NONARCHIMEDEAN FLAG DOMAINS AND SEMISTABILITY

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N.S.W. 2006
AUSTRALIA

december 1998

Introduction.

Let K be nonarchimedean local field. Then K is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$. Let K° be the valuation ring of K. We consider a simply connected absolutely almost simple linear algebraic group G defined over K°. For a parabolic subgroup $P \subset G$ the corresponding projective homogeneous space is denoted by $X := G/P$. The variety X cannot always be defined over K°. For convenience we will assume in the introduction that it is defined over K°.

Let $\mathcal{L}$ be an ample line bundle on X. Again we will assume in this introduction that the embedding in a projective space determined by $\mathcal{L}$ is defined over K°. For each maximal K-split torus $S \subset G \otimes K$ we take the S linearization of $\mathcal{L} \otimes K$ obtained by restricting the unique $G \otimes K$ linearization of $\mathcal{L} \otimes K$ to S. In this paper we will study the analytic subspaces Y^s and Y^{ss} of $X \otimes \mathbb{K}$. Here $\mathbb{K}$ denotes the completion of the separable algebraic closure of K. The space Y^s consists of the points in $X \otimes \mathbb{K}$ that are stable for all maximal K-split tori $S \subset G \otimes K$. Similarly Y^{ss} consists of the points that are semistable for all maximal K-split tori.

Let $\mathcal{B}$ denote the building of $G(K)$. We construct a $G(K)$-equivariant map I , that associates to each point $x \in Y^{ss}$ a convex subset $I(x) \subset \mathcal{B}$. The subset $I(x)$

[1]This work has been supported by the Australian Research Council.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

is bounded if and only if $x \in Y^s$. The map I is defined by using semistability over K°.

For groups G that are split over K° the image of the map I is studied in some detail. The proofs involve some combinatorics and the commutator relations for additive subgroups. In particular we need the existence of polyhedra with certain integrality properties. The existence of such polyhedra is guaranteed only if the root system Φ of G is of type A_ℓ, B_ℓ, C_ℓ, D_ℓ or E_6 (See §4). Using these one gets a partial description of $I(x)$ in §7 for these groups. If the image $I(x)$ solely depends on the "absolute values of the coordinates" of x then this partial description of $I(x)$ is in fact a complete description of $I(x)$ for $x \in Y^s$ (See §8).

It is hoped that using the map I one can understand the spaces Y^{ss} and Y^s better. In particular one can construct affinoid coverings of Y^s (See [V.2] for an example). In the case $Y^s = Y^{ss}$ a pure affinoid covering has been constructed in [PV] and [V1]. If $Y^s = Y^{ss}$ then the quotient Y^s/Γ is proper for a discrete co-compact subgroup $\Gamma \subset G(K)$. If $Y^s \neq Y^{ss}$ then the quotient Y^s/Γ is still separated, but it is not proper anymore.

Our terminology differs a little bit from [PV] and [V1]. We call Y^s a nonarchimedean flag domain. In [V1] one calls Y^s a flag domain if and only if Y^s/Γ is proper. In [PV] such spaces were called symmetric spaces. However it seems better to preserve the name symmetric spaces for those spaces Y^s with automorphic forms for Γ on it and such that $X = G/P$ with P maximal parabolic.

From the point of view of rigid analytic uniformization the space Y^{ss} is not very useful, since the group Γ does not act discretely on Y^{ss} if $Y^{ss} \neq Y^s$. However from a moduli perspective it is quite interesting. (See [Ra], [To] and [RZ]).

§1. Ample Line Bundles

1.1. We describe the relative and absolute root system of G. The connection between dominant weights and ample line bundles is described. Using this one can find the field over which a nonarchimedean flag domain can be defined. We also recall some criteria with which one can determine whether a point is (semi-)stable.

1.2. The absolute root system. Let G be a simply connected absolutely almost simple linear algebraic group defined over K°. Let K_s denote the separable closure of K and K_s° its ring of integers.

Let $S \subset T \subset G$ be such that $S \otimes K$ is a maximal K-split torus and T a maximal torus defined over K°. The torus T and the group G both split over K_s. Let $\mathcal{X}(T)$ be the character group of T. Let Φ be the *absolute root system* of G. We choose a *simple basis* Δ of Φ.

The *Weyl group* of Φ is denoted by W. The group W acts on $\mathcal{X}(T) \otimes \mathbb{R}$. After choosing a W-invariant inner product on $\mathcal{X}(T) \otimes \mathbb{R}$, the Weyl group W is generated by the reflections in the hyperplanes orthogonal to the simple roots $\alpha \in \Delta$. Let N_T and Z_T be the *normaliser* and *centraliser* of the torus T in G. Then $W \cong N_T/Z_T(K_s)$.

Let $\alpha_i, i = 1, \ldots, \ell$ be the simple roots in Δ. Here $\ell = \dim_{K_s}(T \otimes K_s)$ is the *absolute rank* of G. Let $w_i \in W$ denote the reflection in the hyperplane orthogonal to α_i.

The simple basis Δ determines a *Borel subgroup* $B \subset G$. Let Φ^+ denote the set of *positive roots* for Δ. To each character $\alpha \in \Phi$ belongs a T-stable additive subgroup $U_\alpha \subset G$ on which T acts with character α. One has $B = < T, U_\alpha | \alpha \in \Phi^+ >$.

Let $I \subset \{1, \ldots, \ell\}$ be a non-empty subset. The subgroup $W_I \subset W$ is the group generated by the reflections $w_i, i \notin I$. The *parabolic subgroup* $P_I \supset B$ is the group $P_I := BW_IB$. Any parabolic subgroup of G is conjugated to exactly one of the groups P_I.

1.3. The relative root system. We fix an ordering on $\mathcal{X}(S)$ and choose a compatible ordering on $\mathcal{X}(T)$. Let Δ_0 denote the set of simple roots of G with respect to T that vanish on S. The *relative root system* Φ_K of G consists of the roots of $\mathcal{X}(S)$. The *relative Weyl group* W_K acts on Φ_K. One has $W_K \cong N_S/Z_S(K)$. Here N_S and Z_S are the normaliser and centraliser of S in G.

The Galois group $H := Gal(K_s/K)$ acts on $\mathcal{X}(T)$, since T is split over K_s. For any element $h \in H$ the image $h(\Delta)$ of Δ is again a simple basis of Φ. Hence there exists a unique element $w \in W$ such that $wh(\Delta) = \Delta$. We put $h^* := w \circ h$ for this element $w \in W$. We call this action of H on $\mathcal{X}(T)$ the *twisted action*. Let $H^* := \{h^* | h \in H\}$. Then H^* is a finite group acting on $\mathcal{X}(T)$ that preserves the simple basis Δ.

1.4. Ample line bundles. Every ample line bundle $\mathcal{L}$ on $X = G/P$ is very ample. So $\mathcal{L}$ determines an embedding $X \hookrightarrow \mathbb{P}(V)$. Here $\mathbb{P}(V)$ denotes the projectivisation of a G-module V. The module V is irreducible if $Char(K) = 0$, whereas if $Char(K) > 0$ this might not be the case. (See [J]). The G-module V is uniquely determined by its highest weight λ. We will denote this G-module by V_λ.

Let $(-,-)$ be the W-invariant inner product on $\mathcal{X}(T) \otimes \mathbb{R}$. The *fundamental weights* ω_i of our root system Φ are determined by $2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$. The ample line bundles $\mathcal{L}$ on $X = G/P_I$ correspond to the modules V_λ with highest weight $\lambda = \sum_{i \in I} n_i \omega_i$ with $n_i > 0$ for all $i \in I$.

Let Λ_r denote the *root lattice*, i.e. the sublattice of $\mathcal{X}(T)$ generated by the roots $\alpha \in \Phi$. In [T.2] theorems 3.3 and 7.2 the irreducible representations of G that are defined over K are determined. From this one gets the linear representations of G. Furthermore these representations are defined over K°.

For each $\lambda \in \Lambda_r$ one has a representation ρ_λ of G into $GL(\oplus_{\sigma \in H^*} V_{\sigma(\lambda)})$ that is defined over K°. (If the weight λ is not contained in Λ_r one might need a skew field $\mathcal{D}$ defined over K to define ρ_λ). For our purposes the representations $\rho_\lambda, \lambda \in \Lambda_r$ suffice.

Note that ρ_λ is irreducible if $Char(K) = 0$, whereas if $Char(K) = p > 0$ this might not be the case.

1.5. Fields of definition. For each weight $\lambda = \sum_{i \in I} n_i \omega_i \in \Lambda_r$ we have a representation of G defined over K°. Let $v_\lambda \in \oplus_{\sigma \in H^*} V_{\sigma(\lambda)}$ be a vector contained in $V_\lambda \oplus < 0 >$, whose component in V_λ is a highest weight vector. The image X_λ of the orbit $G \cdot v_\lambda$ in $\mathbb{P}(\oplus_{\sigma \in H^*} V_{\sigma(\lambda)})$ is isomorphic to G/P_I. We have a variety $X^\dagger \subset \mathbb{P}(\oplus_{\sigma \in H^*} V_{\sigma(\lambda)})$ defined over K°, whose connected components are $X_{\sigma(\lambda)}$, $\sigma \in H^*$. All the connected components of $X^\dagger$ are isomorphic. The very amply line bundle $\mathcal{L}^\dagger$ associated with this embedding restricts on each connected component $X_{\sigma(\lambda)}$ to the line bundle $\mathcal{L}$ belonging to the weight $\sigma(\lambda)$.

For each $X = G/P_I$ defined over K_s° and ample line bundle $\mathcal{L}$ on X corresponding to some weight $\lambda \in \Lambda_r$, one has a variety $X^\dagger$ and an ample line bundle $\mathcal{L}^\dagger$, both defined over K°, such that one connected component of $X^\dagger$ is isomorphic to X and

$\mathcal{L}^\dagger|_X \cong \mathcal{L}$.

One easily sees that $X^\dagger \otimes L^\circ$ has its connected components defined over L°. Here L is a splitting field for G. Moreover also $\mathcal{L}^\dagger|_{X\otimes L^\circ}$ is defined over L°. One can find a minimal field in L such that this is the case. This field will be called the *field of definition* of $(X, \mathcal{L})$.

1.6. (Semi)-stable points. Let $\mathcal{L}$ be an ample line bundle on X corresponding to a weight λ, then $\mathcal{L}^{\otimes n}$ corresponds to the weight $n \cdot \lambda$. For every weight λ one can find an integer $n > 0$ such that $n \cdot \lambda \in \Lambda_r$. Moreover $\mathcal{L}$ and $\mathcal{L}^{\otimes n}$ determine the same sets of (semi-)stable points. So the assumption that $\lambda \in \Lambda_r$ poses no real restriction.

Let $X^s(T, \mathcal{L})$ and $X^{ss}(T, \mathcal{L})$ denote the varieties consisting of the stable and the semistable points, respectively, in X for the action of T w.r.t. $\mathcal{L}$. The T linearization is chosen to be the restriction of the unique G linearization of $\mathcal{L}$ (or better $\mathcal{L}^\dagger$). These sets can be determined using the criteria given in [MFK].

Let $\mathcal{L}$ give an embedding of X into $\mathbb{P}(V)$ for some G-module $V = V_\lambda$. We have a decomposition $V = \oplus V^\beta, \beta \in \mathcal{X}(T)$ into eigenspaces V^β on which T acts with character β. Each V^β has a free K°-basis $e_1^\beta, \cdots, e_{m(\beta)}^\beta$. These bases determine coordinates $x_{\beta,i}$ for every $x \in X \subset \mathbb{P}(V)$. We may assume that x is normalised, i.e. $|x_{\beta,i}| \leq 1, \forall \beta, i$ and $\max|x_{\beta,i}| = 1$. Then one defines the polyhedron $\mu(x) \subset \mathcal{X}(T) \otimes \mathbb{R}$ as the convex hull of $\{\beta | \exists i \quad |x_{\beta,i}| = 1\}$. One has for $x \in X$:

$$x \in X^s(T, \mathcal{L}) \iff 0 \in \text{ int } \mu(x)$$
$$x \in X^{ss}(T, \mathcal{L}) \iff 0 \in \mu(x)$$

Part a) of the proposition below is proved in [GS] and [FH]. Part b) was proved in [PV] theorem 1.1:

1.7. Proposition. *Let $\mathcal{L}$ be an ample line bundle on $X = G/P$ corresponding to the weight λ. Let $T \subset G$ be a maximal torus. Then one has:*

a) *For any point $x \in X$ the vertices of $\mu(x)$ form a subset of the set $\{w(\lambda) | w \in W\}$. The edges of $\mu(x)$ are parallel to the roots $\alpha \in \Phi$.*

b) $X^s(T,\mathcal{L}) = X^{ss}(T,\mathcal{L})$ *if and only if* λ *is not contained in a hyperplane (through* 0*) spanned by roots.*

1.8. Let $r : \mathcal{X}(T) \longrightarrow \mathcal{X}(S)$ denote the restriction map. The sets of (semi-) stable points $X^s(S,\mathcal{L})$ and $X^{ss}(S,\mathcal{L})$ can be described using the polyhedron $r(\mu(x)) \subset \mathcal{X}(S) \otimes \mathbb{R}$. Below we recall some results and their proof from [V.1].

1.9. Proposition. *Let* $\mathcal{L}$ *be an ample line bundle on* $X = G/P$*, corresponding to the weight* $\lambda \in \Lambda_r$ *and let* $S \subset T \subset G$ *as before. Then one has:*

a) *For any point* $x \in X$ *the vertices of* $r(\mu(x))$ *form a subset of* $\{r(w(\lambda))|w \in W\}$*. The edges of* $r(\mu(x))$ *are parallel to roots* $\alpha \in \Phi_K$*.*

b) $X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L})$ *if for all* $w \in W$ *the weight* $r(w(\lambda))$ *is not contained in a hyperplane (through* 0*) spanned by roots* $\alpha \in \Phi_K$*.*

c) $X^s(S,\mathcal{L}) \neq X^{ss}(S,\mathcal{L})$ *if* λ *is contained in a hyperplane* $V \subset \mathcal{X}(T) \otimes \mathbb{R}$ *spanned by roots* $\alpha \in \Phi$ *and moreover there exists an element* $w \in W$ *such that* $r(w(V))$ *is contained in a hyperplane (spanned by roots* $\beta \in \Phi_K$*).*

Proof. Since r maps Φ into $\Phi_K \cup \{0\}$, part (a) of the proposition is clear. Furthermore (b) follows from (a).

To prove (c) one constructs for λ in a hyperplane V spanned by roots $\alpha \in \Phi$ a point $x \in X$ such that $\mu(x) = V \cap$ conv $(\{w(\lambda)|w \in W\})$. This has been done in [PV] 1.4. Then one chooses an element $w \in W$ such that $r(w(V))$ is contained in a hyperplane in $\mathcal{X}(S) \otimes \mathbb{R}$. Then $0 \in r(w(\mu(x)) = r(\mu(w(x))) \subset r(w(V))$ and the point $w(x)$ is in $X^{ss}(S,\mathcal{L}) - X^s(S,\mathcal{L})$. This proves (c).

1.10. For the convenience of the reader we recall the following theorem from [V1] without proof:

1.11 Theorem. *There exists an ample line bundle* $\mathcal{L}$ *on* $X = G/P$ *such that* $X^{ss}(S,\mathcal{L}) = X^s(S,\mathcal{L})$ *if and only if one of the following holds:*

(1) $P = B$ *and* G *is any group.*

(2) $P = P_J$, $J = \{1, \ldots, \ell - 1\}$ *and* G *is a non-split group with absolute root system* Φ *of type* C_ℓ*. Here* α_ℓ *is the unique long root in the simple basis* Δ*.*

(3) $P = P_I$ *and* $G = SL_{s+1}(\mathcal{D})$ *with g.c.d.*$(i \in I, s+1) = 1$*. Here* $\mathcal{D}$ *is a skew*

field (of dimension d^2) defined over K (we allow $\mathcal{D} = K$) and the $P_{\{i\}}$ are such that $G/P_{\{i\}} \cong Gr(i, (s+1)\cdot d)$.

§2. Equivariant maps from the (semi-) stable points to the apartment

2.1. We fix a group G, a projective homogeneous variety $X = G/P$ and an ample line bundle $\mathcal{L}$ on X corresponding to some weight $\lambda \in \Lambda_r$. Let $S \subset G$ be a maximal K°-split torus and $A \subset \mathcal{B}$ the apartment corresponding to S.

The completion of the separable closure of K will be denoted by $\mathbb{K}$. In this section we construct an $S(\mathbb{K})$-equivariant map $I_A : X^{ss}(S, \mathcal{L})(\mathbb{K}) \longrightarrow \{$ convex subsets of A $\}$. The map I_A will be studied in some detail.

First we recall the construction of the apartment A belonging to $S(\mathbb{K})$.

2.2. The apartment. Let $\mathcal{X}_*(S)$ be the space of one-parameter subgroups (1-ps), i.e. the dual of the character space $\mathcal{X}(S)$. Let $V := \mathcal{X}_*(S) \otimes \mathbb{R}$. The perfect pairing between characters and 1-ps will be denoted by $< -, - >$. We extend it to a pairing of V and $\mathcal{X}(S) \otimes \mathbb{R}$.

One defines a map $\nu : S \longrightarrow V$ by $< \chi, \nu(s) >= -v(\chi(s))$ for all $\chi \in \mathcal{X}(S)$. Here v is the additive valuation of K, normalized such that $v(\pi) = 1$ for a generator π of the maximal ideal. We regard V as an affine space. The torus S acts on V by translation, as follows: $s \cdot z = z + \nu(s)$ for $z \in V$.

To G and S belongs an affine root system Φ_{aff}. The affine roots are functions $\alpha + n : V \longrightarrow \mathbb{R}$ with $\alpha \in \Phi_K$ and $n \in \Gamma_\alpha$. Here $(\alpha + n)(z) = \alpha(z) + n$ for $z \in V$, where $\alpha(z) :=< \alpha, z >$. Furthermore Γ_α is the discrete group $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$, depending on the particular affine root system Φ_{aff} and the root α.

The affine root system gives a simplicial decomposition of V. The maximal simplices or *chambers* are the closures of the connected components of $\{z \in V | \forall(\beta \in \Phi_{aff}) \quad \beta(z) \neq 0\}$. The affine space V endowed with this simplicial decomposition is the *apartment* A for the torus $S \otimes K$. The map ν defined above extends uniquely to a map from $S(\mathbb{K})$ to A. Hence we have a well defined action of $S(\mathbb{K})$ on A.

2.3. Analytification of the sets of (semi-) stable points. To each algebraic variety corresponds a rigid analytic variety that has the same set of closed points.

(See [BGR] or [FP]). We denote the analytifications of $X^s(S,\mathcal{L})\otimes\mathbb{K}$ and $X^{ss}(S,\mathcal{L})\otimes\mathbb{K}$ by Y^s_A and Y^{ss}_A, respectively. For notational convenience we have dropped the line bundle from the notation (though these spaces do depend on $\mathcal{L}$). The subscript corresponds to the apartment A belonging to $S(K)$.

To each point $z \in A$ we want to associate an analytic subspace $Y^s_{z,A}$ (resp. $Y^{ss}_{z,A}$) of Y^s_A (resp. Y^{ss}_A). One defines $Y^s_{0,A}$ (resp. $Y^{ss}_{0,A}$) as being the completion of $X^s(S,\mathcal{L})\otimes\mathbb{K}^\circ$ (resp. $X^{ss}(S,\mathcal{L})\otimes\mathbb{K}^\circ$) along the closed fiber. Here $\mathbb{K}^\circ$ is the ring of integers of $\mathbb{K}$. One has by construction:

$$\begin{aligned} Y^s_{0,A}(\mathbb{K}) &= X^s(S,\mathcal{L})(\mathbb{K}^\circ) \\ Y^{ss}_{0,A}(\mathbb{K}) &= X^{ss}(S,\mathcal{L})(\mathbb{K}^\circ)\ . \end{aligned}$$

If $z \in A$ is a point such that $z = s\cdot 0$ for some $s \in S(\mathbb{K})$ then one defines:

$$\begin{aligned} Y^s_{z,A} := s\cdot Y^s_{0,A} &= \{s\cdot x | x \in Y^s_{0,A}\} \\ Y^{ss}_{z,A} := s\cdot Y^{ss}_{0,A} &= \{s\cdot x | x \in Y^{ss}_{0,A}\}\ . \end{aligned}$$

Since $X^{ss}(S,\mathcal{L})/S$ is a projective algebraic variety, the map $\psi^{ss} : S(\mathbb{K})\times Y^{ss}_{0,A} \longrightarrow Y^{ss}_A$, defined by $\psi^{ss}(s,x) = s\cdot x$, is surjective. The map $\psi^s : S(\mathbb{K})\times Y^s_{0,A} \longrightarrow Y^s_A, (s,x) \longrightarrow s\cdot x$, is surjective if and only if $Y^s_A = Y^{ss}_A$.

Note that both $Y^s_{z,A}$, and $Y^{ss}_{z,A}$ are quasi-compact, i.e. they are the union of finitely many affinoid subspaces.

2.4. Definition. For $x \in Y^{ss}_A$ we define the *interval of S-semistability* $I_A(x)$ as follows:

$$\begin{aligned} I_A(x) :&= \{z \in A | x \in Y^{ss}_{z,A}\} \\ &= \{s^{-1}\cdot 0 \in A | s\cdot x \in Y^{ss}_{0,A}\} \end{aligned}$$

Note that $I_A(s\cdot x) = s\cdot I_A(x)$ for all $s \in S(\mathbb{K})$ and $x \in Y^{ss}_A$.

2.5. Proposition.

a) $I_A(x) \subset A$ *is convex.*

b) $I_A(x)$ *is bounded if and only if* $x \in Y_A^s$.

c) $I_A(x) = \{z\}$ *if and only if* $x \in Y_{z,A}^s$.

