STRATIFICATIONS OF THE VARIETY OF $\mathcal{R}$-TRIVIAL MONOIDS

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ABSTRACT

In this paper we consider the variety $\mathbf{R}$ of $\mathcal{R}$-trivial monoids. After introducing the notion of a stratification of a variety, we introduce and use the catalan monoid of a directed graph to generate a stratification of $\mathbf{R}$. Other stratifications of $\mathbf{R}$ are given and the relationship between them is explored.

1. Introduction

Following the convention of Pin [7] we refer to a class of finite monoids closed under the operations of finite direct product, submonoid and homomorphic image as a *variety* of monoids. The variety of all monoids satisfying a set Σ of equations is denoted $[\![\Sigma]\!]$. If $\mathbf{V}$ is a variety of monoids then the class of all regular languages which are recognized by a monoid of $\mathbf{V}$ is denoted by $\mathcal{V}$ and is referred to as the *variety* of regular languages corresponding to $\mathbf{V}$. The set of all elements of $\mathcal{V}$ over the alphabet A is denoted by $A^*\mathcal{V}$. For definitions of automaton, transition monoid of an automaton and syntactic monoid of a language we refer the reader to [7].

Let $\mathbf{V}_0 \subset \mathbf{V}_1 \subset \ldots$ be a sequence of varieties, where $\mathbf{V}_0$ is the trivial variety and each containment is strict. Then $\bigcup_{n\geq 0} \mathbf{V}_n$ is some variety $\mathbf{V}$. We shall say that $(\mathbf{V}_n)$ is a *stratification* of $\mathbf{V}$ and that $\mathbf{V}_i$ is a *stratum* of $(\mathbf{V}_n)$ for each $i \geq 0$. If $(\mathbf{W}_n)$ is another stratification of $\mathbf{V}$ and $\mathbf{W}_i$ is a stratum of $(\mathbf{V}_n)$ for each i then we say that $(\mathbf{W}_n)$ is *coarser* than $(\mathbf{V}_n)$ and $(\mathbf{V}_n)$ is *finer* than $(\mathbf{W}_n)$. A *maximally fine* stratification is one in which each containment is a cover. In [6, pages 19–24] Neumann mentions examples of certain chains of (Birkhoff) varieties of groups whose union is another variety $\mathbf{V}$, and asks whether these chains are 'properly infinite' – there is no finite subchain whose union is $\mathbf{V}$.

Green's $\mathcal{R}$-relation is defined as follows. If $a, b \in S$ for some semigroup S, then we write $a\mathcal{R}b$ if there are $x, y \in S^1$ such that $ax = b$ and $by = a$. If $a\mathcal{R}b$ implies that $a = b$ for all pairs $a, b \in S$ then we say that S is $\mathcal{R}$-trivial.

Result 1.1. *[7, Chapter 3, Proposition 4.4] The class* $\mathbf{R}$ *of all* $\mathcal{R}$*-trivial monoids is a variety which is ultimately defined by the sequence of equations* $((xy)^n x = (xy)^n)_{n\geq 0}$.

In the sequel we show that $(\mathbf{R}_n)$ is a stratification of $\mathbf{R}$ where $\mathbf{R}_n := [\![(xy)^n x = (xy)^n]\!]$. Another stratification $(\mathbf{C}_n)$ of $\mathbf{R}$ is found by means of the catalan monoids of certain graphs, introduced in Section 2. We continue the work of Pin, Straubing and Therien [8] by exploring the relationship of $(\mathbf{C}_n)$ and $(\mathbf{R}_n)$ in the lattice of monoid varieties under $\mathbf{R}$. For each n we find the language variety corresponding to $\mathbf{C}_n$ as a refinement of the results of Brzozowski and Fich [2].

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2. The Catalan Stratification

In this section we define the catalan monoid of a directed graph and then use the catalan monoid of a directed acyclic graph to define a stratification of **R**.

