TWIN BOB PLAN COMPOSITIONS OF STEDMAN TRIPLES: PARTITIONING OF GRAPHS INTO HAMILTONIAN SUBGRAPHS AS USED IN BELLRINGING

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Abstract. This paper considers finite directed graphs with in-degree 2 and out-degree 2 and uses the fact that every edge e_1 belongs to a unique alternating $2n$-gon $e_1 e_2^{-1} e_3 e_4^{-1} \cdots e_{2n}^{-1} e_1$ for some n.

A *covering* of a directed graph is a collection of disjoint directed circuits which together use each vertex of the graph exactly once. Thus a covering partitions the vertices of a graph, each associated subgraph being Hamiltonian.

The paper shows how to transform one covering into a new one. Each step of the transformation involves all the edges of an alternating $2n$-gon. It is shown how the parity of the number of circuits in the covering at each step changes or not according to the parity of n.

This result is applied to bell-ringing and it is shown that there is no Twin-Bob extent of Stedman Triples.

1. Introduction

Rankin [7] discusses the question of whether the elements of a group generated by two elements α and β can all be arranged in r cycles (of possibly different lengths) in which each element of a cycle is derived from its predecessor by multiplication on the left by either α or β. Let l, m, n and l_1, m_1, n_1 be the orders and indices of the cyclic subgroups generated by α, β, $\alpha^{-1}\beta$ (so that the order of the whole group is $ll_1 = mm_1 = nn_1$). Rankin's Theorem states that, if n is odd, a necessary condition for the above arrangement is that $l_1 \equiv m_1 \equiv r \pmod 2$.

In section 4 of this paper we give a result in graph theory which translates and extends Rankin's result. This work arose from discussions with Bernice Sharp [8], Roger Eyland [3] and Humphrey Gastineau-Hills [4].

The motivation for Rankin's work and this work comes from bell ringing. Section 2 gives an introduction to bell ringing and the particular open problem which

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initiated the work in this paper. Section 3 introduces the Twin Bob Plan, a restriction of the open problem. The question of the existence of a Stedman Triples Twin Bob extent is settled in section 5.

2. Introduction to bell ringing

Ringers work with sequences of linear orders of bells, called *rows*. A *method*, in its most general sense, is a set of rules restricting allowable sequences of rows. Ringers start a method from *rounds* (bells in descending order of pitch) and finish again at rounds. Any ringing, using the rules of the method, in which an internal row is repeated is said to be *false*; otherwise it is *true*. We write a row as a string of integers showing each bell number in its position. For example rounds on eight bells is 12345678 where the bell of highest pitch is numbered 1. There are articles within the mathematical literature which give an introduction to bell ringing and to various methods; in particular see [1],[6],[12],[13],[14].

2.1. Stedman Triples. In this paper we concentrate on the method Stedman Triples, a structure of permutations on seven bells, or seven numbers. In Stedman Triples the rows occur in blocks of six, each block naturally called a *Six*.

Given one row of a Six the others are those generated by the row-to-row transitions $a = (23)(45)(67)$ and $c = (12)(45)(67)$. Sixes have two speeds. A *Quick Six* has row-to-row transitions $acaca$ and a *Slow Six* is $cacac$. Two Sixes are written out in full below.

Quick Six	Slow Six
3215476	3241657
3124567	2346175
1325476	2431657
1234567	4236175
2135476	4321657
2314567	3426175

$(*)$

By convention (to provide consistency with bell ringing notation) we usually start with rounds as the fourth row of a Quick Six. The term *Six End* means the last row of a Six. Hence in our theory we start with 2314567 as a Quick Six End.

Quick and Slow Sixes must alternate. Sixes are joined by row-to-row transitions $(12)(34)(56)$ (called *Plain*), $(12)(34)(67)$ (*Bob*) or $(12)(34)$ (*Single*). Thus a Plain transition joins the Quick Six to the Slow Six in $(*)$. Ringers assume a Plain transition unless the conductor calls “Bob” or “Single”.

All transitions other than a Single are odd. If no Singles are called then rows alternate in parity and so, with the usual start, Six Ends are all even.

