})_{n\in2\mathbb{N}}.$$

As before the kernel K of the below projection:

$$\Pi_1(M)\twoheadrightarrow H_1(M)\twoheadrightarrow H_1(M)/\mathrm{Torsion}\cong\mathbb{Z}\twoheadrightarrow\mathbb{Z}/2\mathbb{Z}$$

is characteristic. Similarly $\langle\delta\rangle=[\mathcal{I}([\Pi_1(M),\Pi_1(M)]),\mathcal{I}([\Pi_1(M),\Pi_1(M)]$ and is therefore characteristic in $\Pi_1(M)$ and hence in K. Abelianization of the quotient of K by the characteristic subgroup $\langle\delta\rangle$ yields $\mathbb{Z}^2\oplus\mathbb{Z}/4n\mathbb{Z}$. Therefore the above family of triples is in correspondence with a distinct family of bundle-spaces.

(2) In this case $\ker(qp)\cong B_3$ So that the possibilities for $\psi\in\mathrm{Out}(B_3)$ are $\mu_1\mu_2$ and $\mu_1\mu_2\mu_3$. By inverting γ we may change the sign of the Euler number, so that we obtain the two representations:

$$(\varphi_1,\varphi_2;\begin{pmatrix}0\\0\end{pmatrix}),\ (\varphi_1,\varphi_2;\begin{pmatrix}2\\0\end{pmatrix}).$$

These bundle-spaces have $\Pi_1(M) \cong B_3 \rtimes \mathbb{Z}$ where $\mathrm{Out}(B_3) \cong (\mathbb{Z}/2\mathbb{Z})^3$. Since $\mu_1\mu_2$ and $\mu_1\mu_2\mu_3$ are distinct elements of $\mathrm{Out}(B_3)$, by the above lemma these triples represent distinct bundle-spaces.

(3) In this case $\ker(qp) \cong B_4$. Changing the representative of $\nu_2 \in \mathrm{Out}(B_4)$ to

$$(\beta,\gamma,\delta) \overset{\hat{\nu}_2 = \varsigma_\beta \nu_2}{\longmapsto} (\beta,\gamma\delta,\delta)$$

where $\varsigma_\beta \in \mathrm{Inn}(B_4)$ denotes conjugation by β; we see that the only possibilities for ψ are $\nu_1\hat{\nu}_2$ and $\nu_1\hat{\nu}_2\nu_3$. Thus we obtain the two representations:

$$(\varphi_1,\varphi_3;\begin{pmatrix} 0 \\ -1 \end{pmatrix}),\ (\varphi_1,\varphi_3;\begin{pmatrix} 2 \\ -1 \end{pmatrix}).$$

Here $\mathrm{Out}(B_4) \cong (\mathbb{Z}/2\mathbb{Z})^3$. Since $\nu_1\hat{\nu}_2$ and $\nu_1\hat{\nu}_2\nu_3$ are distinct elements of $\mathrm{Out}(B_4)$, by the above lemma these triples represent distinct bundle-spaces.

We now show that these bundle-spaces admit geometries as claimed.

The bundle-spaces $(e,e;\begin{pmatrix} (2) \\ 0 \end{pmatrix})$ admit geometries of type E^4.

$$\Pi_1(M) = \left\langle \alpha,\beta,\gamma,\delta \,\middle|\, \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma,\delta), \\ \alpha\beta\alpha^{-1}\beta = \gamma^{(2)}, & \beta(\gamma,\delta)\beta^{-1} = (\gamma,\delta). \end{matrix} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$\gamma \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, I),$$

$$\delta \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & (2) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1,e_2,e_3,e_4\rangle$ to $\mathbb{R}^4 = \langle e_1, e_2-(1)e_3, e_3, e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ 1 \\ (1) \\ 0 \end{pmatrix}, I),$$

$$\delta \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle\alpha^2,\beta,\gamma^2,\delta\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ (1) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(e,e;\begin{pmatrix} (2) \\ 0 \end{pmatrix})$ admits an E^4 geometry.