Proof. Since $I_A(s \cdot x) = s \cdot I_A(x)$ for $s \in S(\mathbb{K})$, we may assume that $0 \in I_A(x)$, i.e. $x \in Y_{0,A}^{ss}$. We assume that $x \in Y_{0,A}^{ss}$ is normalised. Let the coordinates $x_{\beta,i}$ be as in 1.6 and let $r : \mathcal{X}(T) \longrightarrow \mathcal{X}(S)$ be the restriction map.

One has : $s \cdot x \in Y_{0,A}^{ss} \iff |r(\beta)(s) \cdot x_{\beta,i}| \leq 1 \quad \forall \beta, i$. Hence $s \in S(\mathbb{K})$ is such that $s \cdot x \in Y_{0,A}^{ss}$ satisfies:

$$\begin{aligned} &- v(r(\beta)(s)) \leq v(x_{\beta,i}) \quad \forall \quad \beta, i \quad s.t. \quad x_{\beta,i} \neq 0 \ . \\ \iff &- v(r(\beta)(s)) \leq \ \min \{v(x_{\beta,i})| \quad x_{\beta,i} \neq 0\} \quad \forall \beta \quad s.t. \quad \exists i \ \text{with} \ x_{\beta,i} \neq 0 \ . \end{aligned}$$

From this one concludes that $\nu(s)$ has to be in the intersection of a finite number of affine half spaces. The same is true of $I_A(x) = \{s^{-1} \cdot 0 | s \cdot x \in Y_{0,A}^{ss}\}$. Hence $I_A(x)$ is convex. This proves a).

The set $I_A(x)$ is bounded if and only if the convex hull of $\{r(\beta)| \exists i \quad x_{\beta,i} \neq 0\}$ contains 0 in its interior. By definition this is equivalent to $x \in Y_A^s$. This proves b).

Now c) follows from b). Indeed, $I_A(x) = \{0\}$ if and only if $\{r(\beta)| \exists i \quad |x_{\beta,i}| = 1\}$ has the same convex hull as $\{r(\beta)| \exists i \quad x_{\beta,i} \neq 0\}$ and has 0 in the interior of the convex hull. Hence $I_A(x) = \{0\}$ if and only if $x \in Y_{0,A}^s$.

2.6. Proposition. *The interval of semistability* $I_A(x)$ *for* $x \in Y_A^{ss}$ *is bounded by hyperplanes parallel to walls. In particular there exist* $n_\alpha(x) \in \mathbb{R} \cup \{\infty\}, \alpha \in \Phi_K$ *such that:*

$$I_A(x) = \{z \in A | \forall (\alpha \in \Phi_K) \quad \alpha(z) \leq n_\alpha(x)\} \ .$$

Proof. Without loss of generality we may assume that $x \in Y_{0,A}^{ss}$. Furthermore we may assume that 0 is not contained in a face of the polyhedron $I_A(x)$. We normalise x such that $\max \{|x_{\beta,i}|| \quad \beta \in \mathcal{X}(T)\} = 1$. Let $V_1(x, A) \subset \mathcal{X}(S) \otimes \mathbb{R}$ be the linear subspace spanned by the weights $r(\beta), \beta \in \mathcal{X}(T)$ such that for some i one has

$|x_{\beta,i}| = 1$. From proposition 1.9 if follows that either $V_1(x,A) = 0$ or $V_1(x,A)$ is spanned by roots $\alpha \in \Phi_K$.

If $V_1(x,A) = \mathcal{X}(S) \otimes \mathbb{R}$ we are done. Indeed, then $x \in Y^s_{0,A}$ and $I_A(x) = \{0\}$. Hence we can take $n_\alpha(x) = 0$ for all $\alpha \in \Phi_K$.

So let us assume that $V_1(x,A) \neq \mathcal{X}(S) \otimes \mathbb{R}$. Let $S_1 \subset S$ be the subtorus on which all characters $\chi \in V_1(x,A) \cap \mathcal{X}(S)$ vanish. Clearly if for $s \in S$, one has $s \cdot x \in Y^{ss}_{0,A}$, then $s \in S_1$. Hence $I_A(z) \subset \{z \in A | \alpha(z) = 0 \quad \forall (\alpha \in V_1(x,A) \cap \Phi_K)\}$.

Let $s \in S_1$ be such that $s^{-1} \cdot 0$ is in the interior of a codimension one face of $I_A(x)$. Let $F \subset \mathcal{X}(S) \otimes \mathbb{R}$ be the linear subspace by the $r(\beta)$ such that some $|x_{\beta,i}(s \cdot x)| = 1$. Then by proposition 1.9 a) the space F is generated by roots $\alpha \in \Phi_K$. Moreover dim $(F) =$ dim $(V_1(x,A)) + 1$. Hence one can find a root $\alpha \in F \cap \Phi$ that together with $V_1(x,A)$ spans F. The face of $I_A(x)$ containing $s^{-1} \cdot 0$ is the intersection of $I_A(x)$ with a hyperplane defined by $\alpha(z) = \alpha(s^{-1} \cdot 0)$. Furthermore $I_A(x)$ is either contained in the half space $\{z \in A| \quad \alpha(z) \leq \alpha(s^{-1} \cdot 0\}$ or in the halfspace $\{z \in A| \quad \alpha(z) \geq \alpha(s^{-1} \cdot 0\}$. Since we can do this for any codimension one face of $I_A(x)$, the proposition follows.

2.7. Remark. Note that $I_A(x) = A$ if and only if $x \in Y^{ss}_A$ is a fixed point for the action of $S(\mathbb{K})$. Such a fixed point exists if and only if $r(w(\lambda)) = 0$ for some $w \in W$. In particular this never occurs if G is split.

If none of the characters $r(w(\lambda)), w \in W$ vanishes on S, then $I_A(x) \subset A$ has always some codimension > 0.

For later use we describe the codimension one faces of $I_A(x)$ in the lemma below.

2.8. Lemma. *Let G be a split group. Let $V_1(x,A) \subset \mathcal{X}(S) \otimes \mathbb{R}$ be the linear subspace spanned by the set of characters that are constant on $I_A(x)$. Then the codimension one faces of $I_A(x)$ correspond to roots $\alpha \in \Phi$ such that $\alpha \not\subset V_1(x,A)$, that is the faces are given by $\alpha(z) = c_\alpha(x)$.*

Proof. Note that we have on $\mathcal{X}(S) \otimes \mathbb{R}$ a W-invariant inner product that is unique up to a scalar.

Again we may assume that 0 is in $I_A(x)$, but not on a face of $I_A(x)$. Furthermore we may assume that $x \in Y^{ss}_{0,A}$ is normalised. Then $V_1(x,A) =< \mu | \exists i \quad |x_{\mu,i}| = 1 >$. Let $s \in S$ be such that $s^{-1} \cdot 0$ is in a codimension one face H of $I_A(x)$. We may choose s in such a way that $s^{-1} \cdot 0$ is not contained in a codimension two face of $I_A(x)$.

Now $s \cdot x \in Y^{ss}_{0,A}$. Let $F := < \mu | \exists i \quad |x_{\mu,i}(s \cdot x)| = 1 >$ as in the proof of proposition 2.6. Since both F and $V_1(x, A)$ are spanned by roots and dim $(F) =$ dim $V_1(x, A) + 1$, we can find a root α in $F - V_1(x, A)$. Clearly the face H of $I_A(x)$ is given by $\alpha(z) = c_\alpha(x)$, for a suitable constant $c_\alpha(x) \in \mathbb{R}$.

So we only have to show that we can choose $\alpha \not\perp V_1(x, A)$. This follows from the fact that the $w \in W$ with $|x_{w(\lambda)}(sx)| = 1$ are the vertices of a convex polyhedron with edges parallel to roots. Moreover if λ is joined to $w(\lambda)$ by an edge parallel to α then $w = w_\alpha$, the reflection associated to α. In particular $\alpha \not\perp \lambda, w_\alpha(\lambda)$.

§3 Equivariant maps to the building

3.1. Introduction. Let $Y^s := \bigcap_{g \in G(K)} g(Y^s_A)$ and $Y^{ss} := \bigcap_{g \in G(K)} g(Y^{ss}_A)$.

We will construct a $G(K)$-equivariant map I from Y^{ss} to the set of convex subsets of $\mathcal{B}$. Here $\mathcal{B}$ is the Bruhat-Tits building of $G(K)$.

First we recall the definition of the building $\mathcal{B}$. To each point $z \in \mathcal{B}$ we associate some analytic subspaces $Y^s_z, Y^{ss}_z \subset X(\mathbb{K})$. One proves that $Y^s_z \subset Y^s$ and $Y^{ss}_z \subset Y^{ss}$. Then one uses the spaces Y^{ss}_z to define $I(x)$.

3.2. The Building. All maximal K-split tori $S \subset G(K)$ are conjugated. To each maximal K-split torus $gSg^{-1}, g \in G(K)$, belongs an apartment $g(A)$. Each simplex $\sigma \in A$ is stabilized by a parahoric subgroup $P_\sigma \subset G(K)$. The simplex $g(\sigma), g \in G(K)$, is stabilized by $P_{g(\sigma)} := gP_\sigma g^{-1}$.

The building $\mathcal{B}$ is defined as $\mathcal{B} := \bigcup_{g \in G(K)} g(A)/\sim$. The equivalence relation $\sim$ is given by $\sigma_1 \sim \sigma_2$ if and only if $P_{\sigma_1} = P_{\sigma_2}$. Here σ_1, σ_2 are simplices in the building $\mathcal{B}$.

3.3. The intervals of $G(K)$-semistability. Let $z' \in A'$. We can find an element $g \in g(K)$ such that $A' = g(A)$. We put $z := g^{-1}(z') \in A$. Then $Y^s_{z',A'} := g(Y^s_{z,A})$ and $Y^{ss}_{z',A'} := g(Y^{ss}_{z,A})$.

We associate the following analytical subspaces of $X(\mathbb{K})$ to a point $z \in \mathcal{B}$:

$$Y_z^s := \bigcap_{A' \ni z} Y_{z,A'}^s \;=\; \bigcap_{g \in P_z} g(Y_{z,A'}^s) \;.$$

$$Y_z^{ss} := \bigcap_{A' \ni z} Y_{z,A'}^{ss} \;=\; \bigcap_{g \in P_z} g(Y_{z,A'}^{ss}) \;.$$

Here $P_z \subset G(K)$ denotes the stabilizer of $z \in \mathcal{B}$. Note that $P_z = P_\sigma$, where $\sigma \in \mathcal{B}$ is the smallest simplex containing the point z.

For $x \in Y^{ss}$ one defines the *interval of $G(K)$-semistability* $I(x)$ as follows:

$$I(x) := \{z \in \mathcal{B} | x \in Y_z^{ss}\} \;.$$

Since $Y_z^{ss} = \bigcap_{A' \in z} Y_{z,A'}^{ss}$ one has:

$$I(x) = \{z \in \mathcal{B} | \forall (A' \in z) \quad z \in I_{A'}(x)\} \;.$$

Note that $I_{gA}(x) = g(I_A(g^{-1}(x))$ for $g \in G(K)$. Clearly one has by construction that $I(g(x)) = g(I(x))$ for $g \in G(K)$.

3.4. Definition. Let $f_1, \ldots, f_m$ be generators of the set of S-invariants $\Gamma(X, \mathcal{L}^{\otimes d})^S$. Here d is chosen in such a way that the homogeneous S-invariants generate $\oplus_{n>0}\Gamma(X, \mathcal{L}^{\otimes dn})^S$ as a K°-algebra.

Let $A_1 = g_1(A)$ and $A_2 = g_2(A)$ be two different apartments and $g_1, g_2 \in G(K)$. As in [PV] section 3.6 we define a function $r_{A_1,A_2} : Y_{A_2}^{ss} \longrightarrow \mathbb{R}$ as follows:

$$r_{A_1,A_2}(x) = r_{g_1(A),g_2(A)}(x) : \; \max_{1 \leq i \leq m}\{|g_1^* f_i(x)|\} / \max_{1 \leq i \leq m}\{|g_2^* f_i(x)|\} \;.$$

This function has the following obvious properties:

a) $r_{A_1,A_2}(x) = r_{A_2,A_1}(x)^{-1}$, if $x \in Y_{A_1}^{ss} \cap Y_{A_2}^{ss}$.

b) $r_{g(A_1),g(A_2)}(g(x)) = r_{A_1,A_2}(x)$ for $g \in G(K)$ and $x \in Y_{A_2}^{ss}$.

Below we will study the map r_{A_1,A_2} in more detail. We follow [PV] closely here.

3.5. Lemma. *Let $x \in Y_{z,A}^{ss}$ and assume that $z \in A \cap A'$. Then $r_{A',A}(x) \leq 1$.*

Proof. Since z is contained in some simplex $\sigma \in A \cap A'$, one can apply the proof of [PV] 3.6 property c).

3.6. Definition. For $x \in X(\mathbb{K})$ we define the function $\widetilde{r} : X(\mathbb{K}) \longrightarrow \mathbb{R}$ by:

$$\widetilde{r}(x) = \begin{cases} 0 & \text{if } x \notin Y_A^{ss} \\ \inf\{r_{gA,A}(x) | g \in G(K)\} & \text{if } x \in Y_A^{ss} \end{cases}$$

3.7. Proposition. *Let $x \in Y_A^{ss}$. There exists an apartment $A' \subset \mathcal{B}$ such that $r_{A',A}(x) = \widetilde{r}(x)$*

Proof. See [PV] 3.6, proof of property d).

3.8. Proposition. *Let $z \in A' \subset \mathcal{B}$ and $x \in Y_{z,A'}^{ss}$. Then $x \in Y_z^{ss}$ if and only if $r_{A',A}(x) = \widetilde{r}(x) > 0$.*

Proof. We first prove that if $\widetilde{r}(x) = 0$ then $x \notin Y_z^{ss}$. If $\widetilde{r}(x) = 0$ then by proposition 3.7 we can find an apartment $A_1 \subset \mathcal{B}$ such that $r_{A_1,A}(x) = 0$. Hence $x \notin Y_{A_1}^{ss}$. Now we can find $z_1 \in A_1$ and an apartment A_2 such that $x \in Y_{z_1,A_2}^{ss}$. Since $A_1 = hA_2$ for some $h \in P_{z_1}$ we can construct $\widetilde{h} \in P_z$ such that $r_{\widetilde{h}A_2,A_2}(x) < \epsilon$. In particular we may assume that $0 < r_{\widetilde{h}A_2,A}(x) < r_{A',A}(x)$. Now $x \in Y_{z_2,\widetilde{h}A_2}^{ss}$ for some $z_2 \in \widetilde{h}A_2$.

Let A'' be such that $z_2, z \in A''$ Using lemma 3.5 one finds that $r_{A'',A}(x) < r_{A',A}(x)$. It follows in particular that $x \notin Y_{z,A''}^{ss}$. Hence $x \notin Y_z^{ss}$.

Now let us assume that $\widetilde{r}(x) > 0$. Let $A_1 \subset \mathcal{B}$ be an apartment such that $r_{A_1,A}(x) = \widetilde{r}(x)$. Suppose $x \in Y_{z_1,A_1}^{ss}$. Let A_2 be an apartment containing both z and z_1. Using lemma 3.5 one finds $r_{A_2,A}(x) = \widetilde{r}(x) > 0$. Again applying 3.5 gives us $x \in Y_z^{ss} \Longleftrightarrow r_{A',A}(x) = \widetilde{r}(x) > 0$.

3.9. Corollary. *Let $z \in \mathcal{B}$. Then $Y_z^{ss} \subset Y^{ss}$.*

Proof. If $x \in Y_z^{ss}$ then $\widetilde{r}(x) > 0$ by proposition 3.8. Hence for all $A' \subset \mathcal{B}$ one has $x \in Y_{A'}^{ss}$. Thus $x \in Y^{ss}$.

3.10. Theorem. $Y^{ss} = \bigcup_{z \in \mathcal{B}} Y_z^{ss}$.

Proof. In the corollary above we have already shown that $Y_z^{ss} \subset Y^{ss}$. Therefore it is sufficient to prove that $Y^{ss} \subset \bigcup_{z \in \mathcal{B}} Y_z^{ss}$.

If $x \in Y^{ss}$ then there exists an apartment $A' \subset \mathcal{B}$ such that $r_{A',A}(x) = \widetilde{r}(x)$. Hence $x \in Y_{z,A'}^{ss}$ for some $z \in A'$. Using proposition 3.8 one concludes that $x \in Y_z^{ss}$. This proves the theorem.

3.11. Proposition. *Let $x \in X(\mathbb{K})$. Then $x \in Y^{ss}$ if and only if $\widetilde{r}(x) > 0$.*

Proof. Clear from proposition 3.8 and theorem 3.10.

3.12. Proposition. *Let $x \in Y^{ss}$. Then $I(x) = \bigcup\{I_{A'}(x) | r_{A',A}(x) = \widetilde{r}(x)\}$*

Proof. This follows immediately from the definition of $I(x)$ and propositions 3.8 and 3.11.

3.13. Corollary. *Let $x \in Y^{ss}$. Then $I(x)$ is convex.*

Proof. Let $z_1, z_2 \in I(x)$. If $A' \subset \mathcal{B}$ is an apartment containing both z_1 and z_2 then $r_{A',A}(x) = \widetilde{r}(x)$. Hence $I_{A'}(x) \subset I(x)$. Now $z_1, z_2 \in I_{A'}(x)$ and $I_{A'}(x)$ is convex. From this the corollary follows.

3.14. Theorem. *Let $x \in Y^{ss}$. Then $x \in Y^s$ if and only if $I(x)$ is bounded.*

Proof. We will first show that if $I(x)$ is not bounded then $x \notin Y^s$. So let us assume $I(x)$ is not bounded. Let $z \in I(x)$. We can find a sequence of points $z_i \in I(x)$ such that the distance dist $(z, z_i) \longrightarrow \infty$ if $i \longrightarrow \infty$. Let $A_i \subset \mathcal{B}$ be an apartment containing both z and z_i.

Then $I_{A_i}(x) \subset I(x)$ and $z, z_i \in I_{A_i}(x)$. There exists $h_i \in P_z$ such that $A_i = h_i A$. Here we assumed that $z \in A$. Since P_z is compact, a subsequence of the sequence h_i converges to an element $h \in P_z$. Clearly $I_{hA}(x) \subset I(x)$ and $I_{hA}(x)$ is not bounded. Hence $x \notin Y^s_{hA}$. So $x \notin Y^s$. This proves the "only if" part of the theorem.

Next we will show that if $x \in Y^{ss} - Y^s$ then the interval $I(x)$ is not bounded. Let $x \in Y^{ss} - Y^s$. There exists an apartment $\widetilde{A}$ such that $x \in Y^{ss}_{\widetilde{A}} - Y^s_{\widetilde{A}}$. In particular $I_{\widetilde{A}}(x)$ is not bounded. Let $z \in I(x)$. Then there exists an apartment A' such that $z \in A'$ and $A' \cap I_{\widetilde{A}}(x)$ is not bounded (see [B] Ch. VI §8). Now $I_{A'}(x)$ is not bounded by lemma 3.15 below. Furthermore $I_{A'}(x) \subset I(x)$, since $z \in A'$. Hence $I(x)$ is not bounded.

3.15. Lemma. *Let x be contained in $Y^{ss}_A \cap Y^{ss}_{A'}$. Suppose that both $I_A(x)$ and $I_A(x) \cap A'$ are not bounded. Then $I_{A'}(x)$ is not bounded.*

Proof. Since $I_A(x)$ and A' are convex, $I_A(x) \cap A'$ is convex. Let $z \in I_A(x) \cap A'$. There exists a $1 - ps.$ $\mu : \mathbb{K} \longrightarrow S(\mathbb{K})$ such that $\mu(s) \cdot z \in I_A(x) \cap A'$ for all $s \in \mathbb{K}$ such that $|s| \leq 1$.

Let $g \in P_z$ be such that $A' = g \cdot A$. We may assume that g stabilises $A \cap A'$. Then $g \in P_{\mu(s)\cdot z}$ for s such that $|s| \leq 1$. Let $\sigma_0 \in A$ be a chamber containing z. Let $\sigma_i := \mu(\pi^i) \cdot \sigma_0, i \geq 0$. Then $g \in P_{\sigma_i}, i \geq 0$.

Since $g \in P_{\sigma_i}$, one has $g = t \cdot \prod_{\alpha \in \Phi_K} u_\alpha(f_\alpha)$, with $|f_\alpha| \leq n_{\alpha,i}$. Here we have taken a fixed order for the roots $\alpha \in \Phi_K$, and $t \in S(\mathbb{K}^\circ)$. One has $P_{\sigma_i} = \mu(\pi^i)P_{\sigma_0}\mu(\pi^i)^{-1}$ and $\mu(\pi^i)u_\alpha(f_\alpha)\mu(\pi^i)^{-1} = u_\alpha(\alpha(\mu(\pi^i)) \cdot f_\alpha)$. Therefore $n_{\alpha,i} \longrightarrow 0$ if $i \longrightarrow \infty$, for any root α such that $\alpha(\mu) > 0$. Hence we have $g = t \cdot \prod_{\alpha \in \Phi_K,\quad <\alpha,\mu> \leq 0} u_\alpha(f_\alpha)$.

By assumption we have $x \in Y^{ss}_{\mu(s)\cdot z,A}$ if $|s| \leq 1$. Therefore $\mu(s)^{-1} \cdot x \in Y^{ss}_{z,A}$ if $|s| \leq 1$. This implies that $x_{\chi,j} = 0$ for all χ such that $r(\chi)(\mu) > 0$.

One has $u_\alpha(f_\alpha)^* x_{\chi,j} = x_{\chi,j} + \sum_\nu a_{\nu,i}(f_\alpha)x_{\nu,i}$, where $r(\nu) = r(\chi) - n\alpha$ for some $n \in \mathbb{Z}_{\geq 0}$.

From this one concludes that $g^* x_{\chi,i} = 0$ if $r(\chi)(\mu) > 0$ for our point x. Hence $g \cdot \mu(s) \cdot g^{-1} \cdot y \in I_{A'}(x)$ if $|s| \leq 1$ and $y \in I_{A'}(x)$. In particular $I_{A'}(x)$ is not bounded. (Note that $I_{A'}(x)$ intersected with A might be empty!).