The bells in positions 4,5,6 and 7 of an even row uniquely determine the six rows of the Six, though not the order of the rows. We will call these last four digits of an even row the *characteristic* of the Six.

A *extent* of Stedman Triples is $7!+1$ rows rung according to the above structure, starting from and ending with rounds and with every other row rung once.

2.2. Open Problems. The classic open problem within bell ringing (outstanding since the invention of Stedman Triples almost three hundred years ago) is the following.

Does there exist an extent of Stedman Triples without Singles?

The question of the existence of an extent within any method is a popular theoretical question for bell ringers. On n bells an extent is $n! + 1$ rows starting and ending at rounds with every other row rung once.

One of the most fruitful composition structures in Stedman Triples is called the Twin Bob Plan. Ringers have composed extents based on the Twin Bob Plan modified by using two or more Singles. Extents of Stedman Triples using Plains and Singles only also exist [2],[11]. In this paper we show that there is no extent strictly within the Twin Bob Plan.

2.3. Notation. In describing a composition of Stedman Triples we must specify an initial Six End, x, with associated speed, and a sequence of transitions determining subsequent Sixes. We write P for a Six End to Six End transition resulting from a Plain, and B for a Six End to Six End transition resulting from a Bob. Each of these represents two possible permutations depending on the speed of the Sixes.

We write from left to right, so PB means a Plain followed by a Bob, and xPB, where x is the Quick Six End 2314567, gives the Quick Six End 3461275.

Starting with any Six End (with an associated speed), the sequence P^{14} gives 14 Sixes all joined by Plain transitions. This is called a Plain Course and returns to the same Six End, that is, $xP^{14} = x$.

In the theory we use xZ^{-1} (where x is a Six End with an associated speed and Z is a sequence of calls or transitions) to mean the Six End y (with associated speed) where $x = yZ$. So, for example, xP^{-1} is the same Six End and with the same speed as xP^{13}.

3. Twin Bob Plan for Stedman Triples

The Twin Bob Plan outlined in this section and the results and techniques involving Natural Course Ends were all developed by ringers [2],[11].

3.1. Courses. In organising Stedman Triples for composition purposes we work within a structure of *courses*, by which we mean 14 successive Sixes and a calling sequence associated with them. The Twin Bob Plan uses only courses with calling sequences of the shape

$$P^2 SHLP^3 QP$$

where each of S, H, L and Q can be either P^2 or B^2. We say we *Bob at* S or that S *is Bobbed* if we use B^2 in place of S etc. (For explanation of the S, H, L, Q notation visit your local bell tower!)

In section 5 we model the Twin Bob Plan as a graph and use the result of section 4 to show that there is no extent of Stedman Triples using only Twin Bob Courses.

A course starts with the Six End of a Quick Six, called the *Course End.* The *first* Six and *first* Six End refer to the first complete Six of the course, that is, the Six following the Course End. The nth Six End of the course is $6n$ rows after the Course End.

For example, the following is the start of a course.

inital Six End (Quick), Course End →	2314567	
row reached by a Plain transition →	3241657	a Slow Six, the first Six of the Course
	2346175	
	2431657	
	4236175	
	4321657	
first Six End of the course, reached from the Course End by P →	3426175	

The fourteenth Six End of a course becomes the Course End for the next course. Starting with the Course End 2314567, Twin Bob courses generate a set of sixty Course Ends. These are known as the Hudson Course Ends [5],[11] and, identifing the row 2314567 with the permutation (132), are the elements of the coset $2314567G = (132)G$ of $G = \langle(16352), (132)(456)\rangle$, a subgroup of A_7 of order 60. The cosets xG, $x(123)G$ and $x(132)G$ give all the even rows of the sixty Sixes represented in xG. We say that two cosets of G are *Six-different* if the 120 elements of the two cosets are from 120 different Sixes, that is, Sixes with different characteristics.

3.2. Natural Course Ends. For any Twin Bob Course we define the *Natural Course End associated with* S to be xP^{-4}, where x is the fourth Six End of the course. Clearly this Natural Course End can be retrieved from the actual Course End by applying P^2SP^{-4}.

The Natural Course Ends associated H, L and Q are defined similarly. For example a Natural Course End associated with Q would be xP^{-13} where x is the thirteenth Six End of the relevant course. This Natural Course End can be retrieved from the actual Course End by applying $P^2SHLP^3QP^{-13}$.

