PRIME TO p FUNDAMENTAL GROUPS, AND TAME GALOIS ACTIONS.

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Abstract. We show that for a local, discretely valued field F, with residue characteristic p, and a variety $\mathcal{U}$ over F, the map $\rho : \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}^{(p)}(\mathcal{U}))$ to the outer automorphisms of the prime to p geometric etale fundamental group of $\mathcal{U}$ maps the wild inertia onto a finite image. We show that under favourable conditions ρ depends only on the reduction of $\mathcal{U}$ modulo a power of the maximal ideal of F.

Introduction

If $\mathcal{U}$ is a variety over a field F, then the geometric etale fundamental group $\pi_{1,\mathrm{geom}}(\mathcal{U})$ gives rise to a map

$$\rho : \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}(\mathcal{U}))$$

from the absolute Galois group of F to the outer automorphism group of $\pi_{1,\mathrm{geom}}(\mathcal{U})$, defined by pulling back etale coverings of $\mathcal{U} \otimes_F F^{\mathrm{sep}}$ by Galois automorphisms.

Grothendieck conjectured that when F is a number field, ρ should be a strong invariant of $\mathcal{U}$, and should determine $\mathcal{U}$, if the group $\pi_{1,\mathrm{geom}}(\mathcal{U})$ was sufficiently far from being abelian. Indeed Mochizuki [Mzk] has shown that when F is a finite extension of $\mathbb{Q}_p$ and $\mathcal{U}$ is a hyperbolic curve, then this map determines $\mathcal{U}$ upto isomorphism, even if one uses only the maximal pro-p quotient of $\pi_{1,\mathrm{geom}}(\mathcal{U})$ instead of the whole group.

The point of this paper is to study the analogous map

$$\rho : \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}^{(p)}(\mathcal{U})),$$

where $\pi_{1,\mathrm{geom}}(\mathcal{U})$ is replaced by its maximal prime to p quotient $\pi_{1,\mathrm{geom}}^{(p)}(\mathcal{U})$, and the field F is assumed to be local with discrete valuation, and residue characteristic p.

Our main results (Corollary 1.5) show that when $\mathcal{U}$ is smooth, has a smooth compactification $\mathcal{X}$, with complement $\mathcal{Z} = \mathcal{X} - \mathcal{U}$ a normal crossings divisor in $\mathcal{X}$, and the pair $(\mathcal{X}, \mathcal{Z})$ has a sufficiently nice integral model then ρ factors through a tame quotient, and in fact can be completely characterised by the reduction of $\mathcal{X}$ and $\mathcal{Z}$ modulo a suitable finite power of a uniformiser π of F. For example, if

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$(\mathcal{X}, \mathcal{Z})$ extends to a semi-stable pair (X, Z) in the sense of [de J 1, 6.3] then ρ can be characterised by the reduction of (X, Z) modulo π^2.

The tameness result can be thought of as a non-abelian version of the tameness of vanishing cycles for a semi-stable scheme [RZ, 2.23]. Our proof is quite different however, since we do not have the machinary of abelian cohomology available. In fact the author was unable to make the proof of [RZ] go through, even though we are dealing only with the non-abelian anaologue of H^1.

As applications of our results, we show that for *any* variety $\mathcal{U}$ over F, the wild inertia has finite image under ρ. Note that unlike the situation of Galois representations on abelian cohomology, this does not seem to be forced by purely group theoretic considerations.

Our results also cast new light on our previous work [Ki]. There we proved that if $f : \mathcal{X} \longrightarrow \mathcal{S}$ is a smooth proper map, of F schemes, $\mathcal{Z} \subset \mathcal{X}$ a normal crossings divisor relative to $\mathcal{S}$, and $\mathcal{U} = \mathcal{X} - \mathcal{Z}$ then the maps

$$\rho_s : \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}^{(p)}(\mathcal{U}_s)),$$

defined for each F rational point s of $\mathcal{S}$, are equal for F rational points which are sufficiently close π-adically. The methods of [Ki] were entirely rigid analytic. Here we obtain a weaker version of this result (Corollary 2.5), however the proof is entirely different, and is deduced, after some technicalities, from the fact, mentioned above, that under suitable circumstances, the map ρ only depends on the reduction of $\mathcal{U}$ modulo a suitable power of π. Unfortunately the proof depends on the existence of a very nice integral model for f, so that it gives the result unconditionally only in the case of curves.

In fact we show that if $\mathcal{X}$ and $\mathcal{Z}$ are as above, and one of the fibres $(\mathcal{X}_s, \mathcal{Z}_s)$ has a nice integral model, then the maps ρ_t are equal for all F- rational points t sufficiently close to s.

We use heavily the notion of scheme equipped with logarithmic structure for which [Ka] is a general reference. In particular, if (X, Z) is a semi-stable pair of $\mathcal{O}_F$ schemes, and $\mathcal{U}$ is the generic fibre of $X - Z$, then a key point is to characterise the group $\pi_{1,\mathrm{geom}}^{(p)}(\mathcal{U})$ in terms of the log. structure on X. Here $\mathcal{O}_F$ is the valutation ring of F. We show that $\pi_{1,\mathrm{geom}}^{(p)}(\mathcal{U})$ is equal to a certain *logarithmic etale fundamental group* of X. Observing that the log. structure on X can be recovered from the geometry of the the pair (X, Z) modulo a power of π, we obtain that $\pi_{1,\mathrm{geom}}^{(p)}(\mathcal{U})$ together with the Galois action on it can be recovered from the reduction of (X, Z) modulo the same power of π.

We understand that K.Kato has proved a more general version of one of our results (Propositions 1.2, 1.3) comparing the fundamental group of $\mathcal{U}$ and the logarithmic fundamental group of X. We include our proof here for the reader's convenience, since it is not too long, and because Kato's work has not yet appeared. We believe that it should not be hard to generalise our method of proof to the general case of X log. smooth over $\mathcal{O}_F$, since this involves considering only a slightly more general type of singularity. However, we restrict our attention to the cases we will need.

Part of the proof of these results (Proposition 1.2) involves giving an etale local characterisation of log. etale coverings for certain nice log. schemes over $\mathcal{O}_F$. If X is a semi-stable $\mathcal{O}_F$ scheme, this characterisation can be used to relate the vanishing cycles functor to the total direct image functor of the projection from the log. etale site to the etale site of the closed fibre of X. The details will appear in a paper in preparation [Ki 3].