3.16. Corollary. *A point $x \in Y^{ss}$ is contained in Y^s_z if and only if $I(x) = \{z\}$. In particular $Y^s_z \subset Y^s$.*

Proof. A point $x \in Y^{ss}$ is in Y^s_z if and only if for all apartments A containing z, one has $I_A(x) = \{z\}$. This happens only when $I(x) = \{z\}$. This proves the first statement.

Let $x \in Y^s_z$. Then $I(x) = \{z\}$ and hence is bounded. By theorem 3.14 the point x is in Y^s. Therefore $Y^s_z \subset Y^s$.

3.17. Remark. Let $F \subset \mathcal{B}$ be a bounded subset. Let $cc(F)$ be the *circumcenter* of F. The circumcenter of F is the center of the closed ball with minimal radius that contains F. It is well defined and unique (See [B] Ch. VI §4). This gives a $G(K)$-equivariant map $\psi : \quad Y^s \longrightarrow \mathcal{B}$, where $\psi(x) = cc(I(x))$. This map cannot be extended to all of Y^{ss} if $Y^{ss} \neq Y^s$. This clarifies somewhat property 3.12 in [Ra].

§4. Ideal polyhedra for split groups

4.1. Introduction. In this section we associate to a linear subspace $V \subset \mathcal{X}(S) \otimes \mathbb{R}$ that is spanned by roots, certain polyhedra $\Delta(V, J, R, z)$ with radius R, center z and a "parameter set" J (See 4.2). Our main interest is in such polyhedra that have certain integrality properties. These polyhedra will be called ideal polyhedra. For the root systems $\Phi = A_\ell, \dots, D_\ell$ and E_6 we prove the existence of such ideal polyhedra. For the other irreducible root systems the existence of ideal polyhedra depends on the choice of the linear space V.

In the next section these polyhedra will be used to give a stratification of $I_A(x)$.

4.2. Definition. Let the group G be split. Let $S = T$ be a maximal K° split torus in G. We fix a W-invariant inner product $(-,-)$ on $\mathcal{X}(S) \otimes \mathbb{R}$. The inner product is unique up to a scalar. We fix this scalar such that the long roots $\alpha \in \Phi$ satisfy $(\alpha, \alpha) = 2$ if $\Phi \neq C_\ell, \quad \ell \geq 3, \quad G_2$, $(\alpha, \alpha) = 4$ if $\Phi = C_\ell, \quad \ell \geq 3$ and $(\alpha, \alpha) = 3$ if $\Phi = G_2$. Using the inner product we identify $\mathcal{X}(S) \otimes \mathbb{R}$ with the apartment $A = \mathcal{X}(S)_* \otimes \mathbb{R}$ belonging to S.

Let Φ be the root system belonging to G. Let $\Phi^\circ \subset \Phi$ be equal to Φ if $\Phi \neq C_\ell, \quad \ell \geq 3$ and consist of the short roots if Φ is of type $C_\ell, \quad \ell \geq 3$. Let $V \subset \mathcal{X}(S) \otimes \mathbb{R}$ be a linear subspace spanned by roots. We will always assume that $V \neq < 0 >$.

We define the space V^c as being the linear space spanned by the roots that are orthogonal to all roots that are orthogonal to all roots in V, that is $V^c := << V^\perp \cap \Phi >^\perp \cap \Phi >$. If $V^\perp \cap \Phi = \emptyset$, then $V^c := \mathcal{X}(S) \otimes \mathbb{R}$. Since we assume that $V \neq < 0 >$, $< V^\perp \cap \Phi >^\perp \cap \Phi = \emptyset$ does not occur.

For each root $\alpha \in V^c \cap \Phi^\circ$, such that $\alpha \notin V$, we fix an integer $p_\alpha \in \mathbb{Z}_{>0}$. Let $J := \{(p_\alpha, \alpha) \mid \alpha \in \Phi^\circ \cap V^c, \ \alpha \notin V\}$. If $V^c = V$ then $J := \emptyset$. The *root polyhedron* $\Delta(V, J, R, z) \subset A$ with *parameter set* J, *radius* R and *center* z is defined as the union of all the points $y \in A$ that satisfy the following conditions:

1) $\forall (\chi \in V) \quad (\chi, y) = (\chi, z)$
2) $\forall (\alpha \in \Phi^\circ, \alpha \notin V^c, \alpha \notin V^\perp) \quad (\alpha, y - z) \leq R$
3) $\forall (\alpha \in \Phi^\circ, \alpha \notin V, \alpha \in V^c) \quad (\alpha, y - z) \leq p_\alpha \cdot R$

4.3. Remark. Different parameter sets $J,\ J'$ can (and quite often do) determine the same root polyhedron. In many cases the polyhedron, obtained by omitting condition 3) of the definition, is already bounded. In these cases the polyhedron is the same for any choice of the parameters p_α such that $p_\alpha \geq m_\alpha$ for suitably choosen integers $m_\alpha > 0$.

The lemma below explains our choice of Φ° for the root system of type $C_\ell, \quad \ell \geq 3$.

4.4. Lemma. *Let $\Phi \subset \mathbb{R}^\ell$ be a root system of type $C_\ell, \quad \ell \geq 3$. Let $V \subset \mathbb{R}^\ell$ be a linear subspace spanned by roots. Let $\alpha \in \Phi$ be a root such that $\alpha \not\perp V$ and $\alpha \notin V$. Then there exists a short root $\beta \in \Phi$, such that $\alpha \equiv \beta$ mod V.*

Proof. Let Φ consist of the roots $\pm 2e_i, \quad \pm e_i \pm e_j, \quad i \neq j, \quad 1 \leq j \leq \ell$. Here the e_i, $i = 1, \ldots, \ell$ form an orthonormal basis of $\mathbb{R}^\ell$.

Let $\alpha \in \Phi$ be such that $\alpha \not\perp V$ and $\alpha \notin V$. If α is short then we can take $\beta = \alpha$ and we are done. So let us assume that α is long. Then there exists a root $\gamma \in V$ such that $(\alpha, \gamma) > 0$. Clearly γ has to be a short root. Then $\beta := \alpha - \gamma$ is again a short root and satisfies the statement of the lemma.

4.5. Definition. Let $V \subset \mathcal{X}(S) \otimes \mathbb{R}$ be a linear subspace spanned by roots. We put $c_{\alpha,V,J} := max\{(\alpha, y)/R| \quad y \in \Delta(V, J, R, 0)\}$. We are mainly interested in root polyhedra that have the property that $c_{\alpha,V,J} \geq 1$ for all $\alpha \in \Phi^\circ$ such that $\alpha \notin V,\ V^\perp$. A root polyhedron that has this property will be called a *decent root polyhedron* . A decent root polyhedron $\Delta(V, J, R, z)$ is called *minimal* if every decent polyhedron $\Delta(V, J', R, z)$ contained in $\Delta(V, J, R, z)$ is equal to $\Delta(V, J, R, z)$ itself. A minimal decent root polyhedron will be called an *ideal polyhedron.*

4.6. Proposition. *Let $\Phi \neq F_4$ or G_2. Let $\alpha \in \Phi^\circ, \quad \alpha \notin V^c$ and $\alpha \notin V^\perp$. Then $c_{\alpha,V,J} = 1$.*

Proof. From the definition of $\Delta(V, J, R, 0)$ it is immediate that $c_{\alpha,V,J} \leq 1$ for α as above. So to proof the proposition it is sufficient to show that there exists for a root $\alpha \notin V^c, V^\perp$ a point $y \in \Delta(V, J, R, 0)$ such that $(\alpha, y) = R$.

Since $\alpha \notin V^c$, there exists a root $\beta \in V^\perp$ such that $(\alpha, \beta) \neq 0$. Let $y := (R/(\alpha, \beta)) \cdot \beta$. Then $(\alpha, y) = R$. Since $(\gamma, y) \leq R$ for any root $\gamma \in \Phi^\circ$, $\gamma \neq \pm\beta$, it follows that $y \in \Delta(V, J, R, 0)$.

4.7. Remark. If $\alpha \in \Phi^\circ$, $\alpha \notin V$ and $\alpha \in V^c$, the value of $c_{\alpha,V,J}$ can be different for different roots α. Consider the following example.

Let $\Phi = D_7$, consisting of the roots $\pm e_i \pm e_j, \quad i \neq j, \quad 1 \leq i,j \leq 7$. Here the $e_i, \quad i = 1, \ldots, 7$ form an orthonormal basis of $\mathbb{R}^7$. Let $\alpha_i := e_i - e_{i+1}, \quad i = 1, \ldots, 6$ and $\alpha_7 := e_6 + e_7$. The $\alpha_i, \quad 1 \leq i \leq 7$ form a simple basis of Φ. Let V be the hyperplane generated by the $\alpha_i, \quad i \neq 3$. The parameter set J is determined by taking $p_\alpha = 2$ for all roots $\alpha \notin V$. This determines a root polyhedron $\Delta(V, J, R, z)$.

The linear space $V^\perp$ is spanned by the fundamental weight $\omega_3 = e_1 + e_2 + e_3$. One has $V^c = \mathbb{R}^7 \neq V$. Let $\alpha \in V^c - V$ be a root. Then $\alpha = \Sigma_i n_i \cdot \alpha_i$, with $n_3 \neq 0$. In fact n_3 is either ± 1 or ± 2. Therefore $|(\omega_3, \alpha)|$ is either 1 or 2. Hence $\Delta(V, J, R, 0) = \{t \cdot \omega_3 | \quad -R \leq t \leq R\}$. It follows that $c_{\alpha,V,J} = 2$ if $n_3 = \pm 2$ and that $c_{\alpha,V,J} = 1$ if $n_3 = \pm 1$.

4.8. Lemma. *Let $\alpha \in \Phi, \quad \alpha \notin V, \quad \alpha \notin V^\perp$. We moreover assume that $p_\alpha = 2$ for all roots $\alpha \in \Phi^\circ \cap V^c$, $\alpha \notin V$. Then the following statements hold:*

i) *If $\Phi = B_\ell$ and α is short then $c_{\alpha,V,J} = 1$.*
ii) *If $\Phi = C_\ell, \quad \ell > 2$ and α is long then $c_{\alpha,V,J} = 2$.*

Proof. Since the proof of both statements is rather similar, we only give a proof of statement (i). Let $\Phi = \{\pm e_i \pm e_j, \quad \pm e_i | \quad i \neq j, \quad 1 \leq i,j \leq \ell\}$ be the root system of type B_ℓ, $\ell > 1$. Here the e_i form an orthonormal basis of $\mathbb{R}^\ell$. We may assume that $\alpha = e_1$. Since $\alpha \notin V$ and $\alpha \notin V^\perp$, there exists a root $\beta \in V$ such that $(\alpha, \beta) \neq 0$. We may assume that $\beta = e_1 - e_2$.

From the fact that $e_1 \notin V$, one concludes that the irreducible component $\Phi^\dagger$ of $\Phi \cap V$ that contains β is a root system of type A_m for some $m \geq 1$. Moreover, this is the only irreducible component of the root system $\Phi \cap V$ that is not orthogonal to $\alpha = e_1$. Without loss of generality, we may assume that $\Phi^\dagger$ consists of the roots $\pm(e_i - e_j)$ with $i \neq j$ and $1 \leq i,j \leq m+1 \leq \ell$.

If $m = 1$, then $e_1 + e_2 \in V^\perp$ and $\alpha \notin V^c$. One verifies that $y := R \cdot (e_1 + e_2) \in \Delta(V, J, R, 0)$. Moreover, $(\alpha, y) = R$. Hence $c_{\alpha,V,J} = 1$ in this case.

Now let us consider the case where $m > 1$. Then $\alpha \in V^c$. Let $y := R \cdot (e_1 + e_2 +$

$\cdots + e_{m+1})$. One verifies that $y \in \Delta(V, J, R, 0)$. One has $(\alpha, y) = R$. We leave it now to the reader to verify that $c_{\alpha,V,J} = 1$.

4.9. Proposition. *Let $V \subset \mathcal{X}(S) \otimes \mathbb{R}$ be a linear subspace spanned by roots. We assume that $V^\perp \cap \Phi = \emptyset$ and that $V^\perp \neq < 0 >$. Then there exists an ideal polyhedron $\Delta(V, J, R, z)$ with radius R and center z.*

Proof. It is sufficient to prove the proposition for the case where $R = 1$ and $z = 0$. We construct an ideal polyhedron.

Choose a simple basis Δ of Φ, consisting of the simple roots $\alpha_1, \dots, \alpha_\ell$, such that V is spanned by simple roots. Let $W' \subset W$ be the subgroup consisting of the elements $w \in W$ such that $w(\Delta \cap V) = \Delta \cap V$. Let W_V be the Weyl group of $\Phi \cap V$. Then $W' \cdot W_V$ is the normaliser $N_W(W_V)$ of W_V in W (See [Ho] Corr. 3).

Let $O_1, \dots, O_m$ be the orbits of $N_W(W_V)$ on roots $\alpha \notin \Phi$. Let $\Delta_{i,n}$ be the polyhedron defined by $(\beta, z) \leq n$ for all $\beta \in O_i$ and $(\alpha, z) = 0$ for all $\alpha \in V$. Now let us define n_i, $i = 1, \dots, m$ inductively as being the smallest integer in $\mathbb{Z}_{>0}$ such that $max\{\ (\alpha, z) \mid z \in \Delta_{1,n_1} \cap \cdots \Delta_{i,n_i}\} \geq 1$ for all roots $\alpha \in \Phi^\circ$ that are not in V. Now we define J by taking $p_\alpha = n_i$ for $\alpha \in O_i$. Then $\Delta(V, J, 1, 0) = \Delta_{1,n_1} \cap \cdots \Delta_{m,n_m}$.

It is clear from the construction that $p_\alpha = c_{\alpha,V,J} \geq 1$ for any root $\alpha \in \Phi - \Phi \cap V$. Hence $\Delta(V, J, 1, 0)$ is a decent root polyhedron.

If we replace any p_α by a smaller number, then it is clear that the thus obtained root polyhedron is not decent anymore. From this one concludes that our polyhedron does not contain any smaller decent root polydra. Therefore $\Delta(V, J, 1, 0)$ is an ideal polyhedron.

4.10. Lemma. *Let Φ be a root system of type A_ℓ. Let J be given by putting for all $\alpha \in \Phi \cap V^c$, $\alpha \notin V$ the parameter p_α equal to $p_\alpha = 1$. Then for all roots $\alpha \notin V, \quad V^\perp$ one has that $c_{\alpha,V,J} = 1$.*

Proof. If $\ell = 1$ there is nothing to prove. So let us consider the case where Φ is of type $A_\ell, \quad \ell > 1$. We choose a simple basis $\alpha_1, \dots, \alpha_\ell$ of the root system Φ, such that the linear space V is spanned by simple roots. Let α_i be a simple root that is not contained in V. Then the fundamental weight ω_i is orthogonal to V. For any root α one has $(\alpha, \omega_i) = 0, \pm 1$. Indeed, this follows from the fact that $\alpha = \Sigma n_j \cdot \alpha_j$

with $n_j \in \{0, \pm 1\}$. Therefore $\pm R \cdot \omega_i \in \Delta(V, J, R, 0)$ for any i such that $\alpha_i \notin V$. From this it easily follows that for any root α that is not contained in V and $V^\perp$ one has that $c_{\alpha,V,J} = 1$.

4.11. Lemma. *Let Φ be a root system of type B_ℓ, C_ℓ with $\ell \geq 2$ or D_ℓ with $\ell > 3$. We put $p_\alpha = 2$ for all $\alpha \in \Phi^\circ \cap V^c$, $\alpha \notin V$. Then for all roots $\alpha \notin V$, $V^\perp$ one has that $c_{\alpha,V,J} \geq 1$.*

Proof. Since the roots $\alpha \notin V^c, V^\perp$ have been treated in proposition 4.6, we only have to consider roots α that are contained in V^c but not in V.

Let us first consider the root system D_ℓ with $\ell > 3$. Let Φ consist of the roots $\pm e_i \pm e_j$ with $i \neq j$ and $1 \leq i, j, \leq \ell$. Here the e_i form an orthonormal basis of $\mathbb{R}^\ell$. First we look at the case where $V^\perp \cap \Phi \neq \emptyset$. Then $V^c \neq V$ if and only if $\Phi \cap V$ has at least one irreducible component of type A_i with $i > 1$. If $i = 3$ then we moreover have to assume that the component of type A_3 is contained in a root system of type A_4 in D_ℓ. Let us choose such a component of type A_i. We may assume that it consists of the roots $\pm(e_n - e_m)$ with $1 \leq n < m \leq i + 1$. One verifies that the point $y := R \cdot (e_1 + \cdots e_{i+1})$ is contained in the polyhedron $\Delta(V, J, R, 0)$. Hence for all roots $\alpha \notin V, V^\perp$ that are not orthogonal to our component of type A_i one has $(y, \alpha) = \pm R$, $\pm 2R$. In particular $c_{\alpha,V,J} \geq 1$.

To each irreducible component Φ_j of $\Phi \cap V$ that is of type A_{i_j}, we can associate a point y_j as above. If $i_j = 3$ then we make the same assumption as above. Let $\alpha \in V^c$ be a root. Then either α is not orthogonal to exactly one component of type $A_{i_j}, i_j > 1$, or to exactly two components of type $A_{i_j}, i_j > 1$, or to one component of type $A_{i_j}, i_j > 1$ and one component of type $D_{i_j}, i_j > 1$. (Here $D_2 = A_1 \times A_1$ and $D_3 = A_3$ are such that they are not contained in a system of type A_4 inside Φ.) Then $c_{\alpha,V,J}$ is 2, 2 and 1, respectively.

Now let us consider the case where $\Phi = D_\ell$ and $V^\perp \cap \Phi = \emptyset$. We choose a simple basis $\alpha_1, \ldots, \alpha_\ell$ of Φ such that V is spanned by simple roots. Let $\alpha_i \notin V$ and ω_i be the corresponding fundamental weight. For each root $\alpha \in \Phi$ one has $(\alpha, \omega_i) = 0, \pm 1, \pm 2$. Hence $y_i := R \cdot \omega_i$ is contained in $\Delta(V, J, R, 0)$. Since this is true for all roots α_i that are not contained in V, it follows that $c_{\alpha,V,J} \geq 1$ for all roots $\alpha \notin V$.

For the root systems of type B_ℓ, respectively C_ℓ, the short respectively long roots have been treated above. The proof for the remaining roots is similar to that for the root system D_ℓ and we will omit it.

4.12. Lemma. *Let $\Phi \subset \mathbb{R}^6$ be a root system of type E_6. Let $V \subset \mathbb{R}^6$ be a linear space spanned by roots such that $V^\perp \cap \Phi \neq \emptyset$. For each root $\alpha \in \Phi \cap V^c - \Phi \cap V$ we take $p_\alpha = 1$. Then $c_{\alpha,V,J} = 1$.*

Proof. We fix a simple basis $\alpha_1, \ldots, \alpha_6$ of Φ. The ordering of the roots is such that $(\alpha_i, \alpha_{i+1}) = (\alpha_1, \alpha_4) = -1$ for $i = 2, \ldots, 5$. Since $V^\perp \cap \Phi \neq \emptyset$, we may assume that the highest root α_0 is contained in $V^\perp$. We may moreover assume that V is spanned by simple roots. Then $\Phi \cap V$ is a root system that is contained in the root system of type A_5 that is generated by the roots $\alpha_2, \ldots, \alpha_6$.

If $\Phi \cap V$ is a root system of type A_5, then $V^\perp =< \alpha_0 >$. Therefore $V = V^c$ and the proposition follows from proposition 4.6. Now let us assume that $rank(\Phi \cap V) = 4$. Then a simple basis of $\Phi \cap V$ is obtained by ommitting a simple root α_j from the basis of the root system of type A_5. We will denote the corresponding space V by V_j. If $j = 4$, then $V^\perp =< \alpha_0,\ \alpha_1 >$. Hence $V_4 = V_4^c$ and we are done.

Let us now assume that $j \neq 4$. Let the highest root α_0 in E_6 be $\alpha_0 = \Sigma n_i \cdot \alpha_i$. Define μ_j as being $\mu_j := n_1 \cdot \omega_j - n_j \cdot \omega_1 = 2 \cdot \omega_j - n_j \cdot \omega_1$. Here the ω_i are the fundamental weights belonging to our simple basis. Note that $\mu_j \perp \alpha_0$ and that $\omega_1 = -\alpha_0$. We have $n_j = 2$ if $j = 3, 5$ and $n_j = 1$ if $j = 2, 6$. We will now show that $1/2 \cdot \mu_j$ is in $\Delta(V_j, J, 1, 0)$.

If $\alpha \in \Phi^+$ and $\alpha \notin V^c,\ V^\perp$, then $\alpha = \Sigma m_i \cdot \alpha_i$ with $m_1 = 1$ and $m_j \leq n_j$. Then $(\alpha, \mu_j) = 2 \cdot m_j - n_j$. So $(\alpha, \mu_j) \in \{0, \pm 2\}$ if $n_j = 2$ and $(\alpha, \mu_j) \in \{0, \pm 1\}$ if $n_j = 1$. Now let $\alpha \in \Phi^+$ be such that $\alpha \in V^c - V$. Then $\alpha = \Sigma m_i \cdot \alpha_i$ with $m_1 = 0,\ m_j = 1$ and $m_i = 0, 1$ if $i \neq 1, j$. From this it follows that $(\alpha, \mu_j) = 2$. Therefore $1/2 \cdot \mu_j \in \Delta(V_j, J, 1, 0)$. Now it is clear that $c_{\alpha, V_j, J} = 1$ for all roots $\alpha \notin V$.

Let us now consider the case where $rank(\Phi \cap V) = 3$. Then $\Phi \cap V$ is of type A_3, $A_1 \times A_2$ or of type $A_1 \times A_1 \times A_1$. If $\Phi \cap V$ is of type A_3 , then $\Phi \cap V^\perp$ is of type $A_1 \times A_1$. In this case one has $V = V^c$, even though $dim(V) + dim(V^\perp) = 5$. So in this case we are done by using proposition 4.6.

