The Stochastically Subordinated Log Normal Process Applied to Financial Time Series and Option Pricing

David Edelman
University of Wollongong
Department of Accounting and Finance
Northfield Avenue
Wollongong NSW 2522
Australia
davide@dedalus.uow.edu.au

Thomas Gillespie
University of Sydney
Department of Mathematics and Statistics
Carslaw Building F07
University of Sydney NSW 2006
Australia
tgillesp@postbox.usyd.edu.au

Abstract

The method of stochastic subordination, or random time indexing, has been recently applied to Wiener process price processes to model financial returns. Previous emphasis in stochastic subordination models has involved explicitly identifying the subordinating process with an observable quantity such as number of trades. In contrast, the approach taken here does not depend on the specific identification of the subordinated time variable, but rather assumes a class of time models and estimates parameters from data. In addition, a simple Markov process is proposed for the characteristic parameter of the subordinating distribution to explain the significant autocorrelation of the squared returns. It is shown in particular, that the proposed model, while containing only a few more parameters than the commonly used Wiener process models, fits selected financial time series particularly well, characterising the autocorrelation structure and heavy tails, as well as preserving the desirable self-similarity structure present in popular chaos-theoretic models, and the existence of risk-neutral measures necessary for objective derivative valuation.

I. Introduction

Since the publication of the seminal paper Black & Scholes (1973), the so called Black-Scholes model for the pricing of financial options has met with wide approval by most practitioners as a basic model to value financial options. This model makes some strong assumptions about the form and stationarity of the stochastic process or probability distribution for the stock price. Many studies have shown these assumptions to be inaccurate and are thought to be one of the reasons for the fact that actual option prices trade away form the theoretical Black-Scholes values.

Stochastic subordination is the use of a stochastic time variable as opposed to calendar time to subordinate the price process. This concept was introduced by

Clark (1971) and elaborated upon by German & Ané (1996). Both of these authors identified some physically observed variable such as number of trades or trading volume as the stochastic subordinating variable. While for parameter fitting this has an obvious appeal, for prediction it is just as difficult to predict the subordinating variable as prediction of the asset price process. Additionally, statistics such as volume traded are either not available for particular markets (for example foreign exchange) or can be grossly inaccurate measures due to trading off market, arbitrage activity and large corporate transactions in equity markets.

The main contribution of this paper is to extend the technique of stochastic subordination by assuming a suitable process for the subordinating variable without having to identify it with a physically observed variable. This paper will also test the fit of the proposed process to empirically and derive option pricing formulae that more accurately model the way options are priced in practice.

This section will examine the evidence for rejecting the assumptions of Black & Scholes. Issues such as non-normality of returns, autocorrelation of squared returns and volatility smiles will be discussed. A brief review is presented of currently popular models.

Section II will present an alternative stochastic subordination model for the dynamics of security price returns that assumes a class of subordinating random processes without identifying with a particular observed quantity.

Section III will discuss the fitting of parameters to the proposed stochastic subordination process and show the improvement in fit over the more usual Wiener process with empirical data. The model will be used to fit the parameters to a total of 46 financial time series including equities, equity indices, foreign exchange rates, interest rates and commodity prices. The case of the Standard & Poors 500 index of the US equity market will be examined in detail.

Section IV will use the stochastic subordination model developed in Section II to present an alternative option pricing model. This option pricing model will be used to generate indicative option prices that more closely match those observed in most traded markets.

Section V will conclude the paper with a brief description of the main results of the work presented and suggest further possible avenues of work.

A. Non-Normality of Returns

The non-normality of security price returns has attracted a large number of studies, for example Kendell (1953), Mandelbrot (1963), Fama (1965) and recently in Zangari (1996). The observed distributions are commonly called leptokurtic because of the narrower body of the distribution and fatter tails.

Figure 1 below shows the empirical probability distribution function for daily Standard and Poors 500 (S&P 500) index returns over a 7 year period from 1988 to 1995. Also shown on this chart is smoothed kernel based estimate of the distribution due to Thompson & Tapia (1978) and a normal approximation. While this is one of many securities used in this study (a total of 46 different financial securities), the empirical distributions have similar forms. Generally, empirical distributions are noted to have significantly heavier tails, narrower body and may be skewed when compared to a normal distribution.

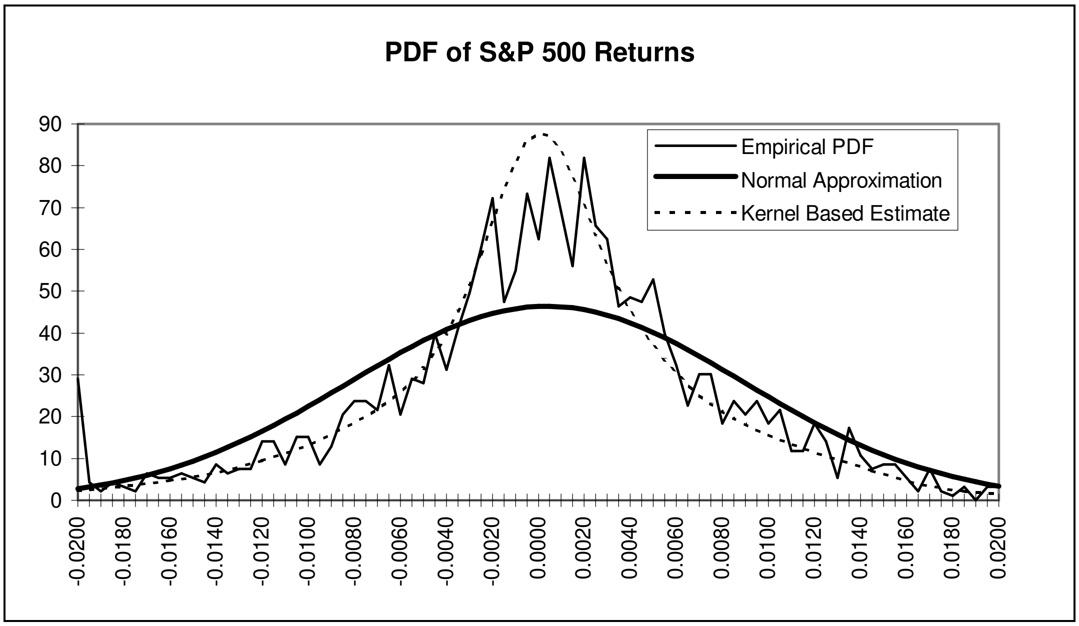

Figure 1

Figure 1 may be log transformed as in Figure 2 below, to show the deviations in the tails more accurately. Note that large outliers on the downside have artificially enlarged the lowest histogram value.

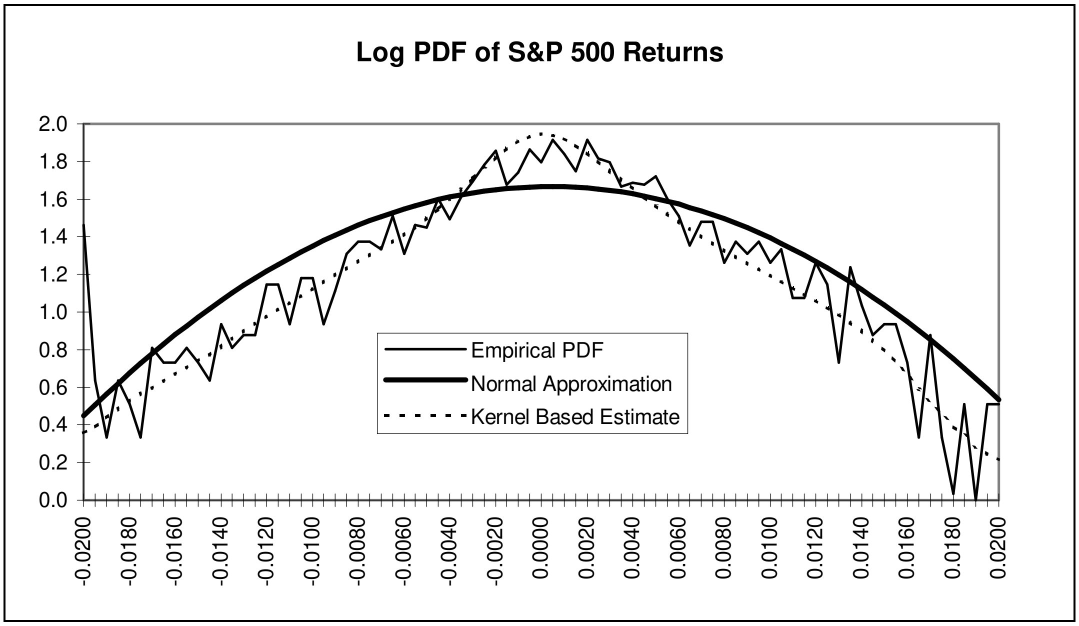

Figure 2

Another form of graphical representation is known as a the QQ plot which plots the percentiles of the empirical distribution on the verticle axis with those of the approximating normal distribution on the horozontal axis. The QQ plot for the S&P 500 index is shown in Figure 3 below. If the empirical distribution were normal then the QQ plot would represent a perfectly straight line (see bold line in Figure 3). Again QQ plots are used to investigate the departures of the empirical distribution from the normal distribution in the tails.