As a consequence one can show, that if $f : \mathcal{X} \longrightarrow \mathcal{S}$ is a proper smooth map of F-schemes, which extends to a flat map of proper flat $\mathcal{O}_F$ schemes $X \longrightarrow S$, and if the fibre over some $\mathcal{O}_F$ valued point $\tilde{s}$ of S is semi-stable, then the Galois representations on the l-adic ($l \neq p$) cohomology of the fibres $\mathcal{X}_t$ of f are isomorphic for all F-valued points t sufficiently close to the generic point of $\tilde{s}$. This gives a different approach to the results of [Ki 2], although as in the non-abelian case, the results are much weaker since they require the existence of the nice integral models X and S.

Other, possibly more interesting, applications are the subject of [Ki 3]. There we use the realisation of the vanishing cycles functor as a direct image functor to prove results about the local terms in the Lefschetz Trace Formula, and apply this to the construction of the Artin representation for characteristic 0 local fields.

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§1 Specialisation and Tame Fundamental Group.

Let F be a discretely valued local field of residue characteristic p, $\mathcal{O}_F$ its ring of integers, and π a uniformiser for F. We give Spec $(\mathcal{O}_F)$ the structure of a fine log. scheme, induced by the map of monoids $\mathbb{N}^+ \longrightarrow \mathcal{O}_F, 1 \mapsto \pi^{p^k}$, where k is a non-negative integer. We assume k fixed once and for all.

$$A = \mathcal{O}_F[x_1, \dots x_r, y_1, \dots y_s, z_1, \dots z_t]/(y_1 \dots y_s - \pi^{p^k l}),$$

where l is an integer which is prime to p.

We are going to consider pairs (X, Z), where X is a finite type $\mathcal{O}_F$ scheme, and Z is a closed subscheme such that locally on X there is an etale map $\phi : X \longrightarrow$ Spec (A) with $Z = \phi^{-1}(\text{Spec }(A/(x_1 \dots x_r))$. We call such a pair (X, Z) $p^k l-$ semi-stable. One reason we consider these pairs instead of just semi-stable pairs is that our notion is stable under base extension (though the "$p^k l$" will change).

Denote by P the monoid

$$\langle e_1, \dots e_{r+s}, u; e_{r+1} \dots e_{r+s} = u^{p^k l} \rangle.$$

Spec (A) can be made into a fine log. scheme over $\mathcal{O}_F$, by giving it the log. structure induced by

$$P \longrightarrow A,\ e_i \mapsto x_i\ (1 \leqslant i \leqslant r),\ e_{i+r} \mapsto y_i\ (1 \leqslant i \leqslant s),\ u \mapsto \pi.$$

If (X, Z) and ϕ are as above, we define a log. structure locally on X by pulling back the one on Spec (A). One checks immediately that the induced log. structure is independent of the choice of ϕ, and that the log. structures thus defined glue and give X the structure of a fine log. scheme over $\mathcal{O}_F$ with the log. structure defined above.

For an F scheme Y we denote by $\mathbf{Et}^{(p)}(Y)$ the category of etale coverings $Y' \longrightarrow Y$ whose connected components are Galois of prime to p order, and quotient coverings of these. We set $\mathbf{Et}^{(p)}_{\mathrm{geom}}(Y) = \varinjlim_{F'} \mathbf{Et}^{(p)}(Y \otimes_F F')$, where F' runs through finite seperable extensions of F. Note that the Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on $\mathbf{Et}^{(p)}_{\mathrm{geom}}(Y)$ in a natural way.

If (Y, M) is a fine, saturated, log. smooth log. scheme over $\mathcal{O}_F$, denote by $U \subset Y$ the open subscheme where the log. structure on Y is trivial.We call a finite log. etale map $(Y', M') \longrightarrow (Y, M)$ *prime to* p if its restriction U is in $\mathbf{Et}^{(p)}(U)$. We denote by $\mathbf{Et}^{(p)}_{\mathrm{log.}}(Y, M)$ the category of fine saturated log. schemes $(Y', M') \longrightarrow (Y, M)$ over (Y, M) which are finite (on the underlying schemes) log. etale, and prime to p. We set $\mathbf{Et}^{(p)}_{\mathrm{log.geom}}(Y, M) = \varinjlim_{F'} \mathbf{Et}^{(p)}_{\mathrm{log.}}((Y, M) \otimes_{\mathcal{O}_F} \mathcal{O}_{F'})$, where F' runs through finite, seperable, tame extensions of F. As above, there is a natural Galois action on this category. Here $Y \otimes_{\mathcal{O}_F} \mathcal{O}_{F'}$ has the log. structure generated by the pull back of M, and a uniformiser of F'.

The main point of this section is to prove the following

Theorem 1.1. *Suppose that (X, Z) is as above and set $U = X - (X_s \cup Z)$, where X_s denotes the closed fibre of X. There is are equivalence of categories, induced by restriction,*

$$\mathbf{Et}^{(p)}_{\mathrm{log.geom}}(X, M_X) \xrightarrow{\sim} \mathbf{Et}^{(p)}_{\mathrm{geom}}(U).$$

This equivalence is compatible with the Galois action on both sides. In particular the Galois action on the right hand side factors through a tame quotient. Here M_X denotes the log. structure on X discussed above.

The proof consists of two parts. The first, contained in Propositions 1.2 and 1.3, shows that there is an equivalence $\mathbf{Et}^{(p)}_{\mathrm{log.}}(X, M_X) \xrightarrow{\sim} \mathbf{Et}^{(p)}(U)$, and that a similar statement remains true after a tame extension of scalars. This is one reason it is not enough to work with semi-stable schemes. The second is Proposition 1.4, and shows that every object of $\mathbf{Et}^{(p)}_{\mathrm{geom}}(U)$ is in fact defined over a tame extension of F.

The importance of Theorem 1.1 is that it allows one to show that, for example, if $p^k l = 1$ the category $\mathbf{Et}^{(p)}_{\mathrm{geom}}(U)$ depends only on the first nilpotent neighbourhood of the special fiber X_s of X. That is $\mathbf{Et}^{(p)}_{\mathrm{geom}}(U)$ depends only on the reduction of X modulo π^2. The reason for this is that if $(X', M') \hookrightarrow (X, M)$ is an exact closed nilpotent immersion of fine log. schemes, then the log. etale topology of (X', M') and (X, M) are equivalent [Ka 3.14], this being the analogue of the topological invariance of etale morphisms.

