A Category of Concrete Monoids

Andrew Solomon*

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Abstract

A concrete monoid over a category **C** is a subset of the endomorphisms of an object of **C**, containing the identity and closed under composition. To contrast, an abstract monoid is just a one object category.

There is a natural notion of morphism between concrete monoids distinct from the usual morphism of abstract monoids. This type of morphism is identified via an example, and then defined, giving rise to the category of concrete monoids over **C**.

The utility of these definitions is explored via applications to the Theories of Semigroups, Matrices, Vines and Automata. With these definitions, it is possible for the first time to make the action monoid construction into a functor whose domain is the usual category of automata.

1 Introduction

In a category of abstract algebras (such as the category **Mon** of monoids and monoid homomorphisms) division generalizes the notion of embedding: crudely speaking, the algebra B divides the algebra A if B can be viewed as a subalgebra of A when one ignores some of the structure of A. (Formally, B divides A if there is a subalgebra C of A with B a quotient of C.) Birkhoff's Variety Theorem for abstract algebras — and in the particular case of finite monoids, the Krohn-Rhodes Theorem — attests to the fundamental importance of the division relationship.

This paper develops the notion of a *filter*, which is analogous to division, between concrete monoids over **C**.

By understanding filters, we are able to define a category **Conc**(**C**) whose objects are concrete monoids over **C** and whose arrows are simply arrows of **C** (see Definition 7.4).

To establish a heuristic notion of a filter between concrete monoids over **C**, and to motivate the developments in Part I of this paper, we begin with an instructive

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example in the case $\mathbf{C} = \mathbf{Set}$. Recall that a concrete monoid over a category $\mathbf{C}$ is one whose elements are endomorphisms of an object of $\mathbf{C}$. Thus the concrete monoids over **Set** are precisely the transformation monoids, which are of importance in the Theories of Groups (as permutation groups) and Semigroups.

A monoid A of transformations of a set X will be represented as a table — each row of the table represents an element of the monoid and each column represents an element of the set. The ith entry in the row corresponding to $a \in A$ is $a(i)$, the value of a at i. The top row represents the identity element. Consider the transformation monoids

$$A = \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 4 \\ \hline 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ \hline 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 4 \end{array} \quad \text{and} \quad B = \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 2 \end{array}.$$

We find B by 'filtering' A — by first deleting some of the rows of the table (not the top row), in such a way that the set of remaining rows is closed under composition, and then deleting some of the columns, ensuring that the remaining entries of the table consist only of elements whose columns have not been deleted. (In the table above, we have deleted column 4 and rows 5–7.) In short, we have found a submonoid N of A and a subset Y of X, such that for each $n \in N, y \in Y$, $n(y) \in Y$. To make the remaining part of the table into the transformation monoid B, we collapse identical rows.

If this process is repeated with B to give a transformation monoid C, then C may be derived from A by a single step of deleting rows and columns and collapsing identical rows. This defines a composition of filters.

The first part of this paper is devoted to making this notion of a filter precise, and to defining the category $\mathbf{Conc}(\mathbf{C})$ whose objects are concrete monoids over $\mathbf{C}$ and whose arrows are filters (in the example above, there would be an arrow from B to A).

Part I – Theory

This part is a systematic approach to the definition of the category $\mathbf{Conc}(\mathbf{C})$ of concrete monoids over $\mathbf{C}$. In Section 2, we recall the definition of a bicategory and illustrate it in Section 3, with some relevant examples.

The category $\mathbf{Reps}(\mathbf{C})$ whose objects are monoid representations over $\mathbf{C}$ is defined in Section 4, and several useful theorems are proved. We take a detour in Section 5 to explain how the well established notions of *morphism*, *relational morphism* and *division* of monoid representations over **Set** fit into the framework constructed so far.

Finally, in Section 6 we study the bicategory of filters, whose objects are concrete monoids and whose arrows are the filters described in the introduction. Through a thorough understanding of this bicategory, we derive the category **Conc**(**C**) which is applied to various diverse situations in Part II.

2 Bicategories

In his seminal work [3] Bénabou defines a bicategory. For completeness, we repeat this definition and an example almost verbatim. A *bicategory* S is determined by the following data:

B–I A set $Ob(\mathsf{S})$ called the set of *objects* of S.

