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Catalan Monoids, Monoids of Local Endomorphisms, and their Presentations

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Abstract

The Catalan monoid and partial Catalan monoid of a directed graph are introduced. Also introduced is the notion of a local endomorphism of a tree, and it is shown that the Catalan (resp. partial Catalan) monoid of a tree is simply its monoid of extensive local endomorphisms (resp. partial endomorphisms) of finite shift. The main results of this paper are presentations for the Catalan and partial Catalan monoids of a tree. Our presentation for the Catalan monoid of a tree is used to give an alternative proof for a result of Higgins. We also identify results of Aĭzenštat and Popova which give presentations for the Catalan monoid and partial Catalan monoid of a finite symmetric chain.

1. Introduction

This paper is entirely concerned with the Catalan monoid of a directed graph, introduced by the author in [16]. We take this opportunity, in Section 3, to give some justification for its naming, and to define the partial Catalan monoid of a directed graph. Certain Catalan monoids have already been studied extensively, in particular, those which consist of order-preserving transformations of $[n]$, where $[n]$ denotes the set $\{1,\ldots,n\}$. Monoids of order-preserving transformations of $[n]$ are studied by Aĭzenštat [1] and Popova [14] who find presentations; Gomes and Howie [4], Howie [10] and Higgins [5] who give combinatorial results; and Higgins [6] and Vernitskii and Volkov [18] who discuss their divisors.

2. Preliminaries

Let ϕ be a (partial) transformation of a set X. Denote by $\Delta(\phi)\subseteq X$ the domain of ϕ. If $\Delta(\phi)=X$ we shall say that ϕ is a *full* transformation of X. The *shift* of ϕ is the set $\mathrm{Sh}(\phi)=\{x\in X \mid x\notin\Delta(\phi) \text{ or } x\phi\neq x\}$.

A *directed graph* G is a pair $(V(G),E(G))$ where $E(G)\subseteq V(G)\times V(G)$. We write $x\to y$ if $(x,y)\in E(G)$. Call $V(G)$ the set of *vertices* of G and $E(G)$ the set of *edges* of G. Say that G is a *symmetric* graph if $\to$ is a symmetric relation on $V(G)$.

Unless otherwise stated, the word 'graph' shall mean a directed graph, and in the sequel we will consider only graphs G which have the following properties : $V(G)$ is countable; each vertex of G has finite in–degree and out–degree; and G has no loops, where a *loop* is an edge $x\to x$ for some $x\in V(G)$.

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A *path* from x to y for $x, y \in V(G)$ is a finite sequence of edges $x \to x_1 \to \cdots \to x_n = y$ where $n \geq 0$. We say that the path thus described has *length* n. If $n = 0$ we say that the path is *trivial*. This should be distinguished from a loop on $x \in V(G)$ which is a path of length 1 from x to x. A *cycle* on $x \in G$ is a path from x to x of length $n > 1$ where $x_i \neq x$ for any $1 \leq i < n$.

Say that a graph is *acyclic* if it has no cycles. It is clear that an acyclic graph gives rise to a partial order on the set of vertices by defining $x \geq y$ if there is a path from x to y.

Let G be a graph and ϕ a transformation of $V(G)$. We say that ϕ is *extensive* if, for each $x \in \Delta(\phi)$, there is a (possibly trivial) path from x to $x\phi$. In an acyclic graph, extensive transformations are also called *non-increasing* since $x \geq x\phi$ for all $x \in \Delta(\phi)$. If G is acyclic and for each $x, y \in \Delta(\phi)$ such that $x \geq y$ we also have $x\phi \geq y\phi$, then we say that ϕ is *order-preserving* or an *order-endomorphism.*

3. Catalan Monoids

Definition 3.1. For each edge $s = (a, b) \in E(G)$, let f_s label the transformation $\overline{f_s}$ of $V(G)$ defined by

$$\begin{aligned} a &\longmapsto b \\ x &\longmapsto x \text{ if } x \neq a, \end{aligned}$$

called the *elementary transition* of s. Let the set of all such labels be denoted by Σ_G. We define the *Catalan monoid* $\mathcal{C}(G)$ of G to be the monoid of transformations of $V(G)$ generated by the elementary transitions.

Remark 3.2. In [12] Molchanov cites the work of Filippov [3] who studies the partial groupoid $\mathrm{Ex}(G)$ of transformations ϕ of $V(G)$ such that for each $x \in V(G)$ either $x\phi = x$ or $(x, x\phi) \in E(G)$. Note that the Catalan monoid of G is not simply the monoid closure of $\mathrm{Ex}(G)$. For example, $\mathrm{Ex}(1 \leftrightarrow 2)$ contains the permutation which interchanges vertices 1 and 2, where $\mathcal{C}(1 \leftrightarrow 2)$ does not.

Example 3.3. It is clear that if G is the graph $1 \leftarrow 2 \leftarrow \cdots \leftarrow n$ then every element of $\mathcal{C}(G)$ is an order-preserving, non-increasing transformation of $[n]$. In fact we shall see in the next section (Corollary 4.11) that $\mathcal{C}(G)$ is precisely the monoid of order-preserving, non-increasing transformations of $[n]$. Under this assumption, we are able to offer an alternative proof of the following proposition, which appears in [5], and justifies the name 'Catalan monoid'.

Proposition 3.4. *Let G be the graph $1 \leftarrow 2 \leftarrow \cdots \leftarrow n$. Then the cardinality of $\mathcal{C}(G)$ is the nth Catalan number (defined in [15], page 101).*

Proof. If we write an element ϕ of $\mathcal{C}(G)$ as the pair of sequences $M_1 < M_2 < \cdots < M_k$ and $q_1 < q_2 < \cdots < q_k$ where M_i is the top element of the kernel class of points which map to q_i under ϕ (an example of which is illustrated below)

then it is easy to see that $\mathcal{C}(G)$ is in bijection with the set of balanced n-bracketings by

$$\phi \mapsto (^{M_1})^{q_2-q_1}(^{M_2-M_1})^{q_3-q_2}\ldots)^{q_k-q_{k-1}}(^{M_k-M_{k-1}})^{n+1-q_k}$$

where $(^q$ and $)^q$ denote $\underbrace{(\ldots(}_{q}$ and $\underbrace{)\ldots)}_{q}$ respectively. Thus the number of elements of $\mathcal{C}(G)$ is the nth Catalan number. ∎

Definition 3.5. For each vertex $x \in V(G)$, let e_x label the (partial) transformation $\overline{e}_x : V(G) \to V(G)$ such that $\overline{e}_x$ is the identity mapping on the domain $V(G) \setminus \{x\}$, and is called the *elementary annihilator* of x. Denote by Ξ_G the set $\{e_x : x \in V(G)\}$. We define the *partial Catalan monoid* $\mathcal{PC}(G)$ of G to be the monoid of transformations of $V(G)$ generated by the elementary transitions and elementary annihilators.

We extend the map $\bar{\cdot}$ to a map from $(\Sigma_G \cup \Xi_G)^*$, the free monoid on letters f_s and e_x, for $s \in E(G), x \in V(G)$, onto $\mathcal{PC}(G)$. Notice that this restricts to a map from ${\Sigma_G}^*$ onto $\mathcal{C}(G)$.

4. Extensive Local Endomorphisms of Trees

In the sequel, unless otherwise stated, all graphs will be acyclic, every transformation will have finite shift and 'endomorphism' will be taken to mean 'order-endomorphism' or 'order-preserving transformation'.

Remark 4.1. Contrast the definition of an order-endomorphism of an (acyclic) graph with the usual notion of *graph-endomorphism* : a transformation ϕ of $V(G)$ such that for all $x, y \in \Delta(\phi)$, $(x, y) \in E(G)$ implies that $(x\phi, y\phi) \in E(G)$.

Definition 4.2. A graph whose underlying undirected graph has no loops or cycles is a *tree*. Notice that, by this definition, a tree may be disconnected.
In this section we give alternative descriptions of the Catalan monoid and partial Catalan monoid of a tree. In order to do this we will require some further notation concerning trees.

Let x and y be vertices of a tree T. If $x \geq y$ then, by the uniqueness of paths in a tree, there is a unique subchain of T of finite length, beginning in x and ending in y, say $x = x_1 \to x_2 \to \cdots \to x_k = y$. Define a subset of $V(T)$ by

$$[x, y] = \begin{cases} \{x_1, x_2, \ldots, x_k\} & \text{if } x \geq y \\ \emptyset & \text{otherwise.} \end{cases}$$

The same set, without the endpoints is denoted (x, y). The sets $(x, y]$ and $[x, y)$ are defined in the obvious way. When $x \geq y$ as above, we are able to define the word $v(x, y)$ in ${\Sigma_T}^*$ to be $f_{(x_1,x_2)}f_{(x_2,x_3)}\cdots f_{(x_{k-1},x_k)}$ if $k > 1$ or the empty word if $k = 1$. A vertex in a tree which has in-degree 0 is called a *root*, and a vertex with out-degree 0 is called a *leaf*.