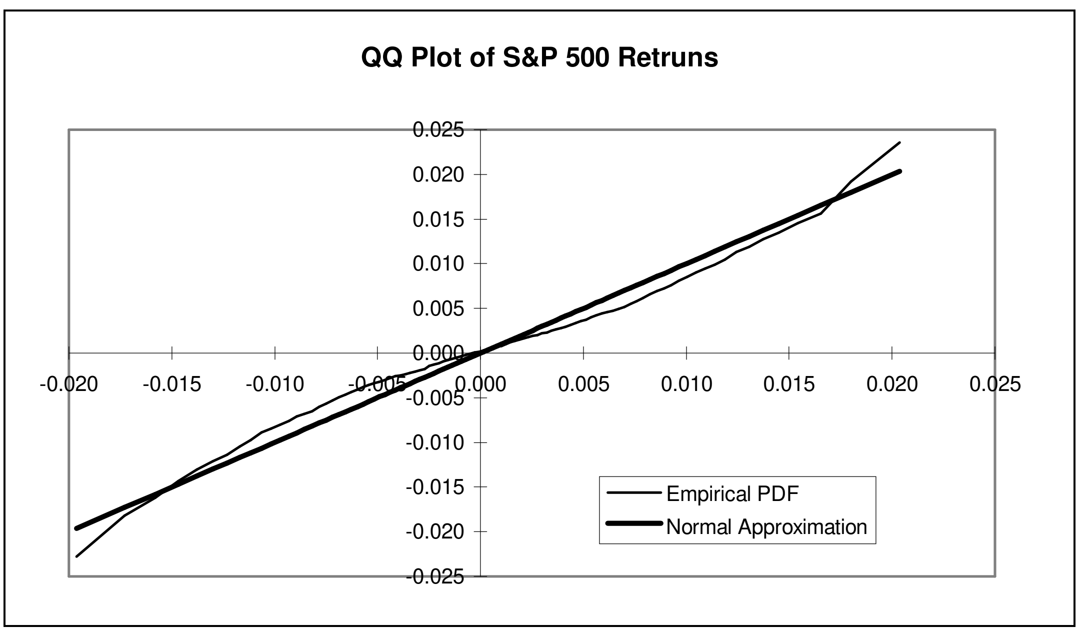

Figure 3

Alternatively, the departures of the empirical distribution from the normal distribution may be plotted directly as in Figure 4 below. This figure shows the differences noted above.

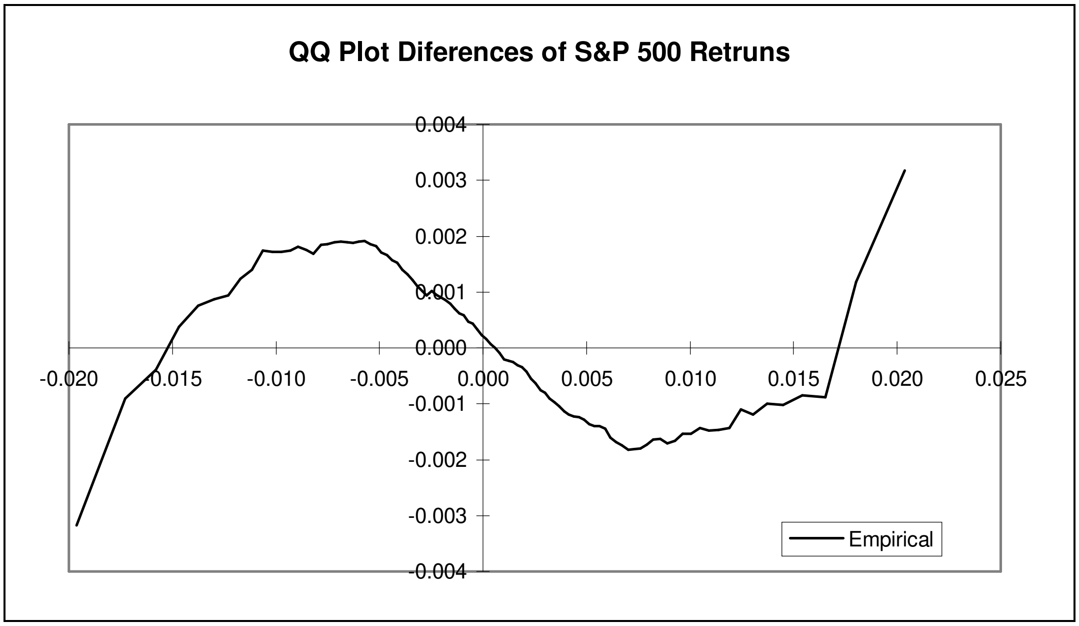

Figure 4

B. Correlation of Returns and Returns Squared

As early as 1900, Bachelier (1900) showed that the movements in security prices are uncorrelated. This fact is borne out by Figure 5, which shows the autocorrelation function of the S&P 500 returns. From this figure it can be seen that only one point deviates significantly from zero at the 5% level. It is entirely reasonable to expect one out of 20 to deviate significantly form zero at the 5% level.

Turning to the squared returns, there is no such arbitrage argument to show that the autocorrelation function should be zero. Figure 5 shows significant positive autocorrelation for the squared returns of the S&P 500 index at most time lags[1]. This heteroscedasticity is typical of many security price series and has motivated a lot of the GARCH modelling described below.

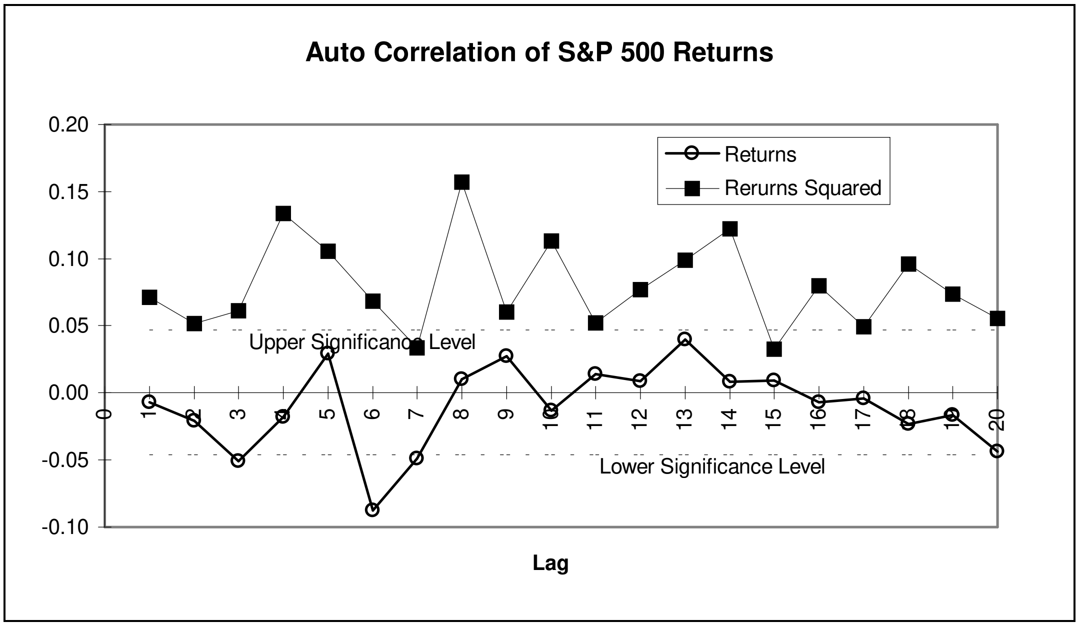

Figure 5

These results are characteristic of those found by other researchers, see Zangari (1996) for an up to date review of results.

C. Implied Volatility Smile

The deviation of option prices from the Black-Scholes values are a well documented phenomena. See MacBeth & Merville (1979), Rubenstein (1985) or (Hull (1993), section 17.6). This phenomena is usually called the "volatility smile" because when the implied volatilities for a fixed expiry date are plotted against the strike, they take on a "smile" shape being lower for the at-the-money options and higher for the in-the-money and out-of-the-money options.

Again, concentrating on the S&P 500 index, the implied volatilities of the offer price of call options is reproduced in Figure 6. The "smile" characteristic is particularly obvious from the 31 day option prices[2].