Consider the pair $(\mathrm{Spec}\ (A), \mathrm{Spec}\ (A/(x_1 \dots x_r)))$ discussed above. Choose an integer q coprime to p, and divisible by l and set

$$A_q = A[x_1^{1/q}, \dots, x_r^{1/q}, y_1^{1/q}, \dots y_s^{1/q}]/(y_1^{1/l} \dots y_s^{1/l} - \pi^{p^k}).$$

Denote by P_q the monoid

$$\langle f_1, \dots, f_{r+s}, u; f_{r+1}^{q/l} \dots f_s^{q/l} = u^{p^k} \rangle.$$

There is a natural log. structure on $\mathrm{Spec}\ (A_q)$ induced by

$$P_q \longrightarrow A_q, f_i \mapsto x_i^{1/q} (1 \leqslant i \leqslant r), f_{i+r} \mapsto y_i^{1/q} (1 \leqslant i \leqslant s), u \mapsto \pi.$$

The inclusion $P \subset P_q$ sending e_i to f_i^q makes the natural map Spec $(A_q) \longrightarrow$ Spec (A) into a log. etale morphism of log. schemes [Ka 3.5]. The underlying map of schemes is flat, and if F contains the q^{th} roots of unity μ_q, then this covering has automorphism group $G = \mu_q^{r+s}$. In this case, if $H \subset G$ is a subgroup, Spec (A_q^H) has a natural log. structure induced by the monoid $(P_q)^H$. The map Spec $(A_q^H) \longrightarrow$ Spec (A) is then log. smooth, and we call it a *standard Kummer covering* of Spec (A).

Now let (X, Z) be as above, and choose locally on X a map $\phi : X \longrightarrow$ Spec (A), such that $Z = \phi^{-1}(\text{Spec }(A/(x_1 \dots x_r)))$. We call a finite map of fine log. schemes $(Y, N) \longrightarrow (X, M)$ *a Kummer covering* if etale locally on X there is a standard Kummer covering Spec (A_q^H) of Spec (A) such that Y is disjoint union of schemes of the form $X \times_{\text{Spec }(A)} \text{Spec }(A_q^H)$, and such that the log. structure on (the components of) Y, induced by pulling back the log. structure on Spec (A_q^H) via the etale map $Y \longrightarrow$ Spec (A_q^H), coincides with N.

We then have

Proposition 1.2. *Let (Y, N) be a fine, saturated log. scheme over $\mathcal{O}_F$, and f : $(Y, N) \longrightarrow (X, M)$ a finite, log. etale map over $\mathcal{O}_F$. Then f is a finite Kummer covering. In particular the underlying scheme Y is normal, and the map $Y \longrightarrow X$ is flat.*

With the notation of Theorem 1.1, any object $V \longrightarrow U$ of $\mathsf{Et}^{(p)}(U)$ extends uniquely to a finite Kummer covering of (X, M).

Proof. We prove the second statement first, as it is somewhat easier.

By uniqueness the claim is etale local on X. V extends to a normal scheme $Y \longrightarrow X$, which is finite over X and tamely ramified at the generic points of the divisor $Z \cup X_s$. Locally on X we have an etale map $X \longrightarrow$ Spec (A). By Abhyankar's Lemma ([SGA 1, 3.6]), for sufficiently large q, divisible by l, pulling Y back by Spec $(A_q) \longrightarrow$ Spec (A) (notation as above) kills the ramification at the generic points of $X_s \cup Z$.

Set

$$A_{p^k q} = A_q[y_1^{1/p^k q}, \dots, y_s^{1/p^k q}]/(y_1^{1/p^k l} \dots y_s^{1/p^k l} - \pi),$$

$X_{p^k q} = X \times_{\text{Spec }(A)} \text{Spec }(A_{p^k q})$, and $Y_{p^k q}$ equal to the normalisation of $Y \times_X X_{p^k q}$. We have $Y_{p^k q} \longrightarrow X_{p^k q}$, etale over the height 1 primes, with $Y_{p^k q}$ normal, and $X_{p^k q}$ regular, as $A_{p^k q}$ is regular. Hence $Y_{p^k q}$ is etale over $X_{p^k q}$ by the Zariski-Nagata purity theorem ([SGA 2, X Thm 3.4]).

Etale locally on Spec (A), the etale map $Y_{p^k q} \longrightarrow X_{p^k q}$ splits, so that Y becomes a quotient of $X_{p^k q}$. Since the degree of Y over X is prime to p, Y is actually a quotient of X_q.

This shows that etale locally on Spec (A), $Y \longrightarrow X$ is the pull back of a standard Kummer covering of Spec (A), and we saw above that it is therefore equipped with a canonical log. structure. Since any two standard Kummer coverings of Spec (A) are dominated by a third, one checks easily that this log. structure is independent of the choice of q, and that the log. structures constructed locally glue into a fine, saturated log. structure N on Y.

To prove the first statement, we introduce a little notation. If W is a scheme, and S a monoid, we denote by S_W the etale sheaf of monoids with constant value

W. Fix the map (defined etale locally) $X \longrightarrow \text{Spec}\,(A)$, and let P be the monoid giving the log. structure on A. We apply [Ka 3.5] which tells us that we can find a chart

$$(P_X \longrightarrow M, Q_Y \longrightarrow N, P \longrightarrow Q),$$

with Q a finitely generated, integral monoid, such that $P^{\text{gp}} \longrightarrow Q^{\text{gp}}$ is injective with finite cokernel of prime to p order, and the map $Y \longrightarrow X \times_{\text{Spec}\,\mathbb{Z}[P]} \text{Spec}\,\mathbb{Z}[Q]$ is etale.

For a monoid S containing P, we will write $\mathcal{O}_F[S]$ for the ring $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[S]$, where A is a $\mathbb{Z}[P]$ algebra via the map $P \longrightarrow A$ described above.

Since Q is integral $Q \hookrightarrow Q^{\text{gp}}$. Let Q^{tor} be the torsion subgroup of Q, and consider the group of characters $\chi : Q^{\text{tor}} \longrightarrow \mu_{p'}$, where $\mu_{p'}$ are the prime to p roots of unity. If we make an unramified extension of $\mathcal{O}_F$, then χ induces a map $\mathcal{O}_F[\chi] : \mathcal{O}_F[Q] \longrightarrow \mathcal{O}_F[Q/Q^{\text{tor}}]$. This corresponds to taking a connected component in Spec $(\mathcal{O}_F[Q])$ so that, if Y is connected, then the map $Y \longrightarrow \text{Spec}\,(\mathcal{O}_F[Q])$ factors through Spec $(\mathcal{O}_F[\chi])$: Spec $(\mathcal{O}_F[Q/Q^{\text{tor}}]) \longrightarrow$ Spec $(\mathcal{O}_F[Q])$ for some χ, and we may replace Q by Q/Q^{tor}, since this preserves all the hypotheses.Thus we may assume Q is torsion free.