Definition 4.3. Let T be any tree and ϕ a transformation of $V(T)$ with finite shift. For any chain C of T define the subset

$$X_\phi^C := \{x \in \Delta(\phi) \cap V(C) \mid x\phi \in V(C)\}$$

of $V(C)$. Let ϕ_C be the restriction of ϕ to X_ϕ^C. Then, of all transformations of $V(C)$ which are restrictions of ϕ, ϕ_C has the largest domain, and we can call it the *localization* of ϕ to C. If the localization of ϕ to every chain of T is a (partial) endomorphism, then we say that ϕ is a *local endomorphism* of T.

Define $\mathcal{LE}(T)$ to be the monoid of all extensive full transformations of $V(T)$ which are local endomorphisms and let $\mathcal{LPE}(T)$ be the monoid of all extensive full or partial transformations of $V(T)$ which are local endomorphisms. It is clear that for any tree T the elementary transitions of T are in $\mathcal{LE}(T)$ and the elementary transitions and annihilators are in $\mathcal{LPE}(T)$ so that, $\mathcal{C}(T) \subseteq \mathcal{LE}(T)$ and $\mathcal{PC}(T) \subseteq \mathcal{LPE}(T)$. The rest of this section is devoted to showing that these containments are actually equalities.

Remark 4.4. The notation $\mathcal{LE}(T)$ and $\mathcal{LPE}(T)$ may be regarded as extending the notation of [13], where the monoid of all extensive full transformations of the chain $[n] = 1 \leftarrow 2 \leftarrow \cdots \leftarrow n$ is denoted $\mathcal{E}_n$. For any graph G, we denote by $\mathcal{E}(G)$ ($\mathcal{PE}(G)$) the monoid of all extensive full (full or partial) transformations of $V(G)$.

We now make a definition which ensures that every element of $\mathcal{LPE}(T)$ can be regarded as a transformation of some finite subtree of T.

Definition 4.5. Let T be a tree and $\phi \in \mathcal{LPE}(T)$. The *cambium* T^ϕ of T under ϕ is a tree with vertex set

$$\{x \in V(T) \mid x \in \mathrm{Sh}(\phi) \text{ or, for some } y \in \mathrm{Sh}(\phi), y \to x \in E(T)\}.$$

We define the edges of T^ϕ to be $(V(T^\phi) \times V(T^\phi)) \cap E(T)$. In a sense which is made precise in the following proposition, the cambium is the subtree of T which is 'changed' by ϕ.

Proposition 4.6. *Let T be a tree and $\phi \in \mathcal{LPE}(T)$. Then:*
(a) T^1 is the empty tree;
(b) $V(T^\phi)$ is finite;
(c) for each $w \in (\Sigma_T \cup \Xi_T)^*$, $w \in (\Sigma_{T^{\overline{w}}} \cup \Xi_{T^{\overline{w}}})^*$.
Thus we can regard ϕ quite naturally as an element of $\mathcal{LPE}(T^\phi)$.

Proof. Part (a) is immediate and part (b) follows from the fact that $\mathrm{Sh}(\phi)$ is finite and each vertex in T has finite out-degree. To see (c), suppose that $f_{(x,y)}$ is a letter of w for some $(x,y) \in E(T)$. Clearly $x \neq x\overline{w}$ so that $x \in \mathrm{Sh}(\overline{w}) \subset V(T^{\overline{w}})$. Since $(x,y) \in E(T)$, $y \in V(T^{\overline{w}})$ as well.

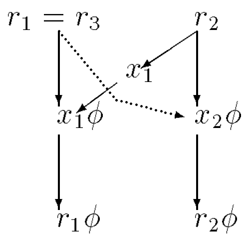
Suppose e_x is a letter of w for some $x \in V(T)$. Write $w = ue_xv$. If $x \in \Delta(\overline{w})$ then $x\overline{u} \neq x$ so that $x \in \mathrm{Sh}(\overline{w}) \subset V(T^{\overline{w}})$. If $x \notin \Delta(\overline{w})$ then $x \in \mathrm{Sh}(\overline{w}) \subset V(T^{\overline{w}})$ as required. ∎

Definition (simple root). Let $\phi \in \mathcal{LPE}(T)$ and r be a root of T. Then we say that r is a *simple root* for ϕ if $r \notin \Delta(\phi)$ or $[r, r\phi) \cap \mathrm{Im}(\phi) = \emptyset$.

Lemma 4.7. *Every $\phi \in \mathcal{LPE}(T)$ has a simple root in each connected component of T^ϕ.*

Proof. Suppose to the contrary that in some connected component of T^ϕ, for every root vertex r_1, $r_1 \in \Delta(\phi)$ and there is another vertex x_1 such that $r_1\phi < x_1\phi \le r_1$. If $x_1 < r_1$ then $r_1\phi < x_1\phi \le x_1 < r_1$ which contradicts the fact that ϕ is an endomorphism when localized to a chain containing $r_1\phi, x_1\phi, x_1$ and r_1. Thus $x_1 \le r_2$ for some root $r_2 \ne r_1$. By assumption, there is another vertex x_2 such that $r_2\phi < x_2\phi \le r_2$, which lies under a root $r_3 \ne r_2$, and so on, yielding a sequence of roots. By finiteness of T^ϕ, there must be some positive integers k and $l > 1$ such that $r_k = r_{k+l}$, which implies that there is a non-trivial cycle in the underlying undirected graph of T^ϕ, contradicting the fact that T^ϕ is a tree. ∎

For example, such a cycle occurs when $k = 1$ and $l = 2$, as depicted below. (Arrows indicate paths of 0 or more edges.)



In the sequel, as above, we use the symbols $<, \le, >, \ge$ to refer to the natural partial order on $V(T)$. For another order θ on $V(T)$ we will write $x >_\theta y$ for $(x, y) \in \theta$.

Let θ be a total order on $V(T)$ which extends the natural partial order. Then we say that θ is a *natural total order* on $V(T)$. Fix a natural total order θ on $V(T)$.

Definition (sequence decomposition). Let $\phi \in \mathcal{LPE}(T)$. Then we define the *sequence decomposition* $Q(\phi)$ of ϕ to be the sequence $(\phi_i)_i$, constructed as follows. Let y be the θ-maximum simple root of T^ϕ for ϕ. Define ϕ' to be the map with domain $\Delta(\phi) \cup \{y\}$, and action given by:

$$x\phi' = \begin{cases} x & \text{if } x = y \\ x\phi & \text{otherwise.} \end{cases}$$

If ϕ is the identity, then $Q(\phi)$ is the empty sequence, otherwise $Q(\phi)$ is the sequence $Q(\phi')$ with ϕ appended. That this recursive definition makes sense is a result of Proposition 4.8 (b) below, which assures us that $\phi' \in \mathcal{LPE}(T)$.

In general we will write the sequence decomposition of ϕ as $(\phi_1, \ldots, \phi_n)$ where $\phi_n = \phi$, and we will write the θ-maximum simple root of T^{ϕ_i} for ϕ_i as x_i.

For any element $\phi \in \mathcal{LPE}(T)$ we can now define $f_\theta(\phi) \in (\Sigma_T \cup \Xi_T)^*$ to be the empty word if $\phi = 1$, otherwise $f_\theta(\phi) = v_1 \ldots v_n$, where

$$v_i = \begin{cases} e_{x_i} & \text{if } x_i \notin \Delta(\phi) \\ v(x_i, x_i\phi) & \text{otherwise.} \end{cases}$$

Proposition 4.8. *Let $\phi \in \mathcal{C}(T)$. We have the following facts about the sequence decomposition of ϕ:*
(a) $\mathrm{Sh}(\phi) = \{x_1, \ldots, x_n\}$*;*
(b) for each $1 \leq i \leq n$, $\phi_i \in \mathcal{LPE}(T)$ and the sequence decomposition of ϕ_i consists of the first i terms of the sequence decomposition of ϕ;
(c) $\overline{f_\theta(\phi)} = \phi$.

Proof. (a) Recall the definition of T^{ϕ_i}. Since x_i is a root of T^{ϕ_i}, there is no vertex y such that $y \to x_i$ is an edge of T^{ϕ_i}. Thus x_i must be in $\mathrm{Sh}(\phi_i) \subseteq \mathrm{Sh}(\phi)$.

Conversely, suppose $x \in \mathrm{Sh}(\phi) = \mathrm{Sh}(\phi_n)$. Let i be minimal such that $x \in \mathrm{Sh}(\phi_i)$. Now, either $x \notin \Delta(\phi_i)$ or $x\phi_i \neq x$. If $x \notin \Delta(\phi_i)$ then, by assumption, $x \in \Delta(\phi_{i-1})$, whence $x\phi_{i-1} = x$ and $x = x_i$. Otherwise, $x\phi_i \neq x$ while $x\phi_{i-1} = x$. Once again, by the definition of the sequence decomposition of ϕ, $x = x_i$.