2.1. The Catalan Monoid

In the sequel, by 'graph' we shall mean a finite directed graph. Denote the set of vertices of a graph G by $V(G)$ and by $E(G) \subseteq V(G) \times V(G)$ the set of edges of G. For each edge $(a, b) \in E(G)$ define

$$\begin{array}{rcl} f_{(a,b)} : V(G) & \longrightarrow & V(G) \\ a & \longmapsto & b \\ x & \longmapsto & x \text{ if } x \neq a. \end{array}$$

For each edge e we call f_e the *elementary transition* of e. We define the *catalan monoid* of G to be the monoid of transformations of $V(G)$ generated by the elementary transitions and denote it by $\mathcal{C}(G)$. In [5] Molchanov cites the work of Filippov [3] who studies the partial groupoid $\mathrm{Ex}(G)$ of transformations ϕ of $V(G)$ such that for each $x \in V(G)$ either $x\phi = x$ or $(x, x\phi) \in E(G)$. The catalan monoid of G is simply the monoid closure of $\mathrm{Ex}(G)$. The catalan monoid is so named because when $G = [n]$ the chain of n vertices, the cardinality of $\mathcal{C}(G)$ is the nth Catalan number, a fact which is proved in [4] and again in [9]. See [9] for presentations of various catalan monoids.

In what follows, we will restrict our attention to acyclic graphs. An acyclic graph gives rise to a partial order in a natural way : $x \geq y$ if there is a (possibly trivial) directed path from x to y.

Lemma 2.1. *The catalan monoid of an acyclic graph whose chains have at most $n+1$ vertices satisfies the set $\{(xy)^{n-i}x^i = (xy)^{n-i}x^{i+1} : 0 \leq i \leq n\}$ of equations.*

Proof. Suppose by way of contradiction that for some such G and some $i \leq n$, $\mathcal{C}(G)$ doesn't satisfy $(xy)^{n-i}x^{i+1} = (xy)^{n-i}x^i$. Then there is some vertex α of G, and $x, y \in \mathcal{C}(G)$ for which

$$\alpha(xy)^{n-i}x^i \neq \alpha(xy)^{n-i}x^{i+1} \tag{1}$$

therefore we must have

$$\alpha(xy)^{n-i}x^j > \alpha(xy)^{n-i}x^{j+1} \text{ for all } 0 \leq j \leq i \tag{2}$$

otherwise,

$$\alpha(xy)^{n-i}x^j = \alpha(xy)^{n-i}x^{j+1}$$

for some $j < i$, and by repeated right multiplication by x, for $j = i$, contradicting (1). Thus

$$\alpha(xy)^{n-i} > \alpha(xy)^{n-i}x > \cdots > \alpha(xy)^{n-i}x^{i+1}.$$

Suppose that $\alpha(xy)^{n-i-1} = \alpha(xy)^{n-i}$ then

$$\alpha(xy)^{n-i-1}x = \alpha(xy)^{n-i}x.$$

But since

$$\alpha(xy)^{n-i-1}x \geq \alpha(xy)^{n-i} \geq \alpha(xy)^{n-i}x$$

we have $\alpha(xy)^{n-i} = \alpha(xy)^{n-i}x$ contradicting (2). Thus $\alpha(xy)^{n-i-1} > \alpha(xy)^{n-i}$, and inductively,

$$\alpha > \alpha(xy) > \cdots > \alpha(xy)^{n-i} > \cdots > \alpha(xy)^{n-i}x^{i+1}$$

which is a chain of $n+2$ vertices, which is impossible, so the identity must hold. □

By Result 1.1 and setting $i = 0$ in the lemma above, we have

Corollary 2.2. *The catalan monoid of an acyclic graph G is $\mathcal{R}$-trivial.*

Of course we can observe this directly by noting that if $x\mathcal{R}y$, for some $x, y \in \mathcal{C}(G)$, then there are $a, b \in \mathcal{C}(G)$ such that $xa = y$ and $yb = x$, so for all $\alpha \in V(G)$ we have $\alpha x \geq \alpha y \geq \alpha x$, whence $x = y$.

2.2. The Catalan Stratification

In this section we refine the following theorem of Eilenberg to give a stratification of $\mathbf{R}$.

