MULTICHAINS AND TABLEAUX IN FIBONACCI LATTICES

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Abstract. An explicit bijection between multichains in the two Fibonacci lattices $\mathrm{Fib}(r)$ and $\mathrm{Z}(r)$ is described. Central to the bijection is the notion of generalized box-tableaux for $\mathrm{Fib}(r)$ and $\mathrm{Z}(r)$ that count multichains in the respective lattices, analogous to semi-standard generalized Young tableaux in Young's lattice. The generalized box-tableaux afford a simple combinatorial proof of Stanley's algebraically derived result on the equinumeracy of multichains, as well as providing another bijection between the elements in the intervals $[\emptyset, x]$ in $\mathrm{Fib}(r)$ and $\mathrm{Z}(r)$, in answer to a query of Stanley.

1. Introduction

For lattice and poset terminology we refer the reader to [8, Chapter3]. Let $\mathbb{P}$ be the set of positive integers. Let $A(r)$ be an alphabet of $r+1$ symbols defined by $A(r) := \{1, 2, ..., r, *\}$. We refer to the letters $1, 2, ..., r$ as *digits*. We consider the language $A(r)^*$ of all finite words on this alphabet, including the empty word $\emptyset$. By imposing two different partial orderings on the words in $A(r)^*$ we construct two posets, denoted by $\mathrm{Fib}(r)$ and $\mathrm{Z}(r)$. $\mathrm{Fib}(1)$ first appeared in a different context in [7], whilst the general $\mathrm{Fib}(r)$ and $\mathrm{Z}(r)$ constructions were first described in [9]. In particular, $\mathrm{Fib}(r)$ was used by Stanley to facilitate the counting of Hasse walks in the $\mathrm{Z}(r)$ lattice. Suppose u and v are elements of $\mathrm{Fib}(r)$ with u covered by v. We can obtain u from v by either changing a $*$ to a digit i (where $i \in [r] := \{1, 2, ..., r\}$) or by deleting the last letter of v if it is not a $*$. If u is covered by v in $\mathrm{Z}(r)$ then we obtain u from v by either deleting the first digit or changing a $*$ to a digit provided there are no digits to the left of the $*$. We refer to these rules as the *order relations* for the respective lattices. For example, $*12*1 \in \mathrm{Fib}(2)$ covers $112*1$, $212*1$, $*1211$, $*1221$ and $*12*$. However, $*12*1 \in \mathrm{Z}(2)$ covers $112*1$, $212*1$ and $*2*1$. Equipped with this partial ordering we have the following:

Proposition 1.1 ([1, Proposition 1]). *For $r \in \mathbb{P}$, $Z(r)$ is a modular lattice whilst $Fib(r)$ is an upper semimodular lattice.*

The result for $\mathrm{Z}(r)$ was first proved in [9, Proposition5.4] whilst the proof for $\mathrm{Fib}(r)$ is straightforward and so is omitted. Given a word $x = x_1 x_2 \cdots x_n \in A(r)^*$

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where $x_i \in A(r)$ we define the *length* $l(x)$ of x by

$$l : A(r)^* \to \mathbb{N} : x_1 x_2 \cdots x_n \mapsto n.$$

Note that $l(\emptyset) = 0$. Following Brenti [1, p465] we let

$$S(x) := \{i | x_i = *\}.$$

Note that the *rank* of x in either $\mathrm{Fib}(r)$ or $\mathrm{Z}(r)$ is given by $\rho(x) = l(x) + |S(x)|$. Following [10] we let $M_n(x,r)$ (respectively, $N_n(x,r)$) denote the number of multichains of length n from $\emptyset$ to x in $\mathrm{Fib}(r)$ (respectively, $\mathrm{Z}(r)$). The following key result of Stanley (a generalisation of the result in [9, Proposition 5.7]) illustrates the remarkable connection between these two lattices.

Theorem 1.2 ([10, Theorem 2.4])**.** *Given $x \in A(r)^*$ and $r \in \mathbb{P}$,*

$$M_n(x,r) = N_n(x,r). \quad \square$$

Stanley's proof in [10] is largely algebraic in nature. Using generalised box-tableaux (defined below) we provide a combinatorial proof of this important result, as well as an explicit bijection between the multichains. In a simplified form, this provides a bijection between the elements in the intervals $[\emptyset, x]$ in $\mathrm{Fib}(r)$ and $\mathrm{Z}(r)$, as sought by Stanley in [10].