(b) To prove that each element in the sequence decomposition is in $\mathcal{LPE}(T)$, we simply show that if ϕ_i is then so is ϕ_{i-1}. Suppose not, then there is a chain C in T containing x_i and some vertex y such that ϕ_i is order preserving on C but ϕ_{i-1} reverses the order of x_i and y. If $x_i > y$ then $y\phi_{i-1} > x_i = x_i\phi_{i-1}$ and $y\phi_{i-1} = y\phi_i > y$, which is impossible since $\phi_i \in \mathcal{LPE}(T)$. Otherwise $y > x_i$ and $y\phi_{i-1} < x_i = x_i\phi_{i-1}$, whence $y \in \mathrm{Sh}(\phi_{i-1}) \subset \mathrm{Sh}(\phi_i) \subseteq V(T^{\phi_i})$. Now since x_i is a root of a connected component of T^{ϕ_i}, y must be in a different connected component. But $x_i \in (y, y\phi_i = y\phi_{i-1})$ and since T is a tree, $[y, y\phi_i) \subseteq \mathrm{Sh}(\phi_i)$. Thus, x_i and y belong to the same connected component, which is a contradiction. The second part of (b) is now immediate from the recursive definition of the sequence decomposition.

We prove (c) by induction on n. If $n = 0$ the result is immediate, so assume $n > 0$. From (b) we have $f_\theta(\phi_n) = f_\theta(\phi_{n-1})v_n$ so that $\overline{f_\theta(\phi_n)} = \phi_{n-1}\overline{v_n}$, by the inductive hypothesis. We proceed by calculating $\overline{f_\theta(\phi_n)}$. First we calculate its domain. If $x \in \Delta(\phi_n)$ then certainly $x \in \Delta(\phi_{n-1})$. Were $x\phi_{n-1} \notin \Delta(\overline{v_n})$ then $x\phi_{n-1}$ would be x_n, and $v_n = e_{x_n}$. But since x_n is a root of T^{ϕ_n}, the only vertex which maps to x_n under ϕ_{n-1} is x_n itself. That $x = x_n$ would then contradict the fact that $x \in \Delta(\phi_n)$, thus $x \in \Delta(\overline{f_\theta(\phi_n)})$. Conversely, if $x \notin \Delta(\phi_n)$ then either $x \notin \Delta(\phi_{n-1})$ or $x = x_n$ and $v_n = e_{x_n}$. In either case, $x \notin \Delta(\overline{f_\theta(\phi_n)})$. Thus we have shown that $\Delta(\overline{f_\theta(\phi_n)}) = \Delta(\phi_n)$.

Now we show that $x\phi_n = x\overline{f_\theta(\phi_n)}$ for each $x \in \Delta(\phi_n)$. If $x = x_n$, then $x\phi_{n-1}\overline{v(x_n, x_n\phi)} = x\overline{v(x_n, x_n\phi)} = x\phi_n$. Suppose, therefore, that $x \neq x_n$, so that $x\phi_{n-1} = x\phi_n$. If $x \in V(T^{\phi_n})$ then $x\phi_n \notin [x_n, x_n\phi_n)$ since x_n is a simple root of T^{ϕ_n} for ϕ_n, whence $x\overline{f_\theta(\phi_n)} = x\phi_n\overline{v_n} = x\phi_n$. Otherwise $x \notin V(T^{\phi_n})$ so that $x\phi_n = x$. Certainly $x\phi_{n-1} = x$ and if $x \in [x_n, x_n\phi_n)$ then $x \in V(T^{\phi_n})$ — a contradiction. Thus, $x\overline{f_\theta(\phi_n)} = x\phi_{n-1}\overline{v_n} = x\overline{v_n} = x = x\phi_n$, as required. ∎

Theorem 4.9. $\mathcal{PC}(T) = \mathcal{LPE}(T)$ *and* $\mathcal{C}(T) = \mathcal{LE}(T)$*. In other words, the (partial) Catalan monoid of a tree consists of all extensive (partial) transformations of finite shift which are local endomorphisms.*

Proof. As noted above, it is clear that $\mathcal{PC}(T) \subseteq \mathcal{LPE}(T)$ and $\mathcal{C}(T) \subseteq \mathcal{LE}(T)$. The reverse inclusions are a result of Proposition 4.8 (c) and the fact that $f_\theta(\phi) \in \Sigma_T{}^*$ if $\phi \in \mathcal{LE}(T)$. ∎

Remark 4.10. The theorem does not hold if T is not a tree. Consider, by way of example, the graph G with vertices $\{1, 2, 3\}$ and edges $\{1 \to 2, 2 \to 3, 1 \to 3\}$. The generator $\overline{f_{(1,3)}}$ of $\mathcal{C}(G)$ is certainly not an endomorphism of the chain $1 \to 2 \to 3$.

As foreshadowed in the previous section we have the following corollary, which previously appeared in [1] for the case of full transformations, and [14] for the case of partial transformations.

Corollary 4.11. *The (partial) Catalan monoid of the graph $1 \leftarrow 2 \leftarrow \cdots \leftarrow n$ consists of all extensive (partial) transformations which are endomorphisms.*

5. Presentations of monoids

Let A be an alphabet and let R be a subset of A^*, referred to as a set of relations on A^*. The smallest congruence on A^* generated by R is denoted $R^\sharp$, and the monoid $A^*/R^\sharp$, is said to be given by the presentation with generating set A and relations R. We will often write this monoid as A^*/R or simply $\langle A \mid R\rangle$. Let u and v be two words in A^*. If they are identical, then we write $u = v$. If one may be obtained from the other by a finite number of applications of the relations R, then we write $u \sim v$, and say that they belong to the same congruence class of A^* with respect to $R^\sharp$. Denote the congruence class containing u by $[u]$. Then the element of $A^*/R^\sharp$ represented by u is $[u]$ and $u \sim v$ by the relations R if and only if $[u] = [v]$ in $A^*/R^\sharp$.

Definition 5.1. Let $f : A^* \to A^*$ be a function such that for all $u, v \in A^*$, $f(u) \sim u$ and $u \sim v$ implies $f(u) = f(v)$. Then we say that f is a *canonical form function* for $A^*/R^\sharp$, and $f(u)$ is the *canonical form* of u.

We will make use of the following well-known general facts about presentations without comment. The proofs are routine, and are omitted.

Theorem 5.2. *Let A and B be disjoint alphabets with $R \subseteq A^* \times A^*$ and $S \subseteq B^* \times B^*$. Then*

$$\langle A \mid R\rangle \coprod \langle B \mid S\rangle \cong \langle A \cup B \mid R, S\rangle.$$

Note that in the theorem above, $\coprod$ denotes the coproduct in the category of monoids which is the same as the free product of monoids.

Theorem 5.3. *Let A be an alphabet and let R and S be subsets of $A^* \times A^*$. Then*

$$\langle A \mid R, S\rangle \cong \langle A \mid R\rangle / \widehat{S}$$

where $\widehat{S} = \{([w_1], [w_2]) \mid (w_1, w_2) \in S\}$, and $[w_1], [w_2] \in \langle A \mid R\rangle$.

6. Presentations for the Catalan monoids of trees

Fix some tree T.

Proposition 6.1. *The submonoid of $\mathcal{PC}(T)$ generated by the elementary annihilators is isomorphic to the semilattice of subsets of $V(T)$ under intersection which has presentation*

$$\mathsf{P}(T) = \langle e_x \in \Xi_T \mid e_x^2 = e_x (\mathsf{P1}), e_x e_y = e_y e_x (\mathsf{P2}) \rangle$$

where e_x represents the subset $V(T) \setminus \{x\} = \Delta(\overline{e_x})$. ∎

If we have a presentation for $\mathcal{PC}(T)$ in generators $f_s \in \Sigma_T$ and $e_x \in \Xi_T$ then it is easy to see that for any vertices a, b, c and d of T the following relations must hold:

$$\begin{array}{lrcl}
\mathsf{PC1} & f_{(a,b)} e_a & = & f_{(a,b)} \\
\mathsf{PC2} & e_a f_{(a,b)} & = & e_a \\
\mathsf{PC3} & f_{(a,b)} e_b & = & e_a e_b \\
\mathsf{PC4} & e_c f_{(a,b)} & = & f_{(a,b)} e_c \text{ if } c \notin \{a, b\}. \\
\mathsf{C1} & f_{(a,b)} f_{(b,c)} f_{(a,b)} & = & f_{(b,c)} f_{(a,b)} f_{(b,c)} = f_{(a,b)} f_{(b,c)} \\
\mathsf{C2} & f_{(a,b)} f_{(a,c)} & = & f_{(a,b)} \\
\mathsf{C3} & f_{(a,b)} f_{(c,d)} & = & f_{(c,d)} f_{(a,b)} \text{ if } \{a, b\} \cap \{c, d\} = \emptyset \\
\mathsf{C4} & f_{(a,c)} f_{(b,c)} & = & f_{(b,c)} f_{(a,c)}
\end{array}$$