Each Natural Course End is one of the sixty Hudson Course Ends and provides a stable reference point to describe the Sixes being rung in a course.

3.3. Proof using Natural Course Ends. For an extent of Stedman Triples using the Twin Bob plan we must have sixty courses and so each Hudson Course End must be rung once. Furthermore, for each of S, H, L and Q, these sixty Course Ends must occur exactly once as a Natural Course End.

Remark. To prove the truth of an extent it is sufficient to check that

(a) the Natural Course Ends associated with S occur exactly once; similarly for H, L and Q.
(b) the Natural Course Ends associated with S corresponding to S Bobbed ($S = B^2$), when transposed by (146)(235), coincide with the Natural Course Ends associated with L corresponding to L not Bobbed ($L = P^2$).
(c) the Natural Course Ends associated with H corresponding to H Bobbed ($H = B^2$), when transposed by (124)(356), coincide with the Natural Course Ends associated with Q corresponding to Q not Bobbed ($Q = P^2$).

Consider a set of (possibly different) Twin Bob ringing sequences starting from different Hudson Course Ends, each ringing sequence ending at the same Course End as it started. Bell ringers would call this a set of *round blocks*. If such a set is to use each of the 7! rows just once (other than the starting Course Ends) then this can be checked according to the above rules.

We give a brief explanation of why the checks (b) and (c) are needed and why, together with (a), they are sufficient.

It is necessary to check the Natural Course Ends associated with S against those associated with L since the cosets xP^3G and xP^7G are not Six-different. Consider the Hudson Course Ends x and $y = x(146)(235)$. Then xP^4B^{-1} and yP^8B^{-1} are rows from the same Six as are xP^4P^{-1} and yP^8P^{-1}.

Similarly, xP^6B^{-1} and $x(124)(356)P^{13}B^{-1}$ are rows from the same Six as are xP^6P^{-1} and $x(124)(356)P^{13}P^{-1}$. So we must also check the Natural Course Ends associated with H against those associated with Q.

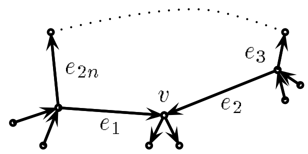
Apart from the 3rd, 5th, 7th and 12th Sixes of a Twin Bob Course, all other Sixes lie in fixed cosets of the Hudson Course Ends all of which are Six-different. That is, xPG, xP^2G, xP^2SG, xP^2SHG etc. are all Six-different and xP^2SHG is independent of whether S and H are Bobbed or not. The only coincidences are $xP^3G = xP^7(132)G$, $xP^2BG = xP^6B(132)G$, $xP^5G = xP^{12}(123)G$, $xP^4BG = xP^{11}B(123)G$. So given the above checks there are no other positions in the courses where Sixes could coincide. Full understanding is achieved through Parker's Table which details the cosets of the Hudson Course Ends and the movement through the cosets by all Stedman Triples ringing sequences [5].

4. Graph Theoretic Result

Definition. A *covering* C of a directed graph is a collection of disjoint directed circuits which together use each vertex of the graph exactly once.

A covering involving only one circuit is a Hamiltonian circuit. A covering of r circuits partitions the vertices of the graph into r sets each corresponding to a Hamiltonian subgraph.

Consider C, a covering of a finite directed graph in which every vertex has in-degree 2 and out-degree 2. Let e_1 and e_2 be the two edges into a vertex v. The edge e_1 belongs to a (necessarily unique) $2n$-gon of the form $e_1e_2^{-1}e_3e_4^{-1}\cdots e_{2n}^{-1}e_1$, where e_i^{-1} means the edge e_i used in the opposite direction. Call such a $2n$-gon, an *alternating $2n$-gon.*



Suppose that e_1 is part of the original covering C; then e_2 cannot be, so e_3 must be, etc. Suppose we remove e_1 from C but still want a covering of the graph. In this case we must use e_2 so as to still use vertex v, cannot use e_3, etc. Making all such forced changes we obtain a new covering of the graph. All segments of circuits not involving $e_1, \ldots, e_{2n}$ will be unaffected. We call this procedure a *one-step transformation* using the alternating $2n$-gon $e_1e_2^{-1}e_3\cdots e_{2n}^{-1}e_1$. A more precise description of the procedure is contained in the proof of the following theorem.