If $\Phi \cap V$ is of type $A_1 \times A_2$, then $\Phi \cap V^c$ is of type $A_2 \times A_2$. We may choose $V =< \alpha_3, \alpha_5, \alpha_6 >$. Using the weight ω_2 one now shows that $c_{\alpha,V,J} = 1$.

If $\Phi \cap V$ is of type $A_1 \times A_1 \times A_1$, then $\Phi \cap V^c$ is of type A_5. One may choose $V =< \alpha_2, \alpha_4, \alpha_6 >$. Now one can use the weights $\omega_3 - \omega_1$ and $\omega_5 - \omega_1$ to show that $c_{\alpha,V,J} = 1$ in this case.

If $rank(\Phi \cap V = 2)$ then $\Phi \cap V = A_2$ (and $\Phi \cap V^\perp = A_2 \times A_2$) or $\Phi \cap V = A_1 \times A_1$ (and $\Phi \cap V^\perp = A_3$). In both cases $V = V^c$. If $rank(\Phi \cap V) = 1$, then $\Phi \cap V^\perp = A_5$. So again we have $V = V^c$ and we are done. We have now treated all the cases.

4.13. Theorem. *Let $\Phi \subset \mathbb{R}^\ell$ be an irreducible root system. Let $V \subset \mathbb{R}^\ell$ be a linear space spanned by roots. We assume that $V \neq < 0 >, \mathbb{R}^\ell$. If Φ is of type A_ℓ, B_ℓ, C_ℓ, D_ℓ or E_6 then ideal polyhedra $\Delta(V, J, R, z)$ exist. If $\Phi \cap V^\perp \neq \emptyset$, then these ideal polyhedra are unique.*

Proof. The existence of the ideal polyhedra in these cases has been proved in the lemmas above. So we only have to prove the uniqueness in the case where $\Phi \cap V^\perp \neq \emptyset$.

If $V = V^c$, then the uniqueness is clear (See also lemma 4.6). So let us assume that $V \neq V^c$. If there exist ideal polyhedra with $c_{\alpha,V,J} = 1$ for all roots $\alpha \in \Phi^\circ$, $\alpha \notin V$, $V^\perp$, then the uniqueness is obvious. This holds for $\Phi = A_\ell$, E_6 by lemmas 4.10 and 4.12. Moreover it follows from the proof of lemma 4.11 that this also holds for $\Phi = D_\ell$ if $\Phi \cap V$ contains no component of type D_i (Here we follow the same conventions as in that proof).

In the other cases the ideal polyhedron is obtained by taking $p_\alpha = 2$ for all roots $\alpha \in \Phi^\circ$, $\alpha \in V^c$, $\alpha \notin V$. One sees that in these cases the ideal polyhedron in fact is entirely determined by the properties 1) and 2) of definition 4.2. From this the uniqueness follows.

4.14. Remark. For the irreducible root systems not treated in the proposition above ideal polyhedra do not always exist. The existence depends on V.

Let W_V be the Weyl group of $\Phi \cap V$. The normaliser $N_W(W_V)$ of W_V preserves the ideal polyhedra $\Delta(V, J, R, 0)$ for the root systems $A_\ell, \dots, D_\ell, E_6$. If $\Phi \cap V^\perp = \emptyset$ then it follows from the construction in proposition 4.9. If $\Phi \cap V^\perp \neq \emptyset$ then it follows from the uniqueness.

4.15. Remark. In the paragraphs that follow we will only consider ideal polyhedra. In vieuw of the uniqueness of these polyhedra if $V^\perp \cap \Phi \neq \emptyset$, we will drop the parameter set J from the notation. If $V^\perp \cap \Phi = \emptyset$, then we fix a parameter set (which has the same properties as those constructed in proposition 4.9). So from now on we will write $\Delta(V, R, z)$ and $c_{\alpha,V}$ instead of $\Delta(V, J, R, z)$ and $c_{\alpha,V,J}$.

From now on we will restrict ourselves to the root systems Φ of type $A_\ell, B_\ell, C_\ell, D_\ell$ and E_6. The proofs given are still valid for the root systems E_7 and E_8, provided that Y^{ss} is such that for all the linear spaces $V_i(x, A)$ (See def. 5.3) that occur, ideal polyhedra $\Delta(V_i(x, A), R, z)$ exist. For the root systems of type F_4 and G_2 it is not clear, wether our definition of ideal polyhedra is the correct one.

§5. A STRATIFICATION OF $I_A(x)$ FOR SPLIT GROUPS

5.1. Introduction. In this section we study $I_A(x)$ in some detail for split groups G with root system Φ of type $A_\ell, \dots, D_\ell$ or E_6. In particular we give a stratification $I_A^i(x)$ of $I_A(x)$. Here $I_A^i(x) \subset I_A^{i-1}(x)$ is a certain convex subset if $i > 1$ and $I_A^1(x) := I_A(x)$.

The stratification is defined by using suitable ideal polyhedra $\Delta(V, R, z) \subset I_A^i(x)$. Here V is spanned by the roots that are constant on $I_A^i(x)$ and the radius R is taken maximal. These ideal polyhedra $\Delta(V, R, z)$ resemble $I_A^i(x)$ itself, but are in general more symmetric. If $z = 0$ then the group $N_W(W_V)/W_V$ acts on the ideal polyhedron. As $I_A^{i+1}(x)$ one takes the union of the centres z of all polyhedra $\Delta(V, R, z)$ with maximal radius R that are contained in $I_A^i(x)$.

Using this stratification one is able to compare $I_A(x)$ and $I_{A'}(x)$ for apartments A and A'. This will be done in §7, where we describe $I(x)$ in some detail.

5.2. Definition. Let $F \subset A$ be a convex polyhedron. We admit polyhedra that are not bounded. Let $V(F) \subset \mathcal{X}(S) \otimes \mathbb{R}$ be the linear subspace spanned by the characters χ that are constant on F. We assume that $V(F)$ is spanned by roots.

The *radius* $R(F)$ of F is defined as $R(F) := \max \{R | \Delta(V(F), R, z) \subset F\}$. If the maximum does not exist we say that $R(F) = \infty$.

If $R(F) < \infty$ then the set of centres of F is given by $Cent(F) :=$

$\{z|\Delta(V(F),R(F),z)\subset F\}$. If $R(F)=\infty$, then the set $Cent(F)$ is not defined.

The set $Cent(F)$ is again a convex polyhedron, if it exists. Moreover the dimension of $Cent(F)$ is less than that of F. If the faces of F are parallel to walls (i.e. if the faces are of the form $\{y\in F|(\alpha,y)=c_\alpha\}$ for some $c_\alpha\in\mathbb{R}$ and $\alpha\in\Phi$) then $V(Cent(F))$ is again spanned by roots, if $R(F)<\infty$.

We will now apply this to $I_A(x)$.

5.3. Definition. We again fix a projective homogeneous variety X for G and an ample line bundle $\mathcal{L}$ on it. Let $x\in Y_A^{ss}$. We construct a strictly increasing flag $V_\bullet(x,A)=(V_1(x,A)\subset\cdots\subset V_{m(x,A)}(x,A))$ of linear subspaces of $\mathcal{X}(S)\otimes\mathbb{R}$. The length $m(x,A)\geq 1$ depends on x (and A). Associated to each $V_i(x,A)$ is a polyhedron $I_A^i(x)$ that is contained in $I_A(x)$. Their definition will be done inductively.

First we define $V_1(x,A):=V(I_A(x))$ and $I_A^1(x):=I_A(x)$. Let us now assume that $V_i(x,A)$ and $I_A^i(x)$ are defined. If $V_i(x,A)=\mathcal{X}(S)\otimes\mathbb{R}$ or $R(I_A^i(x))=\infty$, then we put $m(x,A):=i$ and we are done.

So let us assume that $V_i(x,A)\neq\mathcal{X}(S)\otimes\mathbb{R}$ and that $R(I_A^i(x))<\infty$. Then we define $I_A^{i+1}(x):=Cent(I_A^i(x))$ and $V_{i+1}(x,A):=V(I_A^{i+1}(x))$.

For $i<m(x,A)$ the subset $I_A^{i+1}(x)$ of $I_A^i(x)$ is a polyhedron of smaller dimension. Therefore everything is well defined.

The linear space $V_{m(x,A)}(x,A)$ is equal to the entire space $\mathcal{X}(S)\otimes\mathbb{R}$ if and only if $I_A(x)$ is bounded. This is the case if and only if $x\in Y_A^s$.

5.4. Proposition. *Let $x\in Y_{0,A}^{ss}$ be such that 0 is not contained in a face of $I_A(x)$. We assume that x is normalised and that $|x_\lambda|=1$. Let $1\leq i\leq m(x,A)$. Then there exists in each irreducible component of the root system $\Phi\cap V_i(x,A)$ a root α such that $(\alpha,\lambda)\neq 0$. In particular if $i>1$ then each irreducible component of $\Phi\cap V_i(x,A)$ contains at least one irreducible component of $\Phi\cap V_{i-1}(x,A)$.*

Proof. First we consider the case $i=1$. Then $V_1(x,A)$ is spanned by the weights $w(\lambda),\quad w\in W$ such that $|x_{w(\lambda)}|=1$. Since $\lambda\in V_1(x,A)$, the weights $w(\lambda)$ that occur are such that the element $w\in W$ is in fact an element of W'. Here W' is the Weyl group of the root system $\Phi\cap V_1(x,A)$. Therefore the weights $w(\lambda),\quad w\in W'$

span $V_1(x,A)$. From this it follows that λ is not orthogonal to any irreducible component of $\Phi \cap V_1(x,A)$.

Next we consider the case where $i > 1$. The linear space $V_i(x,A)$ is generated by $V_{i-1}(x,A)$ and the roots α that correspond to certain faces of $I_A(x)$. By lemma 2.8 these roots α are never othogonal to $V_1(x,A)$. Using induction it follows that each irreducible component of $\Phi \cap V_i(x,A)$ contains at least one irreducible component of $\Phi \cap V_{i-1}(x,A)$. In particular each irreducible component contains at least one irreducible component of $\Phi \cap V_1(x,A)$. Therefore λ is not orthogonal to any irreducible component of $\Phi \cap V_i(x,A)$ for $1 \leq i \leq m(x,A)$.

5.5. Definition. We make the correspondence between faces $I_A^i(x)$ and roots a little bit more precise. A codimension one face F of $I_A^i(x)$ is said to *correspond* to a root α if there exists a constant $c \in \mathbb{R}$ such that $F = \{z \in I_A^i(x) | (\alpha, z) = c\}$ and if for all $y \in I_A^i(x)$ one has $(\alpha, y) \geq c$.

Now we are able to state:

5.6. Proposition. *Let us fix for each face F of $I_A^i(x)$ a root $\beta_F \in \Phi^\circ$ that corresponds to the face F. We assume that $R(I_A^i(x)) < \infty$. Let Φ_i° consist of the roots β_F corresponding to faces F that have the property that every $\Delta(V_i(x,A), R(I_A^i(x)), z) \subset I_A^i(x)$ has a non-empty intersection with F.*

Then the $\beta_F \in \Phi_i^\circ$ together with $V_i(x,A)$ generate $V_{i+1}(x,A)$. Moreover there exist $n_\beta > 0, \quad \beta \in \Phi_i^\circ$ such that $\displaystyle\sum_{\beta \in \Phi_i^\circ} n_\beta \cdot \beta = 0 \ mod \ V_i(x,A)$.

Proof. Let Φ^i consist of the roots β that correspond to a face F of $I_A^i(x)$. The root $\beta \in \Phi^i$ is constant on $I_A^{i+1}(x)$ if and only if the face F of $I_A^i(x)$ corresponding to β has the same distance to every center z of an ideal polyhedron $\Delta(V_i(x,A), R(I_A^i(x)), z) \subset I_A^i(x)$. This is clearly the case for the $\beta \in \Phi_i^\circ$. Hence $V_{i+1}(x,A)$ contains Φ_i°.

If β is not in the span of $\Phi_\circ^i$ and $V_i(x,A)$, then one can find two different polyhedra $\Delta(V_{i+1}(x,A), R(I_A^i(x)), z_j), j = 1,2$, such that the corresponding centers $z_j, j = 1,2$ have a different distance to the face corresponding to β. Hence $V_{i+1}(x,A)$ is spanned by $V_i(x,A)$ and Φ_i°.

Next we prove the second statement of the proposition. Suppose there do not exist $n_\beta > 0$ for the $\beta \in \Phi_i^\circ$ such that $\sum n_\beta \cdot \beta = 0 \bmod V_i(x,A)$. Let $H \subset \Phi_i^\circ$ be the largest subset such that there do exist $n_\beta > 0, \beta \in H$ such that $\sum n_\beta \cdot \beta = 0 \bmod V_i(x,A)$. Clearly H is unique and by assumption we have that $H \neq \Phi_i^\circ$. For each $\beta \in \Phi^i, \beta \notin < \Phi_i^\circ, V_i(x,A))$ one can find an ideal polyhedron $\Delta(V_i(x,A), R(I_A^i(x)), z_\beta) \subset I_A^i(x)$ such that this polyhedron has a non-empty intersection with the face F corresponding to β. Let z be in the convex hull of the z_β. We may assume that z is not on a face of the polyhedron formed by the convex hull of the z_β. Then $\Delta := \Delta(V_i(x,A), R(I_A^i(x)), z) \subset I_A^i(x)$. Moreover the only faces F that have non-empty intersection with Δ are the faces F corresponding to roots $\beta_F \in \Phi_i^\circ$.

Let $s \in S(\mathbb{K})$ be such that $\chi(s) = 1$ for all $\chi \in < V_i(x,A), H >$. We choose s in such a way that $|\beta(s)| > 1$ for $\beta \in \Phi_i^\circ - H$ and $\epsilon \leq |\beta(s)| \leq \epsilon^{-1}$ for some $0 < \epsilon < 1$. By choosing ϵ close enough to 1, we can make sure that $s \cdot \Delta \subset I_A(x)$, since no relation $\sum n_\beta \cdot \beta = 0 \quad mod < V_i(x,A), H >, \quad n_\beta > 0, \quad \beta \in \Phi_i^\circ - H$ exists. This cannot be. Therefore we must have $H = \Phi_\circ^i$. This proves the second statement of the proposition.

5.7. Definition. We define a function $\rho_A : Y_A^{ss} \longrightarrow \mathbb{R}^\ell$ that measures the volume of $I_A(x)$ for $x \in Y_A^{ss}$. Here ℓ is the rank of G. Let $\rho_A(x) = (\rho_A(x)_i | i = 1, \dots, \ell)$ and $R_j := R(I_A^j(x)), \quad j = 1, \dots, m(x,A)$. Then we define:

$$\rho_A(x)_i := \begin{cases} 1 & \text{if } 0 < i \leq \dim(V_1(x,A)). \\ \exp\ (-\sum_{s=1}^{j} R_s) & \text{if } \dim(V_j(x,A)) < i \leq \dim(V_{j+1}(x,A))\ . \\ 0 & \text{if } \dim(V_{m(x,A)}(x,A)) < i. \end{cases}$$

Here exp denotes the exponent function.

On $\mathbb{R}^\ell$ we put the lexicographical ordering, i.e. $a = (a_i) < b = (b_i)$ if $\exists (i = 1, \dots, \ell) \quad b_j = a_j, j < i$ and $b_i > a_i$. Now $I_A(x) \subset I_A(y)$ implies that $\rho_A(x) \geq \rho_A(y)$.

5.8. Definition. For $x \in Y_A^{ss}$ the *normalisation* $\mathbb{I}_A^i(x)$ of $I_A^i(x)$ for $1 \leq i \leq m(x,A)$ is defined as follows. Let $m := m(x,A)$. Then we put $\mathbb{I}_A^m(x) := I_A^m(x)$. If

$m > 1$, then we define $\mathbb{I}^i_A(x)$ for $1 \le i < m$ inductively as follows:

$$\mathbb{I}^i_A(x) := \bigcup \{y \in \Delta(V_i(x,A), R(I^i_A(x)), z) | \quad z \in \mathbb{I}^{i+1}_A(x)\}$$

We will often write $\mathbb{I}_A(x)$ instead of $\mathbb{I}^1_A(x)$. From the definition it is clear that $\mathbb{I}_A(x)$ is the smallest polyhedron that has the same flag $V_\bullet(x,A)$, the same "inverse volume" $\rho_A(x)$ and the same center $I_A^{m(x,A)}(x)$ as $I_A(x)$.

§6. Group theoretical description of $I_A(x)$ for split groups.

6.1. For use in §7 below, we will now give a description of the interval of semistability $I_A(x)$ in more group theoretical terms. The root system Φ of G is again assumed to be of type $A_\ell, \dots, D_\ell$ or E_6.

The parabolic subgroup $P \subset G$ that corresponds to the highest weight λ is generated by $S = T$ and the additive subgroups $U_\alpha \subset G$ with $(\alpha, \lambda) \ge 0$. Let $x \in Y^{ss}_A$. If $x_\lambda \ne 0$, one can write x uniquely as $x = \Pi u_\alpha(a_\alpha) \cdot P$ with $u_\alpha(a_\alpha) \in U_\alpha$, where the roots $\alpha \in \Phi$ are such that $(\alpha, \lambda) < 0$ and one has choosen a fixed order of the roots $\alpha \in \Phi$. If x is in $Y^{ss}_{0,A}$ and after normalisation is such that $|x_\lambda| = 1$ then for all $\alpha \in \Phi$ one has $|a_\alpha| \le 1$.

Let us assume that x as above is in $Y^{ss}_{0,A}$. If $I_A(x) \ne \{0\}$, then we assume moreover that 0 is not on a face of $I_A(x)$. Then the α with $|a_\alpha| = 1$ span $V_1(x,A)$. Now for a $s \in S(\mathbb{K})$ with $\chi(s) = 1$ for all $\chi \in V_1(x,A)$ and such that $|a_\alpha \alpha(s)| \le 1$ for all $\alpha \in \Phi$ with $(\alpha, \lambda) < 0$ one has $s \cdot x \in Y^{ss}_{0,A}$. So $s^{-1} \cdot 0 \in I_A(x)$. In fact these $s \in S(\mathbb{K})$ give us all of $I_A(x)$.

Since the roots α with $|a_\alpha| = 1, \quad (\lambda, \alpha) < 0$ span $V_1(x,A)$ and $\lambda \in V_1(x,A)$, our conditions on $s \in S(\mathbb{K})$ are equivalent with $\lambda(s) = 1$ and $|a_\alpha \cdot \alpha(s)| \le 1$ for the roots α with $(\lambda, \alpha) < 0$. Moreover, $|a_\alpha \cdot \alpha(s)| \le 1$ is equivalent to $v(a_\alpha) + v(\alpha(s)) = v(a_\alpha) + (\alpha, s^{-1} \cdot 0) \quad \ge 0$. This in turn means that $u(a_\alpha)$ stabilises the point $s^{-1} \cdot 0 \in I_A(x) \subset A$. This proves the following proposition:

6.2. Proposition. *Let $x \in Y^{ss}_{0,A}$ and $\lambda \in V_1(x,A)$. We assume that x is normalised and has $|x_\lambda| = 1$. We fix an order on the roots $\alpha \in \Phi$ with $(\lambda, \alpha) < 0$. Let*

$x = h \cdot P$ *with* $h = \prod_{(\alpha,\lambda)<0} u_\alpha(a_\alpha)$ *with* $|a_\alpha| \leq 1$. *Then* $I_A(x)$ *consists of the points* $y \in A$ *satisfying the following two equivalent conditions:*

i) $(\lambda, y) = 0$ *and* $v(a_\alpha) + (\alpha, y) \geq 0$ *for all roots* α *such that* $(\lambda, \alpha) < 0$.

ii) $(\lambda, y) = 0$ *and* $h(y) = y$.

6.3. Definition. We relate the values $v(a_\alpha)$ in the expression of x as $x = \prod u_\alpha(a_\alpha) \cdot P$ to the constants $c_{\alpha,V}$ defined above. Let α be a root such that $\alpha \not\perp V_1(x, A)$ (We do allow $\alpha \in V_1(x, A)$). Then we define $c_\alpha^1(x, A) := 0$ and for $1 < i \leq m(x, A)$ we define $c_\alpha^i(x, A) := \sum_{j=1}^{i-1} c_{\alpha,V_j(x,A)} \cdot R(I_A^j(x))$. We furthermore put $c_\alpha(x, A) := c_\alpha^m(x, A)$.

6.4. Lemma. *Let* $x \in Y_{0,A}^{ss}$ *and* $\lambda \in V_1(x, A)$. *We assume that* x *is normalised and has* $|x_\lambda| = 1$. *We fix an order on the roots* $\alpha \in \Phi$ *with* $(\lambda, \alpha) < 0$. *Then* $x = h \cdot P$ *with* $h = \prod_{(\alpha,\lambda)<0} u_\alpha(a_\alpha)$. *Let* $1 \leq i \leq m(x, A)$. *Then* $0 \in I_A^i(x)$ *if and only if for all the roots* α *with* $(\lambda, \alpha)) < 0$ *one has* $v(a_\alpha) \geq c_\alpha^i(x, A)$.

Proof. Let $0 \in I_A^i(x)$. We put $V_j := V_j(x, A)$ and $R_j := R(I_A^j(x))$ for $j = 1, \ldots, i-1$. For any root $\alpha \not\perp V_1(x, A)$, we construct a point y_α^1 in $I_A(x)$ such that $(y_\alpha^1, \alpha) = c_\alpha^i(x, A)$. From this it then follows that $v(a_\alpha) \geq c^i(x, A)$. We put $y_\alpha^i = 0$. If $i > 1$ then we define the points y_α^j for $j = i-1, \ldots, 1$ inductively as follows. Let us assume that we have constructed the point $y_\alpha^{j+1} \in I_A^{j+1}(x)$. Since $\Delta(V_j, R_j, y_\alpha^{j+1}) \subset I_A^j(x)$, there exists a point $y_\alpha^j \in I_A^j(x)$ such that $(\alpha, y_\alpha^j) = (\alpha, y_\alpha^{j+1}) + c_{\alpha,V_j} \cdot R_j$. Then $(\alpha, y_\alpha^1) = c_\alpha^i(x, A)$. Therefore $v(a_\alpha) \geq c_\alpha^i(x, A)$.