Finally, if a_1, a_2, b_1, b_2, c_1, $c_2 \in V(T)$ with $a_1 > c_1$, $b_1 > c_1$, $c_1 \geq c_2$, $c_2 > a_2$, $c_2 > b_2$ and $[a_1, a_2] \cap [b_1, b_2] = [c_1, c_2]$ then we have the relation scheme:

$$\mathsf{C5} \quad v(c_1, t) v(a_1, a_2) v(b_1, b_2) \quad = \quad v(c_1, t) v(b_1, b_2) v(a_1, a_2) \text{ for every } t \in (c_2, a_2].$$

Denote by C the set of all relations in $\mathsf{C1}$–$\mathsf{C5}$, define the sets PC and P similarly. Define the monoid $\mathsf{PC}(T)$ to be the quotient of the free monoid $(\Sigma_T \cup \Xi_T)^*$ by the congruence $(\mathsf{P} \cup \mathsf{C} \cup \mathsf{PC})^\sharp$, and define $\mathsf{C}(T)$ to be the quotient of $\Sigma_T{}^*$ by $\mathsf{C}^\sharp$. Theorems 5.2 and 5.3 then imply that

$$\mathsf{PC}(T) \cong (\mathsf{P}(T) \coprod \mathsf{C}(T)) / \widehat{\mathsf{PC}}.$$

From these observations we have

Theorem 6.2. *The map $\Gamma_{\mathsf{C}} : \mathsf{C}(T) \to \mathcal{C}(T)$ defined by $[w] \mapsto \overline{w}$ for each $w \in \Sigma_T{}^*$ is a homomorphism. Similarly, $\Gamma_{\mathsf{PC}} : \mathsf{PC}(T) \to \mathcal{PC}(T)$ defined by $[w] \mapsto \overline{w}$ for each $w \in (\Sigma_T \cup \Xi_T)^*$ is a homomorphism.* ∎

Remark 6.3. Notice that the submonoid of $\mathsf{PC}(T)$ generated by Σ_T is a homomorphic image of $\mathsf{C}(T)$ as extra relations between words of $\Sigma_T{}^*$ may result from P and PC. So if $[u] = [v]$ as elements of $\mathsf{C}(T)$ then $[u] = [v]$ as elements of $\mathsf{PC}(T)$.

In the sequel we prove that the homomorphisms of Theorem 6.2 are isomorphisms.

Lemma 6.4. *Suppose that $x \to y$ is an edge in T, $u \in (\Sigma_T \cup \Xi_T)^*$ and $x \notin \mathrm{Im}(\overline{u})$. Then $[u] = [u f_{(x,y)}] = [u e_x]$, as elements of $\mathsf{PC}(T)$. Further, if $u \in \Sigma_T{}^*$ then $[u] = [u f_{(x,y)}]$ in $\mathsf{C}(T)$.*

Proof. We leave the reader to notice that in the following proof we only use relations from C to prove that $u \sim uf_{(x,y)}$, when $u \in {\Sigma_T}^*$.

Let u be such that $x \notin \mathrm{Im}(\overline{u})$. Certainly $u \neq 1$. If u is a letter, then either $u = e_x$ or $u = f_{(x,z)}$ for some $z \neq x$. If $u = e_x$ then $u \sim ue_x$ by $\mathsf{P1}$ and $u \sim uf_{(x,y)}$ by $\mathsf{PC2}$. Otherwise $u = f_{(x,z)}$ so that $uf_{(x,y)} \sim u$ by $\mathsf{C2}$ and $ue_x = f_{(x,z)}e_x \sim f_{(x,z)}$ by $\mathsf{PC1}$. This starts an induction on the length of u.

Suppose that $u = u'f_{(a,b)}$ for some edge $a \to b$ and $u' \neq 1$. There are only four possible ways in which the distinct edges $a \to b$ and $x \to y$ can be related in the tree T:
(1) $y \leftarrow x = a \to b$;
(2) $a \to b = x \to y$;
(3) $a \to b = y \leftarrow x$ or $\{a,b\} \cap \{x,y\} = \emptyset$;
(4) $x \to y = a \to b$.

In Case (1), $uf_{(x,y)} = u'f_{(a,b)}f_{(a,y)} \sim u'f_{(a,b)} = u$, by $\mathsf{C2}$, while $ue_x = u'f_{(a,b)}e_a \sim u'f_{(a,b)} = u$ by $\mathsf{PC1}$. For Cases (2) to (4) we may assume that $a \neq x$, hence $x \notin \mathrm{Im}(\overline{u'})$, for otherwise $x \in \mathrm{Im}(\overline{u})$.

In Case (2), $a \notin \mathrm{Im}(\overline{u'})$ for otherwise again $x \in \mathrm{Im}(\overline{u})$; so, by successive applications of an inductive hypothesis, $uf_{(x,y)} = u'f_{(a,b)}f_{(x,y)} \sim u'f_{(x,y)} \sim u' \sim u'f_{(a,b)} = u$. Similarly, $ue_x = u'f_{(a,b)}e_x \sim u'e_x \sim u' \sim u'f_{(a,b)} = u$.

In Case (3) $uf_{(x,y)} = u'f_{(a,b)}f_{(x,y)} \sim u'f_{(x,y)}f_{(a,b)} \sim u'f_{(a,b)} = u$, by $\mathsf{C4}$, or $\mathsf{C3}$ if $\{a,b\} \cap \{x,y\} = \emptyset$. We use $\mathsf{PC4}$ and the inductive hypothesis to deduce that $ue_x = u'f_{(a,b)}e_x \sim u'e_xf_{(a,b)} \sim u'f_{(a,b)} = u$.

Finally, in Case (4), $\mathsf{C1}$ and the inductive hypothesis give

$$uf_{(x,y)} = u'f_{(y,b)}f_{(x,y)} \sim u'f_{(x,y)}f_{(y,b)}f_{(x,y)} \sim u'f_{(x,y)}f_{(y,b)} \sim u'f_{(y,b)} = u.$$

That $ue_x \sim u$ in Case (4) follows in the same manner as for Case (3).

We can now assume that $u = u'e_a$ for some $a \in V(T)$ and $u' \neq 1$. There are only three possible ways in which the vertex a and edge $x \to y$ can be related in the tree T:
(1) $a = x \to y$;
(2) $x \to y = a$;
(3) $a \notin \{x,y\}$.

In Case (1) $uf_{(x,y)} = u'e_xf_{(x,y)} \sim u$ by $\mathsf{PC2}$ while $ue_x \sim u$ by $\mathsf{P1}$. In Cases (2) and (3) we may assume that $x \notin \mathrm{Im}(\overline{u'})$ for otherwise $x \in \mathrm{Im}(\overline{u})$. Thus, in Case (2)

$$\begin{aligned}
uf_{(x,y)} &= u'e_af_{(x,a)} \\
&\sim u'f_{(x,a)}e_af_{(x,a)} \text{ (by the inductive hypothesis)} \\
&\sim u'e_ae_xf_{(x,a)} \text{ (by } \mathsf{PC3} \text{ and } \mathsf{P2}) \\
&\sim u'e_ae_x \text{ (by } \mathsf{PC2}) \\
&\sim u'e_a = u
\end{aligned}$$

by $\mathsf{P2}$ and the inductive hypothesis. For the same reasons we also have $ue_x = u'e_ae_x \sim u'e_xe_a \sim u'e_a = u$. In Case (3), $\mathsf{PC4}$ and the inductive hypothesis give $uf_{(x,y)} = u'e_af_{(x,y)} \sim u'f_{(x,y)}e_a \sim u'e_a = u$ and by $\mathsf{P2}$, $ue_x \sim u$ as in Case (2). The lemma now follows by induction. ■

Lemma 6.5. *Let $x_1 \to x_2 \to \cdots \to x_k$ be a chain in some tree T. Then for all pairs $i \leq j \in \{1, \ldots, k\}$ we have $[v(x_i, x_j)v(x_1, x_k)] = [v(x_1, x_k)v(x_i, x_j)] = [v(x_1, x_k)]$ as elements of $\mathsf{C}(T)$.*