Theorem 2.3. *[7, Chapter 4, Theorem 3.3] For every alphabet A, $A^*\mathcal{R}$ consists of languages which can be written as disjoint unions of languages of the form $A_0^* a_1 A_1^* a_2 \dots a_n A_n^*$ where $n \geq 0$, $a_1, \dots, a_n \in A, A_i \subset A \setminus \{a_{i+1}\}$ for $0 \leq i \leq n-1$ and $A_n \subset A$.*

For each $n > 0$ define $\mathbf{C}_n$ to be the variety generated by all monoids $\mathcal{C}(G)$ where G is an acyclic graph with no more than $n+1$ vertices in a chain. We define $\mathbf{C}_0$ to be the trivial variety.

For $n > 0$, say that any language of A^* denoted by an expression of the form $A_0^* a_1 A_1^* a_2 \dots a_m A_m^*$ where $0 \leq m \leq n$, $A_i \subset A \setminus \{a_{i+1}\}$ and $A_m \subset A$, is of *type* X_n if $m < n$ and of *type* Y_n if $m = n$ and $A_m = A$. Define $A^*\mathcal{V}_n$ to be the boolean algebra (under finite union and intersection) of languages of A^* generated by all languages of type X_n and Y_n, and $A^*\mathcal{V}_0$ to be $\{\emptyset, A^*\}$. Let T_n be the rooted tree depicted below.

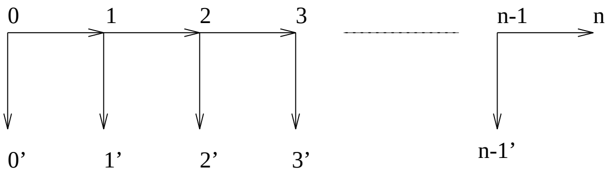

Figure 1: The rooted tree T_n with root 0.

Theorem 2.4. *Fix $n \geq 0$ and some alphabet A. Let L be a language of A^* and let M_L be the syntactic monoid of L, then the following conditions are equivalent.*
(C1) *L is an element of $A^*\mathcal{V}_n$.*
(C2) *M_L is an element of $\langle\mathcal{C}(T_n)\rangle$.*
(C3) *M_L is an element of $\mathbf{C}_n$.*
In particular $\mathbf{C}_n = \langle\mathcal{C}(T_n)\rangle$.

Proof. If $n = 0$ the result is immediate, so assume that $n > 0$.

(C1 $\Rightarrow$ C2) It suffices to prove this for languages of type X_n and Y_n since $M_{L\cup K}$ and $M_{L\cap K}$ both divide $M_L \times M_K$. Automata which accept languages of the form X_n and Y_n are exhibited in Figure 2.

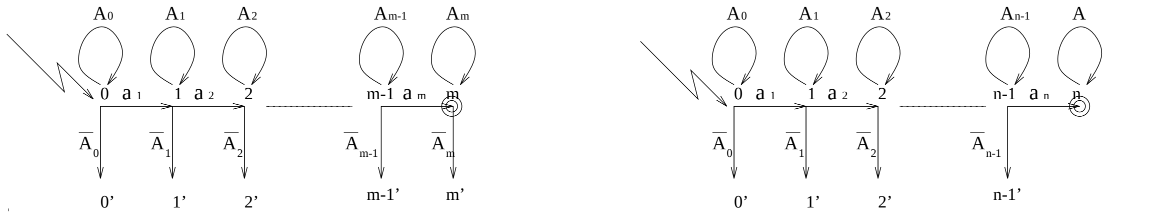

Figure 2: Automata accepting languages of type X_n and Y_n resp. where $\bar{A}_i := A \setminus (A_i \cup \{a_{i+1}\})$, and each i' is a failure state.

Since the transition monoids of the automata are both submonoids of $\mathcal{C}(T_n)$, the result follows.