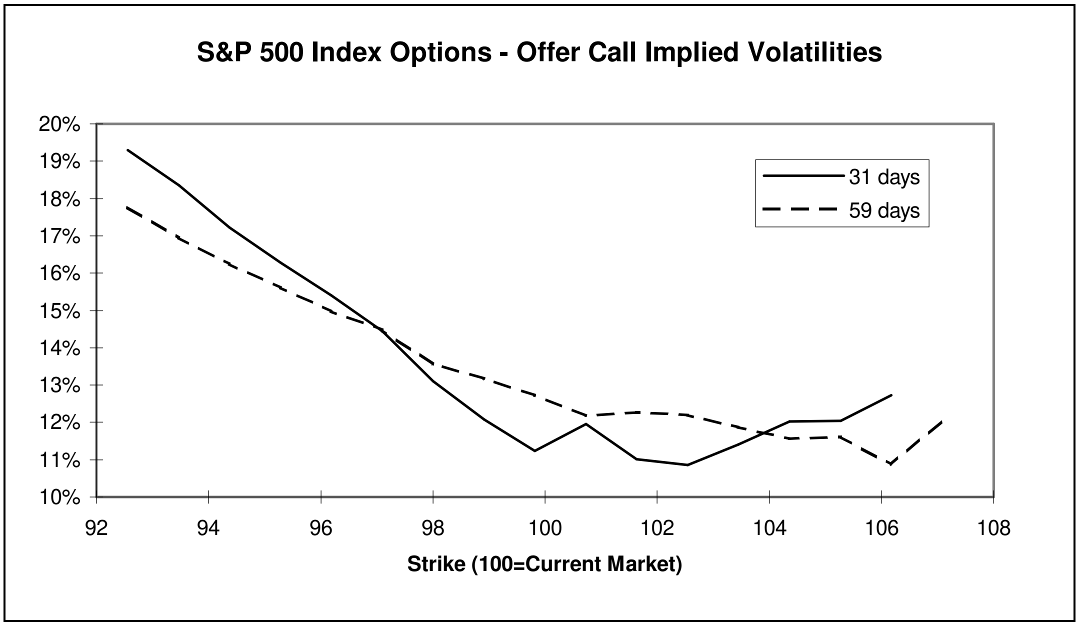

Figure 6

D. Current Approaches

Since the publication of Black & Scholes (1973) a number of approaches have been adopted to tackle the problem of non-normality of returns. By far the greatest body of research has followed the generalised autoregressive conditional hetroskedasticity (GARCH) introduced by Engle (1982) and recently summarised in Duan (1996) and Engle & Mezrich (1995). Under one form of GARCH model, the asset price S_t is assumed to follow

$$\log\left(\frac{S_{t+1}}{S_t}\right) = r + \lambda\sigma_{t+1} - \frac{\sigma_{t+1}{}^2}{2} + \sigma_{t+1}\varepsilon_{t+1}$$

$$\sigma_{t+1}{}^2 = \beta_0 + \beta_1\sigma_t{}^2 + \beta_2\sigma_t{}^2(\varepsilon_t - \theta)^2 \tag{1}$$

$$\varepsilon_{t+1} \sim NID(0,1)$$

While GARCH models successfully model the observed tendency for successive squared price returns to correlate with each other (along with the accompanying heavy-tailed price return behaviour), the lack of fundamental economic motivation for the approach is somewhat disconcerting, and this lack often leads to a class of models which is far too broad to lead to reliably generalisable modelling results. Further, the lack of economic structure of GARCH may not often lead to tradable models, as the emphasis of the approach lies in *describing* rather than *explaining*.

Another approach proposed earlier was the stochastic volatility model of Hull & White (1987). This model assumed the asset price S and asset variance parameter V is governed by the following set of stochastic differential equations

$$
\begin{aligned}
dS &= S\mu_S dt + S\sqrt{V}dW_S \\
dV &= V\mu_V dt + V\xi dW_V
\end{aligned}
\tag{2}
$$

where dW_S and dW_V are standard Wiener processes and ρ_{SV} is the correlation coefficient between dW_S and dW_V. A common implementation of this model is to use a mean reverting process for the variance parameter, $\mu_V = \alpha(\bar{v} - V)$. This model requires an astute choice of the variance drift component μ_V to model observed correlation structures and distribution properties. Option pricing under this general model has no closed form nor a satisfactory analytic approximation. Other stochastic volatility models have been introduced, such as Madan & Seneta (1990).

Time independent models aim to model the non normality of returns by replacing the normal distribution with a wider class of distributions. The autocorrelation structure is implicitly ignored with these approaches. Mandelbrot (1963) introduced the Levy stable distributions into finance literature and later option pricing was discussed by Edelman (1995), jump-diffusion processes have been discussed by Merton (1976), compound normal models covered by Kon (1984) and hyperbolic distributions by Eberlein & Keller (1995). This is an ongoing area of mathematical finance. Apart from ignoring the autocorrelation structure, these models are often not closed under convolution. Closure under convolution is a desirable feature of any model as the distribution for any time period will be of the same class.

Another approach to pricing of options under non-normal stochastic processes was proposed by Jarrow & Rudd (1982). This model proposed using an Edgeworth series approximation to the asset price to model the heavy tails, the autocorrelation structure was ignored. Jarrow & Rudd (1983a) empirically test this model against market data for stock options traded on US stocks. The adjustment factors were found to improve the pricing with the kurtosis term having the largest impact. These authors also found the estimation of the parameters problematic and unstable using a method of moments approach. Again, this model ignores the autocorrelation structure and lacks economic motivation.

II. A General Stochastic Subordination Model

A. Development

Stochastic Subordination might be motivated by imagining that logarithmic price returns are distributed as a Wiener process $W(\cdot)$, which is indexed not by the customary calendar time *t*, but by the random trading time variable T_t, which may be thought of as a measure of cumulative trading activity to time *t*. Most published research involving this approach, Clark (1973) and German & Ané (1996), attempts to relate T_t to some observable, measurable quantity, such as number of trades, volume of trade, etc., and then applies the result to price modelling. The approach taken here involves merely assuming that whatever measure of trading activity T_t should model, it is represented by an inhomogeneous Poisson process:

$$\Pr\{T_t = k\} = \frac{\Lambda(t)^k}{k!} e^{-\Lambda(t)} \tag{3}$$

where

$$\Lambda(t) = \int_{s=0}^{t} ds \lambda(s) \tag{4}$$

and $\lambda(t)$ denotes an "intensity" process, which may stochastic, in which case it will be assumed that the above distribution is conditional on $\Lambda(t)$. Such a model, may be justified by noting that between times t and $t+\varepsilon$, for ε sufficiently small, the occurrence of a trading event may be regarded as a "rare event", and hence modelled by a Poisson variable with intensity which we may write as $\varepsilon\lambda(t)$.

The simplest of such models, of course, occurs when the intensity $\lambda(t) = \lambda_0$, constant over time. Hence $\Lambda(t) = \lambda_0 t$. However, while this simple model is able to describe much of the autocorrelation of squared returns, it is soon seen to not model all of such autocorrelation, nor does it adequately model the observed increase in intensity during some trading periods, which are more than could be explained by chance alone.

The next level of complexity[3] allows $\lambda(t)$ to alternate between two states, which we shall refer to as "cold" or "low" (perhaps the "usual" state), λ_1, and "hot" or "high", λ_2. It will be assumed that the process $\lambda(t)$ is Markovian, in that the distribution of $\lambda(t_3)$ given $\lambda(t_2)$ for $t_2 < t_3$ is independent of $\lambda(t_1)$ if $t_1 < t_2 < t_3$, and is stationary, not having a nature which is evolving over time. Futher, it will be assumed that the Wiener process $W(\cdot)$, the marginally Poisson subordinating variable T_t and the Markov process $\lambda(t)$ are independent.

By discretising time, the process $\lambda(t) = \{\lambda(0), \lambda(\tau), \lambda(2\tau), \ldots\}$ can be thought of as a Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{pmatrix} \tag{5}$$

where

$$p_{ij} = \Pr\left\{\lambda(t+\tau) = \lambda_j \middle| \lambda(t) = \lambda_i\right\} \tag{6}$$

The long run probabilities of being in one state or the other are given by solutions of

$$\mathbf{p} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \mathbf{p} \cdot \mathbf{P} \tag{7}$$

that is

$$\begin{aligned} p_1 &= \frac{1-p_{22}}{2-p_{11}-p_{22}} \\ p_2 &= 1-p_1 \end{aligned} \tag{8}$$

Results from real analysis and probability theory may be used to argue the generality of this class of models for approximating very broad classes of stationary monotone increasing processes $\lambda(t)$, but such a discussion will not be detailed here. Rather, in the sequel the time model will merely be assumed to be general enough for modelling purposes, as may be verified from comparison with actual data.