Given two coverings C and C', C may be transformed into C' by a finite number of step transformations.

Theorem. *Consider a directed graph in which every vertex has in-degree 2 and out-degree 2. Let C be a covering of the graph and consider a one-step transformation of C using the $2n$-gon $e_1e_2^{-1}e_3e_4^{-1}\cdots e_{2n}^{-1}e_1$. If n is odd the parity of the number of* *circuits in the covering will be preserved. If n is even then the parity of the number of circuits will change.*

Corollary. *For such a graph: if n is odd for all alternating $2n$-gons then the parity of the number of circuits in any covering will be fixed; if n is even for all alternating $2n$-gons then the parity of the number of circuits in coverings alternates with each one-step transformation.*

Proof. (following Rankin) Let $e_1e_2^{-1}e_3e_4^{-1}\cdots e_{2n}^{-1}e_1$ be a $2n$-gon in the graph. Let C be a collection of r disjoint circuits partitioning the graph and e_1 used in one of these circuits. Then all the edges $e_1, e_3, e_5, \dots, e_{2n-1}$ are used in the circuits of C. And the edges $e_2, e_4, e_6, \dots, e_{2n}$ are not used in the circuits of C.

Let σ_1 represent the order that the edges $e_1, e_3, e_5, \dots, e_{2n-1}$ are used in C.

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ g_1 & g_2 & g_3 & \cdots & g_n \end{pmatrix}$$

where the first edge of the shape e_i following e_{2k-1} in C is e_{2g_k-1}. (Possibly $g_k = k$.)

σ_1 breaks into as many disjoint cycles, say h, as the number of circuits in C involving the edges $e_1, e_3, e_5, \dots, e_{2n-1}$. There are $r-h$ circuits of C not involving $e_1, e_3, e_5, \dots, e_{2n-1}$.

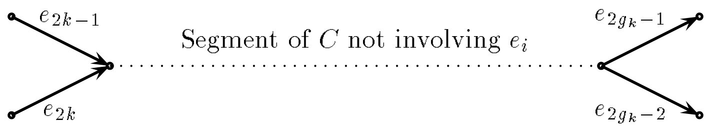
The segments of the circuits in C with $e_1, e_3, e_5, \dots, e_{2n-1}$ removed are joined together by $e_2, e_4, e_6, \dots, e_{2n}$ in a (unique) arrangement C'. Each such segment has a unique edge from $\{e_2, e_4, \dots, e_{2n}\}$ going into it (joining it at the start) and a unique edge from $\{e_2, e_4, \dots, e_{2n}\}$ going into it (joining it at the end). Hence C' must be a new collection of r' circuits partitioning the graph.

There are $r' - h' = r - h$ circuits of C' which do not involve $\{e_2, e_4, \dots, e_{2n}\}$. The h circuits of C involving $e_1, e_3, e_5, \dots, e_{2n-1}$ are reformed as h' circuits of C' involving $\{e_2, e_4, \dots, e_{2n}\}$.

Let

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ g'_1 & g'_2 & g'_3 & \cdots & g'_n \end{pmatrix}$$

represent the order that the edges $\{e_2, e_4, \dots, e_{2n}\}$ are used in C' where the first edge of the shape e_i following e_{2k} in C' is $e_{2g'_k}$.



By inspection, if $k \mapsto g_k$ in σ_1 then $k \mapsto g_k - 1$ in σ_2. (If $k \mapsto 1$ in σ_1 then $k \mapsto n$ in σ_2.)

Thus we can write σ_2 as $\sigma_2 = \sigma_1\tau^{-1}$ where τ is the n-cycle $\tau = (123\cdots n)$.