Proof. We proceed by induction on $j - i$ in order to prove that

$$v(x_i, x_j)v(x_1, x_k) \sim v(x_1, x_k).$$

If $j = i$ then $v(x_i, x_j) = 1$ and we are finished, so assume that $j - i > 0$. If $j = 1$ then $j = i$, so $j = 1$ cannot occur. If $j = 2$ then $v(x_i, x_j) = f_{(x_1, x_2)}$ so that $v(x_i, x_j)v(x_1, x_k) = f_{(x_1,x_2)}v(x_1, x_k) \sim v(x_1, x_k)$ by use of relation $\mathsf{C2}$. Otherwise, $j > 2$. Use relation $\mathsf{C3}$ to commute $f_{(x_{j-1}, x_j)}$ from the end of $v(x_i, x_j)$ as far along $v(x_1, x_k)$ as possible. It will then encounter $f_{(x_{j-2}, x_{j-1})}f_{(x_{j-1}, x_j)}$ and by use of relation $\mathsf{C1}$ we can delete it. Thus $v(x_i, x_j)v(x_1, x_k) \sim v(x_i, x_{j-1})v(x_1, x_k)$ and by appeal to the inductive hypothesis we are done. That $[v(x_1, x_k)v(x_i, x_j)] = [v(x_1, x_k)]$ is a direct consequence of Lemma 6.4. ∎

Lemma 6.6. *Let $x, y, z \in V(T)$. If $x \geq z$ and $y \geq z$ then $[v(x, z)v(y, z)] = [v(y, z)v(x, z)]$ as elements of $\mathsf{C}(T)$.*

Proof. Let $z' \geq z$ be the maximal element of $[x, z] \cap [y, z]$. Then

$$\begin{aligned} v(x, z)v(y, z) &= v(x, z')v(z', z)v(y, z) \\ &\sim v(x, z')v(y, z) \text{ (by Lemma 6.5)} \\ &= v(x, z')v(y, z')v(z', z) \\ &\sim v(y, z')v(x, z')v(z', z) \text{ (by } \mathsf{C3} \text{ and } \mathsf{C4}) \\ &\sim v(y, z)v(x, z) \end{aligned}$$

as required. ∎

7. $\mathsf{C}(T)$ is a presentation for the Catalan monoid of T

In this section $u \sim v$ will be taken to mean $[u] = [v]$ as elements of $\mathsf{C}(T)$.

Lemma 7.1. *Let $\phi, \psi \in \mathcal{C}(T)$, and x the θ-maximum simple root of T^ϕ for ϕ. Write $f_\theta(\psi) = v(z_1, z_1\psi) \ldots v(z_m, z_m\psi)$. Let W be the triple (u, y, i) where $u \in {\Sigma_T}^*$, $y \in [x, x\phi]$ and $i \in \{1, \ldots, m\}$. Let*

$$\sigma(W) = uv(x, y)v(z_i, z_i\psi) \ldots v(z_m, z_m\psi) \in {\Sigma_T}^*.$$

Define $\gamma_W = \overline{uv(x, y)}$ and for each j with $i \leq j \leq m$,

$$X_j = \mathrm{Im}(\gamma_W) \cap (z_j, z_j\psi).$$

If $\overline{\sigma(W)} = \phi$, $x \notin \mathrm{Sh}(\overline{u}) \cup \mathrm{Sh}(\psi)$ and for each $j \in \{i, \ldots, m\}$, $X_j \subseteq \{z_i, \ldots, z_{j-1}\}$ then $\sigma(W) \sim w' = u'v(x, y')$ for some $y' \leq y$ and u' such that $x \notin \mathrm{Sh}(\overline{u'})$.

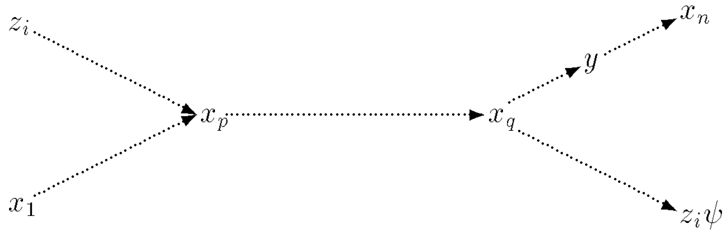
Proof. We proceed by induction on $m-i$. If $m-i=0$, then the result is immediate, so assume that $m>i$. Suppose $[x,y]\cap[z_i,z_i\psi]=\emptyset$ then relation C3 gives $v(x,y)v(z_i,z_i\psi)\sim v(z_i,z_i\psi)v(x,y)$. Set $W'=(uv(z_i,z_i\psi),y,i+1)$ then

$$\sigma(W)\sim\sigma(W')=uv(z_i,z_i\psi)v(x,y)v(z_{i+1},z_{i+1}\psi)\dots v(z_m,z_m\psi),$$

$\gamma_{W'}=\overline{uv(z_i,z_i\psi)v(x,y)}$ and for each j, $i+1\le j\le m$, $X'_j=\mathrm{Im}(\gamma_{W'})\cap(z_j,z_j\psi)$.

We show that W' satisfies the conditions of the inductive hypothesis. Let $t\in X'_j$, and suppose by way of contradiction that $t\notin\{z_{i+1},\dots,z_{j-1}\}$. If $t\in X_j$ then $t=z_i$. But $z_i\notin\mathrm{Im}(\gamma_{W'})$, so $t\in\{z_{i+1},\dots,z_{j-1}\}$. Otherwise $t\notin X_j$ so that $t\in\mathrm{Im}(\gamma_{W'})\setminus\mathrm{Im}(\gamma_W)$, which implies that $t=z_i\psi$. But that $z_i\psi\in(z_j,z_j\psi)$ contradicts the definition of $f_\theta(\psi)$ which states that z_j is a simple root of T^{ψ_j} for ψ_j. Thus in either case we have that $X'_j\subseteq\{z_{i+1},\dots,z_{j-1}\}$, and we can appeal to the inductive hypothesis to give the result.

Write the chain between x and $x\phi$ as $x=x_1\to x_2\to\cdots\to x_n=x\phi$ so that $y=x_r$ for some $r\in[n]$. We can now assume that $[x,y]\cap[z_i,z_i\psi]=[x_p,x_q]$ with $p\le q\le r$, so we have the following diagram.



Suppose that $z_i\notin\mathrm{Im}(\gamma_W)$ then Lemma 6.4 and the fact that X_i is necessarily empty allow us to delete $v(z_i,z_i\psi)$, giving $\sigma(W)\sim\sigma(W')$ where $W'=(u,y,i+1)$. In this case $\gamma_{W'}=\gamma_W$, and $X'_j=X_j$ for each $i+1\le j\le m$. Suppose that $t\in X'_j$ and $t\notin\{z_{i+1},\dots,z_{j-1}\}$ then $t=z_i\notin\mathrm{Im}(\gamma_W)=\mathrm{Im}(\gamma_{W'})$, which is a contradiction. Thus, we may appeal to the inductive hypothesis with W' to give the result.

We can now assume that $z_i\in\mathrm{Im}(\gamma_W)$, so for some $z\ge z_i$, we can write $z\gamma_W=z_i$, and $z_i\psi=z\phi$. Note also that $x_1>x_p$, for if $x_1=x_p$ then $z_i>x_1$ contradicting the fact that x_1 is a root of T^ϕ. There are, however, several other cases where the paths between points in the diagram above are trivial. We deal with each one in turn.

(1) First we deal with the case that $z_i=x_p$. In this case $z_i=x_p=x_q=y$, for otherwise $z_i\notin\mathrm{Im}(\gamma_W)$. Thus we can write

$$\sigma(W)=\sigma(W')=uv(x,y)v(y,z_i\psi)v(z_{i+1},z_{i+1}\psi)\dots(z_m,z_m\psi),$$

where $W'=(u,z_i\psi,i+1)$. We only need to show that $X'_j\subseteq\{z_{i+1},\dots,z_{j-1}\}$. Suppose to the contrary that $t\in X'_j\setminus\{z_{i+1},\dots,z_{j-1}\}$. Since $X'_j\subseteq(X_j\setminus\{y\})\cup\{z_i\psi\}$, t must be $z_i\psi$. But $z_i\psi\in(z_j,z_j\psi)$ contradicts the definition of $f_\theta(\psi)$. Thus $X'_j\subseteq\{z_{i+1},\dots,z_{j-1}\}$, and we are done by appeal to the inductive hypothesis with W'. In the sequel we may assume that $z_i>x_p$.

(2) Consider the case where $x_q=x_r=y$. Then

$$\begin{aligned}\sigma(W)&=uv(x,y)v(z_i,y)v(y,z_i\psi)\dots v(z_m,z_m\psi)\\&\sim uv(z_i,y)v(x,y)v(y,z_i\psi)\dots v(z_m,z_m\psi)\text{ (by Lemma 6.6)}\\&=uv(z_i,y)v(x,z_i\psi)\dots v(z_m,z_m\psi)\\&=\sigma(W')\end{aligned}$$

where $W' = (uv(z_i, y), z_i\psi, i+1)$. We only need to check that $t \in X'_j$ implies that $t \in \{z_{i+1}, \ldots, z_{j-1}\}$. Suppose not, then since $X'_j \subseteq (X_j \setminus \{z_i, y\}) \cup \{z_i\psi\}$, $t = z_i\psi$. (Notice that our argument does not depend on $y \neq z_i\psi$.) As above, this yields a contradiction by the definition of $f_\theta(\psi)$, and we are able to appeal to the inductive hypothesis. Assume now that $z_i > x_p$, $x_1 > x_p$ and $x_q > y$.