The stochastic process for the stock price S_t is given by

$$\frac{dS_t}{S_t} = \mu t + \delta\sqrt{T_t}\, dW \tag{9}$$

By conditioning on the value of $\lambda(t)$ and the value of the Poisson subordinating time variable, the probability density function for the stock returns

$$R_t = \log\left(S_t / S_0\right) \tag{10}$$

is given by

$$f_R(r) = \sum_{i=1}^{2} p_i \sum_{j=0}^{\infty} \frac{\left(t\lambda_i\right)^j}{j!} e^{-t\lambda_i} \frac{1}{\delta\sqrt{2\pi j}} e^{-\frac{(r-\mu t)^2}{2\delta^2 j}} \tag{11}$$

Equation (11) has the obvious practicle problem when *j*=0, replacing the normal distribution function with the Dirac delta function $\delta_D(\cdot)$ 4 in this case yields

$$f_R(r) = \sum_{i=1}^{2} p_i e^{-t\lambda_i} \left(\delta_D\left(r-\mu t\right) + \sum_{j=1}^{\infty} \frac{\left(t\lambda_i\right)^j}{j!} \frac{1}{\delta\sqrt{2\pi j}} e^{-\frac{(r-\mu t)^2}{2\delta^2 j}} \right) \tag{12}$$

The cumulative distribution function may easily be found by integration to be

$$F_R(r)=\sum_{i=1}^{2}p_i e^{-t\lambda_i}\left(H(r-\mu t)+\sum_{j=1}^{\infty}\frac{(t\lambda_i)^j}{j!}\Phi\left(\frac{r-\mu t}{\delta\sqrt{j}}\right)\right) \tag{13}$$

where $\Phi(\cdot)$ is the usual standard normal cumulative distribution function and $H(\cdot)$ is the Heaviside step function[5].

For large values of λ_1 and λ_2, equations (12) and (13) are computationally expensive to calculate as the sum in each question must be iterated until the incremental probability $\sum_{i=1}^{2}p_i e^{-t\lambda_i}(t\lambda_i)^j/j!$ is smaller than a predetermined level, say 10^{-12} with $j>\lambda_1 t$ and $j>\lambda_2 t$. For large values of λ_1 and λ_2 an Edgeworth series approximation was found to be satisfactory.

Conditioning on the value of $\lambda(t)$, the first the mean and the first three central moments of the return distribution were found to be

$$\mathrm{E}\left[R_t|\lambda(t)=\lambda\right]=\mu t \tag{14}$$

$$\mathrm{E}\left[\left(R_t-\mathrm{E}[R_t]\right)^2|\lambda(t)=\lambda\right]=\delta^2 t\lambda \tag{15}$$

$$\mathrm{E}\left[\left(R_t-\mathrm{E}[R_t]\right)^3|\lambda(t)=\lambda\right]=0 \tag{16}$$

$$\mathrm{E}\left[\left(R_t-\mathrm{E}[R_t]\right)^4|\lambda(t)=\lambda\right]=3\delta^4 t\lambda(1+t\lambda) \tag{17}$$

and so the four term Edgeworth approximation to the density function and the distribution function were found to be

$$f_R(r)=\sum_{i=1}^{2}p_i e^{-t\lambda_i}\frac{\exp\left(\frac{-(r-\mu t)^2}{2\delta^2\lambda_i t}\right)}{\delta\sqrt{2\pi\lambda_i t}}\left(1+\frac{(r-\mu t)^4-6\delta^2\lambda_i t(r-\mu t)^2+3\delta^4\lambda_i^{\,2}t^2}{8\delta^4\lambda_i^{\,3}t^3}\right) \tag{18}$$

$$F_R(r)=\sum_{i=1}^{2}p_i e^{-t\lambda_i}\left(\Phi\left(\frac{r-\mu t}{\delta\sqrt{\lambda_i t}}\right)-\frac{\exp\left(\frac{-(r-\mu t)^2}{2\delta^2\lambda_i t}\right)}{\delta\sqrt{2\pi\lambda_i t}}\left(\frac{(r-\mu t)^3-3\delta^2\lambda_i t(r-\mu t)}{8\delta^2\lambda_i^{\,2}t^2}\right)\right) \tag{19}$$

II. Estimating Parameters

A. Maximum Likelihood

With estimation there are two distinct choices to be made, one of what data to use to make the estimation and the other what method to use. With regard to the choice of data, daily closing price data was chosen for the estimation of the parameters. Many workers have adopted the use of ‘tick by tick’ to estimate parameters as the shear number of observations gives great statistical efficiency with a sample taken over a relatively short time. The aim of estimation is to establish parameters of the model that reflect the broad characteristics of a market or security. These broad characteristics should be more or less constant over time and not reflect the exact behaviour of a market for only a very short amount of time. In Section I, it was stated that the aim of this work was to price options with a time to expiry of between one month and two years, for this reason daily data was settled upon as short enough to capture the non normality of price movements and long enough to smooth out abnormal short term fluctuations.

Two methods of estimating parameters are immediately accessible, the method of moments and the method of maximum likelihood. Concentrating on the method moments firstly, the mean and the first three central moments of the return distribution as defined in equations (10) and (12) are

$$\mathrm{E}[R_t] = \mu t \tag{20}$$

$$\mathrm{E}\left[\left(R_t - \mathrm{E}[R_t]\right)^2\right] = \delta^2 t\bar{\lambda} \tag{21}$$

$$\mathrm{E}\left[\left(R_t - \mathrm{E}[R_t]\right)^3\right] = 0 \tag{22}$$

$$\mathrm{E}\left[\left(R_t - \mathrm{E}[R_t]\right)^4\right] = 3\delta^4\left(p_1 t\lambda_1\left(1 + t\lambda_1\right) + \left(1 - p_1\right)t\lambda_2\left(1 + t\lambda_2\right)\right) \tag{23}$$

where

$$\bar{\lambda} = p_1\lambda_1 + \left(1 - p_1\right)\lambda_2 \tag{24}$$

Also the one time lag autocorrelation for the squared returns is given by

$$\frac{\mathrm{E}\left[\left(R_t^{\ 2}-\mathrm{E}\left[R_t^{\ 2}\right]\right)\left(R_{2t}^{\ 2}-\mathrm{E}\left[R_t^{\ 2}\right]\right)\right]}{\mathrm{E}\left[\left(R_t^{\ 2}-\mathrm{E}\left[R_t^{\ 2}\right]\right)^2\right]} = \frac{\delta^4 t^2\left(\lambda_2-\lambda_1\right)p_1\left(p_{11}-p_1\right)}{3\delta^4\left(p_1\lambda_1 t\left(\lambda_1 t-1\right)+\left(1-p_1\right)\lambda_2 t\left(\lambda_1 t-1\right)\right)+4\mu^2\delta^2 t^3\bar{\lambda}-\delta^4 t^2\bar{\lambda}^2} \tag{25}$$

Also, the ratio of the n'th and the n+1'th time lag autocorrelation can be shown to be

$$\frac{\mathrm{E}\left[\left(R_t^{\ 2}-\mathrm{E}\left[R_t^{\ 2}\right]\right)\left(R_{(n+1)t}^{\ 2}-\mathrm{E}\left[R_t^{\ 2}\right]\right)\right]}{\mathrm{E}\left[\left(R_t^{\ 2}-\mathrm{E}\left[R_t^{\ 2}\right]\right)\left(R_{nt}^{\ 2}-\mathrm{E}\left[R_t^{\ 2}\right]\right)\right]} = \frac{p_{11}-p_1}{1-p_1} \tag{26}$$

Hence, equations (20), (21), (23), (25), (26) as well as one other must be used to solve for the parameters of the model, namely $\{\mu, \delta, \lambda_1, \lambda_2, p_1, p_{11}\}$. The obvious choice for the addition equation is to use the 6th central moment, that is $\mathrm{E}\left[\left(R_t-\mathrm{E}\left[R_t\right]\right)^6\right]$. As Jarrow & Rudd (1983a) found, this would be an inherently unstable method as the high moments of financial time series are known to be unstable and the solution method would not be robust.

For prediction, the parameter p_{11} is of use only if the current state of $\lambda(t)$ is known. If the current state of $\lambda(t)$ is unknown, it is assumed for this first treatment that the probability of being in the low state is the long run probability p_1 and being in the high state takes the probability $p_2 = 1 - p_1$. For this reason knowing the precise value of p_{11} is unimportant, and so allowing the use of maximum likelihood estimation as an alternative to the method of moments. The estimation of the current state of a particular market, and the corresponding influence on option pricing will be the subject of a future paper.

The maximum likelihood approach was used and found to give stable and robust results. Equation (18) was used with the usual statistical definition of likelihood score

$$L = -2\sum_j \log\left(f_R\left(r_j\right)\right) \tag{27}$$

This equation may be numerically minimised with over the parameter space $\{-\infty < \mu < \infty, \delta > 0, 0 < \lambda_1 < \infty, 0 < \lambda_2 < \infty, 0 \le p_1 \le 1\}$.

B. Results

A combination of Australian and international data was used to test the applicability of this model. The data consisted of the 20 most liquid stocks traded on the Australian Stock Exchange, four Australian stock indices, five foreign stock indices, five of the most actively traded currencies and 12 metals and agricultural commodities prices - a total of 46 data series. Daily data was used for the period 1988 to 1995 inclusive.