Recall that h is the number of disjoint cycles of σ_1 and h' is the number of disjoint cycles of $\sigma_2 = \sigma_1\tau^{-1}$. Since τ^{-1} may be written as $n-1$ transpositions we have (by Lemma 2 Rankin [7])

$$h' - h \equiv n - 1 \pmod 2.$$

Thus if n is odd, h' and h (and therefore r and r') have the same parity. If n is even then h' has the opposite parity to h and r' has the opposite parity to r. $\square$

The corollary follows from the fact that if we have any graph with two coverings C and C', C may be transformed into C' by a finite number of step transformations. Specifically, order the edges used in C'. Take the first edge used in C' but not in C. Apply the one-step transformation, involving the alternating $2n$-gon containing this edge, to C. All changed edges coincide with C'. Any edges previously coinciding with C' will still coincide with C'. Continue in this way until C' is reached.

Thus if C is a covering for our graph and n is odd for every edge in the graph, then the parity of the number of circuits in any covering will be the same as it is in C.

5. Applications

The technique described in this paper of finding a covering of a graph and then applying step transformations in an attempt to lengthen the circuits and hopefully find an extent is a very common technique within many methods in bell ringing. In its most general form the graph has all the possible rows as vertices. In its most useful forms the graph usually has lead ends or other relevant coset representatives as its vertices.

5.1. Grandsire Triples. The main motivation for Rankin's work [7] was to prove that there is no extent of Grandsire Triples (a specific bell-ringing method) using only Plains and Bobs (no Singles). This result had already been proved by Thompson [9],[10] using essentially the same ideas but in a less general setting.

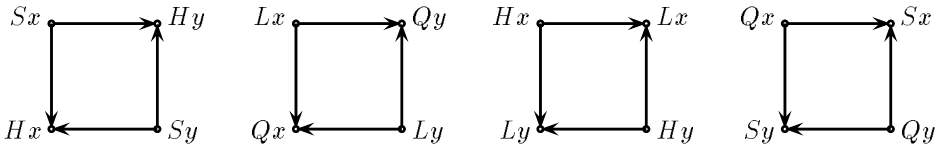
It turns out that it is sufficient to model the Grandsire Triples problem by considering the *Lead Ends*, that is, the even rows of 7 bells with bell 1 in position 1. Clearly if an extent exists then it must use each Lead End. These $6!/2$ rows form the vertices of our graph. We can ring from one vertex in two ways, so every vertex has out-degree 2. Similarly, every vertex has in-degree 2. In Grandsire Triples the permutation from Lead End to Lead End is (34675) (in a Plain Course), or (247)(365) (if a Bob is called). Thus each edge is part of an alternating $2n$-gon with $n = 3$. The graph can be covered by an even number of circuits (circuits of length 5 based on the permutation (34675), that is Plain Courses, for example.) Since $n = 3$ the parity of the number of circuits is always preserved and so we can never get a single circuit covering the graph.

One of the interesting features of Grandsire is that even if we had found a circuit through the Lead Ends we would have still had to check that there was no repetition of rows between the Lead Ends. This contrasts to Plain Bob methods where the leads form cosets [1],[13],[14] and to the Twin Bob Plan of Stedman Triples as modelled below.

5.2. Application to Stedman Triples (Twin Bob Plan). We will model the Twin Bob Compositions of Stedman Triples as a graph.

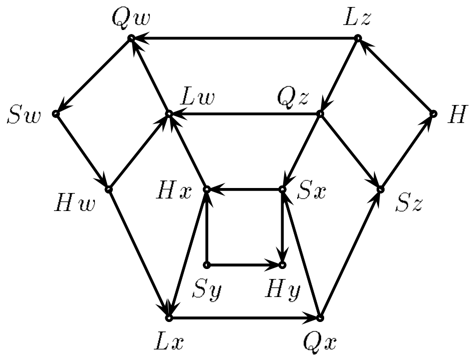
The fourth Six of a course will be represented by a vertex labelled Sx where x is the associated Natural Course End. There are sixty such vertices. Similarly, we have sixty vertices for each of H, L and Q representing the sixth, eighth and thirteenth Sixes respectively. A directed edge joins two vertices if the second vertex follows the first in some Twin Bob Course.

We now have a directed graph of the kind described in the theorem. For example a vertex labelled Sx has edges to two vertices, Hx and Hy, corresponding to the possibilities that H is P^2 or B^2 in the calling sequence. Further, each edge is part of a 4-gon of the shape $e_1 e_2^{-1} e_3 e_4^{-1} e_1$. The eight types of edges are shown in their 4-gons in the diagram below.