(3) Suppose that $x_q = z_i\psi$. Now, by the observation above, $z_i\psi = z\phi$ for some $z \geq z_i$. But $z_i\psi = x_q > y \geq x_1\phi$. Thus we have $x_1 > z\phi > x_1\phi$ contradicting the fact that x_1 is a simple root of T^ϕ for ϕ.

Finally, we are able to assume that $x > x_p$, $z_i > x_p$, $x_q > y$, and $x_q > z_i\psi$.

$$\begin{aligned}
\sigma(W) &= uv(x,y)v(z_i, z_i\psi)\ldots v(z_m, z_m\psi) \\
&\sim uv(x_p, y)v(x,y)v(z_i, z_i\psi)\ldots v(z_m, z_m\psi) \text{ (by Lemma 6.5)} \\
&\sim uv(x_p, y)v(z_i, z_i\psi)v(x,y)v(z_{i+1}, z_{i+1}\psi)\ldots v(z_m, z_m\psi) \text{ (by } \mathsf{C5}) \\
&= \sigma(W')
\end{aligned}$$

where $W' = (uv(x_p, y)v(z_i, z_i\psi), y, i+1)$. As above we prove that $t \in X'_j$ implies that $t \in \{z_{i+1}, \ldots, z_{j-1}\}$. Suppose not, then since $X'_j \subseteq (X_j \setminus \{z_i, x_p\}) \cup \{z_i\psi\}$, $t = z_i\psi$. As above, this yields a contradiction by the definition of $f_\theta(\psi)$. By appeal to the inductive hypothesis, the lemma is proven. ∎

Theorem 7.2. *The function $F_\theta : \Sigma_T{}^* \to \Sigma_T{}^*$ defined by $F_\theta(w) = f_\theta(\overline{w})$ is a canonical form function for $\mathsf{C}(T)$.*

Proof. Since $w_1 \sim w_2$ implies that $\overline{w_1} = \overline{w_2}$, it is certainly true that $F_\theta(w_1) = F_\theta(w_2)$, so it is only necessary to prove that $F_\theta(w) \sim w$.

Let $\phi = \overline{w}$. We go by induction on $n = |\mathrm{Sh}(\phi)|$. If $n = 0$ then $w = 1 = f_\theta(\phi)$ as required, so assume that $n > 0$, and $w \neq 1$. In order to show that $w \sim f_\theta(\phi) = v(x_1, x_1\phi)\ldots v(x_n, x_n\phi)$, it suffices to prove that $w \sim w'v(x_n, x_n\phi)$, with $\overline{w'} = \phi_{n-1}$, for then we are able to proceed by induction to give $w \sim w'v(x_n, x_n\phi) \sim f_\theta(\phi_{n-1})v(x_n, x_n\phi) = f_\theta(\phi)$.

Let the chain between x_n and $x_n\phi$ be $x_n = y_1 \to y_2 \to \cdots \to y_k = x_n\phi$. Write $w = w_1v(x_n, y)w_2$ where $|w_1|$ is maximal containing no letter $f_{x_n,z}$, and $|v(x_n, y)|$ is maximal. Then $w = w_1v(x_n,y)w_2 \sim w_1v(x_n,y)w'_2$ where w'_2 is w_2 with all letters $f_{x_n,z}$ deleted, by repeated application of Lemma 6.4, and the fact that no prefix of w containing $v(x_n, y)$ maps a vertex into x_n, since it is a root of T^ϕ.

Let $\psi = \overline{w'_2}$. By the fact that w'_2 contains no letter $f_{x_n,z}$, $\mathrm{Sh}(\psi) \subseteq (\mathrm{Sh}(\phi) \setminus \{x_n\})$ and we can apply the inductive hypothesis to conclude that $w'_2 \sim f_\theta(\psi)$. Application of Lemma 7.1 to ϕ, ψ and $W = (w_1, y, 1)$ gives $w \sim w'_1v(x_n, y')$ with $x_n \notin \mathrm{Sh}(\overline{w'_1})$. Notice that this implies that $y' = x_n\phi$. Lemma 6.5 then gives $w \sim w'_1v(y_2, x_n\phi)v(x_n, x_n\phi)$. Set $w' = w'_1v(y_2, x_n\phi)$. It is clear that $\overline{w'} = \phi_{n-1}$ and the theorem is proven. ∎

Theorem 7.3. *For any tree T, $\mathcal{C}(T)$ is isomorphic to $\mathsf{C}(T)$. That is to say that the generators $f_s \in \Sigma_T$ with the relations $\mathsf{C1}$–$\mathsf{C5}$ form a presentation for $\mathcal{C}(T)$.*

Proof. Let T be any tree and θ any natural total order on $V(T)$. By Theorem 6.2, there is an epimorphism $\Gamma_{\mathsf{C}} : \mathsf{C}(T) \to \mathcal{C}(T)$ defined by $[w] \mapsto \overline{w}$. Further, this epimorphism is injective, because if $w_1, w_2 \in {\Sigma_T}^*$ are such that $\overline{w_1} = \overline{w_2}$ then by Lemma 7.2 $w_1 \sim F_\theta(w_1) = F_\theta(w_2) \sim w_2$. Hence the homomorphism Γ_{C} of Theorem 6.2 is an isomorphism, and the theorem is proved. ∎

Corollary 7.4. *$\mathsf{C}(T)$ is a submonoid of $\mathsf{PC}(T)$, so that Γ_{C} is just the restriction of Γ_{PC} to $\mathsf{C}(T)$.*

Proof. Let S be the submonoid of $\mathsf{PC}(T)$ generated by Σ_T. Since $\Gamma_{\mathsf{PC}} : \mathsf{PC}(T) \to \mathcal{PC}(T)$ maps S onto $\mathcal{C}(T) \leq \mathcal{PC}(T)$, and by Remark 6.3, we can factor $\Gamma_{\mathsf{C}} : \mathsf{C}(T) \to \mathcal{C}(T)$ through S. But $\Gamma_{\mathsf{C}} : \mathsf{C}(T) \to \mathcal{C}(T)$ was shown to be an isomorphism so we must have that $S \cong \mathsf{C}(T)$. ∎

8. $\mathsf{PC}(T)$ is a presentation for the partial Catalan monoid of T

Henceforth, $u \sim v$ shall mean $[u] = [v]$ in $\mathsf{PC}(T)$.

Lemma 8.1. *For every $u \in {\Sigma_T}^*, e_x \in \Xi_T$, $ue_x \sim \epsilon u'$ for some $\epsilon \in {\Xi_T}^*, u' \in {\Sigma_T}^*$ such that $|u'| \leq |u|$.*

Proof. We show this by induction on $|u|$. If $|u| = 0$ the result is immediate, beginning the induction. Otherwise we may write $u = vf_{(a,b)}$. Now there are three cases to consider: $x \notin \{a, b\}; x = a$; or $x = b$. In the first case, $ue_x = vf_{(a,b)}e_x \sim ve_xf_{(a,b)}$ by $\mathsf{PC4}$ and we are finished by induction. If $x = a$ then $ue_x = vf_{(a,b)}e_a \sim vf_{(a,b)} = u$ by $\mathsf{PC1}$. Finally, if $x = b$ then $ue_x = vf_{(a,b)}e_b \sim ve_ae_b$ by $\mathsf{PC3}$, and we are done by two applications of the inductive hypothesis. ∎

Lemma 8.2. *If $w \in (\Sigma_T \cup \Xi_T)^*$, then $w \sim \epsilon v$ for some $v \in {\Sigma_T}^*$ and $\epsilon = \prod_{x \notin \Delta(\overline{w})} e_x$.*

Proof. By repeated application of Lemma 8.1 we have that every word $w \in (\Sigma_T \cup \Xi_T)^*$ there is a word δv, with $\delta \in {\Xi_T}^*, v \in {\Sigma_T}^*$ which represents the same element of $\mathsf{PC}(T)$.