B–II For each pair (A, B) of objects of S, a category $\mathsf{S}(A, B)$. An object S of $\mathsf{S}(A, B)$ is called an *arrow* of S and written $A \xrightarrow{S} B$. If S and T are objects of $\mathsf{S}(A, B)$, an arrow s of $\mathsf{S}(A, B)$ between S and T is called a 2-*cell* of S and will be written $S \overset{s}{\Rightarrow} T$ or more usually

$$\begin{tikzcd} A \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "s\Downarrow"] & B \end{tikzcd}$$

Composition in the category $\mathsf{S}(A, B)$ will thus correspond to the pasting

$$\begin{tikzcd} A \arrow[r, bend left=50, "S"] \arrow[r, "T" description] \arrow[r, bend right=80, "U"'] \arrow[r, phantom, shift left=3, "s\Downarrow"] \arrow[r, phantom, shift right=4, "t\Downarrow"] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, bend left, "S"] \arrow[r, bend right, "U"'] \arrow[r, phantom, "t.s\Downarrow"] & B \end{tikzcd}$$

which is referred to as *vertical* composition of 2-cells.

B–III For each triple (A, B, C) of objects of S, a functor $c_{A,B,C} : \mathsf{S}(A, B) \times \mathsf{S}(B, C) \to \mathsf{S}(A, C)$ which is called the *composition functor* of S. For $A \xrightarrow{S} B$ and $B \xrightarrow{T} C$, write $T \circ S$ for $c_{A,B,C}(S, T)$. If $S \overset{s}{\Rightarrow} S'$ is an arrow of $\mathsf{S}(A, B)$ and $T \overset{t}{\Rightarrow} T'$ is an arrow of $\mathsf{S}(B, C)$ then write $t \circ s$ for $c_{A,B,C}(s, t)$. Then the composition functor corresponds to the pasting

$$\begin{tikzcd} A \arrow[r, bend left, "S"] \arrow[r, bend right, "S'"'] \arrow[r, phantom, "s\Downarrow"] & B \arrow[r, bend left, "T"] \arrow[r, bend right, "T'"'] \arrow[r, phantom, "t\Downarrow"] & C \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, bend left, "T\circ S"] \arrow[r, bend right, "T'\circ S'"'] \arrow[r, phantom, "tos\Downarrow"] & C \end{tikzcd}$$

which is referred to as *horizontal* composition of 2-cells.

B–IV For each object A of S an object I_A of $\mathsf{S}(A,A)$ called the *identity arrow* of A. The identity morphism of I_A in $\mathsf{S}(A,A)$ is denoted i_A and called the *identity 2-cell* of A.

B–V For each quadruple (A,B,C,D) of objects of S, a natural isomorphism $a_{A,B,C,D}$ between the two functors $c_{A,B,D}\cdot(id\times c_{B,C,D})$ and $c_{A,C,D}\cdot(c_{A,B,C}\times id)$ from $\mathsf{S}(A,B)\times\mathsf{S}(B,C)\times\mathsf{S}(C,D)$ to $\mathsf{S}(A,D)$. In particular, if $(S,T,U)\in\mathsf{S}(A,B)\times\mathsf{S}(B,C)\times\mathsf{S}(C,D)$ this gives a component isomorphism between $U\circ(T\circ S)$ and $(U\circ T)\circ S$.

B–VI For each pair (A,B) of objects of S, natural isomorphisms $l_{A,B}$ and $r_{A,B}$, called the *left* and *right* identities:

$$\begin{tikzcd}
1\times\mathsf{S}(A,B) \arrow[rr, "\iota_A\times id"] \arrow[dr, "\pi_2"'] & \arrow[d, phantom, "\Nearrow l_{A,B}"] & \mathsf{S}(A,A)\times\mathsf{S}(A,B) \arrow[dl, "c_{A,A,B}"] \\
& \mathsf{S}(A,B) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{S}(A,B)\times 1 \arrow[rr, "id\times\iota_B"] \arrow[dr, "\pi_1"'] & \arrow[d, phantom, "\Nearrow r_{A,B}"] & \mathsf{S}(A,B)\times\mathsf{S}(B,B) \arrow[dl, "c_{A,B,B}"] \\
& \mathsf{S}(A,B) &
\end{tikzcd}$$

Here we write 1 for the trivial category and $1\xrightarrow{\iota_A}\mathsf{S}(A,A)$ for the functor landing on I_A. For each $S\in\mathsf{S}(A,B)$, the left and right identities give rise to component isomorphisms $I_A\circ S\cong S$ and $S\circ I_B\cong S$.