$y = x(12)(56)$ $\quad y = x(24)(36)$ $\quad y = x(15)(24)$ $\quad y = x(15)(36)$

Part of the graph around a vertex Sx is shown below.

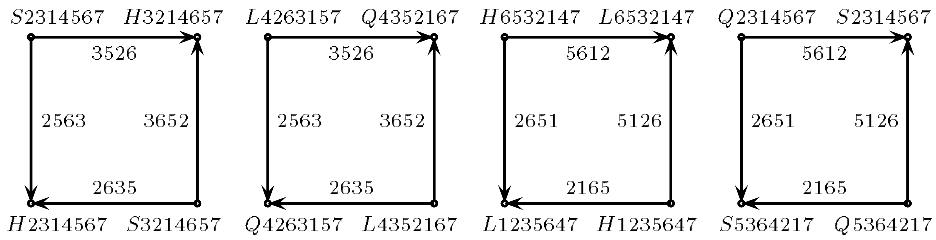


$$\begin{aligned} y &= x(12)(56) \\ w &= x(15)(24) \\ z &= w(24)(36) \\ &= x(15)(36) \end{aligned}$$

A one-step transformation within this graph will clearly change the parity of the number of circuits (since $n = 2$ is even). However a one-step transformation will force a second as described below. Hence the parity of the number of circuits is preserved.

Since P^4B^{-1} and $(146)(235)P^8B^{-1}$ give rows within the same Six (and other similar relationships as described in section 3.3) we need to incorporate this information in our graph. We will do this by labelling the edges of the graph according to the Six preceeding each Six represented by a vertex. That is, an edge into an S vertex will be labelled according to the characteristic of the third Six of a course etc. There are 120 characteristics which appear as labels for the 240 edges into S and L vertices, and 120 other characteristics which appear as labels for the edges into H and Q vertices.

The edges used in a covering of the graph must all have distinct labels if different Sixes are to be used. We call such a covering a *valid covering*. The 4-gons above are redrawn below with representative Course Ends labelling the vertices and with the edges labelled by the relevant characteristics.



So each 4-gon $e_1 e_2^{-1} e_3 e_4^{-1}$ is associated with another, $e_5 e_6^{-1} e_7 e_8^{-1}$ and for any valid covering we can say that either e_1, e_3, e_6, e_8 are used or e_2, e_4, e_7, e_9 are used. A one-step transformation therefore forces a second one-step transformation. Thus the parity of the number of circuits in a valid covering is constant.

Some particular valid coverings are known from bell ringers [2],[11]. For example, Thurstans' Four Part Peal gives a valid covering of two circuits of equal length and Thurstans' One Part Peal gives a valid covering which is two circuits, one of length 236 (59 courses) and one of length 4 (one course). Another well known covering is made of circuits corresponding to the calling sequence $(P^2B^4P^8)^5$. Twelve such circuits (each with S and H Bobbed for 5 courses) give a valid covering.

As shown above the number of circuits in all valid coverings have the same parity. Hence all valid coverings have an even number of circuits. In particular, there is no valid covering consisting of only one circuit. Thus there is no extent of Stedman Triples strictly within the Twin Bob Plan.

6. General Stedman Problem: Difficulties

In Stedman Triples (not restricted to the Twin Bob Plan) we have had difficulty in modelling the situation usefully. Starting anywhere, any even row can be reached as a Quick or Slow Six End by a calling sequence of at most 18 Bobs and Plains. There are 2^{840} calling sequences of the length required for an extent (mostly false). This is finite but, in practice, is too large to check by brute force. We hope that by learning more about the structure of Stedman Triples we can refine our techniques to settle the question of whether there is a extent without using Singles.

7. Conclusion

For finite directed graphs with vertices of in-degree 2 and out-degree 2, we have described a method of searching for a Hamiltonian circuit. We have established the connection between the parity of the number of circuits in a covering and the alternating $2n$-gons of the graph. Where the parity of the number of circuits in any covering can be shown to be even we have the result that the graph is not Hamiltonian.

We have applied the graph theoretic result to Stedman Triples to show that no Twin Bob extent exists.

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