A maximum likelihood method was used with equations (27) and both (18) and (12). The downhill simplex method of Press et al (1992) was used to numerically minimise equation (27) over the parameter space $\{-\infty < \mu < \infty, \delta > 0, 0 < \lambda_1 < \infty, 0 < \lambda_2 < \infty, 0 \leq p_1 \leq 1\}$. As this problem is numerically stable, most reasonable minimisation routines can be used.

Table 1 below shows these results. Notice that the constraints $\lambda_1 < \lambda_2 < 10^6$ and $p_1 < 1$ were imposed to ensure reasonableness and uniqueness. Exactly the same results (but much faster solution) were obtained using the Edgeworth approximation of equation (18) as opposed to the original formulation of equation (12). The reason for this is that most values of λ_1 and λ_2 were found to be large.

File	Num	Mu	Delta	Lamda1	Lamda2	p1	Likelihood
ALLORD	1877	0.0822	0.00357	1337.90	19599.01	0.9843	-12577.93
ALLIND	1878	0.0805	0.00395	957.17	12209.48	0.9780	-12794.75
ALLRES	1860	0.0953	0.00522	1033.55	18460.55	0.9903	-11593.16
ALLMIN	1863	0.0432	0.00519	1182.76	12937.24	0.9707	-11238.14
BHP	1720	0.1466	0.00038	251846.15	999703.88	0.9109	-10015.18
NAB	1669	0.1994	0.00123	20454.64	128036.10	0.8879	-9727.88
NCP	1734	0.1081	0.00977	1057.49	19423.60	0.9513	-8203.10
CRA	1713	0.1493	0.00734	1161.06	19322.50	0.9805	-9212.39
WBC	1678	0.0800	0.00645	1404.21	7076.19	0.9340	-9029.96
WMC	1755	0.0666	0.00683	1621.87	7891.41	0.9582	-9104.04
BTR	1679	0.1460	0.00628	1847.63	19379.54	0.9705	-8718.25
ANZ	1679	0.0328	0.00047	230261.89	999993.07	0.8463	-8933.68
AMC	1654	0.1931	0.00555	1187.91	7243.97	0.9262	-9595.04
CML	1650	0.1068	0.00522	1627.61	6990.88	0.9384	-9383.70
CBA	829	0.1012	0.00239	3470.63	17595.00	0.8497	-5170.54
CSR	1687	0.0783	0.00545	1551.75	11978.05	0.9625	-9533.56
BOR	1674	0.0535	0.00828	963.02	8014.42	0.9890	-9055.93
FBG	1553	-0.0095	0.00075	157509.49	999999.62	0.9246	-7578.56
WPL	1570	0.1955	0.00939	835.39	6625.37	0.9249	-7946.00
LLC	1608	0.1188	0.00691	895.36	10646.80	0.9765	-9285.36
PDP	1690	0.0295	0.00379	3448.99	23395.26	0.9446	-9331.26
WOW	375	0.0578	0.00036	339069.28	999999.68	0.9192	-2141.09
MIM	1652	0.0075	0.00084	152837.19	999999.76	0.9527	-7862.48
BIL	1656	0.0590	0.00566	1417.86	12770.67	0.9648	-9379.39
DOWJON	1862	0.1303	0.00446	677.78	5402.71	0.9208	-12596.73
S&P	1857	0.1209	0.00394	726.82	4800.35	0.8747	-12702.97
FTSE	1856	0.0852	0.00039	99844.89	999999.68	0.9593	-12415.38
JAPAN	1825	0.0047	0.00582	614.96	4283.29	0.7636	-10809.95
GERM	1768	0.1045	0.00362	1581.99	23705.67	0.9469	-11137.41
AUDUSD	1646	0.0663	0.00245	1446.31	11649.08	0.9737	-12108.31
GBPUSD	1804	0.0286	0.00019	186331.19	999982.72	0.7121	-12648.10
USDJPY	1862	-0.0242	0.00403	652.24	8163.82	0.9780	-13361.07
USDDEM	1798	-0.0334	0.00271	1356.97	4448.33	0.7820	-12537.83
USDCHF	1776	-0.0141	0.00021	258966.92	996561.16	0.7582	-11955.40
GOLDUK	1792	-0.0133	0.00401	657.76	4913.84	0.9306	-12643.40
CLME	1882	0.0326	0.00914	665.51	3399.04	0.7843	-9631.65
ALUM	1828	-0.0262	0.00970	472.01	5304.75	0.8693	-9761.03
HTOIL	1696	0.0604	0.01288	556.08	7475.98	0.9340	-8174.86
BRENT	1665	0.1162	0.00818	903.99	17528.67	0.9100	-8365.38
RGAS	1700	0.0376	0.01273	696.78	10468.16	0.9391	-7834.53
SUGNY	1602	0.0836	0.01555	445.87	6958.09	0.9654	-7720.62
COTT	1285	0.0726	0.00671	513.28	17174.02	0.9720	-8135.11
WHEAT	1452	0.1007	0.00428	2374.49	124432.37	0.9901	-8360.90
WOOL	743	0.0527	0.00926	385.04	65460.56	0.9841	-4488.86
SOYBEA	1737	0.0980	0.00547	859.51	7583.81	0.8473	-10141.32
CORNNY	530	0.0406	0.00995	380.43	13037.00	0.9946	-3268.25

Table 1

Note that the likelihood given in Table 1 above is the likelihood score given in equation (27).

C. Likelihood Ratio Tests

The model proposed in equation (12) (hypothesis H_1 say) may be tested against the null hypothesis (H_0 say) of the Black-Scholes assumption, namely

$$R \sim N(\mu t, \sigma^2 t) \tag{28}$$

or, alternatively

$$f_R(r) = \frac{\exp\left(\frac{-(r-\mu t)^2}{2\sigma^2 t}\right)}{\sigma\sqrt{2\pi t}} \tag{29}$$

A likelihood ratio test was used for these tests where the test statistic was defined in the usual manner, that is for log likelihoods

$$T = L_{H_0} - L_{H_A} \tag{30}$$

For a large number of observations, this test statistic is distributed as

$$T \sim \chi^2_\nu \tag{31}$$

where the ν is the number of additional parameters, in this case three.

The likelihood ratio test was applied to the 46 data series discussed above and the test statistic of (30) was compared to the critical point of the χ^2_3 distribution at the 0.01 level, $\chi^2_3(0.99) = 11.3449$.

The results in Table 2 below, show an overwhelming case for the Black-Scholes assumptions to be rejected in favour of the model presented in equation (12). Note that only one of the 46 securities failed to show an appropriately large improvement in the likelihood to compensate for the three additional parameters.

A further likelihood ratio test was undertaken to investigate an alternative hypothesis (H_2 say) that the effect of replacing the Poisson process with a constant value, that is

$$\frac{dS_t}{S_t} = \mu t + \delta\sqrt{\Lambda(t)}dW \tag{32}$$

which should be compared to equation (9). The results show a mixed picture with only 19 of the 46 securities rejecting hypothesis H_2 in favour of hypothesis H_1 at the 0.01 level. The addition of the Poisson process added little to the fit of individual equities, while the reverse was true for most equity indices. The aggressiveness of the tests at the 0.01 level could play a part in these results.