The bundle-spaces $(\varphi_1, e; \left(\begin{smallmatrix} (2) \\ 0 \end{smallmatrix}\right))$ admit geometries of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, \gamma, \delta \,\middle|\, \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma\delta, \delta), \\ \alpha\beta\alpha^{-1}\beta = \gamma^{(2)}, & \beta(\gamma,\delta)\beta^{-1} = (\gamma, \delta). \end{matrix} \right\rangle .$$

We define a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto (\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ \frac{(1)}{2} \\ 1 \\ 0 \end{pmatrix}, I),$$

$$\delta \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta, \gamma^2, \delta \rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{(1)}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_1, e; \left(\begin{smallmatrix} (2) \\ 0 \end{smallmatrix}\right))$ admits an E^4 geometry.

The bundle-spaces $(e, \varphi_1; \left(\begin{smallmatrix} (2) \\ -1 \end{smallmatrix}\right))$ admit geometries of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, \gamma, \delta \,\middle|\, \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma, \delta), \\ \alpha\beta\alpha^{-1}\beta = \gamma^{(2)}\delta^{-1}, & \beta(\gamma,\delta)\beta^{-1} = (\gamma\delta, \delta). \end{matrix} \right\rangle .$$

We define a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto (\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{(1)}{2} \\ -\frac{1}{2} \end{pmatrix}, I),$$

$$\delta \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta, \gamma^2, \delta \rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{(1)}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(e, \varphi_1; \left(\begin{smallmatrix} (2) \\ -1 \end{smallmatrix}\right))$ admits an E^4 geometry.

The bundle-space $(\varphi_2, e; \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right))$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha, \beta, \gamma, \delta \,\middle|\, \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma^{-1}, \delta), \\ \alpha\beta\alpha^{-1}\beta = 1, & \beta(\gamma,\delta)\beta^{-1} = (\gamma, \delta). \end{matrix} \right\rangle .$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as follows:

$$\gamma \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, I),$$
$$\delta \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta, \gamma^2, \delta\rangle$ of $\Pi_1(M)$ has finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\n\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\2\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_2, e; \begin{pmatrix} 0\\0 \end{pmatrix})$ admits an E^4 geometry.

The family of bundle-spaces $(\varphi_2, e; \begin{pmatrix} 2n\\0 \end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}$ admit geometries of type $Nil^3 \times E$.

$$\Pi_1(M) = \left\langle \alpha, \beta, \gamma, \delta \,\middle|\, \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma^{-1},\delta), \\ \alpha\beta\alpha^{-1}\beta = \gamma^{2n}, & \beta(\gamma,\delta)\beta^{-1} = (\gamma,\delta). \end{matrix} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(\gamma)(x,y,z,w) &= (x, y, z+1, -w),\\
\text{i.e}\quad \Phi(\gamma) &= (\begin{pmatrix} 1&0&1\\0&1&0\\0&0&1 \end{pmatrix},\ (0,\mathfrak{r})).\\
\Phi(\delta)(x,y,z,w) &= (x,y,z,w+1)\\
\text{i.e}\quad \Phi(\delta) &= (I,\ (1,1)).\\
\Phi(\beta)(x,y,z,w) &= (x, y+b, z, w)\\
\text{i.e}\quad \Phi(\beta) &= (\begin{pmatrix} 1&0&0\\0&1&b\\0&0&1 \end{pmatrix},\ (0,1)).\\
\Phi(\alpha)(x,y,z,w) &= (x+1, -y, -z-y, w)\\
\text{i.e}\quad \Phi(\alpha) &= (\begin{pmatrix} 1&1&0\\0&1&0\\0&0&1 \end{pmatrix},\ (0,1))\rho;
\end{aligned}$$

where b is non-zero and ρ is the automorphism $(x,y,z,w) \mapsto (x,-y,-z,w)$.

Then the relations $\gamma\delta\gamma^{-1}\delta = 1$, $\alpha(\gamma,\delta)\alpha^{-1} = (\gamma^{-1},\delta)$ and $\beta(\gamma,\delta)\beta^{-1} = (\gamma,\delta)$ are preserved by Φ. The remaining relation $\alpha\beta\alpha^{-1}\beta = \gamma^{2n}$ implies that $b = -2n$.