Now for some large prime to p integer q, which is divisible by l, the map $P^{\text{gp}} \longrightarrow P_q^{\text{gp}}$ lifts to Q^{gp}. Indeed this is clear, since $P^{\text{gp}} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} P^{\text{gp}}$ lifts to a map $Q^{\text{gp}} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} P^{\text{gp}}$. Since Q is torsion free, the map $Q^{\text{gp}} \longrightarrow P_q^{\text{gp}}$ is necessarily injective, and we have a commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] & \\ P^{\text{gp}} \arrow[r] & Q^{\text{gp}} \arrow[r] & P_q^{\text{gp}} \end{tikzcd}$$

where all the maps are injective.

For a ring R and a subring S we denote by $n(S, R)$ the integral closure of S in R.

Let

$$Q_1 = Q \cap P_q = Q^{\text{gp}} \cap P_q \subset P_q^{\text{gp}},$$

where the second equality holds as Q is saturated.

Since, $n(\mathbb{Z}[P], \mathbb{Z}[P_q^{\text{gp}}]) = \mathbb{Z}[P_q]$, we have

$$n(\mathbb{Z}[P], \mathbb{Z}[Q^{\text{gp}}]) = \mathbb{Z}[P_q] \cap \mathbb{Z}[Q^{\text{gp}}] = \mathbb{Z}[Q_1].$$

In particular this shows that $\mathbb{Z}[Q_1]$ is a finite $\mathbb{Z}[P]$ module, so that Q_1 is finitely generated.

Next we do a similar analysis after "extending scalars" to $\mathcal{O}_F$ We have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_F[P] \arrow[r] \arrow[d] & \mathcal{O}_F[Q] \arrow[d] & \\ \mathcal{O}_F[P^{\text{gp}}] \arrow[r] & \mathcal{O}_F[Q^{\text{gp}}] \arrow[r] & \mathcal{O}_F[P_q^{\text{gp}}] \end{tikzcd}$$

Since the monoids in question are integral, we see that the vertical maps are injective, and the lower horizontal maps clearly are also, whence all of them are.

Now $\mathcal{O}_F[P_q]$, for example, is a free $\mathcal{O}_F$ module with a basis, the equivalence classes of P under the action of multiplication by u and similarly for $\mathcal{O}_F[P_q^{\mathrm{gp}}]$ except that it is a free F module.

Next we have

$$n(\mathcal{O}_F[P], \mathcal{O}_F[Q^{\mathrm{gp}}]) = \mathcal{O}_F[P_q] \cap \mathcal{O}_F[Q^{\mathrm{gp}}]$$

inside $\mathcal{O}_F[P_q^{\mathrm{gp}}]$, since the first term in the intersection is even regular. We claim that this intersection is equal to $\mathcal{O}_F[Q_1]$. Indeed if $r \in \mathcal{O}_F[P_q], s \in \mathcal{O}_F[Q^{\mathrm{gp}}]$ are equal in $\mathcal{O}_F[P_q^{\mathrm{gp}}]$ then they must have equal components with respect to the bases described above. Thus we may assume $r = ar_0, s = bs_0$, with $a \in \mathcal{O}_F, b \in F, r_0 \in P_q, s_0 \in Q^{\mathrm{gp}}$. Since $r = s$ in $\mathcal{O}_F[P_q^{\mathrm{gp}}]$, there exists an $n \in \mathbb{Z}$ such that $r_0 = u^n s_0 \in Q^{\mathrm{gp}} \cap P_q = Q_1$, which proves the claim.

We remark that if $g \in Q^{\mathrm{gp}}$, then there is a $h \in P$ such that $(gh)^q \in P$, so that $gh \in Q_1$. This shows that $Q^{\mathrm{gp}} = Q_1^{\mathrm{gp}}$.

Put $W = X \times_{\mathrm{Spec}\,(\mathbb{Z}[P])} \mathrm{Spec}\,(\mathbb{Z}[Q_1])$, $W' = X \times_{\mathrm{Spec}\,(\mathbb{Z}[P])} \mathrm{Spec}\,(\mathbb{Z}[Q])$. Since $X \longrightarrow \mathrm{Spec}\,(\mathcal{O}_F[P])$ is etale, and $\mathcal{O}_F[Q_1]$ is normal, W is normal by ([EGA IV, 6.14.4].) We have a commutative diagram with cartesian squares.

$$\begin{tikzcd}
Y \arrow[r] & W' \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & X \arrow[d] \\
& \mathrm{Spec}\,(\mathcal{O}_F[Q]) \arrow[r] \arrow[d] & \mathrm{Spec}\,(\mathcal{O}_F[Q_1]) \arrow[r] \arrow[d] & \mathrm{Spec}\,(\mathcal{O}_F[P]) \arrow[d] \\
& \mathrm{Spec}\,(\mathbb{Z}[Q]) \arrow[r] & \mathrm{Spec}\,(\mathbb{Z}[Q_1]) \arrow[r] & \mathrm{Spec}\,(\mathbb{Z}[P])
\end{tikzcd}$$

Now just as in the classic theory of toroidal embeddings, the fact that Q is saturated implies that $\mathbb{Z}[Q]$ is normal (the proof of [Ful, p29-30] goes through). Since $Y \longrightarrow X$ is finite, and $Y \longrightarrow W'$ is etale, the map $\mathrm{Spec}\,(\mathbb{Z}[Q]) \longrightarrow \mathrm{Spec}\,(\mathbb{Z}[Q_1])$ is quasi-finite, over points in the image of $X \longrightarrow \mathrm{Spec}\,(\mathbb{Z}[P])$. Thus, by Zariski's Main Theorem, [EGA IV, 8.12.6], this map is an open immersion over such a point, since both $\mathbb{Z}[Q]$ and $\mathbb{Z}[Q_1]$ are normal with the same field of fractions.

This shows that $Y \longrightarrow W$ is etale, that Y is normal, and that the restriction to $\mathrm{Spec}\,(\mathbb{Z}[Q])$ of the canonical log. structure on $\mathrm{Spec}\,(\mathbb{Z}[Q_1])$ is equal to the one induced by Q, over points in the image of $X \longrightarrow \mathrm{Spec}\,(\mathbb{Z}[P])$.

This allows us to replace Q by Q_1, and assume that $Q \subset P_q$, and that $\mathbb{Z}[Q]$ and $\mathcal{O}_F[Q]$ are normal. Set $Y'' = X \times_{\mathrm{Spec}\,(\mathbb{Z}[P])} \mathrm{Spec}\,(\mathbb{Z}[P_q])$. Y'' is a Kummer covering of X, having W as quotient, and $Y'' \times_W Y$ has Y as a quotient, and is etale over Y''. Etale locally on X the etale map $Y'' \times_W Y \longrightarrow Y''$ splits, so that Y is a Kummer covering of X as was claimed. $\square$

We remark that we will occasionally need to use a variant of 1.2, where Y and X are formal schemes with log. structure rather than schemes. The proof is identical, and we have restricted ourselves to the case of schemes, only for the sake of clarity. In particular, in the proof of the formal scheme case one still uses the *scheme*

Spec (A), but maps such as $Y \longrightarrow$ Spec (A) become maps of locally ringed spaces rather than schemes.