Now let us assume that $v(a_\alpha) \geq c_\alpha^i(x, A)$ for all roots $\alpha \not\perp V_1(x, A)$. If $i = 1$ then it is clear that $0 \in I_A(x)$. So let us assume that $i > 1$. Let $H_{i-1} := \Delta(V_{i-1}, R_{i-1}, 0)$. If $i > 2$ then we define the H_j, $j = 1, \ldots, i-2$ inductively. Suppose H_{j+1} is defined. Then we put $H_j := \cup\{u \in \Delta(V_j, R_j, y)| \quad y \in H_{j+1}\}$. From the construction of the H_j it follows that $max\{|(\alpha, y)|| \quad y \in H_1\} = c_\alpha^i(x, A)$ for all roots $\alpha \not\perp V_1(x, A)$. Since $v(a_\alpha) \geq c_\alpha^i(x, A)$, it follows that $u_\alpha(a_\alpha)$ stabilises H_1. Therefore $H_1 \subset I_A(x)$. From this one concludes that $0 \in I_A^i(x)$. This proves the lemma.

6.5. Lemma. *Let* $x \in Y_A^{ss}$ *and* $1 \leq i \leq m(x, A)$. *Let* $\alpha \in \Phi$ *be a root not contained in* $V_i(x, A)^\perp$. *If* $i > 1$, *then we also assume that* $\alpha \notin V_{i-1}(x, A)$. *Let* $d_1(x, A) := 0$

and if $i > 1$ let $d_i(x,A) := \sum_{j=1}^{i-1} R(I_A^j(x))$ *for* $1 < i \leq m(x,A)$. *Then one has:*

i) $c_\alpha^i(x,A) \geq d_i(x,A)$.

ii) *If* $\alpha \notin V_i(x,A)^c$, *then* $c_\alpha^i(x,A) = d_i(x,A)$, *except when* $\Phi = C_\ell, \quad \ell > 2$ *and* α *is long.*

iii) $c_\alpha^i(x,A) = 2 \cdot d_i(x,A)$, *if* $\Phi = C_\ell,\ \ell > 2,\ \alpha \notin V_i(x,A)^c$ *and* α *is long.*

Proof. This follows directly from the definitions, proposition 4.6 and lemma 4.8.

6.6. Definition. We say that a point $x \in Y_{0,A}^{ss}$ is *almost in standard position* if $\lambda \in V_1(x,A)$ and $|x_\lambda| = 1$ after normalisation. We say that x is *in standard position* if x is almost in standard position, $0 \in I_A^{m(x,A)}(x)$ and moreover 0 is not contained in a face of $I_A^{m(x,A)}(x)$.

6.7. Lemma. *Let* $x \in Y_{0,A}^{ss}$ *be in standard position. Let* $m := m(x,A)$. *Let* x *be* $x = \Pi u_\alpha(a_\alpha) \cdot P$ *for some fixed order of the roots* $\alpha \in \Phi$ *with* $(\alpha, \lambda) < 0$. *Then the following statements hold:*

i) *For all* $\alpha \in \Phi$ *one has* $v(a_\alpha) \geq c_\alpha(x,A)$

ii) *The* $\alpha \in \Phi$ *with* $v(a_\alpha) = c_\alpha^i(x,A)$ *span* $V_i(x,A)$.

iii) *For all* $\alpha \in \Phi, \alpha \notin V_m(x,A)$ *one has* $v(a_\alpha) > c_\alpha(x,A) = c_\alpha^m(x,A)$. *This can only occur if* $V_m(x,A) \neq \mathcal{X}(S) \otimes \mathbb{R}$, *that is if* $x \in Y_A^{ss} - Y_A^s$.

Proof. This follows directly from lemma 6.5 above.

6.8. Definition. Let $x \in Y_{0,A}^{ss}$ be normalised and be such that $|x_\lambda| = 1$. Let $m := m(x,A)$. We order the roots $\alpha \in \Phi$ in such a way that $\alpha_j \in V_1(x,A)$ if $1 \leq j \leq j_1$ and if $m > 1$ then for $1 < i \leq m$ we have $\alpha_j \in V_i(x,A) - V_{i-1}(x,A)$ if $j_i \geq j > j_{i-1}$. Here $j_1 < j_2 < \cdots < j_m$. If $x \in Y_A^{ss} - Y_A^s$, then $\alpha_j \notin V_m(x,A)$ for $j = j_m + 1, \ldots, j_{m+1}$. We write $x = u_{\alpha_{j_m}}(a_{j_m}) \cdot \cdots \cdot u_{\alpha_1}(a_1) \cdot P$ if $x \in Y_A^s$, and we write $x = u_{\alpha_{j_{m+1}}}(a_{j_{m+1}}) \cdot \cdots \cdot u_{\alpha_1}(a_1) \cdot P$ if $x \in Y_A^{ss} - Y_A^s$. For $1 \leq i \leq m$ we define $h_i := u_{\alpha_{j_i}}(a_{j_i}) \cdot \cdots \cdot u_{\alpha_1}(a_1)$ and $y_i := h_i \cdot P$.

Both h_i and y_i depend (uniquely) on x and A. It will always be clear from the context which x and A define the h_i and y_i.

6.9. Proposition. *Let $x \in Y_{0,A}^{ss}$ be normalised and be such that $|x_\lambda| = 1$ and $m(x, A) > 1$. Let $1 \leq i < m(x, A)$ and $z \in I_A^i(x)$. Then one has:*

i) $I_A(x) \subset I_A(y_i)$.

ii) *If $i > 1$ then $R(I_A^j(y_i)) = R(I_A^j(x))$ for $j = 1, \ldots, i-1$.*

iii) $I_A^i(y_i) = \{s \cdot z | s \in S(\mathbb{K}) \quad \chi(s) = 1, \quad \forall(\chi \in V_i(x, A))\}$.

Proof. After translation by a suitable element $s \in S$, we may assume that $0 \in I_A^{m(x,A)}(x)$. If $I_A^{m(x,A)}(x)$ is not a point, then we may moreover assume that 0 is not on a face of $I_A^{m(x,A)}(x)$.

Now i) follows from the fact that if $s \cdot x \in Y_{0,A}^{ss}$ then $|a_\alpha \cdot \alpha(s)| \leq 1$ for all $\alpha \in \Phi$. Hence also $s \cdot y \cdot \in Y_{0,A}^{ss}$. Therefore $I_A(x) \subset I_A(y_i)$.

Next we prove statement ii). Let F_β be a face of $I_A(x)$ corresponding to a root $\beta \in V_j(x, A), j < i$. Let $s \in S(\mathbb{K})$ be such that $s^{-1} \cdot 0 \in F_\beta$ and such that $s^{-1} \cdot 0$ is not in a face of F_β. One easily sees that the reductions of $s \cdot x$ and $s \cdot y$ coincide. Hence the hyperplane $(\beta, y) = c_\beta$, defined by F_β gives also a face of $I_A(y_i)$. From this ii) follows.

For all $s \in S$ such that $\chi(s) = 1$ for all $\chi \in V_i(x, A)$ one has $sh_i s^{-1} = h_i$. Hence $s \cdot y_i = s \cdot h_i P = sh_i s^{-1} P \in Y_{0,A}^{ss}$. From this iii) follows.

§7. Description of $I(x)$ for split groups I

7.1. Introduction. Let G, X and λ be fixed. We also assume that G is split and has a root system of type $A_\ell, \ldots, D_\ell$ or E_6. In this section we will give a partial description of $I(x)$.

The proofs will be mainly group theoretical. Instead of using our line bundle $\mathcal{L}$ we will write x as $x = h \cdot P \in X(\mathbb{K}) = G(\mathbb{K})/P(\mathbb{K})$. The main ingredient of the proofs will be the well known commutator relations for elements $u_\alpha(a) \in U_\alpha$ and $u_\beta(b) \in U_\beta$ (See 7.2 below). These relations will enable us to relate $I_{hA}(x)$, for $h \in U_\alpha(K) \cap P(\mathbb{I}_A^i(x))$ with $\alpha \perp V_{i-1}(x, A)$ (if $i > 1$) to $I_A(x)$.

7.2. Commutator relations. Let α and $\beta \neq \pm\alpha$ be roots. The groups U_α and U_β are subgroups of G that are isomorphic to K°. Let $u_\alpha(a) \in U_\alpha$ and $u_\beta(b) \in U_\beta$. Then the commutator of these two elements is given by $[u_\alpha(a), u_\beta(b)] =$

$\prod_{n,m>0} u_{n\alpha+m\beta}(d_{n,m}a^n b^m)$, where $n\alpha + m\beta \in \Phi$ and $d_{n,m} \in K^\circ$ (See [Hu] §32 and §33 or [S] proposition 10.1.4). Though the constant $d_{n,m}$ does depend on the roots α and β, we will not make this explicit in the notation. It will always be clear fom the context which roots are involved.

The isomorphisms between the groups U_α and U_β and K° can be fixed in such a way that all the constants $d_{n,m}$ are integers (See [S] proposition 11.2.7 or [Hu] §33). One can choose the isomorphisms such that the only integers occurring are $0, \pm 1, \pm 2$ and ± 3 (See [Hu] §33). Therefore in $Char(K) = 2$ and 3 some exceptional commutations occur (See [BoT] 4.2 and 4.3).

7.3. Definition. Let $V \subset \mathcal{X}(S) \otimes \mathbb{R} = \mathbb{R}^\ell$ be a hyperplane spanned by roots. Then we write $\alpha \perp\!\!\!\perp V$ and say that α is *extremely orthogonal* to V if and only if the additive group $U_\alpha \subset G$ commutes with all the groups U_β, where the roots β are contained in V. Note that this in some cases also depends on the characteristic of the field K.

In the proposition below we describe in detail when $\alpha \perp\!\!\!\perp V$ occurs. We omit the proof, since it consists of applying the commutator relations. For the exceptional commutations occurring when $Char(K) = 2$ we refer to [BoT] sections 4.2 and 4.3(i) and the references given there.

7.4. Proposition. *Let $V \subset \mathcal{X}(S) \otimes \mathbb{R}$ be a linear subspace spanned by roots. Then $\alpha \perp\!\!\!\perp V$ if and only if $\alpha \perp V$ and moreover one of the following statements holds:*

(1) *All roots in Φ have the same length.*

(2) *$\Phi = B_\ell, C_\ell$ or F_4 and $Char(K) = 2$.*

(3) *$\Phi = B_\ell, C_\ell$ or F_4 and α is a long root.*

(4) *$\Phi = B_\ell$ or F_4, α is a short root and V contains no short roots.*

(5) *$\Phi = C_\ell$, α is a short root and V does not contain a short root γ (unique upto sign) such that $\alpha \pm \gamma$ is a long root.*

(6) *$\Phi = G_2$.*

7.5. Proposition. *Let $x \in Y_A^{ss}$ be such that $m(x, A) > 1$ and let $1 \leq i < m(x, A)$. Let $\alpha \perp\!\!\!\perp V_i(x, A)$. Let $z \in I_A^{i+1}(x)$ and $h \in U_\alpha(K) \cap P_z$. Then one has:*

i) *If $i > 1$ then $V_j(h^{-1} \cdot x, A) = V_j(x, A)$ and $R(I^j_{hA}(x)) = R(I^j_A(x))$ for*

$j = 1, \dots, i-1$.

ii) $I^i_A(x) \cap h \cdot A \subset I^i_{hA}(x)$. *In particular* $z \in I^i_{hA}(x)$.

iii) $h(\Delta(V_i(x,A), R(I^i_A(x)), z)) \subset I^i_{hA}(x)$. *In particular* $R(I^i_{hA}(x)) \geq R(I^i_A(x))$.

Proof. Since $\alpha \perp\perp V_i(x,A)$, we have $\alpha \perp V_i(x,A)$. In particular the element $h \in U_\alpha(K) \cap P_z$ does not act on the coordinates $x_{w(\lambda)}$, $w \in W$, with $w(\lambda) \in V_1(x,A)$. From this and the fact that $x \in Y^{ss}_{z,A}$, one concludes that $x \in Y^{ss}_{z,hA}$. Since $I_{hA}(x) = h(I_A(h^{-1}(x)))$, we only need to study $I_A(h^{-1}(x))$ to prove the lemma. We can now translate by an element $s \in S(\mathbb{K})$ such that $s \cdot z = 0$. Then we have to replace h by shs^{-1}, which is not an element of $U_\alpha(K)$ anymore, but is contained in $U_\alpha(\mathbb{K})$. In fact $shs^{-1} \in U_\alpha(\mathbb{K}^\circ)$, since $h \in U_\alpha(K) \cap P_z$.

So let us assume that $z = 0$. Let $h^{-1} = u_\alpha(a) \in U_\alpha(\mathbb{K}^\circ)$. Then $x \in Y^{ss}_{0,A}$ and $0 \in I^{i+1}_A(x)$. We may assume that x is normalised and has $|x_\lambda| = 1$. Then $\lambda \in V_1(x,A)$. Since $\alpha \perp \lambda$, we have $h^{-1} = u_\alpha(a) \in P$. So $u_\alpha(a) \cdot x = u_\alpha(a)\Pi_\beta u_\beta(b_\beta) \cdot P = \Pi_\beta(u_\alpha(a)u_\beta(b_\beta)u_\alpha(a)^{-1}) \cdot P$. To prove the proposition we look at the expressions $f_\beta := u_\alpha(a)u_\beta(b_\beta)u_\alpha(a)^{-1}$. It follows from the commutator relations for U_α and U_β (see 7.2) that $f_\beta = \Pi_{n,m>0} u_{n\alpha+m\beta}(d_{n,m}a^n b^m_\beta) \cdot u_\beta(b_\beta)$, where $n\alpha + m\beta \in \Phi$ and $d_{n,m} \in K^\circ$.

Since $0 \in I^{i+1}_A(x)$, one has $v(b_\beta) \geq c^{i+1}_\beta(x,A)$ if $\beta \notin V_i(x,A)$. If $\beta \in V_j(x,A), j \leq i$ then $v(b_\beta) \geq c^j_\beta(x,A)$. We will show that similar statements hold for the $u_{n\alpha+m\beta}(d_{n,m}a^n b^m_\beta)$ occurring in the f_β. From this the proposition will follow.

If $\beta \in V_i(x,A)$ then $f_\beta = u_\beta(b_\beta)$ and there is nothing to prove. So let us assume $\beta \notin V_i(x,A)$. We only have to consider β such that U_α and U_β do not commute.

First we consider the case where all the roots in Φ have the same length. If the root β is in $V_i(x,A)^c$ then the group U_α commutes with U_β. Therefore it is enough to consider only roots β that are not in $V_i(x,A)^c$ in this case.

Let γ be one of the roots $n\alpha + m\beta$ with $n, m > 0$. Only $n = m = 1$ occurs. Since $(\beta,\lambda) < 0$ and $(\alpha,\lambda) = 0$, one has $(\gamma,\lambda) < 0$. Since $v(a) \geq 0$ and $v(b_\beta) \geq c^{i+1}_\beta(x,A)$, it follows that $v(d_{1,1}ab_\beta) \geq v(b_\beta) \geq c^{i+1}_\beta(x,A)$. Since $\beta \notin V_i(x,A)^c$, one has by lemma 6.5 that $c^{i+1}_\beta(x,A) = d_{i+1}(x,A)$. Since $\alpha \not\perp \gamma$, it is clear that $\gamma \notin V_i(x,A)^c$. Therefore $c^{i+1}_\gamma(x,A) = d_{i+1}(x,A)$ by lemma 6.5. Hence $v(d_{1,1}ab_\beta) \geq c^{i+1}_\gamma(x,A)$.

Next we consider the case where $\Phi = C_\ell$, $\ell > 2$. We will assume that

$Char(K) \neq 2$. If $Char(K) = 2$, then the only difference is that some more roots commute. So the proof in that case follows easily from the case where $Char(K) \neq 2$.

Let us first consider the case where the root α is short. If the root β is also short and does not commute with α, then either $\gamma = \alpha + \beta$ is a short root or $\gamma = \alpha + \beta$ is a long root. Let us first assume that $\gamma = \alpha + \beta$ is a short root. From $\gamma \not\perp \alpha$ it follows that $\gamma \notin V_i(x, A)^c$. Therefore $c_\gamma^{i+1}(x, A) = c_\beta^{i+1}(x, A) = d_{i+1}(x, A)$. Consequently, we have $v(d_{1,1} a b_\beta) \geq c_\gamma^{i+1}(x, A)$.

Now we will show that $\gamma = \alpha + \beta$ cannot be a long root, if α and β are short roots. Let us assume that it does occur. Let Φ consist of the roots $\pm 2e_i, \quad \pm e_i \pm e_j, \quad i \neq j, \quad 1 \leq i, j \leq \ell$. Here the e_i form an orthonormal basis of $\mathbb{R}^\ell$. By assumption our root β is short. So we must have $\beta + \alpha = \gamma$ and $\beta \perp \alpha$. We may choose $\alpha = e_1 + e_2$ and $\beta = e_1 - e_2$. Since $\alpha \perp V_i(x, A)$, none of the roots $\pm 2e_1, \quad \pm 2e_2, \quad \pm e_1 \pm e_j, \quad \pm e_2 \pm e_j, \quad j \neq 1, 2$ is contained in $V_i(x, A)$. From this it follows that either $\beta \perp V_i(x, A)$ or $\beta \in V_i(x, A)$. This contradicts our assumptions on β. Therefore γ cannot be a long root.

Next we consider the case where β is a long root that does not commute with the short root α. Then $\gamma_1 := \beta + \alpha$ is a short root and $\gamma_2 := \beta + 2\alpha$ is a long root. For both these roots one has that $c_{\gamma_n}^{i+1}(x, A) \leq c_\beta^{i+1}(x, A)$, $n = 1, 2$. Therefore $v(d_{n,1} a^n b_\beta) \geq c_{\gamma_n}^{i+1}(x, A)$ for $n = 1, 2$.

Let us now assume that the root α is long. If the root β does not commute with α, then the root β is short. Then $\gamma_1 := \alpha + \beta$ is a short root and $\gamma_2 := \alpha + 2\beta$ is a long root. For γ_1 one has again that $c_{\gamma_1}^{i+1}(x, A) \leq c_\beta^{i+1}(x, A)$. For γ_2 one has $c_{\gamma_2}^{i+1}(x, A) \leq 2 \cdot c_\beta^{i+1}(x, A)$. Hence also in this case we have $v(d_{1,n} a b_\beta^n) \geq c_{\gamma_n}^{i+1}(x, A)$ for $n = 1, 2$.

We will omit the proof for the case where $\Phi = B_\ell$. In this case one again proves, by looking at each possible root γ occurring in the expressions f_β, that $v(d_{n,m} a^n b_\beta^m) \geq c_\gamma^{i+1}(x, A)$ holds.

For all the roots $\gamma = n \cdot \alpha + m \cdot \beta$ occurring, we have now shown that $v(d_{n,m} a^n b_\beta^m) \geq c_\gamma^{i+1}(x, A)$ holds. For $\gamma \in V_i(x, A)$ we have in fact shown that $v(d_{n,m} a^n b_\beta^m) > c_\gamma^{i+1}(x, A)$. Let us rearrange the roots and write $h^{-1}(x) = \Pi_\beta u_\beta(b'_\beta) \cdot P$. For all roots β with $v(b_\beta) = c_\beta^i(x, A)$ one also has that $v(b'_\beta) = c_\beta^i(x, A)$. Moreover, for the roots

$\beta \in V_i(x,A)$ such that $v(b_\beta) > c^i_\beta(x,A)$ one has that $v(b'_\beta) > c^i_\beta(x,A)$. Furthermore for the roots $\alpha \notin V_i(x,A)$ one has that $v(b'_\beta) \geq c^{i+1}_\beta(x,A)$. From this one concludes that if $i > 1$ then $V_j(h^{-1}(x),A) = V_j(x,A)$ and $R(I^j_A(h^{-1}(x))) = R(I^j_A(x))$ for $j = 1,\dots,i-1$. This proves (i).

Let $y \in I^i_A(x) \cap h \cdot A$. Let $s \in S(\mathbb{K})$ be such that $s \cdot y = 0$. Then $s \cdot x = \Pi_\beta u_\beta(b_\beta \cdot \beta(s)) \cdot P$. Moreover $s \cdot h^{-1} \cdot x = \Pi_\beta(b'_\beta \cdot \beta(s)) \cdot P$. Since $y \in I^i_A(x)$, we have that $v(b_\beta \cdot \beta(s)) \geq c^i_\beta(x,A)$. The same arguments as above used to show that $v(b'_\beta) \geq c^{i+1}_\beta(x,A)$ give us that $v(b'_\beta \cdot \beta(s)) \geq c^i_\beta(x,A)$. From this one concludes that $y \in I^i_{hA}(x)$. This proves (ii).

From (ii) it follows that $z \in I^i_A(h^{-1}(x))$. For all $\beta \notin V_i(x,A)$ one has that $v(b'_\beta) \geq c^{i+1}_\beta(x,A)$. From this one concludes that $\Delta(V_i(x,A), R(I^i_A(x),0) \subset I^i_A(h^{-1}(x))$. This proves (iii).

7.6. Proposition. *Let $x \in Y^{ss}_A$ and $m(x,A) > 1$. Let $1 \leq i < m(x,A)$. Let $\alpha \perp\perp V_i(x,A)$ and $h \in U_\alpha(K) \cap P(\mathbb{I}^{i+1}_A(x))$. Then $h(\mathbb{I}_A(x)) \subset I_{hA}(x)$.*

Proof. This follows from the previous proposition.