Since the prefix δ determines the domain of $\overline{w}$, and by the fact (Proposition 6.1) that $\mathsf{P}(T)$ is a presentation for the semilattice of subsets of $V(T)$ under intersection, we are able to transform δ into ϵ, the required word. ∎

Lemma 8.3. *Let $w = \epsilon u$ with ϵ as in Lemma 8.2, $\phi = \overline{w}$, $\psi = \overline{u}$ and $u \in {\Sigma_T}^*$ in the canonical form of Theorem 7.2 (i.e. $u = F_\theta(u)$).*

Write $u = v(x_1, x_1\psi) \dots v(x_k, x_k\psi)$ and let $x_{i_1}, \dots, x_{i_m}$ be the subsequence of $x_1, \dots, x_k$ which are in $\Delta(\phi)$. Then

$$\begin{aligned} w &\sim \epsilon v(x_{i_1}, x_{i_1}\psi) \dots v(x_{i_m}, x_{i_m}\psi) \\ &= \epsilon v(x_{i_1}, x_{i_1}\phi) \dots v(x_{i_m}, x_{i_m}\phi). \end{aligned}$$

Proof. Let the subsequence of $x_1, \ldots, x_k$ which is the complement of $x_{i_1}, \ldots, x_{i_m}$ be written $x_{j_1}, \ldots, x_{j_n}$. Notice that $\{x_{j_p} \mid p \in [n]\} = \mathrm{Sh}(\psi) \setminus \Delta(\phi)$. We prove the lemma by induction on n. If $n = 0$ then w is already the word required. Otherwise, let $x = x_{j_n}$ and write $w = \epsilon u_1 v(x, x\psi) u_2$. Since, by definition of the canonical form, $x \notin V(T^{\psi_{j_{n-1}}})$, relations $\mathsf{P1}$, $\mathsf{P2}$ and $\mathsf{PC4}$ ensure that we can commute the letter e_x from the front, to write $w \sim \epsilon u_1 e_x v(x, x\psi) u_2$. If we write the chain between x and $x\psi$ as $x \to y_1 \to y_2 \to \cdots \to y_q = x\psi$ then

$$\begin{aligned} \epsilon u_1 e_x v(x, x\psi) u_2 &= \epsilon u_1 e_x f_{(x,y_1)} \cdots f_{(y_{q-1},y_q)} u_2 \\ &\sim \epsilon u_1 e_x f_{(y_1,y_2)} \cdots f_{(y_{q-1},y_q)} u_2 \text{ (by } \mathsf{PC2}) \\ &\sim \epsilon u_1 u_2 \end{aligned}$$

by Lemma 6.4 and the fact that x is a simple root for ψ_{j_n} on $T^{\psi_{j_n}}$. By Proposition 4.8 (b) $u_1 = F_\theta(u_1)$, so that ϵu_1 satisfies the conditions for the inductive hypothesis and the lemma is proven.

■

Definition 8.4. Say that a word in the form $\epsilon v(x_{i_1}, x_{i_1}\phi) \ldots v(x_{i_m}, x_{i_m}\phi)$ as described in Lemma 8.3 is in *domain simplified form.* Lemmas 8.2 and 8.3 combine to ensure that every word in $(\Sigma_T \cup \Xi_T)^*$ can be written in domain simplified form.

Theorem 8.5. *If T is any tree, then $\mathcal{PC}(T) \cong \mathsf{PC}(T)$.*

Proof. Since we have already established that $\Gamma_{\mathsf{PC}} : \mathsf{PC}(T) \to \mathcal{PC}(T)$ is a homomorphism, we only need to show that it is injective. Assume therefore that w_1 and w_2 are two words in domain simplified form such that $\overline{w_1} = \overline{w_2} = \phi \in \mathcal{PC}(T)$.

Lemma (inductive proof of Theorem 8.5). *Let $\phi \in \mathcal{PC}(T)$. Suppose*

$$u_1 = v(x_1, x_1\phi) \ldots v(x_n, x_n\phi)$$

and

$$u_2 = v(x_{1\pi}, x_{1\pi}\phi) \ldots v(x_{n\pi}, x_{n\pi}\phi)$$

for some $\pi \in \mathrm{Sym}_n$ such that :
(1) $\phi = \overline{\epsilon u_1} = \overline{\epsilon u_2}$;
(2) the distinct elements $x_1, \ldots, x_n$ constitute the set $\mathrm{Sh}(\phi) \cap \Delta(\phi)$;
(3) if $i < j$ then $x_i \not> x_j$ and $x_{i\pi} \not> x_{j\pi}$ in the natural partial order on $V(T)$;
(4) for each i in $\{1, \ldots, j-1\}$, $x_i\phi \notin [x_j, x_j\phi)$ and $x_{i\pi}\phi \notin [x_{j\pi}, x_{j\pi}\phi)$.
Then $\epsilon u_1 \sim \epsilon u_2$.

Since w_1 and w_2 are in domain simplified form, we can write $w_1 = \epsilon u_1$ and $w_2 = \epsilon u_2$, satisfying the preconditions of the lemma. Thus it is sufficient to prove the lemma.

Define a total order θ on $\{x_1, \ldots, x_n\}$ by $x_i <_\theta x_j$ if $i < j$. We proceed to show that ϵu_2 can be transformed into ϵu_1 by repeatedly swapping adjacent subwords $v(x_i, x_i\phi)$ in u_2 which are out of order. If there is no element i of $[n]$ such that $i\pi >_\theta i+1\pi$ then $u_1 = u_2$, which starts an induction on the size of the set $\{(i,j) \mid i < j \text{ and } i\pi >_\theta j\pi\}$. Otherwise, let i be the smallest element such that $i\pi >_\theta i+1\pi$. Notice that neither $x_{i\pi} > x_{i+1\pi}$ nor $x_{i+1\pi} > x_{i\pi}$ for otherwise either u_1 or u_2 would contradict assumption (3).

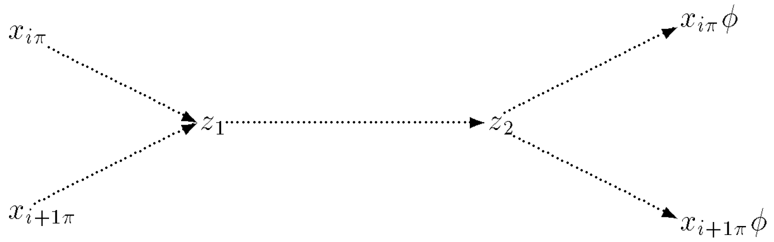
If $[x_{i\pi}, x_{i\pi}\phi] \cap [x_{i+1\pi}, x_{i+1\pi}\phi] = \emptyset$, then relation C3 ensures that $u_2 \sim u_2'$ where we define

$$u_2' = v(x_{1\pi}, x_{1\pi}\phi)\dots v(x_{i-1\pi}, x_{i-1\pi}\phi)v(x_{i+1\pi}, x_{i+1\pi}\phi)v(x_{i\pi}, x_{i\pi}\phi)\dots v(x_{n\pi}, x_{n\pi}\phi).$$

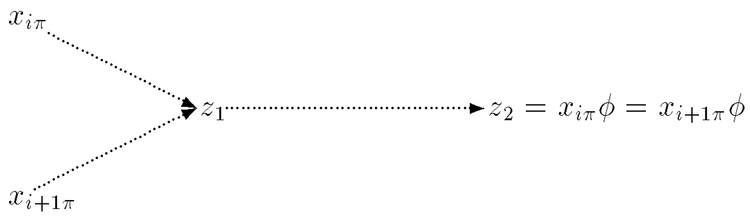
We may now appeal to the inductive hypothesis with ϵu_1 and $\epsilon u_2'$ to give the result.

We can now assume that $[x_{i\pi}, x_{i\pi}\phi] \cap [x_{i+1\pi}, x_{i+1\pi}\phi] = [z_1, z_2] \neq \emptyset$. By assumption (2) $x_{i\pi} \neq x_{i+1\pi}$, so that $x_{i\pi} > z_1$ and $x_{i+1\pi} > z_1$. If $x_{i+1\pi}\phi < z_2$ then $x_{i\pi}\phi \neq z_2$ by assumption (4). If $x_{i\pi}\phi < z_2$ then $x_{i+1\pi}\phi \neq z_2$ for otherwise $x_{i+1\pi}\phi \in [x_{i\pi}, x_{i\pi}\phi)$ while $x_{i\pi} >_\theta x_{i+1\pi}$, contradicting assumption (4) with regard to u_1. Thus there are two possible situations:

(a)



(b)



In Case (b), Lemma 6.6 ensures that $u_2 \sim u_2'$ as above, and once again we appeal to the inductive hypothesis.

