On the monadicity of finitary monads

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Abstract

In 1993, Kelly and Power showed that the category of finitary monads on a locally finitely presentable category $\mathcal{A}$ is of descent type over a power of $\mathcal{A}$; here we establish the stronger result that the forgetful functor in question is monadic. Both their result and ours remain true in the $\mathcal{V}$-enriched case for suitable monoidal categories $\mathcal{V}$. Generalizing further, we obtain a monadicity result for algebras for an operad.

1 Introduction

In their paper [8], Kelly and Power describe as follows a notion of presentation for finitary $\mathcal{V}$-monads on a locally finitely presentable $\mathcal{V}$-category $\mathcal{A}$, where $\mathcal{V}$ is symmetric monoidal closed and moreover *locally finitely presentable as a closed category* [6].

The category $\mathrm{End}_f(\mathcal{A})$ of finitary endo-$\mathcal{V}$-functors of $\mathcal{A}$ and $\mathcal{V}$-natural transformations has a monoidal structure given by composition. A monoid in this monoidal category is of course a finitary $\mathcal{V}$-monad on $\mathcal{A}$; we shall write $\mathrm{Mnd}_f(\mathcal{A})$ for the category of such monoids and their homomorphisms, and write $W : \mathrm{Mnd}_f(\mathcal{A}) \to \mathrm{End}_f(\mathcal{A})$ for the forgetful functor.

If we write $\mathcal{A}_f$ for the full sub-$\mathcal{V}$-category of $\mathcal{A}$ given by the finitely presentable objects, and $J : \mathcal{A}_f \to \mathcal{A}$ for the inclusion, then composition with J induces an equivalence of categories between $\mathrm{End}_f(\mathcal{A})$ and the ordinary category $(\mathcal{A}_f, \mathcal{A})$ of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations; since $\mathcal{A}_f$ is (equivalent to) a small $\mathcal{V}$-category, this $(\mathcal{A}_f, \mathcal{A})$, and hence also $\mathrm{End}_f(\mathcal{A})$, are themselves locally finitely presentable [8, (3.4) and (7.5)]. Now if X is any finitary endo-$\mathcal{V}$-functor of $\mathcal{A}$, then the functor $-X : \mathrm{End}_f(\mathcal{A}) \to \mathrm{End}_f(\mathcal{A})$ sending Y to the composite YX preserves all colimits; in fact it has a right adjoint [3, p.173]. The functor $X- : \mathrm{End}_f(\mathcal{A}) \to \mathrm{End}_f(\mathcal{A})$ does not preserve *all* colimits, but it does preserve *filtered* colimits; it follows [10, Theorem 23.3] that $W : \mathrm{Mnd}_f(\mathcal{A}) \to \mathrm{End}_f(\mathcal{A})$

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has a left adjoint and is monadic. Since the monad in question on $\mathrm{End}_f(\mathcal{A})$ is finitary, it further follows that $\mathrm{Mnd}_f(\mathcal{A})$ is locally finitely presentable.

Choosing a set $|\mathcal{A}_f|$ of representatives of the isomorphism classes of objects of the underlying category $(\mathcal{A}_f)_0$ of $\mathcal{A}_f$, we let $\|\mathcal{A}_f\|$ be the free $\mathcal{V}$-category on the discrete ordinary category $|\mathcal{A}_f|$, giving an isomorphism of categories

$$(\|\mathcal{A}_f\|, \mathcal{A}) \cong (|\mathcal{A}_f|, \mathcal{A}_0).$$

Composition with the canonical $\mathcal{V}$-functor $\|\mathcal{A}_f\| \to \mathcal{A}$ induces a forgetful functor $V : \mathrm{End}_f(\mathcal{A}) \to (\|\mathcal{A}_f\|, \mathcal{A})$; sometimes we prefer to think of it as landing not in $(\|\mathcal{A}_f\|, \mathcal{A})$ but in the isomorphic $(|\mathcal{A}_f|, \mathcal{A}_0)$. This V is conservative, and has left and right adjoints given by ($\mathcal{V}$-enriched) Kan extension; hence is monadic.