7.7. Definition. Let $x \in Y^{ss}_A$ and $1 \leq i < m(x,A)$. Then we write $\alpha \perp\perp_x V_i(x,A)$ if and only if $\alpha \perp V_i(x,A)$ and for all $h \in U_\alpha(\mathbb{K}) \cap P(\mathbb{I}^{i+1}_A(x))_\mathbb{K}$ one has that $\mathbb{I}_A(x) \subset I_A(h^{-1} \cdot x)$. Here $P(\mathbb{I}^{i+1}_A(x))_\mathbb{K}$ denotes the stabiliser of $\mathbb{I}^{i+1}_A(x)$ in $G(\mathbb{K})$.

Note that $\alpha \perp\perp V_i(x,A)$ implies that $\alpha \perp\perp_x V_i(x,A)$. Below we determine for which short roots $\alpha \in \Phi$ one has $\alpha \perp\perp_x V_i(x,A)$.

7.8. Proposition. *Let $\Phi = B_\ell$, $\ell \geq 2$, $Char(K) \neq 2$ and $x \in Y^{ss}_{0,A}$ be such that $m(x,A) > 1$. Let $1 \leq i < m(x,A)$. We assume that x is almost in standard position and that $0 \in I^{i+1}_A(x)$. Let $x = \Pi u_\beta(b_\beta) \cdot P$. Let $\alpha \perp V_i(x,A)$ be a short root.*

Then $\alpha \perp\perp_x V_i(x,A)$ if and only if for all short roots $\beta \in V_i(x,A)$ one has that $v(2 \cdot b_\beta) \geq \sum_{j \leq i} R(I^j_A(x))$.

Proof. From the fact that $\alpha \perp V_1(x,A)$ and $x \in Y^{ss}_{0,A}$, it follows that $h^{-1} \cdot x \in Y^{ss}_{0,A}$. Let $h^{-1} = u_\alpha(a) \in U_\alpha(\mathbb{K}^\circ)$. To prove the proposition we again look at the expressions f_β in $u_\alpha(a) \cdot x = u_\alpha(a) \prod_\beta u_\beta(b_\beta) \cdot P$. For all the long roots $\beta \in V_i(x,A)$

we have $f_\beta = u_\beta(b_\beta)$. For all the $f_\beta, \beta \notin V_i(x,A)$ one has again that in the $u_{n\alpha+m\beta}(d_{n,m}a^n b_\beta^m)$ occurring in f_β the inequalities $v(d_{n,m}a^n b_\beta^m) \geq c^{i+1}_{n\alpha+m\beta}(x,A)$ hold.

So we only have to look at the f_β with β short, $(\beta,\lambda) < 0$ and $\beta \in V_i(x,A)$. One has $f_\beta = u_{\alpha+\beta}(d_{1,1}ab_\beta)u_\beta(b_\beta)$. If $i > 1$ then one concludes from the fact that $\alpha \perp\!\!\perp_x V_{i-1}(x,A)$ that $v(d_{1,1}ab_\beta) \geq c^i_{\beta+\alpha}(x,A)$. Moreover this is true for all the short roots β contained in $V_i(x,A)$. We only have to consider a short root β such that the value $v(b_\beta)$ is minimal. Furthermore we may choose h in such a way that the value $v(a)$ is minimal, that is $v(a) = 0$.

If $v(d_{1,1}ab_\beta) \geq c^{i+1}_{\beta+\alpha}(x,A)$ then statements i), ii) and iii) of proposition 7.5 hold. From this one concludes as in proposition 7.6 that one has $\alpha \perp\!\!\perp_x V_i(x,A)$. If $v(d_{1,1}ab_\beta) < c^{i+1}_{\beta+\alpha}(x,A)$ then $\Delta(V_i(x,A), R(I^i_A(x)), z)$ is not contained in $I^i_A(h^{-1} \cdot x)$. Hence one cannot have $\alpha \perp\!\!\perp_x V_i(x,A)$. (Note that $v(d_{1,1}) = v(2)$).

7.9. Remark. Next we turn to the root system C_ℓ. If $\alpha \in \Phi$ is a short root there exist a short root γ, unique up to sign, such that $\alpha+\gamma$ and $\alpha-\gamma$ are long roots. Moreover $\alpha \perp \gamma$.

Let V be a linear subspace of $\mathcal{X}(S)\otimes\mathbb{R} \simeq \mathbb{R}^\ell$ spanned by roots. Let $\alpha \perp V$. If $\gamma \in V$ then $V = V' \oplus <\gamma>$ with $\gamma \perp V'$. Now suppose $V = V_i(x,A)$ for some $x \in Y^{ss}_A$. If $\gamma \in V_i(x,A)$ then it follows from proposition 5.4 that $\gamma \in V_1(x,A)$.

7.10. Proposition. *Let $\Phi = C_\ell, \ell \geq 3$ and $Char(K) \neq 2$. Let $x \in Y^{ss}_A$ be such that $m(x,A) > 1$. Let $1 \leq i < m(x,A)$. Let $\alpha \perp V_i(x,A)$ be a short root and $\gamma \in \Phi$ be a short root (unique upto sign) such that $\alpha+\gamma$ is a long root. We assume that $\gamma \in V_i(x,A)$. Let $z \in I^{i+1}_A(x)$ and $h \in U_\alpha(K)\cap P_z$. Then the following holds:*

i) *If K is not a finite extension of $\mathbb{Q}_2$ then $I_{hA}(x) = I_A(x)\cap hA$. In particular one never has $\alpha \perp\!\!\perp_x V_1(x,A)$.*

ii) *If $K \supset \mathbb{Q}_2$ then $\alpha \perp\!\!\perp_x V_i(x,A)$ if and only if $v(2) \geq \frac{1}{2}\sum_{j=1}^{i} R(I^j_A(x))$.*

Proof. From $\alpha \perp V_1(x,A)$ and $x \in Y^{ss}_{z,A}$, it follows that $x \in Y^{ss}_{z,hA}$. After translation by an element $s \in S(\mathbb{K})$, we may again assume that $z = 0$. We may moreover assume that after normalisation one has $|x_\lambda| = 1$. Now one considers $u_\alpha(a)\cdot x = u_\alpha(a)\cdot\Pi_\beta u_\beta(b_\beta)\cdot P$, $\quad h^{-1} = u_\alpha(a)$ and β such that $(\beta,\lambda) < 0$. Again one looks at the expressions f_β. The only such expression different from the ones occurring in the proof of proposition 7.5 is f_γ. So we will only consider this expression.

Now let us assume that $\gamma \in V_i(x,A)$. After replacing γ by $-\gamma$, if necessary, we may assume that $(\gamma,\lambda) < 0$. By remark 7.9 one has that $\gamma \in V_1(x,A)$ and $V_1(x,A) = V' \oplus < \gamma >$ with $\gamma \perp V'$. Then one has $f_\gamma = u_{\alpha+\gamma}(d_{1,1}ab_\gamma)u_\gamma(b_\gamma)$ with $v(b_\gamma) = 0$. From [BoT] 4.2 it follows that $d_{1,1}$ is a unit if K is not a finite extension of $\mathbb{Q}_2$ and that $v(d_{1,1}) = v(2) > 0$ if $K \supset \mathbb{Q}_2$.

Hence if K does not contain $\mathbb{Q}_2$ then $v(d_{1,1}ab_\gamma) = 0$ and $I_{hA}(x) = I_A(x) \cap hA$. This proves i). If $K \supset \mathbb{Q}_2$ then $v(d_{1,1}ab_\gamma) = v(2)$. Moreover $c^{i+1}_{\beta+\gamma}(x,A) = 2\sum_{j=1}^{i} R(I^j_A(x))$ and $c^i_{\beta+\gamma}(x,A) = 2\sum_{j=1}^{i-1} R(I^j_A(x))$. From this ii) follows in a way similar to the proof of proposition 7.8.

7.11. Proposition. *Let $x \in Y_A^{ss}$ be such that $m(x,A) > 1$. Let $1 \le i < m(x,A)$. Let $\alpha \perp V_i(x,A)$ be a short root. We assume that not $\alpha \perp\perp_x V_i(x,A)$. If $i > 1$, then we assume that $\alpha \perp\perp_x V_{i-1}(x,A)$. Let $z \in I_A^{i+1}(x)$.*

Then there exists an element $s \in S(\mathbb{K})$ such that for all $h \in U_\alpha \cap P_z$, the translate $H_h := s\cdot(A \cap hA)$ of $A \cap hA$ with $A \cap hA \subset H_h$ is such that $h(\Delta(V_i(x,A), R(I^i_A(x)), y) \cap H_h) = h(\Delta(V_i(x,A), R(I^i_A(x)), y)) \cap I^i_{hA}(x)$. Here s only depends on x and α.

Proof. The root system Φ is of type B_ℓ or of type C_ℓ. The case with $\Phi = C_\ell$ and the characteristic of the residue field of K unequal to two has been treated in proposition 7.10 (i). Note that in that case one always can take $s = 1$. So we only have to prove the proposition for the case $\Phi = C_\ell$, $K \supset \mathbb{Q}_2$ and the case $\Phi = B_\ell$. The proofs in both cases are rather similar. Therefore we will only treat the case where $\Phi = B_\ell$ (and $Char(K) \neq 2$).

We may assume that $z = 0$ and that $x = \prod_\beta u_\beta(b_\beta) \cdot P$ is in standard position. As in the proof of proposition 7.8, we may assume that $h^{-1} = u_\alpha(a)$ with $v(a) = 0$. Let $h^{-1}(x) = u_\alpha(a) \prod_\beta u_\beta(b_\beta) \cdot P = \prod_\beta u_\beta(b'_\beta) \cdot P$. From the proof of proposition 7.8 it follows that $v(b'_\beta) \ge c^{i+1}_\beta(x,A)$ for all roots β that are not of the form $\beta = \alpha + \gamma$. Here γ is a short root contained in $V_i(x,A)$. Let γ_0 be a short root such that $v(b_{\gamma_0})$ is minimal. Then $c^i_{\alpha+\gamma_0}(x,A) \le v(b'_{\gamma_0}) = v(2ab_{\gamma_0}) < c^{i+1}_{\alpha+\gamma_0}(x,A)$. For the other short roots γ in $V_i(x,A)$ one has $v(b'_{\alpha+\gamma}) \ge v(b'_{\alpha+\gamma_0})$.

Since $\alpha + \gamma_0 \equiv \alpha \ mod \ V_i(x, A)$, it follows from the values of the $v(b'_\beta)$ that $\Delta(V_i(x, A), R(I^i_A(x)), z) \cap s \cdot H_h$ is contained in $I^i_A(h^{-1} \cdot x)$ for a suitably choosen $s \in S(\mathbb{K})$. From the fact that $s \in S(\mathbb{K})$ only depends on the value $v(2 \cdot b_{\gamma_0})$ and not on $v(a)$, one concludes that s only depends on x and α.

7.12. Remark. If $\Phi = B_\ell$ then the statement of the proposition above is more or less independent of the short root $\alpha \perp V_i(x, A)$ chosen. Indeed let γ be another short root with $\gamma \perp V_i(x, A)$. Let $g \in U_\gamma \cap P_z$. Again one may assume that $z = 0$ and that z is on the wall bounding $A \cap gA$. Let $w \in W$ be such that $w(\alpha) = \gamma$. Then if not $\alpha \perp\perp_x V_i(x, A)$ then also not $\gamma \perp\perp_x V_i(x, A)$. The corollary above then holds for γ with s replaced by wsw^{-1}.

7.13 Definition. Let $x \in Y^{ss}_A$. Let $1 \leq i \leq m(x, A)$. Let $\alpha \perp V_1(x, A)$. Then we define $U_{(\alpha,x)} \subset U_\alpha(K)$ as being the subgroup consisting of the elements $h \in U_\alpha \cap P(\mathbb{I}^{m(x,A)}_A(x))$ such that $h(\mathbb{I}_A(x)) \subset I_{hA}(x)$.

Let $F_i(x, A)$ be the group generated by 1 and the groups $U_{(\alpha,x)}$ with $\alpha \in \Phi$ such that $\alpha \perp V_i(x, A)$, not $\alpha \perp\perp_x V_i(x, A)$ and if $i > 1$ then $\alpha \perp\perp_x V_{i-1}(x, A)$. It follows from the definition that the roots α occurring are short roots. Moreover from the results obtained above one deduces that for at most one i one has $F_i(x, A) \neq \{1\}$.

Let $G_i(x, A)$ be the following group:

$$G_1(x, A) := P(\mathbb{I}^1_A(x)).$$

$$G_i(x, A) := P(\mathbb{I}^i_A(x)) \cap < U_\alpha | \alpha \perp\perp_x V_{i-1}(x, A) > \quad \text{if } i > 1 .$$

Here $\mathbb{I}^i_A(x)$ is the normalization of $I^i_A(x)$. We denote by $H_i(x, A)$ the group generated by $F_i(x, A)$ and $G_i(x, A)$. We define the group $H(x, A)$ as the group generated by the groups $H_i(x, A), \quad i = 1, \dots , m(x, A)$.

7.14. Definition. We call a triple (Φ, K, Y^{ss}_A) *disturbed* if there exists a point $x \in Y^{ss}_A$ with $m(x, A) > 1$, such that there exists a root $\alpha \in \Phi$ and $1 \leq i < m(x, A)$ with $\alpha \perp\perp_x V_i(x, A)$ but not $\alpha \perp\perp V_i(x, A)$.

Since this definition does not depend on the choice of the department A, we will also call (Φ, K, Y^{ss}) disturbed if (Φ, K, Y^{ss}_A) is disturbed for some apartment A.

Let us restrict ourselves to the root systems $\Phi = A_\ell, \dots , D_\ell$ and E_6 as usual. Then disturbed triples only exist if (Φ, K) is such that either $\Phi = B_\ell, \ \ell \geq 3$ and

$Char(K) \neq 2$ or $\Phi = C_\ell,\ \ell \geq 2$ and $K \supset \mathbb{Q}_2$. The following proposition is a direct consequence of the definitions and the results obtained sofar.

7.15. Proposition. *Let $x \in Y_A^{ss}$ and $1 \leq i \leq m(x, A)$. Then the following holds:*

i) *For at most one i one has $H_i(x, A) \neq G_i(x, A)$.*

ii) *If (Φ, K, Y_A^{ss}) is not disturbed, then $H_i(x, A) = G_i(x, A)$ for all i.*

7.16 Lemma. *Let $x \in Y_A^{ss}$ with $m(x, A) > 1$. Let $1 \leq n < m \leq m(x, A)$. Then the group $H_m(x, A)$ normalises the group of $H_n(x, A)$.*

Proof. The groups $H_i(x, A)$ are generated by the subgroups $U_\gamma \cap H_i(x, A)$. Let $u_\beta(b) \in H_n(x, A)$ and $u_\alpha(a) \in H_m(x, A)$. To prove the lemma it is sufficient to prove that $f_\beta := u_\alpha(a) u_\beta(b) u_\alpha(a)^{-1} \in H_n(x, A)$.

Let us first assume that $n = 1$ and that $H_1(x, A) = G_1(x, A)$. There exist points $y := \prod_\gamma u_\gamma(a_\gamma) \cdot P$ such that $I_A(y) = \mathbb{I}_A(x)$ and $H_j(y, A) = H_j(x, A)$, $j = 1, \ldots, m(x, A)$. After replacing λ by $w(\lambda)$ (or β by $w(\beta)$), where w is a suitable element in the Weyl group of $\Phi \cap V_1(y, A)$, we may assume that $(\beta, \lambda) < 0$. Let $h := u_\alpha(a)$.

In the proofs of the propositions above (especially 7.5, 7.8 and 7.10), we have in fact shown that $f_\beta = h u_\beta(b_\beta) h^{-1} = \prod_\delta u_\delta(d_\delta) \cdot u_\beta(b_\beta), \quad \delta = i \cdot \alpha + j \cdot \beta, \quad i, j > 0$, where the $u_\delta(d_\delta)$ are such that they stabilise $\mathbb{I}_A(y) = \mathbb{I}_A(x)$. Therefore $f_\beta \in P(\mathbb{I}_A(y)) = H_1(y, A) = H_1(x, A)$. Since this holds for all groups $U_\beta \cap H_1(x, A)$, it follows that the group $H_m(x, A)$ normalises $H_1(x, A)$.

Let us now consider the case where $n = 1$ and $H_1(x, A) \neq G_1(x, A)$. One can still construct points $y \in Y_A^{ss}$ such that $I_A(y) = \mathbb{I}_A(x)$ and $H_j(y, A) = H_j(x, A)$ for $j = 1, \ldots, m(x, A)$. We may again assume that $(\beta, \lambda) < 0$ and that $y = \prod u_\gamma(a_\gamma) \cdot P$. If $u_\beta(b) \in G_1(x, A)$, then the argument used above for the case where $H_1(x, A) = G_1(x, A)$ applies and we are done.

So let us assume that $u_\beta(b) \notin G_1(x, A)$. It is sufficient to prove the lemma for the generators of $F_1(x, A)$. Therefore we may assume that β is a short root, such that $\beta \perp V_1(x, A) = V_1(y, A)$ but not $\beta \perp\perp_y V_1(y, A)$. From proposition 7.15 and definition 7.14 it follows that we only have to consider the case where $\Phi = B_\ell, \ell \geq 3$ and $Char(K) \neq 2$ and the case where $\Phi = C_\ell, \ell \geq 2$ and $K \supset \mathbb{Q}_2$.

Let us first consider the case where $\Phi = B_\ell, \ell \geq 3$ and $Char(K) \neq 2$. Now the root α must be long, since there are no short roots γ such that $\gamma \perp\perp_y V_1(y, A)$. If $(\alpha, \beta) \geq 0$, then $f_\beta = u_\beta(b)$ and we are done. So let us assume that $(\alpha, \beta) = -1$. Then $f_\beta = u_{\alpha+\beta}(d_{1,1}ab) \cdot u_{\alpha+2\beta}(d_{1,2}ab^2) \cdot u_\beta(b)$. Each of the terms occurring in the expression f_β is in $H_1(y, A)$ (in fact in $F_1(x, A)$). So the lemma holds in this case.

Let us now consider the case where $\Phi = C_\ell, \ell \geq 2$ and $K \supset \mathbb{Q}_2$. Again we only have to consider short roots $\beta \perp V_1(y, A)$ such that not $\beta \perp\perp_y V_1(y, A)$. One now easily sees that for all roots $\alpha \perp\perp_y V_1(y, A)$ the groups U_α and U_β commute. Hence $f_\beta = u_\beta(b) \in H_1(y, A) = H_1(x, A)$. So the lemma also holds in this case.

Now we look at the case where $n > 1$. One can now construct a point $y \in Y_A^{ss}$ such that $I_A(y) = \mathbb{I}_A^n(x)$, $H_i(y, A) = H_{i+n-1}(x, A)$ for $i = 2, \dots, m(x, A) - n + 1$ and $H_n(x, A) = H_1(y, A) \cap < U_\alpha \mid \alpha \perp\perp_x V_{n-1}(x, A) >$. Now one can apply the proof used for the case where $n = 1$ by replacing x by y and only considering the roots β such that $\beta \perp\perp V_{n-1}(x, A)$. This concludes the proof of the lemma.

7.17. Theorem. *Let $x \in Y_A^{ss}$. Let $h \in H(x, A)$ be such that $r_{h \cdot A, A}(x) = 1$. Then $h(\mathbb{I}_A(x)) \subset I_{hA}(x)$.*

Proof. Let us first assume that $h \in H_i(x, A)$ for some $i = 1, \dots, m(x, A)$. If $h = u_\alpha(a)$, then it follows from the definition of $H_i(x, A)$. So let us assume that $h = \prod_\alpha u_\alpha(a_\alpha)$. If $i > 1$, then it follows from the proofs that $V_j(u_\alpha(a_\alpha) \cdot x, A) = V_j(x, A)$ and $R(I_A^j(u_\alpha(a_\alpha) \cdot x)) = R(I_A^j(x))$ for $j = 1, \dots, i-1$. Moreover one has that $\mathbb{I}_A(x) \subset I_A(u_\alpha(a_\alpha) \cdot x)$. From this one concludes that $\mathbb{I}_A(x) \subset I_A(h^{-1} \cdot x)$.

Let us now assume that h is not contained in $H_i(x, A)$ for any $i = 1, \dots, m(x, A)$. Then we can write h as $h = h_m \cdots h_1$ with $h_i \in H_i(x, A)$. We again look at $I_A(h^{-1}x)$ instead of $I_{hA}(x)$. For $i = 1, \dots, m := m(x, A)$ one has $V_j(h_i^{-1} \cdots h_m^{-1} \cdot x, A) = V_j(x, A)$ and $R(I_A^j(h_i^{-1} \cdots h_m^{-1} \cdot x)) = R(I_A^j(x))$ for $j = 1, \dots, i$. Moreover, $\mathbb{I}_A^i(x) \subset I_A^i(h_i^{-1} \cdots h_m^{-1} \cdot x)$. Hence $\mathbb{I}_A(x) \subset I_A(h_i^{-1} \cdots h_m^{-1} \cdot x)$ for all $i = 1, \dots, m := m(x, A)$. This proves the lemma.

7.18. Remark. The condition $r_{h \cdot A, A}(x) = 1$ in the theorem above is always satisfied if $A \cap I(x) \neq \emptyset$ or if $h = h_m \cdots h_{j+1} \cdot h_j, \quad h_i \in H_i(x, A)$ with $j > 1$.

7.19 Definition. We define a function $\rho : Y^{ss} \longrightarrow \mathbb{R}^\ell$ that measures the maximal volume of $I_A(x)$ for $x \in Y^{ss}$. Here ℓ is the rank of G. On $\mathbb{R}^\ell$ we put the lexicographical ordering. The apartment A with $\rho_A(x)$ minimal is the apartment where $I_A(x)$ is maximal. We define:

$$\rho(x) := \inf \{\rho_A(x) | A \cap I(x) \neq \emptyset\} .$$

We say that A *determines* $I(x)$ if $A \cap I(x) \neq \emptyset$ and $\rho_A(x) = \rho(x)$.

§8. Description of $I(x)$ for split groups II

8.1. Introduction. We continue our description of $I(x)$ for split groups G with root system Φ of type $A_\ell, \dots , D_\ell$ or E_6.