To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1, y_1, z, w)\} \le Isom(Nil^3 \times E)$ in the image of Φ for any fixed $x_1, y_1 \in \mathbb{R}$ is the group $\Phi(K) = \langle \Phi(\gamma), \Phi(\delta)\rangle = \langle (z,w) \mapsto (z+1, -w), (z,w) \mapsto z, w+1)\rangle \cong \mathbb{Z}\tilde{\times}\mathbb{Z}$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient Kb. Similarly $\overline{\Phi}(\Pi_1(M)/K) = \langle (x,y) \overset{[\alpha]}{\mapsto} (x+1, -y), (x,y) \overset{[\beta]}{\mapsto} (x, y-2n)\rangle \cong \mathbb{Z}\tilde{\times}\mathbb{Z}$ acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient Kb so we have the desired representation.

The bundle-space $(\varphi_3, e; \binom{0}{0})$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha,\beta,\gamma,\delta \;\middle|\; \begin{array}{l} \gamma\delta\gamma^{-1}\delta = 1,\;\; \alpha(\gamma,\delta)\alpha^{-1} = (\gamma^{-1}\delta,\delta), \\ \alpha\beta\alpha^{-1}\beta = 1,\; \beta(\gamma,\delta)\beta^{-1} = (\gamma,\delta). \end{array} \right\rangle.$$

We define a representation $\phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ as follows:

$$\gamma \mapsto (\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&-1\end{pmatrix}),\qquad \beta \mapsto (\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ I),$$

$$\delta \mapsto (\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I),\qquad \alpha \mapsto (\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}).$$

We may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4\rangle$ to $\mathbb{R}^4 = \langle 2e_1+e_4, e_2, e_3, e_4\rangle$ to obtain a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto (\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}),\qquad \beta \mapsto (\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ I),$$

$$\delta \mapsto (\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I),\qquad \alpha \mapsto (\begin{pmatrix}\frac{1}{2}\\0\\0\\-\frac{1}{2}\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta, \gamma^2, \delta\rangle$ of $\Pi_1(M)$ has finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_3, e; \binom{0}{0})$ admits an E^4 geometry.

The family of bundle-spaces $(\varphi_3, e; \binom{2n}{0})_{n\in\mathbb{N}\setminus\{0\}}$ admit geometries of type $Nil^3 \times E$.

$$\Pi_1(M) = \left\langle \alpha,\beta,\gamma,\delta \;\middle|\; \begin{array}{l} \gamma\delta\gamma^{-1}\delta = 1,\qquad \alpha(\gamma,\delta)\alpha^{-1} = (\gamma^{-1}\delta,\delta), \\ \alpha\beta\alpha^{-1}\beta = \gamma^{2n},\; \beta(\gamma,\delta)\beta^{-1} = (\gamma,\delta). \end{array} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3 \times E)$ as follows:

$$\begin{aligned}
\Phi(\gamma)(x,y,z,w) &= (x,y,z+1,-w),\\
\text{i.e}\quad \Phi(\gamma) &= (\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix},\ (0,\mathfrak{r})).\\
\Phi(\delta)(x,y,z,w) &= (x,y,z,w+1)\\
\text{i.e}\quad \Phi(\delta) &= (I,\ (1,1)).\\
\Phi(\beta)(x,y,z,w) &= (x,y+b,z,w)\\
\text{i.e}\quad \Phi(\beta) &= (\begin{pmatrix}1&0&0\\0&1&b\\0&0&1\end{pmatrix},\ (0,1)).\\
\Phi(\alpha)(x,y,z,w) &= (x+1,-y,-z-y,w+a)\\
\text{i.e}\quad \Phi(\alpha) &= (\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\ (a,1))\rho;
\end{aligned}$$

where b is non-zero and ρ is the automorphism $(x,y,z,w)\mapsto(x,-y,-z,w)$.

Then the relations $\gamma\delta\gamma^{-1}\delta=1$, $\alpha\delta\alpha^{-1}=\delta$ and $\beta(\gamma,\delta)\beta^{-1}=(\gamma,\delta)$ are preserved by Φ. The relation $\alpha\gamma\alpha^{-1}=\gamma^{-1}\delta$ implies that $a=-1/2$ and the remaining relation $\alpha\beta\alpha^{-1}\beta=\gamma^{2n}$ gives $b=-2n$.