2. Box-tableaux

Box-tableaux were first introduced as a means of enumerating saturated chains in Fibonacci lattices in [4]. We will follow the more developed treatment given them in [3], to which the reader is referred for more details.

Consider an $(r+1) \times l$ grid. We number the columns of the grid from left to right and the rows from bottom to top. Let $x = i_1 i_2 ... i_l$ be a word in $A(r)^*$ (where $i_j \in A(r)$). We form a unique pictorial representation of x by shading appropriate boxes (or cells) of the $(r+1) \times l$ grid. The letter i_j corresponds to the jth column of the grid. If i_j is a digit then we shade the box in the i_jth row in the j-th column. If i_j is a $*$, we shade the whole of the jth column. For example, $2{*}13 \in A(3)^*$ corresponds to the shaded grid in Fig.2.1. We call such a shaded grid a *box-diagram* of x. A column of a box-diagram containing $r+1$ shaded boxes is called a '*-column'; a column containing a single shaded box in one of its bottom r rows is a 'digit column'.

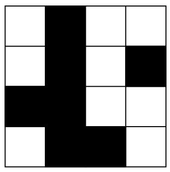


FIG.2.1. The element $2{*}13 \in A(3)^*$.

It is an easy exercise to restate the order relations for $\mathrm{Fib}(r)$ and $\mathrm{Z}(r)$ entirely in terms of box-diagrams. One advantage of this description is that elements of $\mathrm{Fib}(r)$ are ordered by (left-justified) inclusion of their box-diagrams. The inclusion for $\mathrm{Z}(r)$ is slightly more complex, and requires a notion of 'embedding'.

Given $a, b \in Z(\mathrm{r})$ we say that (the box-diagram of) a *embeds* into (the box-diagram of) b if there are columns $j_1 < j_2 < \ldots < j_{l(a)}$ of b such that a_k can be included in column j_k. Moreover, we require that if a_k and column j_k of b are *both* digits (clearly they must be the same digit for an inclusion to be possible) then $a_k a_{k+1} \cdots a_{l(a)}$ must *equal* columns j_k, $j_k + 1$, ..., $l(b)$ of b. This requirement is a consequence of the order relations for $\mathrm{Z}(r)$ which prevent us from changing or inserting any letters in a word to the right of a digit. By way of example, Fig.2.2 shows two illegitimate embeddings in $\mathrm{Z}(r)$.

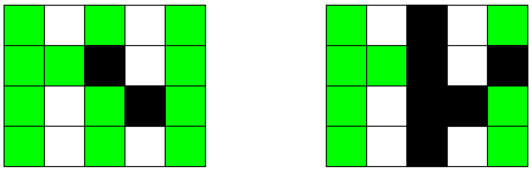


FIG.2.2. Illegitimate embeddings of 32 in $*3{*}2*$ and $*23$ in $*3{*}2*$ in $\mathrm{Z}(3)$.

Given two box-diagrams a and b where a is embedded into b, columns of b that contain embedded boxes of a are called 'embedded'. Columns of b that contain no embedded boxes of a are called 'empty' (in that they are 'empty' of boxes from a). If a digit-column of a embeds into a digit-column of b, or a $*$-column of a embeds into a $*$-column of b, we say the embedded column of b is 'full' (since every shaded box in the column of b contains an embedded box of a). Employing this terminology, if a embeds into b, columns of b are either embedded or empty, and embedded columns are either full or not full.

Given $a \leq b$ in $\mathrm{Z}(r)$, it was noted above that if a_k and column j_k of b were both digits and a_k embedded into column j_k, then $a_k a_{k+1} \cdots a_{l(a)}$ equals columns j_k, $j_k + 1$, ..., $l(b)$ of b. Using the terminology of this section, this condition can be expressed:

(I): if b has an embedded digit-column, then all the columns to the right in b are embedded and full.