Of primary interest in [8] is the composite $U = VW : \mathrm{Mnd}_f(\mathcal{A}) \to (|\mathcal{A}_f|, \mathcal{A}_0)$; of course this has a left adjoint as V and W do so, but a composite of monadic functors need not be monadic or even of descent type [12]. The main theorem of [8] is that *this U is* of descent type (which guarantees that every finitary $\mathcal{V}$-monad has a presentation of the form in question); in proving it, Kelly and Power are led to prove some results of independent interest concerning composites of functors of descent type. At the same time, Kelly and Power raised the question whether U here is in fact monadic, but declared themselves unable to answer it; our principal aim below is to show that U is indeed monadic.

In the special case $\mathcal{A} = \mathcal{V} = \mathbf{Set}$, the finitary monads in question are well known to be equivalent to *Lawvere theories* [11], which are easily seen, as in [2], to be monadic over $\mathbf{Set}^{\mathbf{N}}$. For more general $\mathcal{A}$, however, our theorem seems to be new, even when $\mathcal{V}$ is $\mathbf{Set}$.

2 Theorem

We shall obtain this result as a special case of a more general one, which we now discuss.

We suppose given a monoidal right-closed category $\mathcal{E} = (\mathcal{E}, \otimes, I)$ for which the forgetful functor $W : \mathrm{Mon}(\mathcal{E}) \to \mathcal{E}$, from the category $\mathrm{Mon}(\mathcal{E})$ of monoids in $\mathcal{E}$, has a left adjoint; then W is monadic by a simple and well-known application of the Beck theorem. We further suppose given a monadic functor $V : \mathcal{E} \to \mathcal{B}$, for which there exists a functor $\diamond : \mathcal{E} \times \mathcal{B} \to \mathcal{B}$ and a natural isomorphism

$$\begin{tikzcd} \mathcal{E} \times \mathcal{E} \arrow[r, "\otimes"] \arrow[d, "1 \times V"'] \arrow[dr, phantom, "\cong"] & \mathcal{E} \arrow[d, "V"] \\ \mathcal{E} \times \mathcal{B} \arrow[r, "\diamond"'] & \mathcal{B} . \end{tikzcd}$$

In these circumstances, we shall prove that the composite functor $U = VW : \mathrm{Mon}(\mathcal{E}) \to \mathcal{B}$ is monadic. Our desired result will then follow on taking $\mathcal{E}$ to be $\mathrm{End}_f(\mathcal{A})$ and $V : \mathcal{E} \to \mathcal{B}$ to be the functor $V : \mathrm{End}_f(\mathcal{A}) \to (\|\mathcal{A}_f\|, \mathcal{A})$ of the Introduction; for here we have $V(X \otimes Y) = X \diamond VY$, where $\diamond : \mathrm{End}_f(\mathcal{A}) \times (\|\mathcal{A}_f\|, \mathcal{A}) \to (\|\mathcal{A}_f\|, \mathcal{A})$ is composition.

Let $(-)^{\otimes n} : \mathcal{E} \to \mathcal{E}$ be the "tensor power" functor, which takes an object $E \in \mathcal{E}$ to its n-fold iterated tensor product $E^{\otimes n} = E \otimes \ldots \otimes E$. The key step in the proof is the following lemma:

Lemma 1 *For all $n \in \mathbb{N}$, the functor $(-)^{\otimes n} : \mathcal{E} \to \mathcal{E}$ preserves V-absolute coequalizers of reflexive pairs.*

PROOF:Let

$$\begin{tikzcd} R \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & S \arrow[r, "q"] & T \end{tikzcd} \qquad (*)$$

be such a coequalizer. Trivially $(-)^{\otimes n}$ preserves this coequalizer for $n = 0$ and $n = 1$; we shall prove the result by induction.

For any $X \in \mathcal{E}$, the functor $- \otimes X$, being a left adjoint, preserves the coequalizer. We shall show that $X \otimes -$ preserves the coequalizer. Now

$$\begin{tikzcd} V(R) \arrow[r, shift left, "V(f)"] \arrow[r, shift right, "V(g)"'] & V(S) \arrow[r, "V(q)"] & V(T) \end{tikzcd}$$

is an absolute coequalizer, hence so too is

$$\begin{tikzcd} X \diamond V(R) \arrow[r, shift left, "X \diamond V(f)"] \arrow[r, shift right, "X \diamond V(g)"'] & X \diamond V(S) \arrow[r, "X \diamond V(q)"] & X \diamond V(T); \end{tikzcd}$$

but the isomorphism $X \diamond V- \cong V(X \otimes -)$ gives an absolute coequalizer