To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1,y_1,z,w)\}\le Isom(Nil^3\times E)$ in the image of Φ for any fixed $x_1,y_1\in\mathbb{R}$ is the group $\Phi(K)=\langle\Phi(\gamma),\Phi(\delta)\rangle=\langle(z,w)\mapsto(z+1,-w),(z,w)\mapsto z,w+1)\rangle\cong\mathbb{Z}\tilde{\times}\mathbb{Z}$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient Kb. Similarly $\overline{\Phi}(\Pi_1(M)/K)=\langle(x,y)\overset{[\alpha]}{\mapsto}(x+1,-y),(x,y)\overset{[\beta]}{\mapsto}(x,y-2n)\rangle\cong\mathbb{Z}\tilde{\times}\mathbb{Z}$ acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient Kb so we have the desired representation.

The bundle-spaces $(e,\varphi_1;\left(\begin{smallmatrix}(2)\\-1\end{smallmatrix}\right))$ admit geometries of type E^4.

$$\Pi_1(M)=\left\langle\alpha,\beta,\gamma,\delta\;\middle|\;\begin{array}{ll}\gamma\delta\gamma^{-1}\delta=1, & \alpha(\gamma,\delta)\alpha^{-1}=(\gamma,\delta),\\ \alpha\beta\alpha^{-1}\beta=\gamma^{(2)}\delta^{-1}, & \beta(\gamma,\delta)\beta^{-1}=(\gamma\delta,\delta).\end{array}\right\rangle.$$

We define a representation $\Phi\colon\Pi_1(M)\to\mathbb{R}^4\rtimes O(4)$ as shown below:

$$\gamma\mapsto(\begin{pmatrix}0\\0\\\frac12\\0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}),\qquad \beta\mapsto(\begin{pmatrix}0\\\frac12\\\frac{(1)}{2}\\-\frac12\end{pmatrix},I),$$
$$\delta\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},I),\qquad \alpha\mapsto(\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}).$$

The subgroup $\mathcal{A}\langle\alpha^2,\beta,\gamma^2,\delta\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle\begin{pmatrix}2\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\\frac12\\\frac{(1)}{2}\\-\frac12\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\rangle\cong\mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(e,\varphi_1;\left(\begin{smallmatrix}(2)\\-1\end{smallmatrix}\right))$ admits an E^4 geometry.

The bundle-spaces $(e,\varphi_2;\left(\begin{smallmatrix}(2)\\0\end{smallmatrix}\right))$ admit geometries of type E^4.

$$\Pi_1(M)=\left\langle\alpha,\beta,\gamma,\delta\;\middle|\;\begin{array}{ll}\gamma\delta\gamma^{-1}\delta=1, & \alpha(\gamma,\delta)\alpha^{-1}=(\gamma,\delta),\\ \alpha\beta\alpha^{-1}\beta=\gamma^{(2)}, & \beta(\gamma,\delta)\beta^{-1}=(\gamma^{-1},\delta).\end{array}\right\rangle.$$

We define a representation $\phi\colon\Pi_1(M)\to\mathbb{R}^4\rtimes\mathrm{GL}(4,\mathbb{Z})$ as follows:

$$\gamma\mapsto(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}),\qquad \beta\mapsto(\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\-(2)&0&-1&0\\0&0&0&1\end{pmatrix}),$$
$$\delta\mapsto(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},I),\qquad \alpha\mapsto(\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix});$$

If $(2)=0$ the image of ϕ is contained in $\mathbb{R}^4\rtimes O(4)$. Furthermore in this case the subgroup $\mathcal{A}\langle\alpha^2,\beta^2,\gamma^2,\delta\rangle$ of $\Pi_1(M)$ has finite index and maps faithfully into $\mathbb{R}^4$

under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\2\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\2\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(e, \varphi_2; \begin{pmatrix} 0\\0 \end{pmatrix})$ admits an E^4 geometry.

If $(2) = 2$ we may change the ordered basis to $\mathbb{R}^4 = \langle e_1 - e_3, e_2, e_3, e_4 \rangle$ to obtain a representation $\Phi: \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto (\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}),$$
$$\delta \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, \gamma^2, \delta \rangle$ of $\Pi_1(M)$ has finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2\\0\\2\\0 \end{pmatrix}, \begin{pmatrix} 0\\2\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\2\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(e, \varphi_2; \begin{pmatrix} 2\\0 \end{pmatrix})$ admits an E^4 geometry.