(C2 $\Rightarrow$ C3) is immediate, so we shall prove (C3 $\Rightarrow$ C1) borrowing heavily from the proof in [7] of Theorem 2.3. Assume that M_L lies in $\mathbf{C}_n$ then there is some $k \geq 0$ such that M_L divides $S := \mathcal{C}(G_1) \times \dots \times \mathcal{C}(G_k)$, for some graphs G_i, so S recognizes L. Let $\varphi : A^* \to S$ and $P \subset S$ be such that $L = P\varphi^{-1}$. Since $L = \bigcup_{p\in P} p\varphi^{-1}$, it is enough to show that $p\varphi^{-1} = \bigcap_{i=1}^k p_i\varphi_i^{-1} \in A^*\mathcal{V}_n$ where $p = (p_1, \dots, p_k) \in P$, whence it is sufficient to show that each $p_i\varphi_i^{-1} \in A^*\mathcal{V}_n$. Thus in order to simplify notation, let G be some acyclic graph whose chains have no more than $n+1$ vertices, $L = p\varphi^{-1}$ for some fixed $p \in \mathcal{C}(G)$ and

$$\begin{array}{rcl}\varphi : A^* & \longrightarrow & \mathcal{C}(G) \\ w & \longmapsto & \bar{w}\end{array}$$

For each pair $i > j$ of vertices of G define $K_{i,j} = \{w \in A^* | i\bar{w} = j\}$, so that

$$p\varphi^{-1} = \bigcap_{i \in V(G)} K_{i,ip},$$

$$S(i,j) = \{(i_0, \ldots, i_k) | k \leq n, i = i_0 > i_1 > \cdots > i_k = j\}$$

and

$$A_{i,j} = \{a \in A | i\bar{a} = j\},$$

noting that $A_{i,i} = A$ if i is a *terminal* vertex, that is to say, there is no vertex j such that $(i,j) \in E(G)$. For $s = (i_0, \ldots, i_k) \in S(i,j)$ define

$$K_s = A^*_{i_0,i_0} A_{i_0,i_1} A^*_{i_1,i_1} \ldots A_{i_{k-1},i_k} A^*_{i_k,i_k}$$

which is the union of the languages $A^*_{i_0,i_0} a_1 A^*_{i_1,i_1} \ldots a_k A^*_{i_k,i_k}$ where $(a_1, \ldots, a_k)$ runs over $A_{i_0,i_1} \times \cdots \times A_{i_{k-1},i_k}$. Now if $k < n$ then this expression denotes a language of type X_n. On the other hand, when $k = n$, i_k must be terminal, so that $A_{i_k,i_k} = A$ and L is of type Y_n. Simply observing that

$$K_{i,ip} = \bigcup_{s \in S(i,ip)} K_s$$

completes the proof of the theorem. □

Theorems 2.3 and 2.4 immediately yield

Corollary 2.5. $\mathbf{R} = \bigcup_{n \geq 0} \mathbf{C}_n$

To see that $(\mathbf{C}_n)$ is a stratification of $\mathbf{R}$ it only remains to show that $\mathbf{C}_n \subset \mathbf{C}_{n+1}$ is strict containment. It is sufficient to notice that the element $f_{(n,n+1)} f_{(n-1,n)} \ldots f_{(0,1)}$ of $\mathcal{C}(T_{n+1})$ has index $n+1$, which is impossible for any element of a monoid in $\mathbf{C}_n$ by Lemma 2.1.

It is interesting to compare the corollary above with Higgins' Theorem [4] which states that $\mathbf{R} = \langle PC_n | n \geq 0\rangle$, where PC_n is the monoid of all partial, non-increasing, order preserving transformations of $[n]$. Another similar result is a consequence of Simon's Theorem on piecewise testable languages in [7] which states that $\langle \mathcal{C}([n]) | n \geq 0\rangle = \mathbf{J}$, the variety of $\mathcal{J}$-trivial monoids. By an argument which is almost identical to the proof of Theorem 2.4 we have

Scholium 2.6. *Fix $n \geq 1$ and some alphabet A. Let L be a language of A^* and let M_L be the syntactic monoid of L, then the following conditions are equivalent.*
(J1) *L is an element of the boolean algebra generated by languages of the form $A^* a_1 A^* a_2 \ldots a_m A^*$, where $m < n$.*
(J2) *M_L is an element of $\langle \mathcal{C}([n])\rangle$.*
(J3) *M_L is in the variety generated by all monoids $\mathcal{C}(G)$ where G is an acyclic graph with no more than n vertices in any chain, and no vertex with out-degree greater than 1.*

As with $\mathbf{R}$ this gives rise to a stratification of $\mathbf{J}$.