$$\begin{tikzcd} V(X \otimes R) \arrow[r, shift left, "V(X \otimes f)"] \arrow[r, shift right, "V(X \otimes g)"'] & V(X \otimes S) \arrow[r, "V(X \otimes q)"] & V(X \otimes T)\ ; \end{tikzcd}$$

whence

$$\begin{tikzcd} X \otimes R \arrow[r, shift left, "X \otimes f"] \arrow[r, shift right, "X \otimes g"'] & X \otimes S \arrow[r, "X \otimes q"] & X \otimes T \end{tikzcd}$$

is a coequalizer by monadicity of V. Thus the coequalizer $(*)$ is preserved by both $X \otimes -$ and $- \otimes X$. (It is easy to see that the same remains true when the coequalizer $(*)$ is replaced by *any* V-absolute colimit.)

Suppose now that $(-)^{\otimes n} : \mathcal{E} \to \mathcal{E}$ preserves the coequalizer $(*)$ for $n = k$. Then in the diagram

$$\begin{tikzcd}
R^{\otimes k} \otimes R \arrow[r, shift left, "R^{\otimes k} \otimes f"] \arrow[r, shift right, "R^{\otimes k} \otimes g"'] \arrow[d, shift right, "f^{\otimes k} \otimes R"'] \arrow[d, shift left, "g^{\otimes k} \otimes R"] & R^{\otimes k} \otimes S \arrow[r, "R^{\otimes k} \otimes q"] \arrow[d, shift right, "f^{\otimes k} \otimes S"'] \arrow[d, shift left, "g^{\otimes k} \otimes S"] & R^{\otimes k} \otimes T \arrow[d, shift right, "f^{\otimes k} \otimes T"'] \arrow[d, shift left, "g^{\otimes k} \otimes T"] \\
S^{\otimes k} \otimes R \arrow[r, shift left, "S^{\otimes k} \otimes f"] \arrow[r, shift right, "S^{\otimes k} \otimes g"'] \arrow[d, "q^{\otimes k} \otimes R"'] & S^{\otimes k} \otimes S \arrow[r, "S^{\otimes k} \otimes q"] \arrow[d, "q^{\otimes k} \otimes S"'] & S^{\otimes k} \otimes T \arrow[d, "q^{\otimes k} \otimes T"] \\
T^{\otimes k} \otimes R \arrow[r, shift left, "T^{\otimes k} \otimes f"] \arrow[r, shift right, "T^{\otimes k} \otimes g"'] & T^{\otimes k} \otimes S \arrow[r, "T^{\otimes k} \otimes q"'] & T^{\otimes k} \otimes T
\end{tikzcd}$$

each row and each column is the coequalizer diagram in $\mathcal{E}$ of a reflexive pair. We conclude from the "3-by-3 lemma" [5, Lemma 0.17] that the diagonal

$$R^{\otimes(k+1)} \underset{g^{\otimes(k+1)}}{\overset{f^{\otimes(k+1)}}{\rightrightarrows}} S^{\otimes(k+1)} \xrightarrow{q^{\otimes(k+1)}} T^{\otimes(k+1)}$$

is a coequalizer in $\mathcal{E}$. The result now follows by induction. $\square$

We are now ready to prove the monadicity of $U : \mathrm{Mon}(\mathcal{E}) \to \mathcal{B}$, using Paré's "absolute coequalizer" version [14] of the Beck theorem. According to this version, $U : \mathrm{Mon}(\mathcal{E}) \to \mathcal{B}$ will be monadic provided that:

(i) U has a left adjoint;

(ii) U reflects isomorphisms;

(iii) $\mathrm{Mon}(\mathcal{E})$ has coequalizers of those reflexive pairs whose image under U has an absolute coequalizer, and U preserves these coequalizers.

Now (i) and (ii) are clearly satisfied by any composite of monadic functors, and so it remains only to prove (iii).

Suppose therefore that $R = (R, m_R, i_R)$ and $S = (S, m_S, i_S)$ are objects of $\mathrm{Mon}(\mathcal{E})$, and that

$$\begin{tikzcd} R \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & S \arrow[l, bend left=60, "j"] \end{tikzcd}$$

is a reflexive pair in $\mathrm{Mon}(\mathcal{E})$ such that the pair (Uf, Ug) in $\mathcal{B}$ has an absolute coequalizer. Since V is monadic, there is a coequalizer

$$R \underset{g}{\overset{f}{\rightrightarrows}} S \xrightarrow{q} T$$

in $\mathcal{E}$ which is preserved by V and is V-absolute. We must show that there is a monoid structure on T such that $q : S \to T$ is a monoid morphism, and such that if T' is a monoid and $x.q : S \to T'$ is a monoid morphism, then $x : T \to T'$ is a monoid morphism; for then q will be the coequalizer in $\mathrm{Mon}(\mathcal{E})$.