Proposition 1.3. *With (X, M) as above, there is an equivalence of categories $\Phi_{F'} : \mathsf{Et}^{(p)}_{\log.}(X', M') \xrightarrow{\sim} \mathsf{Et}^{(p)}(U)$.*

If (Y, N) is an object of $\mathsf{Et}^{(p)}_{\log.}(X, M)$ then the underlying scheme Y is normal, and the map $Y \longrightarrow X$ is flat.

Proof. Everything in the proposition follows from 1.2, except the full faithfulness of the restriction functor Φ_F. Since we know that objects in $\mathsf{Et}^{(p)}_{\log.}(X, M)$ are normal, we know that restriction is faithful.

Consider two finite, saturated, log. etale coverings $(Y_1, N_1), (Y_2, N_2)$ of (X, M). By descent, the full faithfulness is local in the etale topology of X, so we may assume that, in our previous notation, $X =$ Spec (A), and $(Y_1, N_1), (Y_2, N_2)$ are both quotients of the standard Kummer covering $X_q =$ Spec (A_q) of X.

First we claim

$$\mathrm{Hom}_{(X,M)}(X_q, X_q) \xrightarrow{\sim} \mathrm{Hom}_X(X_q, X_q) \xrightarrow{\sim} \mathrm{Hom}_U(X_q|U, X_q|U).$$

Here the first term means maps as log. schemes.

The second equality follows immediately from the faithfullness by a counting argument, and the first from the explicit description of the automorphisms (as a classical scheme) of X_q over X. $X_q \longrightarrow X$ comes via base change from Spec $(\mathbb{Z}[P_q]) \longrightarrow$ Spec $(\mathbb{Z}[P])$. Any automorphism of X_q/X (as classical schemes) comes via base change from an automorphism of $\mathbb{Z}[P_q]$ over $\mathbb{Z}[P]$. These all involve multiplying homogeneous components of $Z[P_q]$ by suitable roots of unity, and therefore automatically respect log. structures.

Now suppose we have a map $h_U : Y_1|U \longrightarrow Y_2|U$ over U. This is covered by a map $X_q|U \longrightarrow X_q|U$, which extends to map of log. schemes $X_q \longrightarrow X_q$. On the other hand, since Y_1 and Y_2 are normal, h_U extends to a map of schemes $h : Y_1 \longrightarrow Y_2$. Thus, we have a commutative diagram of schemes.

$$\begin{tikzcd} X_q \arrow[r] \arrow[d] & X_q \arrow[d] \\ Y_1 \arrow[r, "h"] & Y_2 \end{tikzcd}$$

The vertical maps come from maps of log. schemes, by construction, and the top horizontal map does also, by the discussion above. It follows, by the construction of the log. structures on Y_1 and Y_2 that h comes from a map of log. schemes, which proves the full faithfulness. $\square$

Proposition 1.4. *With the notation above let F' be a finite Galois extension of F of p power order. The base change functor $\mathsf{Et}^{(p)}(U) \longrightarrow \mathsf{Et}^{(p)}(U) \otimes_F F'$ is an equivalence of categories.*

Proof. We reduce immediately to the case where F' has degree p over F (though this is mainly for notational convenience). Since the objects of $\mathsf{Et}^{(p)}(U)$ are Galois coverings of prime to p order, the functor is fully faithful. Thus essential surjectivity becomes a local question.

We let $A = \mathcal{O}_F[P]$ be as above, and set $A' = A \otimes_{\mathcal{O}_F} \mathcal{O}_{F'}$.

Let π' be a uniformiser for F'. We give $\mathcal{O}'_F$ the log. structure defined by $\pi^{p^{k+1}}$. Set $w = (\pi')^{p^{k+1}l}/\pi^{p^k l}$, which is a unit in F', and put $y'_s = wy_s$. Thus

$$A' = \mathcal{O}_{F'}[x_1, \dots x_r, y_1, \dots y_{s-1}y'_s, z_1, \dots z_t]/(y' - (\pi')^{p^{k+1}l}),$$

where $y' = y_1 \dots y_{s-1}y'_s$.

We assume that there is an etale map $X \longrightarrow \text{Spec }(A)$, defining the log. structure on X, as usual. Write $U' = U \times_{\text{Spec } A} \text{Spec } A'$, and $X' = X \times_{\text{Spec }(A)} \text{Spec } A'$.

If V is an object of $\mathsf{Et}^{(p)}(U')$, then by 1.2, it extends uniquely to a log. etale covering (Y, N) of (X', M'), where M' is the canonical log. structure on X' inherited from A'.

Choose an integer q which is prime to p and divisible by l, and write

$$A'_q = \mathcal{O}_{F'}[x_1^{1/q}, \dots x_r^{1/q}, y_1^{1/q}, \dots, y_{s-1}^{1/q}, y_s'^{1/q}, z_1, \dots z_t]/((y')^{1/l} - (\pi')^{p^{k+1}}).$$

By 1.3 Y is a quotient of $X' \times_{\text{Spec }(A')} \text{Spec } A'_q$, for suitable q. Hence, by full faithfulness, it is enough to prove the result for the restriction to U' of $X' \times_{\text{Spec }(A')}$ Spec A'_q.

Since we may work etale locally, we may assume that F contains a q^{th} root of w. Then we clearly have

$$X \times_{\text{Spec }(A)} \text{Spec } A_q \otimes_{\mathcal{O}_F} F' \xrightarrow{\sim} X' \times_{\text{Spec }(A')} \text{Spec } A'_q|_{U'}.$$

□

finally we come to

Proof of Theorem 1.1. Write $\mathsf{Et}^{(p)}_{\text{tame}}(U) = \varinjlim_{F'} \mathsf{Et}^{(p)}(U) \otimes_F F'$, where F' runs through tame Galois extensions of F. We have equivalences of categories

$$\mathsf{Et}^{(p)}_{\text{log.geom}}(X, M) \xrightarrow{\sim} \mathsf{Et}^{(p)}_{\text{tame}}(U) \xrightarrow{\sim} \mathsf{Et}^{(p)}_{\text{geom}}(U).$$

Indeed the first equivalence follows from 1.3, and the second from 1.4. □

As a consequence of Theorem 1.1 we have

Corollary 1.5. *Retaining the notation above, assume also that X is a proper $\mathcal{O}_F$ scheme. The group $\pi_1^{(p)}(U)$ and the category $\mathsf{Et}^{(p)}_{\text{geom}}(U)$, together with its Galois action depend only on the reduction of X modulo $\pi^{p^k l+1}$. In particular, the map $Gal(\bar{F}/F) \longrightarrow \text{Out}(\pi_1^{(p)}(U))$ depends only on X modulo $\pi^{p^k l+1}$, and this map factors through a tame quotient.*

Proof. We have already seen above that the Galois action on $\mathsf{Et}^{(p)}_{\text{geom}}(U)$ is tame.