8.2. Lemma. *Let* $x = \prod_\beta u_\beta(b_\beta) \cdot P \in Y_A^{ss}, (\beta, \lambda) < 0$ *and* $1 \leq i < m(x, A)$. *Let* $\alpha \notin V_i(x, A)$ *be a root such that* $\alpha \not\perp V_i(x, A)$ *(and* $\alpha \perp\perp_x V_{i-1}(x, A)$ *if* $i > 1$*). We assume that* $0 \in I_A^{i+1}(x)$. *Then one of the following statements holds:*

i) $(\lambda, \alpha) < 0$ *(and then* $i = 1$*).*

ii) *There exists a root* $\beta \in V_i(x, A)$, $(\beta, \lambda) < 0$ *such that* $(\beta, \alpha) < 0$ *and* $v(b_\beta) = c^i_\beta(x, A)$ *(and* $\beta \notin V_{i-1}(x, A)$ *if* $i > 1$*).*

Proof. Let $\Psi := \{\beta \in \Phi | \beta \in V_i(x, A), (\beta, \lambda) < 0$ and $v(b_\beta) = c^i_\beta(x, A)\}$. If $i = 1$ then the $\beta \in \Psi$ span $V_1(x, A)$, otherwise the $\beta \in \Psi$ span $V_i(x, A)$ mod $V_{i-1}(x, A)$.

First we consider the case where $(\lambda, \alpha) \neq 0$. We will show that if statement ii) is not true, then statement i) holds. Since $(\lambda, \alpha) \neq 0$ and $\lambda \in V_1(x, A)$ we must have $i = 1$. Let us assume that for all $\beta \in \Psi$ one has $(\beta, \alpha) \geq 0$. Since $x \in Y^{ss}_{0,A}$ one has that 0 is in the cone $\lambda + \sum \mathbb{R}_{\geq 0} \cdot \beta, \beta \in \Psi$ (see [PV] proposition 4.10). Hence there exist $n_\beta \leq 0$ such that $\lambda = \sum n_\beta \cdot \beta$. Therefore $(\lambda, \alpha) < 0$ and i) holds.

Next we consider the case where $(\lambda, \alpha) = 0$. Again let us assume that $(\beta, \alpha) \geq 0$ for all $\beta \in \Psi$. By duality α defines a one parameter subgroup of S. We choose an element s of this $1 - ps$ such that $v(\beta(s)) \geq 0$ for all $\beta \in \Psi$. Since $v(b_\gamma) \geq c^{i+1}_\gamma(x, A) > c^i_\gamma(x, A)$ for all roots $\gamma \notin V_i(x, A), (\gamma, \lambda) < 0$, we can choose s in such a way that $v(\gamma(s)b_\gamma) > c^i_\gamma(x, A)$ for all $\gamma \notin V_i(x, A)$.

Then $s \cdot x \in Y^{ss}_{0,A}$. Clearly $\Psi' := \{\beta \in \Phi | \beta \notin V_{i-1}(x, A), (\beta, \lambda) < 0$ and $v(b_\beta \cdot \beta(s)) = c^i_\beta(x, A)\}$ spans a proper (possibly empty) subset of $V_i(x, A)$ (mod $V_{i-1}(x, A)$ if $i > 1$). From $I_A(s \cdot x) = s \cdot I_A(x)$, it follows that $< \Psi', V_{i-1}(x, A) >$ contains $V_i(s \cdot x, A) = V_i(x, A)$. This cannot be. Hence there must exist a root $\beta \in \Psi$ such that $(\beta, \alpha) < 0$. This proves that statement ii) holds if $(\lambda, \alpha) = 0$.

8.3. Lemma. *Let $x \in Y^{ss}_A$ and $1 \leq i < m(x, A)$. Let α be a root such that $\alpha \notin V_i(x, A)$ and $\alpha \not\perp V_i(x, A)$. If $i > 1$ then we moreover assume that $\alpha \perp\perp_x V_{i-1}(x, A)$. If $i = 1$ then we assume that $A \cap I(x) \neq \emptyset$. Let $z \in I^i_A(x)$ and $h \in U_\alpha \cap P_z$. We assume that $h \notin P(I^i_A(x))$. Then $I^i_{hA}(x) = I^i_A(x) \cap hA$. If $i > 1$ then $R(I^j_{hA}(x)) = R(I^j_A(x))$ for $j = 1, \ldots, i - 1$.*

Proof. We may assume that z is on the boundary of $A \cap hA$. Instead of considering $I_{hA}(x)$, we will look at $I_A(h^{-1}(x))$. We may moreover (after possibly translating by an element $s \in S(\mathbb{K})$) assume that $z = 0$. Let $h^{-1} := u_\alpha(a) \in U_\alpha(\mathbb{K}^\circ)$. Since 0 is on the boundary of $A \cap hA$, one has $|a| = 1$. We may assume that x is normalised and has $|x_\lambda| = 1$. Now we again write x as $x = \prod_\beta u_\beta(b_\beta) \cdot P$.

If $i = 1$, then it follows from $A \cap I(x) \neq \emptyset$, that $I_{hA}(x) = I(x) \cap hA \supset I_A(x) \cap hA$. Similarly for $i > 1$, it follows from $\alpha \perp\perp_x V_{i-1}(x, A)$, that $I^{i-1}_{hA}(x) \supset I^{i-1}_A(x) \cap hA$. Moreover, the condition that $R(I^i_A(x)) = R(I^i_{hA}(x))$ ensures us that $z \in I^i_{hA}(x)$. Therefore we have in fact that $I^i_A(x) \cap h \cdot A \subset I^i_{hA}(x)$. Hence we only have to prove that the point $z = 0$ is on the boundary of $I^i_{hA}(x)$.

First we assume that $(\lambda, \alpha) \neq 0$. Then $i = 1$. If $(\lambda, \alpha) < 0$, then one easily sees that our point 0 is on the boundary of $I_A(h^{-1} \cdot x)$, where $h^{-1} \cdot x = u_\alpha(a) \prod_\beta u_\beta(b_\beta) \cdot P$.

Next we consider the case where $(\lambda, \alpha) > 0$. Note that in this case $u_\alpha(a) \in P$. There exists for the roots $\beta \in V_1(x, A)$ with $v(b_\beta) = 0$ positive real numbers n_β such that $\Sigma n_\beta \cdot \beta + \lambda = 0$. Hence some root $\beta \in V_1(x, A)$ with $v(b_\beta) = 0$ satisfies $(\beta, \alpha) < 0$. Again one considers the expressions $f_\beta := u_\alpha(a) u_\beta(b_\beta) u_\alpha(a)^{-1}$. For our root β with $(\beta, \alpha) < 0$, we have that $\gamma := \alpha + \beta$ is a root. Clearly $v(d_{1,1} a b_\beta) = 0$. From this one concludes that our point $z = 0$ is on the boundary of $I_A(h^{-1}x)$.

Next we consider the case where $i = 1$ and $(\lambda, \alpha) = 0$. Again there exists a root $\beta \in V_1(x, A)$ with $v(b_\beta) = 0$, such that $(\beta, \alpha) < 0$. From this one again concludes by considering the expressions f_β that $z = 0$ is on the boundary of $I_A(h^{-1}x)$.

Finally we consider the case where $i > 1$. Since $\alpha \perp V_{i-1}(x, A)$, the roots $\beta \in V_i(x, A)$ with $(\beta, \alpha) \neq 0$ are such that $\beta \notin V_{i-1}(x, A)^c$. Hence $c^i_\beta(x, A) = d_i(x, A)$, unless $\Phi = C_\ell, \quad \ell > 2$ and the root β is long. In that case we have $c^i_\beta(x, A) = 2 \cdot d_i(x, A)$. Now we again consider the expressions $f_\beta := u_\alpha(a)u_\beta(b_\beta)u_\alpha(a)^{-1} = \prod_{n\alpha+m\beta} u_{n\alpha+m\beta}(d_{n,m}a^n b^m_\beta) \cdot u_\beta(b_\beta)$. We consider the expression f_β for a root $\beta \in V_i(x, A)$, such that $v(b_\beta) = c^i_\beta(x, A)$.

Let us first consider the case where $\Phi \neq C_\ell, \quad \ell > 2$. Then $v(b_\beta) = d_i(x, A)$. We look at the root $\gamma := \alpha + \beta$ that occurs in the expression f_β. The root γ has the same length as β. Hence $c^i_\gamma(x, A) = d_i(x, A)$. Clearly $v(d_{1,1}ab_\beta) = c^i_\gamma(x, A)$. From this we conclude that our point $z = 0$ is on the boundary of $I^i_A(h^{-1}x)$.

Let us now consider the case where $\Phi = C_\ell, \quad \ell > 2$. We now choose our root γ occurring in the expression f_β as follows. If both α and β are short then we take $\gamma := \alpha + \beta$, if α is long and β is short then we take $\gamma := \alpha + 2 \cdot \beta$ and if α is short and β is long then we take $\gamma := \alpha + \beta$. One verifies that in all these cases $v(d_{n,m}a^n b^m_\beta) = m \cdot v(b_\beta) = m \cdot c^i_\beta(x, A) = c^i_\gamma(x, A)$. From this it follows that also in this case our point $z = 0$ is on the boundary of $I^i_A(h^{-1}x)$.

8.4. Definition. Let $x \in Y^{ss}_A$ be such that $m(x, A) > 1$. Let $1 \leq i < m(x, A)$.

Let $\alpha \in \Phi$ be a root such that $\alpha \notin V_i(x, A)$ and $\alpha \not\perp V_i(x, A)$. Let $\Phi^i_\alpha(x, A) := \Phi \cap < V_i(x, A), \alpha >$. We define the subset $\Psi^i_\alpha(x, A)$ as being the set $\Psi^i_\alpha(x, A) := \{\beta \in \Phi | \; \exists (n > 0) \quad \beta = n \cdot \alpha \mod V_i(x, A)\}$. Then $\Psi^i_\alpha(x, A) \subset \Phi^i_\alpha(x, A)$. By $H^i_\alpha(x, A)$ we will denote the group $H^i_\alpha(x, A) :=< U_\beta | \beta \in \Psi^i_\alpha(x, A) >$. Before describing $I_{hA}(x), h \in H^i_\alpha(x, A)$ we need some lemmas.

8.5. Lemma. *Let the notation be as in definition 8.4 above.*

i) *Then* $H^i_\alpha(x, A) = \prod_\beta U_\beta$ *for any fixed order of the roots* $\beta \in \Psi^i_\alpha(x, A)$.

ii) *If* $\Psi^i_\alpha(x, A)$ *contains a root* $\gamma \perp V_i(x, A)$, *then* γ *is unique and the group* U_γ *commutes with all the groups* U_β *with* $\beta \in \Psi^i_\alpha(x, A)$.

Proof. We first prove i). Let $H_{i,\alpha} :=< U_\beta | \; \beta \in \Phi^i_\alpha(x, A) >$. Let $\mu \in < V_i(x, A), \alpha >$ be a weight such that $\mu \perp V_i(x, A)$ and $(\mu, \alpha) > 0$. Let $P_{i,\alpha}$ be the maximal parabolic subgroup of $H_{i,\alpha}$ corresponding to the weight μ. Then $P_{i,\alpha}$ is generated by the groups U_β with $(\beta, \mu) \geq 0$ and the connected component of the intersection $S \cap H_{i,\alpha}$ that contains the identity.

From the definition it follows that $H^i_\alpha(x, A)$ is the unipotent radical of $P_{i,\alpha}$. In particular it is clear that i) holds.

Let us assume that the set $\Psi^i_\alpha(x, A)$ contains a root $\gamma \perp V_i(x, A)$. It is clear that γ must be unique. One verifies that $(\beta, \gamma) > 0$ for all roots $\beta \in \Psi^i_\alpha(x, A)$. From this it follows that the group U_γ commutes with the group U_β for any $\beta \in \Psi^i_\alpha(x, A)$.

8.6. Remark. Let us fix a simple basis of the root system $\Phi^i_\alpha(x, A)$ such that the linear space generated by the roots in $\Phi^i_\alpha(x, A) \cap V_i(x, A)$ is spanned by simple roots. Let us moreover assume that the root α is such that there exists a root γ in $\Phi^i_\alpha(x, A)$ that is orthogonal to $V_i(x, A)$. Then γ is a fundamental weight for our simple basis. In particular γ is the highest long or the highest short root in the irreducible component of $\Phi^i_\alpha(x, A)$ that contains α. Moreover this irreducible component cannot be of type A_j, $j \geq 1$.

8.7. Proposition. *Let $x \in Y^{ss}_A$ be such that $m(x, A) > 1$. Let $1 \leq i < m(x, A)$. Let A be such that $\rho_A(x) = \rho(x)$. Let $z \in I^i_A(x)$. Let $h \in H^i_\alpha(x, A) \cap P_z$ be such that $h \notin P(I^i_A(x))$ and such that $I^i_{hA}(x) \neq I^i_A(x) \cap hA$.*

Then there exists a point $y \in Y^{ss}$ with the following properties:

i) $\rho_A(y) = \rho(y)$.

ii) $V_j(y, A) = V_j(x, A)$ *for* $j = 1, \dots, i$.

iii) *If* $i > 1$ *then* $R(I^j_A(y)) = R(I^j_A(x))$ *for* $j = 1, \dots, i-1$.

iv) $I^i_A(y) = \{s \cdot z \mid s \in S(\mathbb{K}),\ \chi(s) = 1,\ \forall \chi \in V_i(x, A)\}$.

v) $h(y) = y$.

Proof. Note that it follows from lemma 8.3 above that one always has that $I^i_A(x) \cap h \cdot A \subset I^i_A(x)$. We consider the point y_i associated to x and A as defined in 6.8 above.

By construction the point y_i has properties ii), iii) and iv). Moreover it is clear that in our case the point y_i also satifies property i).

Let us first consider the case where $i = 1$. Without loss of generality we may assume that z is on the boundary of $A \cap hA$. We look at the reduction of y_i at z

(If $z = 0$ then the reduction $\overline{y_i} \in X \otimes \overline{\mathbb{K}^\circ}$, where $\overline{\mathbb{K}^\circ}$ is the residue field. Otherwise one defines this by translating z to 0 by an element of $S(\mathbb{K})$.). Let Z be the variety consisting of the points $x \in X$ with $x_\lambda = 0$ for all $\lambda \notin V_1(x, A)$. Let Z° be the connected component of Z that contains y_i. Let Z_h° be the variety consisting of the points in Z° that are fixed by h. Then Z_h° is a proper variety defined over $\mathbb{K}^\circ$.

By construction $y_i = g(x)$ for some $g \in P(I_A(x))_{\mathbb{K}}$. From this one concludes that y_i also has the property that $I^i_{hA}(y_i) \neq I^i_A(y_i) \cap hA$. Hence the reduction of y_i is in the reduction of Z_h°. Therefore there exists a point $y \in Z_h^\circ$ that has the same reduction as y_i. One verifies that this point y has all the required properties.

If $i > 1$, then the construction of the point y is basically the same. The only difference is that one has to define the variety Z_h° in a different way over $\mathbb{K}^\circ$ (without changing $Z_h^\circ \otimes \mathbb{K}$). The change is such that one can lift the reduction of y_i to a point y with the required properties. We omit the details.

8.8. Proposition. *Let x, y, A and h be as in proposition 8.7 above. Then the unipotent element h is contained in a semisimple group H defined over K that has K-rank one and stabilises y.*

Proof. To proof the proposition we consider the stabiliser of the point y in the group $H_{i,\alpha} :=< U_\beta |\ \beta \in \Phi^i_\alpha(x, A) >$. Since the coordinates y_λ of y are zero for all $\lambda \notin V_i(x, A)$, we may replace X by the subvariety Z on which $x_\lambda = 0$ for all $\lambda \notin < V_i(x, A), \alpha >$. We moreover only consider the connected component Z° that contains y. Then $Z^\circ = H_{i,\alpha}/H_{i,\alpha} \cap P$ is a projective homogeneous variety.

We fix a simple basis of $\Phi^i_\alpha(x, A)$ such that $V_i(x, A)$ is spanned by simple roots. We may moreover assume that α is the simple root α_j that is not contained in $V_i(x, A)$. Then y is stabilised by the one dimensional torus S_j that is the connected component of $< s \in S | \chi(s) = 1,\ \forall \chi \in V_i(x, A) >$ that contains the identity. By assumption y is also stabilised by the unipotent element h. Let F be the group generated by h and $S_j(K)$. Then F is a subgroup of $H_{i,\alpha}(K)$.

We now consider $I(y)$ inside the building of $H_{i,\alpha}(K)$. Inside the buiding of $H_{i,\alpha}(K)$ one has that $I^i_A(y)$ is the line $S_j(\mathbb{K}) \cdot z$ for some suitable point $z \in I^i_A(y)$. Since F stabilises the point y, it follows that $I^i_{fA}(y) = f(I^i_A(y))$ for all elements $f \in F$. In particular the tree $F(I^i_A(y))$ is a subset of $I(y)$, since $\rho_A(y) = \rho(y)$.

Let us fix a Borel subgroup B of $H_{i,\alpha}$ that stabilises a sector that has a halfline in common with $I^i_A(x)$. We can choose B in such a way that $H^i_\alpha(x,A)$ is contained in B. The opposite Borel group B^- also stabilises a sector that has a halfline in common with $I^i_A(y)$. Moreover B^- contains the group $H^i_{-\alpha}(x,A)$.

Let $f \in B \cap F$ be such that $I^i_{fA}(y) \cap I^i_A(y)$ is a halfline. We can find a unipotent element $g \in B^-$ such that gA and fA have a sector in common. In fact we can choose g in such a way that $f(I^i_A(y)) \cap gA$ contains a halfline (which is the complement in $I^i_{fA}(y)$ of $I^i_{fA}(y) \cap I^i_A(y)$).

Since $g \in B^-$ we can write g uniquely as $g = g_1 \cdot g_2$ with $g_1 \in H^i_{-\alpha}(x,A)$ and $g_2 = \prod u_\gamma(b_\gamma)$ with $\gamma \in \Phi^i_\alpha(x,A)^- - \Psi^i_{-\alpha}(x,A)$. We will now show that g_1 stabilises y.

Since the roots γ occurring in the expression for g_2 above are all contained in $V_i(x,A)$, it follows that $g_2(I^i_A(y)) = I^i_A(y)$. Hence $I_{g_1 A}(y)$ contains the line $g(I^i_A(y))$. Using the commutator relations we will show that this implies that g_1 stabilises the point y.

Let g_1 be $g_1 = u_{\delta_m}(d_m) \cdots u_{\delta_1}(d_1)$ with $\delta_j \in \Psi^i_{-\alpha}(x,A)$ for $j = 1, \dots, m$. We may assume that y has the form $y = \prod u_\beta(b_\beta) \cdot P$ with $\beta \in \Phi \cap V_i(x,A)$.

We first move $u_{\delta_1}(d_1)$ using the commutator relations. If $(\delta_1, \lambda) < 0$ we do not move $u_{\delta_1}(d_1)$. If $(\delta_1, \lambda) \geq 0$, then we move the term $u_{\delta_1}(d_1)$ to the right until it disappears into P. Now some terms $u_{\beta+\delta_1}$ might have appeared because of the commutator relations. If $(\beta + \delta_1, \lambda) < 0$ then we move this term to the left until we arrive at the term $u_{\delta_2}(d_2)$. If $(\beta + \delta_1, \lambda) \geq 0$, then we move this term to the right until it disappears into P. After a finite number of steps this proces stops. Now we do the same with the term $u_{\delta_2}(d_2)$.

After a finite number of steps we are finished. Now we find that $g_1(y) = \prod u_\eta(c_\eta) \cdot y$, where the roots η are such that $(\eta, \lambda) < 0$ and moreover $\eta \in \Psi^i_{-\alpha}(x,A)$. Since $g(I^i A(y)) \subset I_{g_1 A}(y)$, it follows that for all η one must have $c_\eta = 0$. Hence $g_1(y) = y$.

The group F^- generated by g_1 and $S_j(K)$ stabilises y. One observes that the group generated by F and F^- acts transitively on the tree $F(I^i_A(y))$. In fact the union of all the lines $I^i_{A'}(y)$ for apartments A' with $\rho_{A'}(y) = \rho(y)$ gives us a tree on which the stabiliser $\widetilde{H}$ of y acts transitively. Now the groups $< S_j(K),\ H^i_\alpha(x,A) \cap \widetilde{H} >$ and $< S_j(K),\ H^i_{-\alpha}(x,A) \cap \widetilde{H} >$ generate a semisimple group H that acts transitively on this tree. In fact these two groups form a pair of opposite Borel groups inside H. It follows from the construction that this group has K-rank one.

8.9. Remark. It follows from the fact that all elements of the group H stabilise the point y, that for any element g of $H \cap H^i_\alpha(x,A) \cap P_z$ one has: $g \notin P(I^i_A(x)) \Longrightarrow I^i_{gA}(x) \neq I^i_A(x) \cap gA$. Moreover the same statement is also true for elements $g \in H \cap H^i_{-\alpha}(x,A) \cap P_z$.

8.10. Remark. We can now apply the classification of conjugacy classes of unipotent elements in semisimple groups as described in [C] Ch. 5 and §13.1 to our situation. In particular we can use the theory of weighted Dynkin diagrams. Even though the group H in the proposition above might not be $PSL_2(K)$ or $SL_2(K)$, since it might be a non split form of some other group. However it always contains subgroups $SL_2(K)$ or $PSL_2(K)$. Therefore we may assume, after replacing h by another unipotent element of H if necessary, that h is contained in a group $SL_2(K)$.

Let Ψ be the irreducible component of the root system $\Phi^i_\alpha(x,A)$ that contains the root α. Let us fix a simple basis of Ψ such that $\Psi \cap V_i(x,A)$ is spanned by simple roots. Then we may moreover assume that α is the simple root α_j that is not contained in $V_i(x,A)$. It now follows from the construction of H in the proof of proposition 8.8 above, that the maximal $K-$split torus of H (and hence of our group $SL_2(K)$ containing h) is the 1-ps of S that corresponds to a multiple of the fundamental weight ω_j. Hence the weighted Dynkin diagram belonging to our unipotent element h is the Dynkin diagram of Ψ with all weights equal to zero except one. The unique non-zero weight corresponds to the vertex that represents the simple root α_j.