Since

$$R \otimes R \underset{g\otimes g}{\overset{f\otimes f}{\rightrightarrows}} S \otimes S \xrightarrow{q\otimes q} T \otimes T$$

is a coequalizer in $\mathcal{E}$, there is a unique $m_T : T \otimes T \to T$ satisfying $m_T(q \otimes q) = q.m_S$. If we now set $i_T = q.i_S$, then (T, m_T, i_T) is a monoid: the unit laws follow from the unit laws for S and the fact that q, being a coequalizer, is epimorphic; and the associativity law follows from the associativity law for S and the fact that $q \otimes q \otimes q$, being a coequalizer, is epimorphic. The arrow q is a monoid morphism by our choices of m_T and i_T; and if T' is another monoid and $x : T \to T'$ an arrow such that xq is a monoid morphism, then the equation making x a monoid morphism holds since $q \otimes q$ is epimorphic. Thus $q : (S, m_S, i_S) \to (T, m_T, i_T)$ is the desired coequalizer, and we have proved:

Theorem 2 *Suppose that $\mathcal{E}$ is a right-closed monoidal category, and that $V : \mathcal{E} \to \mathcal{B}$ is a monadic functor such that there exists a functor $\diamond : \mathcal{E} \times \mathcal{B} \to \mathcal{B}$ with isomorphisms $X \diamond VY \cong V(X \otimes Y)$, natural in X and Y. If the forgetful functor $W : \mathrm{Mon}(\mathcal{E}) \to \mathcal{E}$ has a left adjoint, then the composite $U = VW : \mathrm{Mon}(\mathcal{E}) \to \mathcal{B}$ is monadic. In particular, since W does admit a left adjoint when $\mathcal{E}$ is complete and cocomplete and for some regular cardinal α each $E \otimes - : \mathcal{E} \to \mathcal{E}$ preserves α-filtered colimits, U is monadic in this case.* $\square$

As a special case we have as promised above:

Corollary 3 *If $\mathcal{V}$ is a symmetric monoidal closed category which is locally finitely presentable as a closed category, and $\mathcal{A}$ is a locally finitely presentable $\mathcal{V}$-category, then $U : \mathrm{Mnd}_f(\mathcal{A}) \to (\|\mathcal{A}_f\|, \mathcal{A})$ is monadic.* $\square$

Remark 4 Of course we could equally fix a regular cardinal α and consider a symmetric monoidal closed category $\mathcal{V}$ which is locally α-presentable as a closed category, and a locally α-presentable $\mathcal{V}$-category $\mathcal{A}$. Then if $\mathcal{A}_\alpha$ is the full sub-$\mathcal{V}$-category of $\mathcal{A}$ comprising the α-presentable objects, there is, just as above, a forgetful functor U from the category $\mathrm{Mnd}_\alpha(\mathcal{A})$ of monads on $\mathcal{A}$ of rank α to the $\mathcal{V}$-functor category $(\|\mathcal{A}_\alpha\|, \mathcal{A})$; from our theorem again we deduce the monadicity of this U.

3 Generalizations

The argument of the previous section clearly admits generalization; we shall now describe what seems to be a reasonable setting. We first consider $\mathcal{E}$ and $V : \mathcal{E} \to \mathcal{B}$ as above, but now replace *monoids* in $\mathcal{E}$ by more general algebraic structures.

The idea is to consider a monad T on the monoidal category $\mathcal{E}$ for which T-algebra structure on an object $E \in \mathcal{E}$ comprises "basic operations" $E^{\otimes n} \to E$, subject to equations between "derived operations", also of the form $E^{\otimes n} \to E$. Writing $W : \mathcal{E}^T \to \mathcal{E}$ for the forgetful functor from the Eilenberg-Moore category of T, we show the composite $U = VW : \mathcal{E}^T \to \mathcal{B}$ to be monadic, using once more the Beck condition.