Let $\mathfrak{X}$ be the π -adic completion of X. We give $\mathfrak{X}$ a log. structure $M_{\mathfrak{X}}$ by pulling back the log. structure on X via the map of locally ringed spaces $\mathfrak{X} \longrightarrow X$. We claim that the pullback functor $\mathbf{Et}^{(p)}_{\log.}(X,M) \longrightarrow \mathbf{Et}^{(p)}_{\log.}(\mathfrak{X},M_{\mathfrak{X}})$ is an equivalence of categories. The claim is etale local on X. Since X is proper, any F^{sep} valued point of X extends to an $\mathcal{O}_{F^{\text{sep}}}$ valued point, thus we only need to check the claim over etale neighbourhoods of X containing an $\mathcal{O}_{F^{\text{sep}}}$ valued point. However now the claim follows easily from 1.2, and its formal scheme analogue, (see the remark following the proof of 1.2.)

By [Ka 3.14] the category $\mathbf{Et}^{(p)}_{\log.}(\mathfrak{X},M_{\mathfrak{X}})$ depends only on the reduction of the formal log. scheme $(\mathfrak{X},M_{\mathfrak{X}})$ modulo π. Thus we are reduced to showing that the log. structure on $X\otimes_{\mathcal{O}_F}\mathcal{O}_F/\pi$ obtained by reducing the log. scheme (X,M) modulo π depends only on the underlying scheme X modulo $\pi^{p^k l+1}$.

To recover M modulo π from $X\otimes_{\mathcal{O}_F}\mathcal{O}_F/\pi^{p^k l+1}$, consider the ideal generated by $\pi^{p^k l}$ on this scheme. Locally this ideal factors as a product of principle primary ideals $J_{r+1},\dots,J_{r+s}$ with distinct radicals. Denote by $J_1,\dots,J_r$ the ideals corresponding to the irreducible components of $Z\subset X$ modulo $\pi^{p^k l+1}$. Let Q be the monoid

$$\langle e_{J_1}\dots e_{J_{r+s}},u; e_{J_1}\dots e_{J_{r+s}}=u^{p^k l}\rangle.$$

If $x_1,\dots x_r,y_1,\dots,y_s$ are generators for $J_1,\dots J_r,J_{r+1},\dots,J_s$ respectively, then the log. structure on $X/\pi^{p^k l}$ is given by

$$Q\oplus\mathcal{O}_X^*\longrightarrow\mathcal{O}_X, e_{J_i}\mapsto x_i(1\leqslant i\leqslant r), e_{J_{i+r}}\mapsto y_i(1\leqslant i\leqslant s), u\mapsto\pi.$$

We have recovered (X,M) modulo $\pi^{p^k l+1}$ whence also modulo π. □

§2 Applications

Theorem 2.1. *Let $\mathcal{U}$ be a variety over F. Under the map*

$$Gal(F^{\text{sep}}/F)\longrightarrow\mathrm{Out}(\pi^{(p)}_{1,\text{geom}}(\mathcal{U})),$$

the image of the wild inertia group is finite.

Proof. By Nagata's compactification theorem, we may embed $\mathcal{U}$ as an dense open subvariety of a proper variety $\mathcal{X}$. Put $\mathcal{Z}=\mathcal{X}-\mathcal{U}$.

By [deJ 1,6.5], after replacing F, by a finite extension, we may assume that there exists a smooth, proper variety $\mathcal{X}'$, and a proper dominant map $f:\mathcal{X}'\longrightarrow\mathcal{X}$ such that f is generically finite, $\mathcal{Z}'=f^{-1}(\mathcal{Z})$ is a normal crossings divisor in $\mathcal{X}'$, and the pair $(\mathcal{X}',\mathcal{Z}')$ extends to a semi-stable (or, in our language 1-semi-stable) pair of $\mathcal{O}_F$ schemes (X',Z').

Let $\mathcal{W}$ be a geometrically connected object of $\mathbf{Et}^{(p)}_{\text{geom}}(\mathcal{U})$, which is also Galois. We claim that $\mathcal{W}$ is actually defined over a tame extension of F, which will prove the theorem.

By 1.4 $f^*(\mathcal{W})$ is defined over a tame extension of F. Thus, replacing F by a tame extension, we may assume that there is an etale, prime to p Galois covering $\mathcal{V}$ of $\mathcal{U}'=\mathcal{X}'-\mathcal{Z}'$ which becomes isomorphic to $f^*(\mathcal{W})$ after a finite extension of F.

Put $\mathcal{X}''=\mathcal{X}'\times_{\mathcal{X}}\mathcal{X}'$, and denote by p_1 and p_2 the two projections of $\mathcal{X}''$ to $\mathcal{X}'$. $p_1^*(\mathcal{V})$ and $p_2^*(\mathcal{V})$ become isomorphic after a finite extension of scalars, and this isomorphism satisfies the usual cocycle condition. Since these are prime to p coverings, this isomorphism must actually exist after a tame extension. Thus, replacing F by a tame extension, we may assume that $p_1^*(\mathcal{V}) \xrightarrow{\sim} p_2^*(\mathcal{V})$, satisfying the usual cocycle condition.

Using [SGA 1, IX 4.12] we may descend $\mathcal{V}$ to a finite etale covering $\mathcal{W}^+$ of $\mathcal{U}$. By construction there is a finite seperable extension E/F such that $\mathcal{W}$ is defined over E, and $\mathcal{W}^+ \otimes_F E \xrightarrow{\sim} \mathcal{W}$.

Thus $\mathcal{W}^+$ is a finite, connected etale covering of $\mathcal{U}$, of prime to p order. By descent, the Galois automorphisms of $\mathcal{V}$ descend to $\mathcal{W}^+$ since this is so after a finite extension of F. $\square$

As a second application of the results of §1, we recover some of the results of [Ki]. Let $f : \mathcal{X} \longrightarrow \mathcal{S}$ be a smooth proper map of finite type F schemes, with geometrically connected fibres. Let $\mathcal{Z} \subset \mathcal{X}$ be a normal crossings divisor relative to $\mathcal{S}$ - that is $\mathcal{Z}$ is flat over $\mathcal{S}$ and its fibres are normal crossings divisors in the fibres of $\mathcal{X}$. Put $\mathcal{U} = \mathcal{X} - \mathcal{Z}$.