We now give some examples of the group H that stabilises a point y as constructed in proposition 8.8.

8.11. Example. Let $Char(K) \neq 2$. Let us take a basis $e_1, \dots, e_4,\ f_1, \dots, f_4$. of K^8. On K^8 we define an inner product g by $g(e_i,f_j) = \delta_{ij}$ and $g(f_i,f_j) = g(e_i,e_j) = 0$. Let $G := SO(g)$. Then $\Phi = D_4$. Let X consist of the points $x \in \mathbb{P}^7$ such that $g(x,x) = 0$. Even though the group G is not simply connected our construction still applies in this case. Let us choose coordinates such that a general point x is of the form $x = x_1 \cdot e_1 + \cdots + x_4 \cdot e_4 + y_1 \cdot f_1 + \cdots + y_4 \cdot f_4$.

We take a point y such that $x_4 = y_4 = 0,\ x_2 = y_2 = a,\ x_3 = y_3 = b,\ x_1 = c,\ y_1 = d$. Here $a, b, c, d \in \mathbb{K}$ and all have absolute value equal to one. We moreover assume that there are no $\overline{K}$-rational linear relations between $\overline{a}, \overline{b}, \overline{c}$ and $\overline{d}$. Here $\overline{K}$ denotes the residue field of K and $\overline{a}, \overline{b}, \overline{c}$ and $\overline{d}$ are the images of a, b, c and d in $\overline{K} = K^\circ/\pi K^\circ$. Let $A \subset \mathcal{B}$ be the apartment corresponding to the torus in G that acts diagonally w.r.t. our basis.

We will now determine the stabiliser of y in $G(K)$. The stabiliser of y also stabilises the variety of the K-rational points in the variety $y^\perp \cap X$. We determine these points. From $g(z, y) = 0$ it follows that $dx_1 + cy_1 + a(x_2 + y_2) + b(x_3 + y_3) = 0$. Since we assume z to be a K valued point and there are no K- rational relations between a, b, c and d, it follows that one has that $x_1 = y_1 = 0,\ x_2 + y_2 = 0$ and $x_3 + y_3 = 0$. So $z = z_2(e_2 - f_2) + z_3(e_3 - f_3) + z_4 e_4 + z_5 f_5$. Since $g(z, z) = 0$, the coordinates z_i satisfy the relation $-z_2^2 - z_3^2 + z_5 z_4 = 0$. Therefore the stabiliser H of y is equal to the automorphism group of this quadratic form. The K-rank of H is one if and only if this form has Witt index one over K. In that case it follows that $\rho_A(y) = \rho(y)$. Otherwise $\rho_A(y) > \rho(y)$.

8.12. Example. Let $\Phi = A_\ell$ consist of the roots $e_i - e_j$ with $i \neq j$ and $1 \leq i, j \leq \ell + 1$. Let $\alpha_i := e_i - e_{i+1},\ i = 1, \ldots, \ell$ be a simple basis of Φ. We assume that $\ell = 2s + 1$ for some integer $s \geq 1$. Let λ be the weight $\lambda = \sum_{n=0}^{s} \alpha_{2n+1}$. Let $P \subset G := SL_{\ell+1}$ be the parabolic subgroup determined by λ. Let $X := G/P$. Let $x := \prod_{n=0}^{s} u_{-\alpha_{2n+1}}(x_n) \cdot P \in X$. We assume that $|x_n| = 1$ for $n = 0, \ldots, s$. Moreover we require that $ax_n + b \neq 0\ mod\ \pi$ for all $a, b \in (K^\circ)^*$ for all n. Then $V := V_1(x, A)$ is spanned by the roots $\alpha_{2n+1},\ n = 0, \ldots, s$. The conditions also ensure us that $\rho_A(y) = \rho(y)$ holds.

Let us also assume that $x_n/x_m \in K^*$ for $0 \leq n, m \leq s$. We will now describe the stabiliser of y in $SL_{\ell+1}(K)$. Let $h_{n,m}(a, b)$ for $n \neq m$ and $0 \leq n, m \leq s$ be the element $u_\beta(a) \cdot u_\gamma(b)$, where $\beta = e_{2n+1} - e_{2m+1}$ and $\gamma = e_{2n+2} - e_{2m+2}$. Note that $\gamma \equiv \beta\ mod\ V$. Using the commutator relations one verifies that $h_{n,m}(a, b) \cdot y = y$

if and only if $d_1 a x_n + d_2 b x_m = 0$. Here d_1 (resp. d_2) is the structure constant occurring in the commutator relations for β and $-\alpha_{2n+1}$ (resp. γ and $-\alpha_{2m+1}$). We may assume that they are ± 1.

Let $H_{n,m}$, $n \neq m$ be the group generated by the elements $h_{n,m}(a,b)$ that stabilise y. Our conditions ensure us that $H_{n,m} \neq \{1\}$. In fact $< H_{n,m},\ H_{m,n} >$ is a group isomorphic to $SL_2(K)$. The stabiliser H of y is the group generated by the groups $H_{n,m},\ n \neq m,\ 0 \leq n, m \leq s$. One verifies that $H = SL_{s+1}(K)$. Note that the root system A_s of the group H is the quotient root system $\Phi\ mod\ \Phi \cap V$ in the sense of [Ho] theorem 6 or [C] §10.4.

8.13. Definition. Let (Φ, K, Y^{ss}) not be disturbed. Let $x \in Y^s$. We say that $I(x)$ *solely depends on the absolute value of the coordinates* if for all apartments A that determine $I(x)$ either $m(x,A) = 1$ or $m(x,A) > 1$ and for all $1 \leq i < m(x,A)$ one has for all $z \in I_A^i(x)$ and for all roots $\alpha \in \Phi$ such that $\alpha \notin V_i(x,A)$ and $\alpha \not\perp V_i(x,A)$, that for all $h \in H_\alpha^i(x,A)^* \cap P_z$ the following holds: $h \notin P(I_A^i(x)) \implies I_{hA}^i(x) = I_A^i(x) \cap hA$.

Here $H_\alpha^i(x,A)^* = H_\alpha^i(x,A)$ if $\Psi_\alpha^i(x,A)$ does not contain a root $\gamma \perp\perp_x V_i(x,A)$. If $\Psi_\alpha^i(x,A)$ does contain a root $\gamma \perp\perp_x V_i(x,A)$, then $H_\alpha^i(x,A)^* := \prod U_\beta$, where the roots β are in $\Psi_\alpha^i(x,A) - \{\gamma\}$. Here we have choosen a fiixed order of the roots. Note that the set $H_\alpha^i(x,A)^*$ depends on the order of the roots choosen. However the property defined above is independent from the order choosen.

Since (Φ, K, Y^{ss}) is not disturbed, it follows that if $\gamma \perp\perp_x V_i(x,A)$ then $\gamma \perp\perp V_i(x,A)$. Let us assume that $\Psi_\alpha^i(x,A)$ contains a root $\gamma \perp\perp V_i(x,A)$. Then the group H constructed in proposition propostion 8.8 as the stabiliser of a point y, always contains a group $SL_2(K)$ that is generated by the groups $U_\gamma(K)$ and $U_{-\gamma}(K)$. The condition above now states that in fact H is equal to this subgroup $SL_2(K)$.

We will now give some examples of spaces Y^s such that $I(x)$ solely depends on the absolute value of the coordinates for all $x \in Y^s$.

8.14. Example. Let $X = G/B$ and let $Y^{ss} \subset X(\mathbb{K})$. The stabiliser of a point $y \in Y^{ss}$ is contained in a Borel group and cannot contain a non-trivial semisimple subgroup. Therefore $I(x)$ is solely determined by the absolute value of the coordinates for all points $x \in Y^s$.

Since $\lambda \in V_1(x, A)$, it follows that there are no roots that are orthogonal to $V_1(x, A)$ for all $x \in Y^{ss}$ and for all apartments $A \subset \mathcal{B}$. Therefore the group $H(x, A)$ defined in 7.13 is the group $P(\mathbb{I}(x))$. The theorem below now implies that $I(x) = I_A(x)$ for some apartment $A \subset \mathcal{B}$. (This remains true for all root systems.)

To be able to give some less trivial examples we collect some results from the literature. They concern the case where the group $H^i_\alpha(x, A)$ is abelian.

8.15. Proposition. *Let $x \in Y^{ss}_A$ be such that $m(x, A) > 1$. Let $1 \leq i < m(x, A)$. Let $H_{i,\alpha} :=< U_\beta \mid \beta \in \Phi^i_\alpha(x, A) >$ and let $L_{i,\alpha} \subset H_{i,\alpha}$ be the Levi subgroup of the maximal parabolic subgroup $P_{i,\alpha}$ that has $H^i_\alpha(x, A)$ as its unipotent radical. We assume that α is such that for all $\beta \in \Psi^i_\alpha(x, A)$ one has that $\beta \equiv \alpha \bmod V_i(x, A)$. If $\Phi^i_\alpha(x, A)$ has roots of different lengths then we moreover assume that $Char(K) \neq 2$. Let $h \in H^i_\alpha(x, A)$.*

Then there exists an element $g \in L_{i,\alpha}(K)$ such that $g \cdot h \cdot g^{-1} = \prod u_{\delta_j}(a_j)$, where the roots $\delta_j \in \Psi^i_\alpha(x, A)$ are mutually orthogonal and are all long roots.

Proof. This has been proved in [RRS] proposition 2.13 for algebraically closed fields. One verifies that the proof remains valid if the field is not algebraically closed. In particular that one then can take g in $L_{i,\alpha}(K)$.

8.16. Corollary. *Let us keep the notation as in proposition 8.15 above. Let $z \in I^i_A(x)$. Let $h \in H^i_\alpha(x, A)$ be such that $h \notin P(I^i_A(x))$.*

Then $I^i_{hA}(x) \neq I^i_A(x) \cap hA$ can only occur if $\sum \delta_j \perp V_i(x, A)$.

Proof. Let us assume that $I^i_{hA}(x) \neq I^i_A(x) \cap hA$. We may moreover assume that the element h is brought into a standard form such that $h = \prod u_{\delta_j}(a_j)$ with the roots δ_j mutually orthogonal and long. Let us now consider the point y constructed in 8.7, that is stabilised by h

We can find a group $SL_2(K)$ containing h using the same variety Z° as in proposition 8.8, but taking the stabiliser of y inside the group $(SL_2)^s =< U_{\delta_j}, U_{-\delta_j} \mid j = 1, \ldots, s >$ similarly to the proof of proposition 8.8. We get a group $H = SL_2(K)$ that is diagonally embedded in the group $(SL_2)^s$. In particular its maximal torus corresponds to the weight $\sum 2\delta_j/(\delta_j, \delta_j)$. If this weight is not a multiple of the weight ω_j (See the proof of proposition 8.8.), then the stabiliser of y (in the group $H_{i,\alpha}$) would contain a torus of K-rank two. This cannot be, since $\rho_A(y) = \rho(y)$. Since all the roots δ_j have the same length, the corollary follows.

8.17. Remark. Le W be the Weyl group of $\Phi^i_\alpha(x, A)$. Let $w \in W$ be the product of the reflections w_{δ_j} in the hyperplanes orthogonal to the roots δ_j, $j = 1, \dots, s$ (See 8.15). Let $V := V_i(x, A)$.

Let us assume that $\sum \delta_j \perp V$. Then w preserves V. Let ω_j be the fundamental weight orthogonal to V. Then $w(\omega_j) = -\omega_j$.

We will now show that w in fact preserves a simple basis of $\Phi^i_\alpha(x, A) \cap V$. There exists a simple basis of $\Phi^i_\alpha(x, A)$, such that the roots δ_j, $j = 1, \dots, s$, are simple. For all roots $\beta \in \Phi^i_\alpha(x, A) \cap V$ one has that β is orthogonal to either all roots δ_j or to all but two roots δ_j. In the latter case, it follows from $\sum \delta_j \perp V$ that the signs must be opposite. From this it follows that $ht(w(\beta)) = ht(\beta)$. Here the height is taken w.r.t our simple basis. Therefore w preserves the simple basis of $\Phi^i_\alpha(x, A) \cap V$ that is determined by the set of positive roots contained in V that is determined by our simple basis of $\Phi^i_\alpha(x, A)$ that contains the roots δ_j.

Therefore we can take a simple basis of $\Phi^i_\alpha(x, A)$ such that V is spanned by simple roots and moreover the set of positive roots that are contained in V is preserved by w. Let w_2 be the longest element of the Weyl group of $\Phi^i_\alpha(x, A)$ and w_1 be the longest element of the Weyl group of $\Phi^i_\alpha(x, A) \cap V$. Then $w = w_2 w_1$.

8.18. Examples. We give some examples of weights λ such that the corresponding flag domain Y^s is such that $I(x)$ is solely determined by the absolute value of the coordinates. Let $x \in Y^s_A$ with $m(x, A) > 1$. Let $1 \leq i < m(x, A)$ and $\rho_A(x) = \rho(x)$. Let $\alpha \in \Phi$ be such that $\alpha \not\perp V := V_i(x, A)$ and $\alpha \notin V$. Let Ψ be the irreducible component of $\Phi^i_\alpha(x, A)$ that contains the root α.

We first consider the root system $\Phi = A_\ell$. Then $H^i_\alpha(x, A)$ is always abelian. Moreover $\Psi \cap V$ consists of at most two irreducible components. The element $w_1 w_2$ (See 8.17) preserves $\Psi \cap V$ if and only if $\Psi \cap V = A_j \times A_j$ and $\Psi = A_{2j+1}$ (See [Ho] lemma 10).

Therefore if the weight λ is such that for all $x \in Y^s$ one has that $\Psi \cap V$ never has two irreducible components A_j, then $I(x)$ is solely determined by the absolute value of the coordinates for all $x \in Y^s$. An example of such a weight is $\lambda = \omega_1 + \omega_\ell$. Then λ is a root. It now follows from proposition 5.4 that $\Psi \cap V$ is alway irreducible, that is of the form A_j.

Next we give an example for the root system $\Phi = C_\ell$. Let Φ consist of the roots $\pm 2e_i$, $\pm e_i \pm e_j$ with $i \leq i, j \leq \ell$ and $i \neq j$. We choose $\lambda = e_1$. Then $X = \mathbb{P}^{2\ell-1}$. It follows from proposition 5.4 that $\Psi \cap V$ is irreducible of type C_j. Here $C_1 = A_1$ is consists of a long root. So we have only one root α to consider in this case.

Now $\Psi^i_\alpha(x, A)$ always contains a long root $\gamma \perp\perp V$. By conjugating $h \in H^i_A(x, A)$ with an element of $L_{i,\alpha}$ (see 8.15), one can write $ghg^{-1} = u_\delta(a) \cdot u_\gamma(b)$. (See also the remark in the introduction of [Rö]). Such an element can only stabilise a point y (as in proposition 8.7) if $a = 0$. Therefore $I(x)$ is solely determined by the absolute value of the coordinates for all $x \in Y^s$.

8.19. Definition. Let $x \in Y^{ss}_A$ and $z \in I^{m(x,A)}_A(x)$. Let B be a Borel group and $U \subset B$ be its maximal unipotent subgroup. Let $h \in U \cap P_z$. We choose an ordering on the set of positive roots determined by B. Let Φ^+ consist of the roots $\alpha_1, \alpha_2, \ldots$.

For $\alpha_j \in \Phi^+$ we define H_j as being the smallest half space bounded by a wall of type α_j that contains $A \cap h \cdot A$. If there exists an integer i, $1 \leq i < m(x, A)$, such that $\alpha_j \perp V_i(x, A)$ (and $\alpha_j \perp\perp_x V_{i-1}(x, A)$ if $i > 1$), but not $\alpha_j \perp\perp_x V_i(x, A)$, then we put $H^\dagger_j := s \cdot H_j$. Here $s \in S(\mathbb{K})$ is as in proposition 7.11. In all other cases we take $H^\dagger_j := H_j$. Note that if (Φ, K, Y^{ss}) is not disturbed then $H^\dagger_j = H_j$ for all roots α_j.

Let $j_0 := 0$ and let $j_i := \sharp\Phi^+ - \sharp\{\alpha \in \Phi^+ \mid \alpha \perp\perp_x V_i(x, A)\}$ if $1 \leq i \leq m(x, A)$. We say that our ordering of the positive roots is *fair dinkum* for x, A and h if the following two statements hold:

i) $\alpha_n \perp\perp_x V_i(x, A)$ if and only if $j_i < n$.

ii) If $j_i < n < m \leq j_{i+1}$ then either $\mathbb{I}^i_A(x) \cap \bigcap_{j=j_i}^{n} H^\dagger_j$ is a proper subset of $\mathbb{I}^i_A(x) \cap \bigcap_{j=j_i}^{m} H^\dagger_j$ or $\mathbb{I}^i_A(x) \cap H^\dagger_n = \mathbb{I}^i_A(x) \cap H^\dagger_m$.

8.20 Lemma. *Let $A_1, A_2 \subset B$ be two apartments. Let $z_i \in A_i, i = 1, 2$. Then there exists an apartment A such that $A \cap A_i, i = 1, 2$ contains a sector S_i with base* *point z_i. In A the sectors S_1 and S_2 are opposite.*

In particular $A = h_i A_i$ with $h_i \in P_{z_i} \cap B_i$. Here B_i is the Borel group associated to S_i.

Proof. This is well known. It is a fairly straightforward consequence of the theorem in [B] Ch. VI §8.

8.21 Theorem. *Let (Φ, K, Y^{ss}) not be disturbed. Let Y^s be such that $I(x)$ is solely determined by the absolute value of the coordinates for all $x \in Y^s$. Let $x \in Y^s$ and $A_1, A_2 \subset B$ be apartments that determine $I(x)$. Then $A_2 = h \cdot A_1$ with $h \in H(x, A_1)$.*

Proof. Let $z_j := I_{Aj}^{m(x,A_j)}(x)$ for $j = 1, 2$. Let A be an apartment containing z_1 and z_2 and such that $A \cap A_j, j = 1, 2$ contains a sector. Then $A = gA_1 = fA_2$, where g (resp. f) is an element of the Borel subgroup B_1 (resp. B_2) of G stabilising the sector in $A \cap A_1$ (resp. $A \cap A_2$).

Let Φ^+ be the set of positive roots, determined by the Borel subgroup B_1. We choose an ordering of the positive roots, that is fair dinkum for x, A_1 and g. Now we can write g as $g = u_{\alpha_{j_{m(x,A_1)}}} \cdot \cdot \cdot u_{\alpha_1}(a_1)$. We let g_i be $g_i := u_{\alpha_{j_i}}(a_{j_i}) \cdots u_{\alpha_{j_{i-1}+1}}(a_{j_{i-1}+1})$. Then $g = g_{m(x,A)} \cdots g_1$. For f we make a similar decomposition $f = f_{m(x,A_2)} \cdots f_1$.

If $g_i \in H_i(x, A_1), i = 1, \ldots, m(x, A_1)$, then $g \in H(x, A_1)$. Then $g(\mathbb{I}_{A_1}(x)) \subset I_A(x)$. Since $\rho_A(x) = \rho(x)$, one has $g(\mathbb{I}_{A_1}(x)) = \mathbb{I}_A(x)$. In particular we have $I_{A_1}^{m(x,A_1)}(x) = I_A^{m(x,A)}(x)$. From $\rho_A(x) = \rho_{A_2}(x) = \rho(x)$, one concludes that $A = fA_2$ with $f \in H(x, A_2) = H(x, A) = H(x, A_1)$. From this it follows that $A_2 = hA_1$ with $h \in H(x, A_1)$.

So let us assume that $g \notin H(x, A_1)$. Let $n \geq 1$ be the smallest integer such that $g_n \notin H_n(x, A_1)$. If $n > 1$, then we can replace A_1 by $g_{n-1} \cdots g_1 A_1$ without changing $\mathbb{I}_{A_1}(x)$. Moreover, if $n > 1$ then also $f_{n-1}, \ldots, f_1 \in H(x, A_2)$. Therefore we can also replace A_2 by $f_{n-1} \cdots f_1 \cdot A_2$.

Since $g_n \notin H_n(x, A_1)$, one has $I_{g_n A_1}^n(x) \subset \mathbb{I}_{A_1}^n(x) \cap g_n A_1$. This follows form the fact that $I(x)$ is solely determined by the absolute value of the coordinates. We now consider the faces of $I_{g_n A_1}^n(x)$ that are in the interior of $\mathbb{I}_{A_1}^n(x)$. The roots that correspond to these faces (See definition 5.5) are positive for the ordering determined by the Borel subgroup B_1. These faces will not be changed by the elements g_j with $j > n$, since these g_j are products of elements of the groups U_α with $\alpha \perp\!\perp_x V_n(x, A)$. However they can add extra faces to $I_A^n(x)$ that are also in the interior of $\mathbb{I}_{A_1}^n(x)$. Again these faces correspond to roots that are positive for the Borel group B_1. Therefore the faces of $I_A^n(x)$ that are in the interior of $\mathbb{I}_{A_1}^n(x)$ are all positive for the ordering determined by B_1.

We can now do the same for A_2 and f_n. We now we consider the faces of $I_A^n(x)$ that are in the interior of $\mathbb{I}_{A_2}^n(x)$. The roots corresponding to these faces are all positive for the Borel group B_2. Hence they are negative for B_1. This cannot be.

Hence we must have $g_n \in H_n(x, A_1)$ and $f_n \in H_n(x, A_2)$. Using induction we find that $g \in H(x, A_1)$ and $f \in H(x, A_2)$. Therefore $A_2 = hA_1$ with $h \in H(x, A_1) = H(x, A) = H(x, A_2)$.

8.22. Remark. Let us keep the assumptions of theorem 8.21. Then it follows from the theorem that for all $x \in Y^s$ the set $\{I_A^{m(x,A)}(x) | \ \rho_A(x) = \rho(x)\}$ consists of a single point. So one can use this to define a $G(K)$- equivariant map from Y^s to the building $\mathcal{B}$. One can not expect that this map coincides with the map defined in 3.17, using the circumcenter of $I(x)$.

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