Security	N	Likelihood Scores		Test Statistic	Result
		H0	H1		
ALLORD	1877	-12385.40	-12577.93	192.53	Reject H0 for H1 at the 0.01 level
ALLIND	1878	-12569.09	-12794.75	225.67	Reject H0 for H1 at the 0.01 level
ALLRES	1860	-11413.98	-11593.16	179.18	Reject H0 for H1 at the 0.01 level
ALLMIN	1863	-11012.31	-11238.14	225.83	Reject H0 for H1 at the 0.01 level
BHP	1720	-9966.93	-10015.18	48.25	Reject H0 for H1 at the 0.01 level
NAB	1669	-9563.40	-9727.88	164.48	Reject H0 for H1 at the 0.01 level
NCP	1734	-7630.89	-8203.10	572.21	Reject H0 for H1 at the 0.01 level
CRA	1713	-8949.41	-9212.39	262.99	Reject H0 for H1 at the 0.01 level
WBC	1678	-8934.48	-9029.96	95.48	Reject H0 for H1 at the 0.01 level
WMC	1755	-9040.13	-9104.04	63.91	Reject H0 for H1 at the 0.01 level
BTR	1679	-8551.13	-8718.25	167.12	Reject H0 for H1 at the 0.01 level
ANZ	1679	-8847.03	-8933.68	86.64	Reject H0 for H1 at the 0.01 level
AMC	1654	-9439.40	-9595.04	155.64	Reject H0 for H1 at the 0.01 level
CML	1650	-9320.60	-9383.70	63.10	Reject H0 for H1 at the 0.01 level
CBA	829	-5103.55	-5170.54	66.98	Reject H0 for H1 at the 0.01 level
CSR	1687	-9401.11	-9533.56	132.44	Reject H0 for H1 at the 0.01 level
BOR	1674	-8974.40	-9055.93	81.53	Reject H0 for H1 at the 0.01 level
FBG	1553	-7460.07	-7578.56	118.49	Reject H0 for H1 at the 0.01 level
WPL	1570	-7708.65	-7946.00	237.34	Reject H0 for H1 at the 0.01 level
LLC	1608	-9075.97	-9285.36	209.39	Reject H0 for H1 at the 0.01 level
PDP	1690	-9203.47	-9331.26	127.78	Reject H0 for H1 at the 0.01 level
WOW	375	-2138.95	-2141.09	2.14	Fail to reject H0 for H1 at the 0.01 level
MIM	1652	-7770.75	-7862.48	91.73	Reject H0 for H1 at the 0.01 level
BIL	1656	-9214.32	-9379.39	165.07	Reject H0 for H1 at the 0.01 level
DOWJON	1862	-12264.25	-12596.73	332.48	Reject H0 for H1 at the 0.01 level
S&P	1857	-12390.53	-12702.97	312.44	Reject H0 for H1 at the 0.01 level
FTSE	1856	-12212.62	-12415.38	202.75	Reject H0 for H1 at the 0.01 level
JAPAN	1825	-10425.05	-10809.95	384.89	Reject H0 for H1 at the 0.01 level
GERM	1768	-10674.91	-11137.41	462.50	Reject H0 for H1 at the 0.01 level
AUDUSD	1646	-12006.46	-12108.31	101.85	Reject H0 for H1 at the 0.01 level
GBPUSD	1804	-12480.05	-12648.10	168.05	Reject H0 for H1 at the 0.01 level
USDJPY	1862	-13110.45	-13361.07	250.62	Reject H0 for H1 at the 0.01 level
USDDEM	1798	-12451.50	-12537.83	86.34	Reject H0 for H1 at the 0.01 level
USDCHF	1776	-11874.44	-11955.40	80.96	Reject H0 for H1 at the 0.01 level
GOLDUK	1792	-12359.62	-12643.40	283.78	Reject H0 for H1 at the 0.01 level
CLME	1882	-9364.62	-9631.65	267.03	Reject H0 for H1 at the 0.01 level
ALUM	1828	-9113.05	-9761.03	647.98	Reject H0 for H1 at the 0.01 level
HTOIL	1696	-7638.23	-8174.86	536.63	Reject H0 for H1 at the 0.01 level
BRENT	1665	-7573.75	-8365.38	791.63	Reject H0 for H1 at the 0.01 level
RGAS	1700	-7283.65	-7834.53	550.88	Reject H0 for H1 at the 0.01 level
SUGNY	1602	-7270.64	-7720.62	449.98	Reject H0 for H1 at the 0.01 level
COTT	1285	-7506.71	-8135.11	628.40	Reject H0 for H1 at the 0.01 level
WHEAT	1452	-7904.77	-8360.90	456.13	Reject H0 for H1 at the 0.01 level
WOOL	743	-3584.53	-4488.86	904.33	Reject H0 for H1 at the 0.01 level
SOYBEA	1737	-9721.37	-10141.32	419.95	Reject H0 for H1 at the 0.01 level
CORNNY	530	-3087.70	-3268.25	180.55	Reject H0 for H1 at the 0.01 level

Table 2

D. Density Functions and QQ Plots

Another more visual method of checking the fit of the model, as opposed to the likelihood ratio tests, is to check the probability density function and QQ plot against the empirical curve. Again, focusing on the case of the S&P 500 returns as a typical sample of the 46 instruments tested, Figure 7 below shows the probability density function of both the model and the empirical data and Figure 8 shows the QQ plot. Both show a good fit against the empirical data, especially in the crucial tail regions.

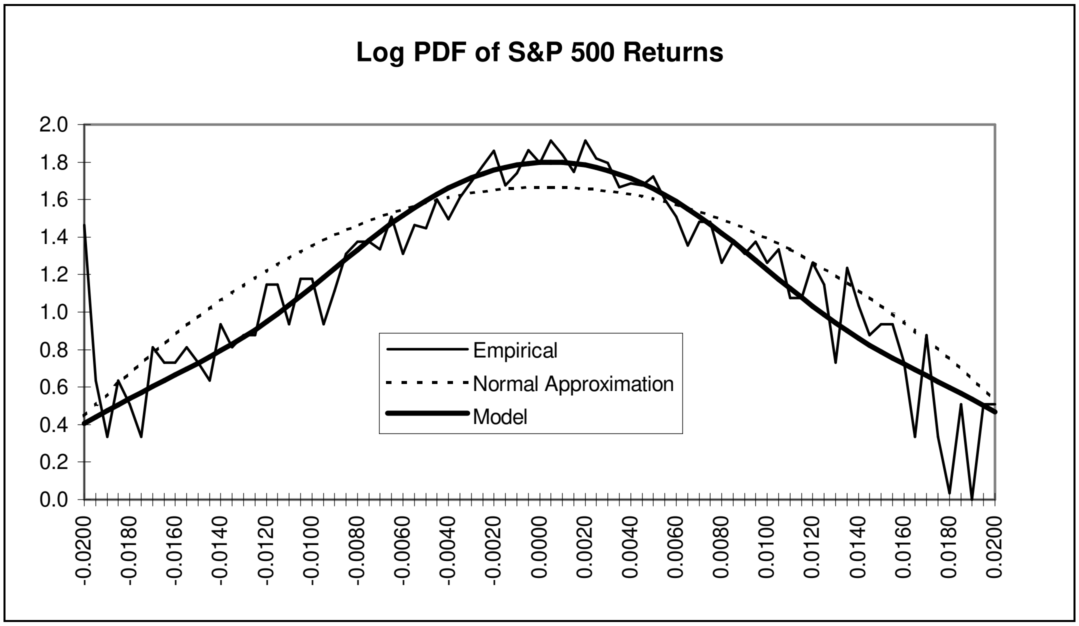

Figure 7

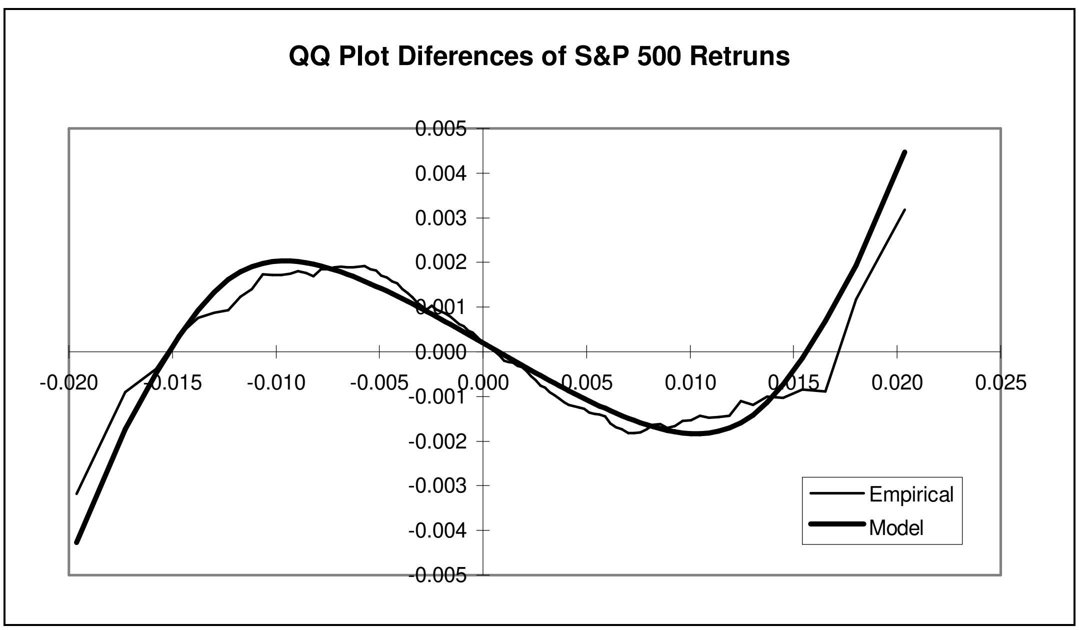

Figure 8

IV. Estimating Option Prices

A. Option Pricing Formulae

The risk neutral valuation formula of Harrison & Kreps (1979) states that the value of a security may be expressed as simply the discounted expected value of the pay off. For a security with payoff V_t at time t and riskless interest rate r, the value v is defined to be

$$v = e^{-rt}\,\mathrm{E}\left[V_t\right] \tag{33}$$

For the stock (or underlying asset) this is simply expressed as

$$\begin{aligned} S &= e^{-rt}\,\mathrm{E}\left[Se^{R}\right] \\ &= e^{-rt}\int_{u=-\infty}^{\infty} du\,Se^{u} f_R(u) \end{aligned} \tag{34}$$

where S is the stock price today, R is the stochastic return of the stock, t is a generic time variable and r is the risk free interest rate. Using equation (12) and a bit of tedious integration gives the equation

$$\mu = r - \frac{1}{t}\log\left(\sum_{i=1}^{2} p_i e^{\lambda_i t\left(e^{\delta^2/2}-1\right)}\right) \tag{35}$$

This selection of μ is not unique as it may be a function of the subordinating variable T_t. Alternatively, we may choose to have marginal risk neutrality, that is

$$S = e^{-rt}\,\mathrm{E}\left[Se^{R}\middle|T_t\right] \tag{36}$$

this gives rise to the more usual equation

$$\mu = r - \frac{\delta^2 T_t}{2t} \tag{37}$$

This latter form of risk neutrality was chosen as it guarantees for every realisation of T_t the market will be risk neutral, as well as unconditionally for every t.