We can uniquely represent a saturated chain from $\emptyset$ to x in $\mathrm{Fib}(r)$ or $\mathrm{Z}(r)$ by an appropriate numbering of the cells of the box-diagram of x. Given a saturated chain of length n from $\emptyset$ to x in either $\mathrm{Fib}(r)$ or $\mathrm{Z}(r)$ (so $\rho(x) = n$)

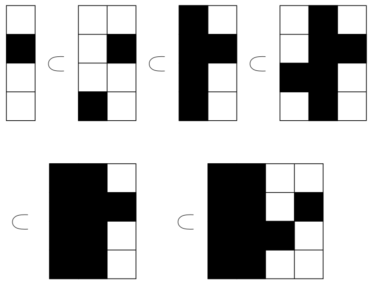
$$\emptyset < c_1 < c_2 < \ldots < c_{n-1} < x$$

we consider the sequence of box-diagrams corresponding to this chain. Since c_i covers c_{i-1}, c_i is formed from c_{i-1} by inserting a digit or changing a digit to a $*$. If we insert a digit, then c_i has an extra column with a single shaded box in one of the bottom r rows (corresponding to the digit added). We number this box with the number i. If we change a digit to a $*$ then there is a single column of c_i that is completely shaded but has only a single shaded box from c_{i-1} embedded into it. We number only the box in the top row of this column with the number i. We call the numbered grid resulting from this process a *box-tableau.*

For example, in Z(3) the chain

$$\emptyset < 3 < 13 < {*3} < 2{*3} < {**3} < {**23}$$

corresponds to the box-diagrams and box-tableau in Fig.2.3.



5	**3**		
			1
4		**6**	
	2		

FIG.2.3. A sequence of box-diagrams and resulting box-tableau in Z(3).

Given a box-tableau we can recover the unique saturated chain represented by considering the sequence of box-diagrams defined by the entries $1, 2, \ldots, i$ for $1 \leq i \leq n$.

We will now define these tableaux formally.

Definition 2.1. A *box-tableau* of size $(r+1)\times l$ is an $(r+1)\times l$ grid with m distinct boxes (numbered $1, 2, \ldots, m$) such that at least one and at most two boxes in each column are numbered. In addition

(1) any column containing only one numbered box must have its top row empty;
(2) any column containing two numbered boxes must have its maximum-valued entry in the top row.

Definition 2.2. The *shape* of a box-tableau λ of size $(r+1)\times l$ is a string $i_1 i_2 \cdots i_l$ where $i_j \in A(r)$ and

(1) $i_j = *$ iff the jth column of λ has two numbered boxes;
(2) $i_j \in [r]$ iff the jth column of λ has a single numbered box in the i_jth row.

Definition 2.3. A *Fib-tableau* is a box-tableau satisfying the following condition:
(FT): the minimum-valued entries from each column increase in value from left to right.

Definition 2.4. A *Z-tableau* is a box-tableau satisfying the following condition:
(ZT): given two entries i and j $(i < j)$ in a column, no entry k satisfying $i < k < j$ can appear in any column to the right of that containing i and j. If i is the only entry in its column then there is no entry $k > i$ in any column to the right of that containing i.

Note that the box-tableau in Fig.2.3 is a Z-tableau of shape $**23$.

We note that conditions (FT) and (ZT) are consequences of the order relations for the two lattices, expressed in terms of box-tableaux. Condition (ZT) first appeared for the $r = 1$ case in [2, Lemma 2.5.8]. Given a shape $x \in A(r)^*$ we write $FT(x, r)$ and $ZT(x, r)$ for the set of Fib-tableaux and Z-tableaux respectively of shape x. Following [9], suppose $e(P, x)$ equals the number of maximal chains in the interval $[\emptyset, x]$ of P. By the one-to-one correspondence outlined in the beginning of this section we have:

Proposition 2.5. *Given $x \in A(r)^*$,*

$$e(Fib(r), x) = \#\, FT(x,\, r),$$
$$e(Z(r), x) = \#\, ZT(x,\, r). \quad \square$$

An explicit bijection between the saturated chains in $[\emptyset, x]$ in Fib(r) and Z(r) using box-tableaux is detailed in [3]. A similar bijection, developed independently by Kremer and O'Hara and expressed in terms of matrices, can be found in [6]. In [9, §5], Stanley derived a formula for the number of saturated chains in an interval $[\emptyset, x]$ of Fib(r) or Z(r). The interested reader is referred to [3] for a reinterpretation of Stanely's formula in terms of 'hook-lengths' of box-diagrams.

3. Generalized Box-tableaux

In a similar vein to the extension of standard Young tableaux to semi-standard generalised Young tableaux, we can extend the definitions of the previous section and form generalised versions of box-tableaux. This enables a combinatorial proof of Stanley's important multichain result described in Theorem 1.2, as well as an explicit bijection between multichains.