2.3. Connections with the work of Brzozowski and Fich

Expanding on Theorem 2.3 of Eilenberg, Brzozowski and Fich [2] characterize the variety of $\mathcal{R}$-trivial monoids, the variety $\mathcal{R}$ of languages corresponding to them, and the reduced finite state automata accepting these languages.

A *partially ordered* automaton with input alphabet A is defined to be one in which the relation $\geq$ on the state set defined by $i \geq j$ if there is some $w \in A^*$ which labels a path from i to j, is a partial order. Now it is clear that an automaton is partially ordered if and only if its state graph is acyclic (when trivial loops are removed). Furthermore one can see that the transition monoid of such an automaton with state graph G is a submonoid of $\mathcal{C}(G)$.

Let L be a language over A and $\mathcal{A}_L$ the reduced automaton accepting L, then in [2] it is shown that $L \in A^*\mathcal{R}$ if and only if $\mathcal{A}_L$ is partially ordered. This is another means of seeing Corollary 2.5. As a result of these observations we have

Proposition 2.7. *If $\mathcal{A}_L$ is partially ordered and its state graph has no chains with more than $n+1$ vertices, then $L \in A^*\mathcal{V}_n$.*

One would not expect the converse to follow since taking the direct product of automata does not preserve maximum chain length of state graphs.

3. The Equational Stratification

Define the variety $\mathbf{R}_n = [\![(xy)^n x = (xy)^n]\!]$ for each $n \geq 0$. To see that $\mathbf{R}_n \subset \mathbf{R}_{n+1}$ is proper containment, simply notice that $\mathcal{C}(T_{n+1})$ is in $\mathbf{R}_{n+1}$ but not in $\mathbf{R}_n$. Result 1.1 simply states that $\mathbf{R} = \bigcup_{n\geq 0}\bigcap_{m\geq n}\mathbf{R}_m$ but since $\mathbf{R}_n \subset \mathbf{R}_{n+1}$, we have $\mathbf{R} = \bigcup_{n\geq 0}\mathbf{R}_n$. Thus, $(\mathbf{R}_n)$ is a stratification of $\mathbf{R}$, each stratum of which is an equational variety. In [7, Chapter 2, Corollary 1.3] it is noted that a variety which is generated by a single monoid is equational, so the strata of $(\mathbf{C}_n)$ are also equational. Noting that $\mathbf{C}_n \subset \mathbf{R}_n$ it is tempting to ask whether $\mathbf{C}_n = \mathbf{R}_n$ for all n. A positive answer is not excluded by Lemma 2.1, for though we cannot deduce the equations which hold in $\mathbf{C}_n$ from the defining equation of $\mathbf{R}_n$, it is not clear that $\mathbf{C}_n$ is properly contained in $\mathbf{R}_n$, as it would be if we were dealing with Birkhoff varieties (see [1]). Observe that $\mathbf{C}_1 = \mathbf{R}_1$ by [8, Theorem 2.6]. However this is the only case where equality holds.

Proposition 3.1. *If $n > 1$ then $\mathbf{C}_n$ is properly contained in $\mathbf{R}_n$.*

Proof. Let $n > 1$. We shall exhibit a monoid $M \in \mathbf{R}_n \setminus \mathbf{C}_n$. Let M be the syntactic monoid of the language $L := \{(ab)^{n-1}a\}$ over the alphabet $A = \{a, b\}$. Now if M were in $\mathbf{C}_n$ then $M \models (xy)^{n-1}x^2 = (xy)^{n-1}x$ by Lemma 2.1. But this would imply that $(ab)^{n-1}a$ and $(ab)^{n-1}a^2$ have the same image under the syntactic morphism, which means that either they are both in, or both not in, L, which is a contradiction.