Using the risk neutrality argument for a European call option with a strike of K, expiring in t years time on a stock with price S, and risk-free interest rate r can simply be written as

$$
\begin{aligned}
C &= e^{-rt}\, \mathrm{E}\left[\left(Se^{R}-K\right)^{+}\right] \\
&= e^{-rt} \int_{u=-\infty}^{\infty} du\left(Se^{u}-K\right)^{+} f_{R}(u) \\
&= e^{-rt} \int_{u=\log(K/S)}^{\infty} du\left(Se^{u}-K\right) f_{R}(u)
\end{aligned}
\tag{38}
$$

Equation (38) may be numerically integrated directly, but may also be manipulated so to be expressed as a sum of Black-Scholes valuations. Rewriting equation (38) using equation (12) and incorporating equation (37) yields

$$
C = e^{-rt}\sum_{i=1}^{2} p_i e^{-\lambda_i t}\left(\begin{array}{l}\displaystyle\int_{u=-\infty}^{\infty} du\left(Se^{u}-K\right)^{+}\delta_D(u-rt) \\ \displaystyle+\sum_{j=1}^{\infty}\frac{(t\lambda_i)^j}{j!}\int_{u=-\infty}^{\infty} du\left(Se^{u}-K\right)^{+}\frac{e^{-\frac{\left(u-\left(rt-\delta^2 j/2\right)\right)^2}{2\delta^2 j}}}{\delta\sqrt{2\pi j}}\end{array}\right)
\tag{39}
$$

Let $C_{BS}(S,r,\sigma,K,t)$ be the Black-Scholes value, that is

$$
\begin{aligned}
C_{BS}(S,r,\sigma,K,t) &= e^{-rt}\int_{u=-\infty}^{\infty} du\left(Se^{u}-K\right)^{+}\frac{e^{-\frac{\left(u-\left(r-\sigma^2/2\right)t\right)^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}} \\
&= S\Phi\left(\frac{\log(S/K)+\left(r+\sigma^2/2\right)t}{\sigma\sqrt{t}}\right)-Ke^{-rt}\Phi\left(\frac{\log(S/K)+\left(r-\sigma^2/2\right)t}{\sigma\sqrt{t}}\right)
\end{aligned}
\tag{40}
$$

where $\Phi(\cdot)$ is the usual standard normal cumulative density function. Then equation (38) can easily be rewritten as

$$
\begin{aligned}
C &= \sum_{i=1}^{2} p_i e^{-\lambda_i t}\left(\sum_{j=0}^{\infty}\frac{(t\lambda_i)^j}{j!}C_{BS}\left(S,r,\delta\sqrt{\frac{j}{t}},K,t\right)\right) \\
&= \sum_{j=0}^{\infty} C_{BS}\left(S,r,\delta\sqrt{\frac{j}{t}},K,t\right)\left(p_1 e^{-\lambda_1 t}\frac{(t\lambda_1)^j}{j!}+(1-p_1)e^{-\lambda_2 t}\frac{(t\lambda_2)^j}{j!}\right)
\end{aligned}
\tag{41}
$$

Note that

$$
C_{BS}(S,r,0,K,t) = \lim_{\sigma\to 0^{+}} C_{BS}(S,r,\sigma,K,t) = \left(S-Ke^{-rt}\right)^{+}
\tag{42}
$$

A similar formula holds for puts

$$P = \sum_{j=0}^{\infty} P_{BS}\left(S, r, \delta\sqrt{\frac{j}{t}}, K, t\right)\left(p_1 e^{-\lambda_1 t}\frac{(t\lambda_1)^j}{j!} + (1-p_1)e^{-\lambda_2 t}\frac{(t\lambda_2)^j}{j!}\right) \qquad (43)$$

where P_{BS} is the Black-Scholes put value.

B. Option Prices

Equation (41) of the previous section was used to generate options prices for a grid of values using the parameters estimated for the S&P 500 return series. These parameters from Table 1 are δ=0.00394, λ_1=726.82, λ_2=4800.35 and p_1=0.8747. For conformity, an initial stock price was set at 100 and option prices were estimated for strikes of 150, 140, 130, 125, 120, 115, 110, 105, 100, 95, 90, 85, 80, 75, 70, 60 and 50 and times to expiry of 1, 2, 3, 4, 5, 6, 9, 12, 18 and 24 months. These values are shown in Table 3 below.

Option Prices: Model and Differences with Black-Scholes

Strike	Time to Expiry (Years)							
	0.0833		0.3333		0.5000		1.0000	
	SS	SS-BS	SS	SS-BS	SS	SS-BS	SS	SS-BS
150	0.0000	0.0000	0.0039	0.0039	0.0191	0.0190	0.1263	0.1179
140	0.0000	0.0000	0.0137	0.0137	0.0469	0.0461	0.2133	0.1724
130	0.0001	0.0001	0.0448	0.0435	0.1107	0.0980	0.3711	0.1936
125	0.0008	0.0007	0.0783	0.0712	0.1686	0.1258	0.5105	0.1595
120	0.0038	0.0038	0.1348	0.1007	0.2626	0.1313	0.7411	0.0736
115	0.0162	0.0158	0.2402	0.1004	0.4413	0.0780	1.1470	-0.0694
110	0.0579	0.0456	0.4921	0.0129	0.8405	-0.0608	1.8700	-0.2474
105	0.2315	0.0104	1.1971	-0.1634	1.7513	-0.2402	3.1066	-0.4026
100	1.4614	-0.1350	2.9255	-0.2657	3.5828	-0.3251	5.0645	-0.4599
95	5.2012	0.0167	6.0830	-0.1462	6.5998	-0.2202	7.8740	-0.3764
90	10.0397	0.0345	10.3712	0.0380	10.6485	-0.0208	11.5053	-0.1866
85	15.0066	0.0065	15.1490	0.0914	15.2812	0.0986	15.7676	0.0109
80	20.0006	0.0006	20.0626	0.0570	20.1349	0.1006	20.4084	0.1276
75	25.0000	0.0000	25.0229	0.0226	25.0626	0.0586	25.2296	0.1467
70	30.0000	0.0000	30.0067	0.0067	30.0255	0.0252	30.1285	0.1101
60	40.0000	0.0000	40.0002	0.0002	40.0024	0.0024	40.0313	0.0310
50	50.0000	0.0000	50.0000	0.0000	50.0001	0.0001	50.0043	0.0043

SS = Stochastic subordination model value.
BS = Black-Scholes value.

Table 3

Additionally, the differences between the model option prices of equation (41) and Black-Scholes values are shown in Table 3. This table shows that at-the-money options are shown to be cheaper than Black-Scholes indicates and out-of-the-money and in-the-money options are more expensive than Black-Scholes indicates. This pricing bias is observed in actual traded markets.

C. Implied Volatility Smile

Another way of studying the pricing bias compared to the Black-Scholes model is to study the implied volatility curves of the valuations. These implied volatility curves are shown numerically in Table 4 and graphically in Figures 9 and 10.