Given a multichain of length n from $\emptyset$ to x in Fib(r) or Z(r)

$$\emptyset \leq c_1 \leq c_2 \leq \ldots \leq c_{n-1} \leq x,$$

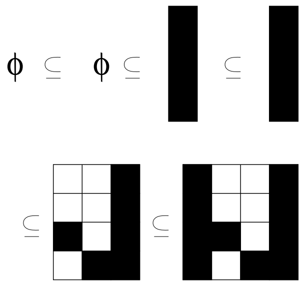
consider the sequence of box-diagrams corresponding to this multichain. Having embedded c_{i-1} into c_i in the manner of the previous section (note that there may be more than one legitimate embedding in the Z(r) case) we number the left-over shaded cells of c_i in two ways. In the $*$-columns of c_i that have a single box from c_{i-1} embedded into them, we number the box in the top row with the number i. In the empty $*$-columns of c_i we number all the shaded boxes with the number i. In the empty digit-columns of c_i we number the single shaded box with the number i.

We call the numbered grids resulting from this process *generalised box-tableaux.* For example, the multichain

$$\emptyset \leq \emptyset \leq * \leq * \leq 21* \leq *21*$$

in Z(3) corresponds to the box-diagrams and generalised box-tableaux in Figure 3.1.

Defintion 3.1. A *generalised box-tableau* of size $(r+1) \times l$ is an $(r+1) \times l$ grid with m distinct boxes each numbered with an integer $i \in \mathbb{P}$ (repetitions allowed).



5			2
			2
4	5		2
		4	2

5			2
5			2
5	4		2
5		4	2

FIG.3.1. Sequence of box-diagrams and resulting g-Z-tableaux.

Each column must have at least one and at most two entries of different value. In addition

(1) any column containing only one numbered box must have its top row empty;
(2) any column containing two entries of different value must have its maximum-valued entry in the top row.
(3) any column containing two entries of the same value i must have *all* the boxes in the column numbered with the same entry $i \in \mathbb{P}$.

Definition 3.2. A *generalised-Fib-tableau* (or *g-Fib-tableau*) is a generalised box-tableau satisfying the following condition:

(GFT): the minimum-valued entries from each column weakly increase from left to right across the columns.

Definition 3.3. A *generalised-Z-tableau* (or *g-Z-tableau*) is a generalised box-tableau satisfying the following condition:

(GZT): given two entries i and j of different value in a column (say, $i < j$), no entry k satisfying $i < k < j$ can appear in any column to the right of that containing i and j. If there is only a single numbered box (with entry i) in the column then there is no entry $k > i$ in any column to the right of that containing i.

The generalised box-tableaux in Figure 3.1 are g-Z-tableaux.

Given a shape $x \in A(r)^*$, we write $GFT(x, r, E)$ and $GZT(x, r, E)$ for the set of g-Fib-tableaux and g-Z-tableaux respectively of shape x and with entry set E. For example, the g-Z-tableaux in Figure 3.1 are members of $GZT(*21*, 3, \{2, 4, 5\})$.

Recall from §2 that there is a one-to-one correspondence between Z-tableaux in $ZT(x, r)$ with entry set $[n]$ and saturated chains of length n from $\emptyset$ to x in $\mathrm{Z}(r)$. However, it is *not* true that there is a one-to-one correspondence between g-Z-tableaux in $GZT(x, r, E)$ and multichains from $\emptyset$ to x in $\mathrm{Z}(r)$. For example, in Figure 3.1 there are two g-Z-tableaux representing the single multichain.

In order to count multichains in $\mathrm{Z}(r)$ we need to introduce an extra condition on g-Z-tableaux.

(UZT): If a column has the same entry i in each of its $(r + 1)$ rows, then there are no entries $j < i$ in any column to the right.

Note that the g-Z-tableau on the left in Figure 3.1 satisfies (UZT), whilst the g-Z-tableau on the right does not. The key result is the following:

Proposition 3.4. *Given a multichain in $Z(r)$, there is a unique g-Z-tableau corresponding to the multichain that satisfies (UZT).*

Proof. Suppose the multichain is

$$\emptyset \leq c_1 \leq c_2 \leq \ldots \leq c_{n-1} \leq x.$$

Consider the embedding of c_i into c_{i+1}. Suppose there is an empty $*$-column in c_{i+1} (i.e., a column of c_{i+1} with $r+1$ shaded boxes and no embedded boxes from a) to the left of an embedded column of c_{i+1}. The corresponding g-Z-tableau will have the entry $i+1$ in every box in the empty $*$-column and there will exist a column to its right containing an entry $j \leq i$. Thus the g-Z-tableau will fail condition (UZT). Therefore there must be no empty $*$-columns to the left of any embedded columns in c_{i+1}. We consider the two cases: when $l(c_i) \leq |S(c_{i+1})|$ and when $l(c_i) > |S(c_{i+1})|$.