The bundle-spaces $(e, \varphi_3; \begin{pmatrix} (2)\\-1 \end{pmatrix})$ admit geometries of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, \gamma, \delta \,\middle|\, \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma,\delta), \\ \alpha\beta\alpha^{-1}\beta = \gamma^{(2)}\delta^{-1}, & \beta(\gamma,\delta)\beta^{-1} = (\gamma^{-1}\delta,\delta). \end{matrix} \right\rangle.$$

We define a representation $\Phi: \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto (\begin{pmatrix} 0\\0\\\frac{1}{2}\\\frac{3}{2} \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}),$$
$$\delta \mapsto (\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1\\0\\(1)\\0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, \gamma^2, \delta \rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2\\0\\(2)\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(e, \varphi_3; \begin{pmatrix} (2)\\-1 \end{pmatrix})$ admits an E^4 geometry.

The bundle-space $(\varphi_2, \varphi_1; \begin{pmatrix} 0\\-1 \end{pmatrix})$ admits an E^4 geometry.

$$\Pi_1(M) = \left\langle \alpha, \beta, \gamma, \delta \,\middle|\, \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma^{-1},\delta), \\ \alpha\beta\alpha^{-1}\beta = \delta^{-1}, & \beta(\gamma,\delta)\beta^{-1} = (\gamma\delta,\delta). \end{matrix} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto \left(\begin{pmatrix}0\\0\\\frac{1}{2}\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\right), \qquad \beta \mapsto \left(\begin{pmatrix}0\\\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix},\ I\right),$$

$$\delta \mapsto \left(\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ I\right), \qquad \alpha \mapsto \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\right).$$

The subgroup $\mathcal{A}\langle\alpha^2,\beta,\gamma^2,\delta\rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\left\langle\begin{pmatrix}2\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\rangle \cong \mathbb{Z}^4$.
Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_2,\varphi_1;\begin{pmatrix}0\\-1\end{pmatrix})$ admits an E^4 geometry.

The family of bundle-spaces $(\varphi_2,\varphi_1;\begin{pmatrix}2n\\-1\end{pmatrix})_{n\in\mathbb{N}\setminus\{0\}}$ admit geometries of type $Nil^3\times E$.

$$\Pi_1(M) = \left\langle \alpha,\beta,\gamma,\delta \;\middle|\; \begin{matrix}\gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma^{-1},\delta),\\ \alpha\beta\alpha^{-1}\beta = \gamma^{2n}\delta^{-1}, & \beta(\gamma,\delta)\beta^{-1} = (\gamma\delta,\delta).\end{matrix}\right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to Isom(Nil^3\times E)$ as follows:

$$\begin{aligned}
\Phi(\gamma)(x,y,z,w) &= (x,y,z+1,-w),\\
\text{i.e}\quad \Phi(\gamma) &= \left(\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix},\ (0,\mathfrak{r})\right).\\
\Phi(\delta)(x,y,z,w) &= (x,y,z,w+1)\\
\text{i.e}\quad \Phi(\delta) &= (I,\ (1,1)).\\
\Phi(\beta)(x,y,z,w) &= (x,y+b,z,w+k)\\
\text{i.e}\quad \Phi(\beta) &= \left(\begin{pmatrix}1&0&0\\0&1&b\\0&0&1\end{pmatrix},\ (k,1)\right).\\
\Phi(\alpha)(x,y,z,w) &= (x+1,-y,-z-y,w)\\
\text{i.e}\quad \Phi(\alpha) &= \left(\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\ (0,1)\right)\rho;
\end{aligned}$$

where b is non-zero and ρ is the automorphism $(x,y,z,w)\mapsto(x,-y,-z,w)$.

Then the relations $\gamma\delta\gamma^{-1}\delta = 1$, $\alpha(\gamma,\delta)\alpha^{-1} = (\gamma^{-1},\delta)$ and $\beta\delta\beta^{-1} = \delta$ are preserved by Φ. The relation $\beta\gamma\beta^{-1} = \gamma\delta$ implies that $k = -1/2$ and the remaining relation $\alpha\beta\alpha^{-1}\beta = \gamma^{2n}\delta^{-1}$ gives $b = -2n$.