To describe more precisely the structures we have in mind, we refer to the notion of (**Set**-based) *operad* [13, 9, 1]. If we write $\mathbb{N}$ for the set of natural numbers, seen as a discrete category, then the functor category $[\mathbb{N}, \mathbf{Set}]$ has a symmetric monoidal closed structure via Day's convolution [4], the n-fold tensor product being given by

$$(X_1 \otimes \ldots \otimes X_n)(m) = \sum_{m_1 + \ldots + m_n = m} X_1(m_1) \times \ldots \times X_n(m_n).$$

This now gives rise, as in [9], to another monoidal structure on $[\mathbb{N}, \mathbf{Set}]$, right-closed, but decidedly non-symmetric. The tensor product is given on objects by

$$(Y \circ X) = \sum_n Y(n) \times X^{\otimes n}$$

and similarly on arrows; while the unit J is given by $\mathbb{N}(1, -)$, so that $J1 = 1$ and $Jn = 0$ for $n \neq 0$. A *monoid* $X = (X, m, i)$ in the monoidal category $([\mathbb{N}, \mathbf{Set}], \circ, J)$ is called a *non-permutative* **Set**-*operad*, and a morphism of operads is just a monoid morphism. Thus to give an operad is to give a set X_n for each $n \in \mathbb{N}$, a designated element of $X1$ called the unit, and a function associating to each sequence $(f, g_1, \ldots, g_n) \in X_n \times X_{m_1} \times \ldots \times X_{m_n}$ an element $f(g_1, \ldots, g_n)$ of $X_{m_1 + \ldots + m_n}$, with these data satisfying the evident associativity and unit laws.

Given a monoidal category $\mathcal{E}$, there is a functor $\langle -, - \rangle : \mathcal{E}^{op} \times \mathcal{E} \to [\mathbb{N}, \mathbf{Set}]$ defined on objects by $\langle A, B \rangle(n) = \mathcal{E}(A^{\otimes n}, B)$. In the case $A = B$, there is a canonical monoid (that is, operad) structure on $\langle A, A \rangle$ for which the unit is the identity arrow 1_A on A, and $f(g_1, \ldots, g_n) = f(g_1 \otimes \ldots \otimes g_n)$. If $\mathcal{E}$ is cocomplete, there is a functor $- \star - : [\mathbb{N}, \mathbf{Set}] \times \mathcal{E} \to \mathcal{E}$ defined on objects by $X \star A = \sum_n X_n \cdot A^{\otimes n}$, where $Y \cdot B$ denotes the coproduct of Y copies of B. Since there is also a natural isomorphism $\mathcal{E}(X \star A, B) \cong [\mathbb{N}, \mathbf{Set}](X, \langle A, B \rangle)$, the functor $- \star A : [\mathbb{N}, \mathbf{Set}] \to \mathcal{E}$ preserves colimits.

We now define an X-*algebra in* $\mathcal{E}$ to be an object A of $\mathcal{E}$ equipped with a morphism of operads $\alpha : X \to \langle A, A \rangle$. Given another algebra (B, β), an X-*morphism* from (A, α) to (B, β) is defined to be an arrow $f : A \to B$ in $\mathcal{E}$ rendering commutative

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \langle A, A \rangle \arrow[d, "\langle A, f \rangle"] \\
\langle B, B \rangle \arrow[r, "\langle f, B \rangle"'] & \langle A, B \rangle .
\end{tikzcd}$$

Clearly the definition of X-algebras and X-morphisms can be expressed in terms of $\star$ rather than $\langle -, - \rangle$. The X-algebras in $\mathcal{E}$ and their morphisms form a category $\mathcal{E}^X$ with a forgetful functor $W : \mathcal{E}^X \to \mathcal{E}$, as in [1].