If $l(c_i) \leq |S(c_{i+1})|$ then c_i must embed into the leftmost $l(c_i)$ $*$-columns of c_{i+1} in order to satisfy condition (I) and the condition that there be no empty $*$-columns to the left of an embedded column in c_{i+1}. If $l(c_i) > |S(c_{i+1})|$ then the same two conditions force each $*$-column of c_{i+1} to be embedded, leaving $l(c_i) - |S(c_{i+1})|$ embedded digit columns of c_{i+1}. To satisfy condition (I), these embedded digit columns must be the rightmost $l(c_i) - |S(c_{i+1})|$ digit-columns of c_{i+1}. Thus to satisfy (UZT), there is a unique embedding of each c_i into c_{i+1}, yielding a unique g-Z-tableau. □

Define a subset $UZT(x, r, E)$ of $GZT(x, r, E)$ by

$$UZT(x, r, E) := \{\, \lambda \in GZT(x, r, E) \mid \lambda \text{ satisfies condition (UZT)} \,\}.$$

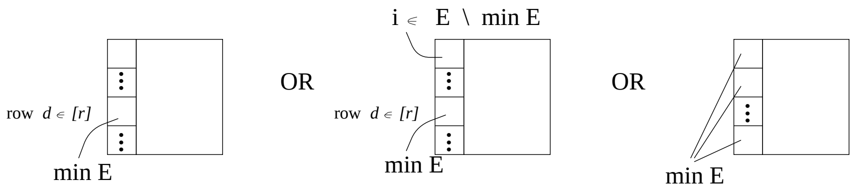
Proposition 3.5. *Given $x \in A(r)^*$ and $r \in \mathbb{P}$*

$$M_n(x, r) = \sum_{E \subseteq [n]} \# GFT(x, r, E),$$
$$N_n(x, r) = \sum_{E \subseteq [n]} \# UZT(x, r, E).$$

Proof. Straightforward application of the one-to-one correspondences established above between multichains in $\mathrm{Fib}(r)$ and $\mathrm{Z}(r)$ and the sets $GFT(x, r, E)$ and $UZT(x, r, E)$ respectively. □

We note the following two results on the form of a tableau λ in either $GFT(x, r, E)$ or $UZT(x, r, E)$.

Lemma 3.6. *If $\lambda_F \in GFT(x, r, E)$ then λ_F has one of the following forms*



Proof. Since $\lambda_F \in GFT(x, r, E)$, it represents a multichain from $\emptyset$ to some $x \in \mathrm{Fib}(r)$. If $x = dw$ where $d \in [r]$ and $w \in A(r)^*$, then the multichain will have the form

$$\emptyset \leq \ldots \leq \emptyset \leq dw_1 \leq \ldots \leq dw = x,$$

where $w_1 \in A(r)^*$ and dw_1 is the first non-empty element of the multichain. On the other hand, if $x = *w$ where $w \in A(r)^*$, then a multichain from $\emptyset$ to x can have one of two forms. First, the multichain could be of the form

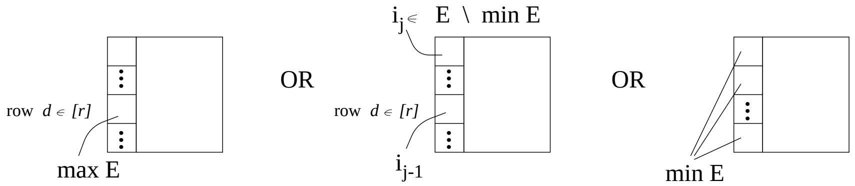
$$\emptyset \leq \ldots \leq \emptyset \leq dw_1 \leq \ldots \leq dw_{k-1} \leq *w_k \leq \ldots \leq *w = x$$

where $d \in [r]$, $w_1, w_{k-1}, w_k \in A(r)^*$ and dw_1 and $*w_k$ are the first elements in the multichain to start with a digit and a $*$ respectively. Alternatively, when $x = *w$ the multichain could be of the form

$$\emptyset \leq \ldots \leq \emptyset \leq *w_1 \leq \ldots \leq *w = x,$$

where $w_1 \in A(r)^*$ and $*w_1$ is the first non-empty element of the multichain. These three alternatives yield the three tableaux in the lemma. $\square$

Lemma 3.7. *If $\lambda_Z \in UZT(x, r, E)$ then λ_Z has one of the following forms*



where i_j is the jth element of E for some j when the elements of E are written in increasing order.