To see that Φ is faithful and discrete with cocompact image note that the stabilizer of $\{(x_1,y_1,z,w)\} \leq Isom(Nil^3\times E)$ in the image of Φ for any fixed $x_1,y_1\in\mathbb{R}$ is the group $\Phi(K) = \langle\Phi(\gamma),\Phi(\delta)\rangle = \langle(z,w)\mapsto(z+1,-w),(z,w)\mapsto z,w+1)\rangle \cong \mathbb{Z}\tilde{\times}\mathbb{Z}$. Clearly K acts on $\mathbb{R}^2\langle z,w\rangle$ with quotient Kb. Similarly $\overline{\Phi}(\Pi_1(M)/K) = \langle(x,y)\overset{[\alpha]}{\mapsto}(x+1,-y),(x,y)\overset{[\beta]}{\mapsto}(x,y-2n)\rangle \cong \mathbb{Z}\tilde{\times}\mathbb{Z}$ acts on $\mathbb{R}^2\langle x,y\rangle$ with quotient Kb so we have the desired representation.

The bundle-spaces $(\varphi_1, \varphi_2; \begin{pmatrix} (2) \\ 0 \end{pmatrix})$ admit geometries of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, \gamma, \delta \,\middle|\, \begin{array}{ll} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma\delta, \delta), \\ \alpha\beta\alpha^{-1}\beta = \gamma^{(2)}, & \beta(\gamma,\delta)\beta^{-1} = (\gamma^{-1}, \delta). \end{array} \right\rangle .$$

We define a representation $\phi \colon \Pi_1(M) \to \mathbb{R}^4 \rtimes \mathrm{GL}(4, \mathbb{Z})$ as follows:

$$\gamma \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -(2) & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}),$$

$$\delta \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

If $(2) = 0$ then we may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle$ to $\mathbb{R}^4 = \langle 2e_1 + e_4, e_2, e_3, e_4 \rangle$ to obtain a representation $\Phi \colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}),$$

$$\delta \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, \gamma^2, \delta \rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_1, \varphi_2; \begin{pmatrix} 0 \\ 0 \end{pmatrix})$ admits an E^4 geometry.

If $(2) = 2$ then we may change the ordered basis of $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle$ to $\mathbb{R}^4 = \langle e_1 - e_3 + e_4, e_2, e_3, e_4 \rangle$ to obtain a representation $\Phi \colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto (\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}), \qquad \beta \mapsto (\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}),$$

$$\delta \mapsto (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, I), \qquad \alpha \mapsto (\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, \gamma^2, \delta \rangle$ of $\Pi_1(M)$ has finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2 \\ 0 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_1, \varphi_2; \begin{pmatrix} 2 \\ 0 \end{pmatrix})$ admits an E^4 geometry.

The bundle-spaces $(\varphi_1, \varphi_3; \begin{pmatrix} (2) \\ -1 \end{pmatrix})$ admit geometries of type E^4.

$$\Pi_1(M) = \left\langle \alpha, \beta, \gamma, \delta \,\middle|\, \begin{matrix} \gamma\delta\gamma^{-1}\delta = 1, & \alpha(\gamma,\delta)\alpha^{-1} = (\gamma\delta, \delta), \\ \alpha\beta\alpha^{-1}\beta = \gamma^{(2)}\delta^{-1}, & \beta(\gamma,\delta)\beta^{-1} = (\gamma^{-1}\delta, \delta). \end{matrix} \right\rangle.$$

We define a representation $\Phi\colon \Pi_1(M) \to \mathbb{R}^4 \rtimes O(4)$ as shown below:

$$\gamma \mapsto \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right), \qquad \beta \mapsto \left(\begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right),$$

$$\delta \mapsto \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, I \right), \qquad \alpha \mapsto \left(\begin{pmatrix} 1 \\ 0 \\ (1) \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right).$$

The subgroup $\mathcal{A}\langle \alpha^2, \beta^2, \gamma^2, \delta \rangle$ of $\Pi_1(M)$ is of finite index and maps faithfully into $\mathbb{R}^4$ under Φ with discrete, cocompact image $\langle \begin{pmatrix} 2 \\ 0 \\ (2) \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle \cong \mathbb{Z}^4$. Thus this representation is both faithful and discrete with cocompact image so that $(\varphi_1, \varphi_3; \begin{pmatrix} (2) \\ -1 \end{pmatrix})$ admits an E^4 geometry.

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