Under fairly mild conditions this W will have a left adjoint and be monadic. Suppose for example that $\mathcal{E}$ is cocomplete and that, for some regular cardinal α, each $E \otimes -$ and each $- \otimes E$ preserves α-filtered colimits. Let P_n be the "tensor power" functor $(-)^{\otimes n} : \mathcal{E} \to \mathcal{E}$; this is easily seen to have rank α, using the finality of the diagonal functor $\Delta : \mathcal{D} \to \mathcal{D}^n$ for an α-filtered $\mathcal{D}$. Since the endofunctor $S : \mathcal{E} \to \mathcal{E}$ given by $X \star -$ is a colimit of these P_n, it too has rank α. Write S-Alg for the category of algebras for the endofunctor S, with $U_S : S\text{-Alg} \to \mathcal{E}$ for the forgetful functor. Since S has rank α, the forgetful functor U_S has a left adjoint F_S and is monadic by [10, Theorem 22.3], the monad $U_S F_S$ further being the free monad $F(S)$ on the endofunctor S and itself having rank α. Similarly the endofunctor $R : \mathcal{E} \to \mathcal{E}$ given by $(X \circ X + J) \star -$ has rank α; so that the forgetful functor $U_R : R\text{-Alg} \to \mathcal{E}$ has a left adjoint F_R and is monadic, and the monad $U_R F_R$ is the free monad $F(R)$ on R and once more has rank α. Now the multiplication m and the unit i of X induce a morphism $(m\ i) : X \circ X + J \to X$, and so, writing h for $(m\ i) \star - : R \to S$, we obtain a monad morphism $F(h) : F(R) \to F(S)$. On the other hand, the universal property of the colimits in the definition of R gives rise to a natural transformation $k : R \to SS + 1_{\mathcal{E}}$, and so there is a functor $P : S\text{-Alg} \to R\text{-Alg}$ sending the S-algebra (A, a) to the R-algebra with underlying object A and with R-action

$$RA \xrightarrow{kA} SSA + A \xrightarrow{Sa+A} SA + A \xrightarrow{(a\ A)} A.$$

It is straightforward to see that $\mathcal{E}^X$ is isomorphic to the full subcategory of S-Alg given by those algebras (A, a) for which

$$\begin{tikzcd}
RA \arrow[r, "kA"] \arrow[d, "hA"'] & SSA + A \arrow[r, "Sa+A"] & SA + A \arrow[d, "(a\ A)"] \\
SA \arrow[rr, "a"'] & & A
\end{tikzcd}$$

commutes. Now P commutes with the forgetful functors U_R and U_S, and so by [11] arises from a unique monad morphism $p : F(R) \to F(S)$. Since the monads $F(R)$ and $F(S)$ have rank α, we can by [10, Theorem 27.1] form the coequalizer $q : F(S) \to T$ of the monad morphisms p and $F(h)$, and now $W : \mathcal{E}^X \to \mathcal{E}$ is monadic, since it coincides with the forgetful $\mathcal{E}^T \to \mathcal{E}$.

Just as in the proof of Theorem 2, we now use Lemma 1 to obtain:

Theorem 5 *Suppose that $\mathcal{E}$ is a right-closed monoidal category, and that X is a non-permutative* **Set**-*operad for which $W : \mathcal{E}^X \to \mathcal{E}$ is monadic. Suppose further that $V : \mathcal{E} \to \mathcal{B}$ is a monadic functor such that there is a functor $\diamond : \mathcal{E} \times \mathcal{B} \to \mathcal{B}$ and natural isomorphisms $X \diamond VY \cong V(X \otimes Y)$. Then the composite $U = VW : \mathcal{E}^X \to \mathcal{B}$ is monadic.* □

There is a more general notion of operad [13, 9], involving not monoidal categories as such but rather symmetric monoidal categories. Suppose that $\mathcal{E}$ is a cocomplete symmetric monoidal closed category, and write $\mathbb{P}$ for the category of finite sets and bijections. Then by Day's convolution once more, there is a symmetric monoidal closed structure on $[\mathbb{P}, \mathcal{E}]$ for which the m-fold tensor product is given on objects by the coend

$$(X_1 \otimes \ldots \otimes X_m)(n) = \int^{n_1, \ldots, n_m \in \mathbb{P}} \mathbb{P}(n_1 + \ldots + n_m, n) \otimes X_1(n_1) \otimes \ldots \otimes X_m(n_m).$$

A functor $X : \mathbb{P} \to \mathcal{E}$, seen as an object $X \in [\mathbb{P}, \mathcal{E}]$, determines an essentially unique strong symmetric monoidal functor $X^{\otimes -} : \mathbb{P}^{op} \to [\mathbb{P}, \mathcal{E}]$ such that $X^{\otimes 1} = X$; its value at an object $n \in \mathbb{P}^{op}$ is the n-fold tensor power $X \otimes \ldots \otimes X$. This allows us to define another monoidal structure on $[\mathbb{P}, \mathcal{E}]$, the tensor product being given on objects by

$$Y \circ X = \int^{m \in \mathbb{P}} Y(m) \otimes X^{\otimes m},$$

and the unit J being given by $\mathbb{P}(1,-)$. A monoid in $([\mathbb{P},\mathcal{E}],\circ,J)$ is called an *$\mathcal{E}$-operad.*