Proof. Similar to Lemma 3.6, a multichain from $\emptyset$ to $x \in \mathrm{Z}(r)$ has one of three possible forms. If $x = dw$ where $d \in [r]$ and $w \in A(r)^*$, then x is join-irreducible in $\mathrm{Z}(r)$ and covers the single element w. So a multichain from $\emptyset$ to $x = dw$ will have the form

$$\emptyset \leq \ldots \leq w_1 \leq dw \leq \ldots \leq dw = x$$

where $w_1 \leq w$. On the other hand, if $x = *w$ where $w \in A(r)^*$ then a multichain from $\emptyset$ to x has one of two forms. First, the multichain could be of the form

$$\emptyset \leq \ldots \leq w_1 \leq dw_2 \leq \ldots \leq dw_2 \leq *w_3 \leq \ldots \leq *w = x,$$

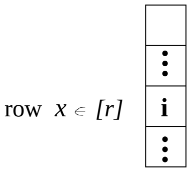
where $w_1, w_2, w_3 \in A(r)^*$ satisfy $w_1 \leq w_2 \leq w_3$. Alternatively, when $x = *w$ the multichain could be of the form

$$\emptyset \leq \ldots \leq w_1 \leq *w_2 \leq \ldots \leq *w = x,$$

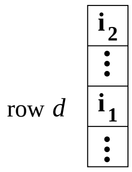
where $w_1, w_2 \in A(r)^*$ satisfy $w_1 \leq w_2$. These three multichain forms correspond to the three tableaux in the lemma. $\square$

We can use generalised box-tableaux to give a combinatorial proof of Stanley's multichain result in [10].

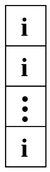
Proof of Theorem 1.2. We show that $\# \, GFT(x,r,E) = \# \, UZT(x,r,E)$, from which the result follows by Proposition 3.5. We use induction on the length, m, of the word x. When $m = 1$, either $x \in [r]$ or $x = *$. If $x \in [r]$, the g-Fib- and g-Z-tableaux of shape x have the form



where i is the only element in the entry set, ie. $E = \{i\}$. If $x = *$ then the g-Fib and g-Z-tableaux of shape x have one of two possible forms. Either they have the form

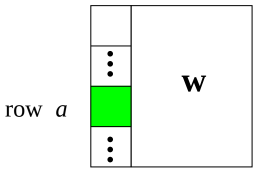


where the entry set is $E = \{i_1, i_2\}$ (there are r of these for a given entry set E, since there are r rows for the entry i_1), or they have the form



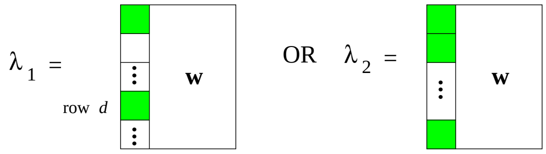
where the entry set is $E = \{i\}$. Thus the result holds when $r = 1$.

Assume the result holds for words $w \in A(r)^*$ of length $m-1$. If $x \in A(r)^*$ has length m, then either $x = aw$ or $x = *w$ where $a \in [r]$ and $w \in A(r)^*$ has length $m-1$. Suppose first that $x = aw$. Then a generalised box-tableau of shape $x = aw$ has the form



Let E be the entry set of this generalised box-tableau. By Lemma 3.6, if $x \in \mathrm{Fib}(r)$ the shaded cell must be numbered with $\min E$. So $\# \, GFT(x,r,E) = \# \, GFT(w,r,E \setminus \min E)$. However, by Lemma 3.7, if $x \in \mathrm{Z}(r)$ the shaded cell must be numbered with $\max E$. So $\# \, UZT(x,r,E) = \# \, UZT(w,r,E \setminus \max E)$. But by a simple renumbering $\# \, UZT(w,r,E \setminus \max E) = \# \, UZT(w,r,E \setminus \min E)$, and by the inductive hypothesis, $\# \, GFT(w,r,E \setminus \min E) = \# \, UZT(w,r,E \setminus \min E)$. Thus $\# \, GFT(x,r,E) = \# \, UZT(x,r,E)$.