In Case (a) transform the subword $v(x_{i\pi}, x_{i\pi}\phi)v(x_{i+1\pi}, x_{i+1\pi}\phi)$ of u_2 as follows. Lemma 6.5 gives

$$\begin{aligned} v(x_{i\pi}, x_{i\pi}\phi)v(x_{i+1\pi}, x_{i+1\pi}\phi) &\sim v(z_1, x_{i\pi}\phi)v(x_{i\pi}, x_{i\pi}\phi)v(x_{i+1\pi}, x_{i+1\pi}\phi) \\ &\sim v(z_1, x_{i\pi}\phi)v(x_{i+1\pi}, x_{i+1\pi}\phi)v(x_{i\pi}, x_{i\pi}\phi) \end{aligned}$$

by C5. We prove that we can delete the subword $u' = v(z_1, x_{i\pi}\phi)$ by induction (*) on its length. Let the prefix of u_2 preceeding u' be u''. If $z_1 \in \mathrm{Im}(\overline{u''})$ then since $z_1 < x_{i\pi}$ and $z_1 < x_{i+1\pi}$, $z_1 = x_{j\pi}\phi$ for some $j < i$. But then the fact that $z_1 \in [x_{i\pi}, x_{i\pi}\phi)$ contradicts assumption (4). Thus $z_1 \notin \mathrm{Im}(\overline{u''})$. By Lemma 6.4 we may delete the first letter of u', and by appeal to (*) we can delete u'. Once again, we have shown that ϵu_2 can be transformed into $\epsilon u_2'$. By appeal to the inductive hypothesis the lemma, and hence the theorem, is proven. ∎

All the relations apart from C5 were easy to find as they involve only two generators. Relation C5 was discovered with the assistance of Magma [2] to count the elements of $\mathcal{C}(T)$ and Walker's presentation enumerator [19] to enumerate $\mathsf{C}(T)$.

9. A review of existing work on Catalan monoids

Theorem 1 of Howie [8] states that

Theorem 9.1. *If X is a finite set, then the subsemigroup of $\mathcal{T}_X$ generated by the idempotents of non-zero defect is $\mathcal{T}_X \setminus \mathcal{G}_X$. In fact, every element of $\mathcal{T}_X \setminus \mathcal{G}_X$ is a product of idempotents of defect 1.*

Here $\mathcal{T}_X$ is the full transformation semigroup on the set X, and $\mathcal{G}_X$ is its subgroup of non-singular transformations. Identifying every idempotent of defect 1 as an elementary transition of the complete graph on X, and since every finite semigroup embeds in $\mathcal{T}_X \setminus \mathcal{G}_X$ for some set X, we are able to deduce from Theorem 9.1 that every finite semigroup embeds in the Catalan monoid of some finite graph.

More specific results are obtained in [16] where it is shown that the pseudovariety of finite $\mathcal{R}$-trivial monoids is generated by all $\mathcal{C}(G)$ where G is a finite acyclic graph. It is also shown that the pseudovariety of finite $\mathcal{J}$-trivial monoids is generated by all $\mathcal{C}(G)$ where G is a finite acyclic graph in which no vertex has out-degree greater than 1.

It is shown in [1] and [17] that when $G_n = 1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow n$, the symmetric chain on n vertices, $\mathcal{C}(G_n)$ is the monoid of all endomorphisms of the set $\{1, \ldots, n\}$ of vertices under the usual total ordering. We shall give a presentation for $\mathcal{C}(G_n)$ via the presentations for the Catalan monoids of the two subtrees $T_n^+ = 1 \to 2 \to \cdots \to n$ and $T_n^- = 1 \leftarrow 2 \leftarrow \cdots \leftarrow n$ of G_n. Let f_i label the generator of $\mathcal{C}(T_n^-)$ which maps $i+1$ to i and fixes the other points, and let g_i label the generator of $\mathcal{C}(T_n^+)$ which maps i to $i+1$, and fixes the other points. Let $\Sigma_- = \{f_i : 1 \le i \le n-1\}$ and $\Sigma_+ = \{g_i : 1 \le i \le n-1\}$. With this notation, by a direct transcription of the presentation for the Catalan monoid of a tree given above,

$$\begin{aligned} \mathcal{C}(T_n^-) \;\cong\; & \mathsf{C}(T_n^-) = \langle \Sigma_- \mid f_i^2 = f_i, \\ & f_i f_{i+1} f_i = f_{i+1} f_i f_{i+1} = f_{i+1} f_i, \\ & f_i f_j = f_j f_i \text{ where } |i-j| > 1 \rangle \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}(T_n^+) \;\cong\; & \mathsf{C}(T_n^+) = \langle \Sigma_+ \mid g_i^2 = g_i, \\ & g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} = g_i g_{i+1}, \\ & g_i g_j = g_j g_i \text{ where } |i-j| > 1 \rangle. \end{aligned}$$

Now since the chain T_n^+ is identical to the chain T_n^- (it is just the mirror image), $\mathcal{C}(T_n^+)$ must be isomorphic to $\mathcal{C}(T_n^-)$. On the other hand, inspection of the presentations above yields the fact that $\mathsf{C}(T_n^-)$ is dual to $\mathsf{C}(T_n^+)$ since we can move from one presentation to the other by simply reversing the words in the relations. Thus we see immediately the result of Higgins [6] that

Corollary 9.2. *$\mathcal{C}(T_n^-)$, which is the monoid of non-increasing endomorphisms of the total order $1 \le 2 \le \cdots \le n$, is self-dual.*

A presentation for $\mathcal{C}(G_n)$ in generators $\Sigma_- \cup \Sigma_+$ will satisfy the following relations, which we shall refer to collectively as O.

$$\begin{array}{lrcll} \mathsf{O}1 & g_i f_i & = & f_i & \\ \mathsf{O}2 & f_i g_i & = & g_i & \\ \mathsf{O}3 & f_i g_j & = & g_j f_i & \text{if } i \notin \{j, j-1\} \\ \mathsf{O}4 & g_{i+1} f_i & = & g_{i+1} & \\ \mathsf{O}5 & f_i g_{i+1} & = & f_i & \end{array}$$

In [1] Aĭzenštat proves that

Theorem 9.3. *The monoid of all (full) endomorphisms of the total order* $1 \leq 2 \leq \cdots \leq n$ *is* $\mathcal{C}(G_n)$ *and has presentation* $\mathsf{C}(T_n^-) \coprod \mathsf{C}(T_n^+)/\widehat{\mathsf{O}}$ *which we denote by* O_n.

The following theorem appears in [6]. Denoting the dual of a monoid M by M^{op}, we have that

Theorem 9.4. *There are embeddings* $\mathsf{O}_n \hookrightarrow {\mathsf{O}_{n+1}}^{\mathrm{op}}$ *and* $\mathsf{O}_n^{\mathrm{op}} \hookrightarrow \mathsf{O}_{n+1}$.

Where the embedding is the homomorphic extension of the map taking $f_i \mapsto f_{n-i}$ and $g_i \mapsto g_{n+1-i}$.

It is observed in [6] that Corollary 9.2 implies that the pseudovariety $\mathbf{J}$ of finite $\mathcal{J}$–trivial monoids, which is generated by the semigroups $\mathcal{C}(T_n^-)$ for $n \geq 1$, is *self-dual* – that is, for each monoid M in $\mathbf{J}$, M^{op} is also in $\mathbf{J}$. It is also observed that Theorem 9.4 implies that the pseudovariety $\mathbf{O}$, generated by $\mathsf{O}_n, n \geq 1$ is self-dual.

With notation as above, Theorem 8.5 implies that $\mathcal{PC}(T_n^-)$ has a presentation consisting of $\mathsf{P}(T_n^-) \coprod \mathsf{C}(T_n^-)/\widehat{\mathsf{T}}$ where T refers to the following relations.

$$\begin{array}{lrcll} \mathsf{T}1 & f_i e_{i+1} & = & f_i & \\ \mathsf{T}2 & e_{i+1} f_i & = & e_{i+1} & \\ \mathsf{T}3 & f_i e_i & = & e_{i+1} e_i & \\ \mathsf{T}4 & e_i f_j & = & f_j e_i & \text{if } i \notin \{j, j+1\} \end{array}$$

Since $\mathsf{P}(T_n^-) \cong \mathsf{P}(T_n^+)$ we know that $\mathcal{PC}(T_n^+)$ has presentation $\mathsf{P}(T_n^-) \coprod \mathsf{C}(T_n^+)/\widehat{\mathsf{T}'}$ where T' will refer to:

$$\begin{array}{lrcll} \mathsf{T}1' & g_i e_i & = & g_i & \\ \mathsf{T}2' & e_i g_i & = & e_i & \\ \mathsf{T}3' & g_i e_{i+1} & = & e_i e_{i+1} & \\ \mathsf{T}4' & e_i g_j & = & g_j e_i & \text{if } i \notin \{j, j+1\} \end{array}$$

In [14] and in [17] it is shown that

Theorem 9.5. $\mathcal{PC}(G_n)$, *the monoid of all partial endomorphisms of* $\{1, \ldots, n\}$ *is isomorphic to* $\mathsf{P}(T_n^-) \coprod \mathsf{C}(G_n)/(\widehat{\mathsf{T} \cup \mathsf{T}'})$.

and in [6] it is shown that the pseudovariety generated by $\mathcal{PC}(G_n), n \geq 0$ is not self–dual.

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