There is an embedding $H:\mathcal{E}\to[\mathbb{P},\mathcal{E}]$, given on objects by

$$H(A)=\mathbb{P}(1,-)\otimes A\ .$$

A straightforward calculation shows that $X\circ H(A)$ lies in the image of H, and so we can define an action $\star:[\mathbb{P},\mathcal{E}]\times\mathcal{E}\to\mathcal{E}$ of the monoidal category $([\mathbb{P},\mathcal{E}],\circ,J)$ on $\mathcal{E}$, with $H(X\star A)=X\circ H(A)$; clearly we have the explicit formula

$$X\star A=\int^{n\in\mathbb{P}}Xn\otimes A^{\otimes n}\ .$$

An $\mathcal{E}$-operad (X,m,i) induces a monad $(X\star-,m\star-,i\star-)$ on $\mathcal{E}$, whose Eilenberg-Moore category is called $\mathcal{E}^X$.

Since $\mathcal{E}$ is symmetric monoidal closed, the functors $E\otimes-$ and $-\otimes E$ preserve arbitrary colimits for each $E\in\mathcal{E}$, and so by the 3-by-3 lemma once again we conclude that the tensor-power functors $(-)^{\otimes n}:\mathcal{E}\to\mathcal{E}$ preserve *all* coequalizers of reflexive pairs. It follows that $X\star-:\mathcal{E}\to\mathcal{E}$ preserves coequalizers of reflexive pairs for any $X:\mathbb{P}\to\mathcal{E}$, so that the forgetful functor $W:\mathcal{E}^X\to\mathcal{E}$ creates coequalizers of reflexive pairs for any $\mathcal{E}$-operad X; compare [7]. Now for an *arbitrary* monadic functor $V:\mathcal{E}\to\mathcal{B}$ the Beck theorem gives:

Theorem 6 *Let $\mathcal{E}$ be a cocomplete symmetric monoidal closed category, and $W:\mathcal{E}^X\to\mathcal{E}$ be the forgetful functor from the category of algebras for an $\mathcal{E}$-operad X. If $V:\mathcal{E}\to\mathcal{B}$ is a monadic functor, then so too is the composite $U=VW:\mathcal{E}^X\to\mathcal{B}$.* □

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References

[1] Mikhail A. Batanin, Homotopy coherent category theory and A_∞-structures in monoidal categories, *Journal of Pure and Applied Algebra*, to appear.

[2] J. Bénabou, Structures algébriques dans les catégories, *Cahiers de Topologie et Géometrie Différentielle*, 10:1–126, 1968.

[3] R. Blackwell, G. M. Kelly, and A. J. Power, Two-dimensional monad theory, *J. Pure Appl. Algebra*, 59:1–41, 1989.

[4] Brian Day. On closed categories of functors, in *Reports of the Midwest Category Seminar*, Lecture Notes in Mathematics 137, pages 1–38. Springer-Verlag, 1970.

[5] P. T. Johnstone, *Topos Theory*, Academic Press, London, 1977.

[6] G. M. Kelly, Structures defined by finite limits in the enriched context I, *Cahiers de Topologie et Géometrie Différentielle*, XXIII:3–42, 1982.

[7] G. M. Kelly, Stephen Lack, and R. F. C. Walters, Coinverters and categories of fractions for categories with structure, *Applied Categorical Structures*, 1:95–102, 1993.

[8] G. M. Kelly and A. J. Power, Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads, *J. Pure Appl. Algebra*, 89:163–179, 1993.

[9] G.M. Kelly, On the Operads of J.P. May, Unpublished notes, 1972.

[10] G.M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, *Bulletin of the Australian Mathematical Society*, 22:1–83, 1980.

[11] F. W. Lawvere, Functorial semantics of algebraic theories, *Proc. Nat. Acad. Sci. U.S.A.*, 50:869–872, 1963.

[12] J. L. MacDonald and A. Stone, The tower and regular decomposition, *Cahiers de Topologie et Géometrie Différentielle*, 23:197–213, 1982.

[13] J.P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics 271, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[14] R. Paré, On absolute colimits, *Journal of Algebra*, 19:80–95, 1971.