Implied Volatilities										
Strike	**Time to Expiry (Years)**									
	0.0833	**0.1667**	**0.2500**	**0.3333**	**0.4167**	**0.5000**	**0.7500**	**1.0000**	**1.5000**	**2.0000**
150	25.71%	24.31%	23.29%	22.47%	21.79%	21.20%	19.81%	18.79%	17.36%	16.41%
140	25.03%	23.38%	22.22%	21.31%	20.56%	19.93%	18.49%	17.48%	16.15%	15.34%
130	23.92%	21.94%	20.61%	19.61%	18.81%	18.17%	16.77%	15.88%	14.85%	14.30%
125	23.05%	20.88%	19.48%	18.46%	17.67%	17.05%	15.78%	15.04%	14.24%	13.83%
120	21.84%	19.49%	18.05%	17.05%	16.33%	15.79%	14.79%	14.25%	13.71%	13.44%
115	20.06%	17.61%	16.27%	15.45%	14.91%	14.54%	13.90%	13.59%	13.27%	13.12%
110	17.36%	15.27%	14.41%	13.97%	13.70%	13.52%	13.24%	13.10%	12.96%	12.89%
105	14.05%	13.34%	13.12%	13.01%	12.95%	12.91%	12.84%	12.80%	12.77%	12.75%
100	12.69%	12.70%	12.70%	12.70%	12.70%	12.70%	12.70%	12.70%	12.70%	12.70%
95	14.19%	13.42%	13.17%	13.05%	12.98%	12.93%	12.85%	12.81%	12.77%	12.75%
90	18.06%	15.80%	14.80%	14.26%	13.94%	13.72%	13.36%	13.19%	13.02%	12.93%
85	21.09%	18.67%	17.25%	16.31%	15.66%	15.19%	14.35%	13.92%	13.49%	13.28%
80	23.05%	20.88%	19.48%	18.46%	17.67%	17.05%	15.78%	15.04%	14.24%	13.83%
75	24.36%	22.50%	21.22%	20.25%	19.47%	18.82%	17.39%	16.44%	15.28%	14.63%
70	25.21%	23.69%	22.57%	21.68%	20.95%	20.33%	18.90%	17.88%	16.51%	15.65%
60		25.15%	24.42%	23.74%	23.15%	22.64%	21.38%	20.41%	18.99%	17.96%
50				25.00%	24.61%	24.22%	23.21%	22.39%	21.10%	20.12%

Table 4

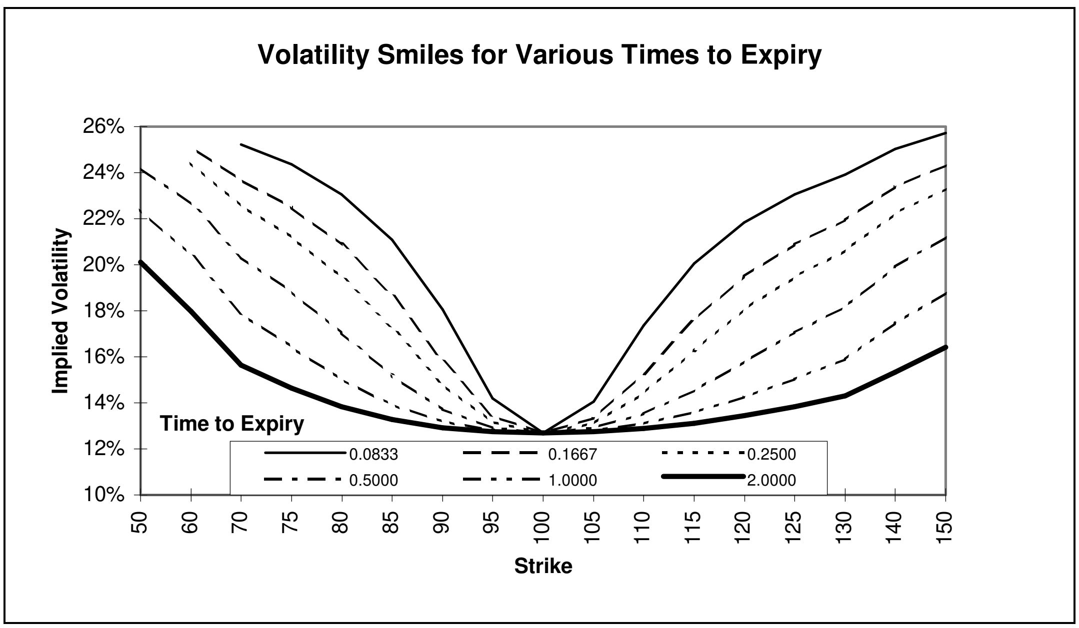

Figure 9

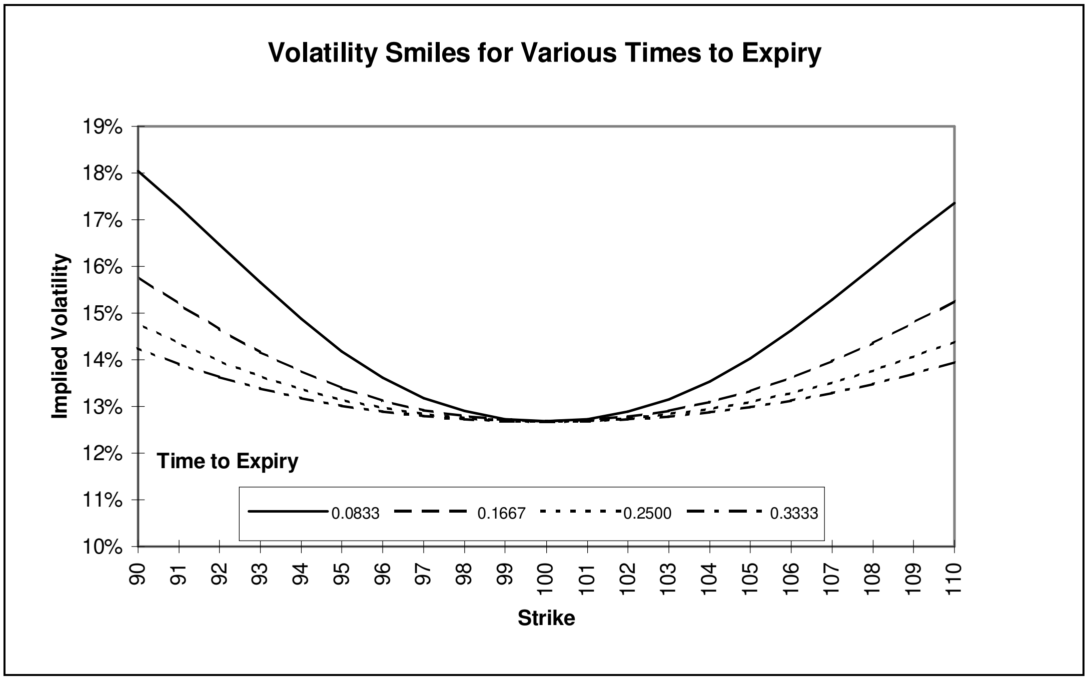

Figure 10

Figure 10 in particular should be compared against that of Figure 6 above. The implied volatility curves show a good fit with those strikes below the current market showing a decreasing curvature for longer times to maturity. For strikes above the current market the picture is less certain as the model implied volatility curves rise more strongly than the actual market observed implied volatility curves.

V Conclusion

The stochastic subordination model proposed in this paper is practically and theoretically apealling for the modelling of financial asset price distributions with the view to the pricing of options. The proposed model is rich enough to model the observed non-normal returns, significant autocorrelation of returns squared and has the desirable feature that it is closed under convolution.

The stochastic subordination model involves three aditional parameters which may easily and quickly be estimated with a maximum likelihood estimation procedure. The fitted model mimics the marked deviations from the normal distribution for asset price returns as well as the observed autocorrelation structure.

The stochastic subordination model has theoretical appeal because of its self similarity, the concept of a random trading time based on "busy" and "quiet" states fits with practicioners views of the way financial markets move and change phase.

The stochastic subordination model can be used to estimate option prices and the resultant implied volatility curves match closely those observed in options markets, particularily for in the money calls. This model may be used to indicate any potential mispricing for different strikes and times to expiry.

A significant advantage of the stochastic subordination model is that option prices will vary depending upon the current state of the market, as experienced practitioners have learnt to correct for the failings of the Black-Scholes model. This first treatment has assumed that the probabilities of being in either state take the long run probabilities. A further paper will investigate the estimation of the current state of the market and hence the adjusted option prices.

The completeness of the market, as discussed thoroughly by Harrison & Pliska (1981), is a highly desirable property for any such model. Moreover, it justifies the use of risk neutral pricing as a valuation technique. The issue of completeness will also give rise to a technique for hedging the option payoff with physical positions in the underlying asset and a riskless security. These issues are of importance to theoreticians and practitioners alike, and will be the subject of a further paper.

The stochastic subordination model does not reflect the observed implied volatility behavior for out-of-the-money call options. Further research is required to investigate market prices for options more thouroghly for a wider spectrum of assets. The model presented in this paper may be slighly modified by replacing the Wiener proces by a maximumly negatively skewed stable process as in Edelman (1995) to match the market implied volatility curves more accurately.

End Notes

1. More formal statistical tests for autocorrelation exits such as the Box-Ljung statistic and more advanced for tests for returns squared as discussed in Li & Mak (1994).

2. The Black-Scholes model was not used to calculate the implied volatilities quoted in this section because of early exercise and dividends. A binomial model similar to that of Jarrow & Rudd (1983b) was used.

3. Introduced by Kijima & Yoshida (1993), but considerably simplified here.

4. The Dirac delta function is defined by $\delta_D(x)=0$ if $x \neq 0$ and $\int_A dxf(x)\delta_D(x)=f(0)$ if $0 \in A$. Note that $\delta_D(0)$ is undefined.

5. The Heaviside step function is defined by $H(x)=\int_{z=-\infty}^{x} dz\delta_D(z)=\begin{cases}0 & \text{if } x<0 \\ 1 & \text{if } x>0\end{cases}$. Note that $H(0)$ is undefined.

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