In [Ki] we showed that under quite general hypotheses the maps

$$\rho_s : \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}^{(p)}(\mathcal{U}_s))$$

were equal for sufficiently nearby F rational points s of $\mathcal{S}$. Here we give a different proof of this sort of result in certain cases.

The idea of the argument is as follows. Suppose that $f : \mathcal{X} \longrightarrow \mathcal{S}$ extended to a proper map $\bar{f} : X \longrightarrow S$ of $\mathcal{O}_F$ schemes, and put Z equal to the closure of $\mathcal{Z}$ in X. Suppose that for each $\mathcal{O}_F$ valued point s of S, the pair (X_s, Z_s) is $p^k l$-semi-stable. By 1.5, we know that if $U = X - Z$, the map $\mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}^{(p)}(\mathcal{U}_s))$ depends only on the reduction of X modulo $\pi^{p^k l+1}$. Thus $\mathcal{O}_F$ valued points which are sufficiently close π- adically will give rise to the same maps.

We will prove the result under the weaker hypothesis that we have the situation described above only after a finite extension of scalars. Unfortunately even this is only known if the fibres of f are one dimensional.

The following lemma shows that if the map $\bar{f}$ above is flat, then the $p^k l$- semi-stability of (X_s, Z_s) implies that of (X_t, Z_t) for points t of S sufficiently close to s.

Lemma 2.2. *Let X be a proper flat $\mathcal{O}_F$ scheme, and $Z \subset X$ a closed subscheme, which is flat over $\mathcal{O}_F$. (X, Z) is $p^k l$-semi-stable if and only if there exists a $p^k l$-semi-stable pair (X', Z') which is isomorphic to (X, Z) modulo $\pi^{p^k l+1}$. That is $p^k l$-semi-stability depends only on the reduction of (X, Z) modulo $\pi^{p^k l+1}$.*

Proof. Let Spec A be an open affine in X. Replacing A by an etale neighbourhood, we may assume that

$$A/\pi^{p^k l+1} = \mathcal{O}_F/\pi^{p^k l+1}[x_1, \dots x_r, y_1, \dots y_s, z_1 \dots z_t]/(y_1 \dots y_s - \pi^{p^k l}),$$

with $Z \cap \mathrm{Spec}\, A$ cut out by $x_1, \dots x_r$ modulo π^{l+1}.

Lift the x_i, y_j and z_k to X_i, Y_j, and Z_k in A. Let $\mathcal{I}$ be the ideal of A corresponding to Z. We may assume that the x_i have been lifted to $X_i \in \mathcal{I}$. Since, $A/\mathcal{I}$ is flat over $\mathcal{O}_F$, one easily deduces from Nakayama's Lemma that the $X_i's$ generate $\mathcal{I}$.

We have $Y_1 \dots Y_s = u\pi^{p^k l}$, with u a unit in A congruent to 1 modulo π. Replacing Y_1 by $Y_1 u^{-1}$, we may assume that $Y_1 \dots Y_s = \pi^{p^k l}$, though Y_1 now only lifts y_1 modulo π.

Put

$$B = \mathcal{O}_F[R_1, \dots R_r, S_1, \dots S_s, T_1, \dots T_t]/(S_1 \dots S_r - \pi^{p^k l}),$$

and define a map $\phi : B \longrightarrow A$ by $R_i \mapsto X_i, S_j \mapsto Y_j, T_k \mapsto Z_k$.

We claim that ϕ is an isomorphism over an open neighbourhood of Spec B which contains its closed fibre. This claim implies the Lemma, by the properness of X.

ϕ is an isomorphism modulo π. Since A and B have no π- torsion A is B flat by [Mat, Ex 22.3]. Similarly if $\mathfrak{p} \in$ Spec $(A/\pi) \subset$ Spec A, $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$, and we complete A and B at $\mathfrak{p}$ and $\mathfrak{q}$ respectively, then the map $\hat{\phi} : \hat{B} \longrightarrow \hat{A}$ makes $\hat{A}$ flat over $\hat{B}$. By [Mat,22.5] $\hat{\phi}$ is injective, as it is modulo π. Since $\hat{B}$ is π-adically complete $\hat{\phi}$ is also surjective, as it is modulo π.

This shows that ϕ is etale over $\mathfrak{q}$, whence it is an isomorphism over $\mathfrak{q}$, as this is true modulo π. $\square$

Next we need the following technical result, on lifting log. structures to a formal neighbourhood. The proof is rather similar to that of the previous lemma.

Lemma 2.3. *If $X \longrightarrow S$ is a flat map of flat $\mathcal{O}_F$ schemes, $Z \subset X$ a closed subscheme, which is flat over S, and that the pair $(X_{\tilde{s}}, Z_{\tilde{s}})$ is $p^k l-$ semi-stable for some $\mathcal{O}_F$ valued point $\tilde{s}$ of S. If Y denotes the special fibre of $X_{\tilde{s}}$ then the usual log. structure on $X_{\tilde{s}}$, lifts canonically to a formal neighbourhood of Y in X, i.e to the completion $\hat{X}$ of X along Y.*

If we write $\hat{X} = \varinjlim X_n$, the X_n being nilpotent neighbourhoods of Y in X, then the maps $Y \longrightarrow X_n$ are induced by exact immersions of log. schemes.

Proof. Working etale locally on X, we may assume that X is affine, and that

$$X_{\tilde{s}} = \text{Spec } (\mathcal{O}_F[x_1, \dots x_r, y_1, \dots, y_s, z_1 \dots z_t]/(y_1 \dots y_s - \pi^{p^k l}),$$

with $Z_{\tilde{s}}$ cut out by $x_1, \dots x_r$. Writing $\hat{X} = \text{Spf } (A)$, we can lift the x_i, y_j, z_k to X_i, Y_j, Z_k in A such that $Y_1 \dots Y_s = \pi^{p^k l}$, assuming however that Y_s only lifts y_s modulo π. Writing $\hat{Z}$ for the completion of Z along Y, we may assume, as in the proof of 2.2 that the X_i' generate the ideal corresponding to $\hat{Z}$.