Second, suppose $x = *w$. Generalised box-tableau of shape $x = *w$ have one of two possible forms, namely,



If x has the form λ_1, then comparing Lemmas 3.6 and 3.7 we note that if $x \in \mathrm{Fib}(r)$ we have

$$\#\ GFT(x,r,E) = (|E|-1).r.\#\ GFT(w,r,E')$$

where $E' = E \setminus \{i \cup \min E\}$. If $x \in \mathrm{Z}(r)$ we have

$$\#\ UZT(x,r,E) = (|E|-1).r.\#\ UZT(w,r,E'')$$

where $E'' = E \setminus \{i_j \cup i_{j-1}\}$. But again by a simple renumbering, $\#\ UZT(w,r,E'') = \#\ UZT(w,r,E')$, and by the inductive hypothesis, $\#\ GFT(w,r,E') = \#\ UZT(w,r,E')$, so we have $\#\ GFT(x,r,E) = \#\ UZT(x,r,E)$.

Finally, if x has form λ_2 then from Lemmas 3.6 and 3.7 we note that the entries in each of the shaded boxes must be $\min E$. So $\#\ GFT(x,r,E) = \#\ GFT(w,r,E')$ and $\#\ UZT(x,r,E) = \#\ UZT(w,r,E')$ where $E' = E \setminus \min E$. Applying the inductive hypothesis, $\#\ GFT(w,r,E') = \#\ UZT(w,r,E')$, so $\#\ GFT(x,r,E) = \#\ UZT(x,r,E)$. $\square$

4. Bijection between Multichains

Using the method of the above proof we can construct an explicit bijection between the sets $GFT(x,r,E)$ and $UZT(x,r,E)$. Let λ be a generalised box-tableau of shape $x \in A(r)^*$ with entry set E. Define sets $X(\lambda, m)$ by

$$X(\lambda,m) := \{\text{entries in the rightmost } m \text{ columns of } \lambda \\ \text{arranged in increasing order}\}.$$

So $X(\lambda,m) \subseteq E$, with equality when $m = l(x)$.

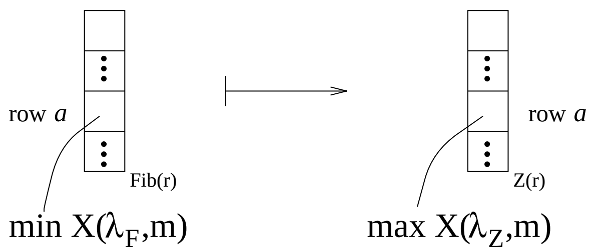
Given a g-Fib-tableau λ_F with entry set E, we construct the corresponding g-Z-tableau $\psi(\lambda_F) = \lambda_Z$ with entry set E, column-wise, starting at the leftmost column and working right. In particular, the jth column of λ_Z is deduced from the jth column of λ_F. The sets $X(\lambda_F, m)$ and $X(\lambda_Z, m)$ indicate the entries that lie in the remaining m columns of λ_F and λ_Z respectively. Initially we set $X(\lambda_Z, l(x)) = X(\lambda_F, l(x))$. For example, suppose we have the g-Fib-tableau $\lambda_F \in GFT(2**\!*, 2, \{1,2,4,5\})$ pictured below,

	4	2	5
1		2	4
	1	2	

to form the corresponding g-Z-tableau $\lambda_Z \in UZT(2**\!*, 2, \{1,2,4,5\})$ we first set $X(\lambda_Z, 4) = X(\lambda_F, 4) = \{1,2,4,5\}$.

As we consider each column, there are three cases to consider, corresponding to the three tableau forms described in Lemmas 3.6 and 3.7.

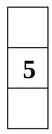
Case (a). ('min-max' rule) The column in λ_F contains a single entry in row $a \in [r]$. We define the bijection ψ by



We then set

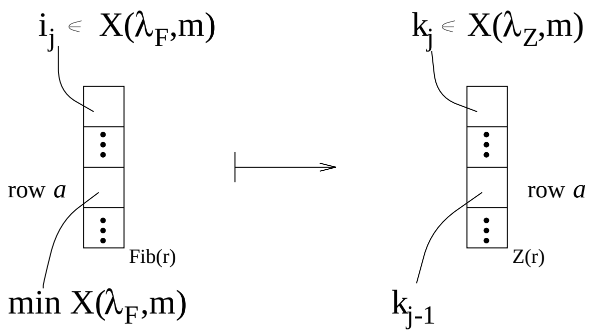
$$X(\lambda_Z, m-1) := \begin{cases} X(\lambda_Z, m), & \\ \qquad \text{if } \min X(\lambda_F, m) \in X(\lambda_F, m-1); & \\ X(\lambda_Z, m) \setminus \max X(\lambda_Z, m), & \\ \qquad \text{if } \min X(\lambda_F, m) \notin X(\lambda_F, m-1). & \end{cases}$$

Applying the 'min-max' rule to the first column of our example above, yields



and $X(\lambda_Z, 3) = \{1, 2, 4, 5\}$.