Now we show that $M \models (xy)^n x = (xy)^n$. Let B be any alphabet, $\varphi : B^* \to A^*$ be any morphism, let $u, v \in A^*$, and let $x, y \in B$. Let S be the statement $u((xy)^n x)\varphi v \in L$ and T be the statement $u((xy)^n)\varphi v \in L$. It is sufficient to show that S if and only if T.

If $x\varphi = 1$ then S *iff* T is immediate. Suppose that $|x\varphi| \geq 1$.

If $|x\varphi| > 1$ or $|y\varphi| \geq 1$ then $u((xy)^n x\varphi)v$ and $u((xy)^n\varphi)v$ both have length strictly greater than $2n-1$ which is the length of $(ab)^{n-1}a$, so S and T are both false.

The only remaining possibility is that $|x\varphi| = 1$ and $y\varphi = 1$. In this case $u((xy)^n x\varphi)v = uc^{n+1}v$ and $u((xy)^n\varphi)v = uc^n v$ for some $c \in A$. But neither can be a word of L since they both have repeated letters, so once again, S and T are both false. $\square$

It remains to determine the precise relationship between the two stratifications of $\mathbf{R}$ in the lattice of varieties. Certainly, by virtue of [8, Theorem 2.6] we have

$$\mathbf{C}_1 = \mathbf{R}_1 \subset \mathbf{C}_2 \subset \mathbf{R}_2 \tag{3}$$

So it is natural to ask whether $\mathbf{R}_{n-1} \subset \mathbf{C}_n \subset \mathbf{R}_n$ for all n, forming a chain in the sublattice of varieties under $\mathbf{R}$. By use of a method similar to that of the proof of Proposition 3.1 we find that in general this is not so.

Proposition 3.2. *The variety $\mathbf{C}_n$ does not contain the variety $\mathbf{R}_{n-1}$ if $n > 4$.*

Proof. Fix $n > 4$. Let $L = \{(ab)^2 a^{n-2}\}$ be the language over $A = \{a, b\}$, consisting of a single word of length $n+2$. Now the syntactic monoid M of L cannot be in $\mathbf{C}_n$, for Lemma 2.1 would again imply, when $i = n-2$ that $(ab)^2 a^{n-1}$ and $(ab)^2 a^{n-2}$ are either both in, or both not in, L. It remains to show that $M \models (xy)^{n-1}x = (xy)^{n-1}$. Let B, φ, x, y, u and v be as in the proof of the previous proposition. Let S be the statement $u((xy)^{n-1}x)\varphi v \in L$ and let T be the statement $u((xy)^{n-1})\varphi v \in L$. Once again we show S *iff* T.

If $x\varphi = 1$ the result is immediate. If $x\varphi$ and $y\varphi$ are both nontrivial, then $|u((xy)^{n-1}\varphi)v| \geq 2n-2$ which is strictly greater than $n+2$. Thus S and T are both false.

If $x\varphi \neq 1$ but $y\varphi = 1$ then either (a) $|x\varphi| = 1$ in which case $u((xy)^{n-1}\varphi)v = uc^{n-1}v$ for some $c \in A$ so that S and T are both false, or (b) $|x\varphi| > 1$ in which case $|u((xy)^{n-1}\varphi)v| \geq 2n-2$ which is strictly greater than $n+2$ when $n > 4$, so that S and T are both false. □

3.1. Open Questions

Problem 3.3. *Does $\mathbf{C}_3$ contain $\mathbf{R}_2$? $\mathbf{C}_4$ contain $\mathbf{R}_3$?*

Problem 3.4. *Fix $n > 1$. Is there any m such that $\mathbf{R}_n \subset \mathbf{C}_m$?*

Problem 3.5. *Is $(\mathbf{C}_n)$ a maximally fine stratification?*

By (3) it is clear that $(\mathbf{R}_n)$ is not maximally fine.

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