If $\hat{S} = \text{Spf } (E)$, is the completion of S at the closed point $\mathfrak{m}$ of $\tilde{s}$, write B for the $\mathfrak{m}$-adic completion of

$$E[P_1, \dots P_r, Q_1, \dots Q_s, R_1, \dots R_t]/(Q_1 \dots Q_s - \pi^{p^k l}).$$

We obtain a map $\phi : B \longrightarrow A$ defined by $P_i \mapsto X_i, Q_j \mapsto Y_j, R_k \mapsto Z_k$. Since A is $\mathfrak{m}$-adically complete, and ϕ is an isomorphism modulo $\mathfrak{m}$, ϕ is surjective. Since A is flat over E, ϕ is injective by [Mat 22.5], as it is modulo $\mathfrak{m}$.

Thus $A = B$, and B has a log. structure defined by the P_i, Q_j, R_k and π in the usual way. As usual one checks that this structure is independent of choices, and has the required properties. $\square$

The previous lemmas allow us to make an improvement to the argument preceding Lemma 2.2.

Proposition 2.4. *Let $f : \mathcal{X} \longrightarrow \mathcal{S}$ be a map of finite type F schemes, and $\mathcal{Z} \subset \mathcal{X}$ a divisor with normal crossings relative to $\mathcal{S}$. Let $s \in \mathcal{S}$ be a rational point, and suppose that for some finite extension F' of F, $\mathcal{X} \otimes_F F' \longrightarrow \mathcal{S} \otimes_F F'$ extends to a flat map of flat $\mathcal{O}_{F'}$ schemes $\bar{f} : X \longrightarrow S$, such that s extends to an $\mathcal{O}_{F'}$ valued point $\tilde{s}$, and the fibre $(X_{\tilde{s}}, Z_{\tilde{s}})$ is $p^k l$-semi-stable.*

Put $\mathcal{U} = \mathcal{X} - \mathcal{Z}$. For all rational points $t \in \mathcal{S}$, which are sufficiently close to s, there is a natural equivalence of categories $\mathsf{Et}^{(p)}_{\text{geom}}(\mathcal{U}_s) \xrightarrow{\sim} \mathsf{Et}^{(p)}_{\text{geom}}(\mathcal{U}_t)$, compatible with the Galois action on both sides.

Proof. Suppose first that $F' = F$. If t is an F- rational point of $\mathcal{S}$, which is sufficiently π-adically close to s, then it lifts to an $\mathcal{O}_F$ valued point $\tilde{t}$ of S, and by 2.3 we may assume that the fibre $(X_{\tilde{t}}, Z_{\tilde{t}})$ is also $p^k l$-semi-stable. Now the result follows from 1.5.

Now suppose $F' \neq F$. By [SGA 1, IX 4.10], we may replace F by $F' \cap F^{p^{-\infty}}$. Increasing F' leaves all our hypothesis intact, except that we may have to increase k and l. Thus, we may assume that F' is Galois over F. We have to show that the equivalence

$$\Phi : \mathsf{Et}^{(p)}_{\text{geom}}(\mathcal{U}_s) = \mathsf{Et}^{(p)}_{\text{geom}}(\mathcal{U}_s \otimes_F F') \xrightarrow{\sim} \mathsf{Et}^{(p)}_{\text{geom}}(\mathcal{U}_t \otimes_F F') = \mathsf{Et}^{(p)}_{\text{geom}}(\mathcal{U}_t)$$

is compatible with the action of $\mathrm{Gal}(F^{\text{sep}}/F)$ on both sides.

To see this we give an alternate description of Φ. Denote by $\hat{X}$ the completion of X along the closed fibre of $X_{\tilde{s}}$, and by $\hat{S}$ the completion of S at the closed point of $\tilde{s}$. We equip $\hat{X}$ with the log. structure given by the previous lemma.

We have equivalences of categories

$$\mathsf{Et}^{(p)}_{\text{geom}}(\mathcal{U}_s) \xrightarrow{\sim} \mathsf{Et}^{(p)}_{\text{log.geom}}(X_{\tilde{s}}, M_{X_{\tilde{s}}}) \xrightarrow{\sim} \mathsf{Et}^{(p)}_{\text{log.geom}}(\hat{X}, M_{\hat{X}}),$$

the first given by Theorem 1.1, and the second by restricting to the closed fibre of $X_{\tilde{s}}$, and then using [Kat 3.14] to lift coverings to $\hat{X}$. Here the $M's$ denote the log. structures on the corresponding schemes and formal schemes. Denote the composite by Γ.

The "generic fiber" of $\hat{X}$ is a rigid analytic space, and similarly for $\hat{S}$, (see [deJ 2, 7.2.5]), so that $\hat{X} \longrightarrow \hat{S}$ gives rise to a map of rigid analytic spaces $\hat{X}_\eta \longrightarrow \hat{S}_\eta$. The fibres of this map are the rigid analytic spaces associated to the fibres $\mathcal{X}_t \otimes_F F'$ of the map f for points t, which are sufficiently close to s. Thus we may restrict the a log. etale covering of $(\hat{X}, M_{\hat{X}})$ to a covering of such a fibre, which is etale over $\mathcal{U}_t \otimes_F F'$. Strictly speaking, we have only obtained a covering of the the rigid space associated to $\mathcal{U}_t \otimes_F F'$, however all such coverings are algebraic [Lut].

Thus for suitable t, we have constructed a functor

$$\Psi : \mathsf{Et}^{(p)}_{\text{log.geom}}(\hat{X}, M_{\hat{X}}) \longrightarrow \mathsf{Et}^{(p)}_{\text{geom}}(\mathcal{U}_t),$$

and one checks easily that $\Phi = \Psi \circ \Gamma$.

If $\sigma \in \mathrm{Gal}(F^{\text{sep}}/F)$, and σ^* denotes pulling back by σ, then we have to check that $\Phi \circ \sigma^* \xrightarrow{\sim} \sigma^* \circ \Phi$.

Since Ψ is simply restriction to a certain rigid analytic fibre, we clearly have $\Psi \circ \sigma^* = \sigma^* \circ \Psi$. Similarly, since Γ has a quasi-inverse, which is restriction to a suitable rigid analytic fibre, we obtain $\Gamma \circ \sigma^* \xrightarrow{\sim} \sigma^* \circ \Gamma$. Thus

$$\Phi \circ \sigma^* = \Psi \circ \Gamma \circ \sigma^* \xrightarrow{\sim} \Psi \circ \sigma^* \circ \Gamma \xrightarrow{\sim} \sigma^* \circ \Psi \circ \Gamma = \sigma^* \circ \Phi.$$

$\square$

From Proposition 2.4, we obtain immediately

Corollary 2.5. *Under the hypotheses of 2.4, for each rational F-rational point t of $\mathcal{S}$ denote by ρ_t the map*

$$Gal(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}^{(p)}(\mathcal{U}_t)).$$

If s is an F rational point of $\mathcal{S}$, $\rho_s = \rho_t$ for all F rational points t which are sufficiently close to s.

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