Case (b). ('position' rule) The column in λ_F contains two entries of different value, one in row $a \in [r]$ and one in the top row. We define the bijection ψ by



where i_j and k_j are the jth elements of the sets $X(\lambda_F, m)$ and $X(\lambda_Z, m)$ respectively. We then set

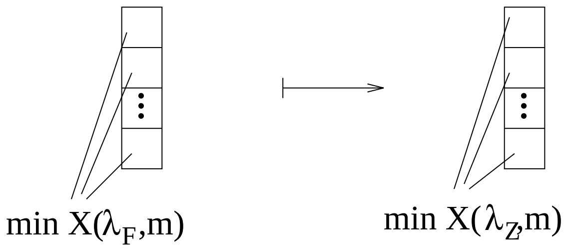
$$X(\lambda_Z, m-1) = \begin{cases} X(\lambda_Z, m), & \\ \quad \text{if } \min X(\lambda_F, m), i_j \in X(\lambda_F, m-1); & \\ X(\lambda_Z, m) \setminus \{k_{j-1}\}, & \\ \quad \text{if } \min X(\lambda_F, m) \notin X(\lambda_F, m-1) \text{ and } i_j \in X(\lambda_F, m-1); & \\ X(\lambda_Z, m) \setminus \{k_j\}, & \\ \quad \text{if } \min X(\lambda_F, m) \in X(\lambda_F, m-1) \text{ and } i_j \notin X(\lambda_F, m-1); & \\ X(\lambda_Z, m) \setminus \{k_j, k_{j-1}, & \\ \quad \text{if } \min X(\lambda_F, m), i_j \notin X(\lambda_F, m-1). & \end{cases}$$

Applying the 'position' rule to the second column of our example yields

4
2

and $X(\lambda_Z, 2) = \{1, 4, 5\}$.

Case (c). ('min' rule) The column in λ_F contains $(r + 1)$ entries of the same value. We define the bijection ψ by



We then set

$$X(\lambda_Z, m-1) := \begin{cases} X(\lambda_Z, m), \\ \qquad \text{if } \min X(\lambda_F, m) \in X(\lambda_F, m-1); \\ X(\lambda_Z, m) \setminus \min X(\lambda_Z, m), \\ \qquad \text{if } \min X(\lambda_F, m) \notin X(\lambda_F, m-1). \end{cases}$$

Applying the 'min' rule to the third column of our example yields

1
1
1

and $X(\lambda_Z, 1) = \{4, 5\}$.

Continuing with our example, we apply the 'position' rule to the fourth column of λ_F, yielding

5
4

Thus the corresponding g-Z-tableau is

	4	1	5
5		1	4
	2	1	

For the reverse mapping, simply reverse the map in each case, taking care to reconstruct the sets $X(\lambda, m)$ correctly. Taking the g-Z-tableau λ_Z resulting from our example, the reader can confirm that applying the reverse maps yields the original g-Fib-tableau λ_F. Interpreting the g-F- and g-Z-tableau from our example in terms of multichains we have

$$\emptyset \leq 21 \leq 21{*} \leq 21{*} \leq 2{*}{*}2 \leq 2{*}{*}{*}$$

in Fib(2) corresponding to the multichain

$$\emptyset \leq {*} \leq 1{*} \leq 1{*} \leq {*}{*}2 \leq 2{*}{*}{*}$$

in Z(2).

In [10, Corollary 2.5] Stanley showed, as a simple consequence of the multichain result that the number of elements in $[\emptyset, x]$ in Fib(r) and Z(r) are the same (place $n = 2$ in Theorem 1.2). Stanley then remarked that it would be interesting to find a simple bijective proof of this result. One such bijection was constructed by Kremer in [5]. By providing a bijective proof of the more general result on multichains of length n we have provided another bijection for the simpler Corollary.

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