B–VII The data above are also required to satisfy the following *coherence conditions:*
(1) If (V,U,T,S) is an object of $\mathsf{S}(A,B)\times\mathsf{S}(B,C)\times\mathsf{S}(C,D)\times\mathsf{S}(D,E)$ then the following diagram commutes:

$$\begin{tikzcd}
((S \circ T) \circ U) \circ V \arrow[rr, "a(S,T,U) \circ Id"] \arrow[d, "a(S \circ T, U, V)"'] & & (S \circ (T \circ U)) \circ V \arrow[d, "a(S, T \circ U, V)"] \\
(S \circ T) \circ (U \circ V) \arrow[dr, "a(S,T,U \circ V)"'] & & S \circ ((T \circ U) \circ V) \arrow[dl, "Id \circ a(T,U,V)"] \\
& S \circ (T \circ (U \circ V)) &
\end{tikzcd}$$

This condition is known as *associativity coherence.*
(2) The following diagram commutes:

$$\begin{tikzcd}
(S \circ I_B) \circ T \arrow[rr, "a_{A,B,B,C}(S, I_B, T)"] \arrow[dr, "r_{A,B}(S) \circ id"'] & & S \circ (I_B \circ T) \arrow[dl, "id \circ l_{B,C}(T)"] \\
& S \circ T &
\end{tikzcd}$$

This condition is known as *identity coherence.*

2.1 From Bicategories to Categories

A consequence of B-V above is that a bicategory is not necessarily a category — composition of arrows is only associative up to isomorphism. However, for any bicategory S, there are two closely related categories which capture much of the structure of S — the *classifying category* and the *Poincaré category* of S.

The *classifying category* of S is the easier to define. Its objects are the objects of S and its arrows are isomorphism classes of arrows of S. If $S \in \mathsf{S}(A,B)$ and $T \in \mathsf{S}(B,C)$ and if $[S]$ denotes the isomorphism class of S, then the composition of $[S]$ and $[T]$ in the classifying category of S is defined to be $[T \circ S]$. The associative law now holds for composition of arrows, since by axiom B-V above, there is a 2-cell isomorphism from $(S \circ T) \circ U$ to $S \circ (T \circ U)$.

To define the Poincaré category, recall from [13, page 86] that a category $\mathbf{C}$ is said to be *connected* if for any two objects A, B of $\mathbf{C}$ there is a finite sequence $A = A_1, A_2, \ldots, A_n = B$ of objects of $\mathbf{C}$, and for each $i < n$, an arrow $A_i \to A_{i+1}$ or an arrow $A_i \leftarrow A_{i+1}$. In [13, page 88, Ex. 7] it is stated that every category is the disjoint union of connected categories, called its *connected components*. We may define an equivalence relation on the objects of a category by $A \sim_\pi B$ if A and B belong to the same connected component.

The objects of the *Poincaré category* are the objects of S and its arrows are $\sim_\pi$ equivalence classes of arrows of S. Composition is defined as in the classifying category.

3 Examples of Bicategories

Let A and B be objects in a category $\mathbf{C}$. A *span* from A to B is a triple (f, S, g) where $A \stackrel{f}{\leftarrow} S \stackrel{g}{\rightarrow} B$ is a diagram of $\mathbf{C}$. One of the many examples of a bicategory cited in [3] is $\mathsf{Span}(\mathbf{C})$, the bicategory of spans in $\mathbf{C}$. Let $\mathbf{C}$ be any category with pullbacks. Choosing a pullback for each diagram in $\mathbf{C}$ of the form $\cdot \rightarrow \cdot \leftarrow \cdot$, we now define the bicategory $\mathsf{Span}(\mathbf{C})$ of spans of $\mathbf{C}$ with respect to this choice of pullbacks. Since a pullback is only unique up to isomorphism, another choice of pullbacks would give another bicategory which is isomorphic to the first.

S–I The objects of $\mathsf{Span}(\mathbf{C})$ are the objects of $\mathbf{C}$.

S–II For each pair (A, B) of objects of $\mathbf{C}$, the category $\mathsf{Span}(A, B)$ has as objects spans (f, S, g). An arrow of $\mathsf{Span}(A, B)$ from $A \stackrel{f}{\leftarrow} S \stackrel{g}{\rightarrow} B$ to $A \stackrel{f'}{\leftarrow} S' \stackrel{g'}{\rightarrow} B$ is an arrow $h : S \rightarrow S'$ such that $f' \cdot h = f$ and $g' \cdot h = g$. Composition in $\mathsf{Span}(A, B)$ is inherited from $\mathbf{C}$.

S–III The composition functor is defined on arrows by taking the chosen pullback as in the following diagram:

$$\begin{tikzcd}
A & & B & & C \\
& S \arrow[ul, "f"] \arrow[ur, "g"'] & & T \arrow[ul, "h"] \arrow[ur, "k"'] & \\
& & S \times_B T \arrow[ul, "\pi_S"] \arrow[ur, "\pi_T"'] & &
\end{tikzcd}$$

and then setting $(h, T, k) \circ (f, S, g) = (f\pi_S, S \times_B T, k\pi_T)$. Horizontal composition of 2-cells is performed using the universal property of pullbacks.

The remaining details of this definition are left to the reader.

3.1 Bicategories from a regular category : $\mathsf{Rel}(\mathbf{C})$, $\mathsf{Rel}^*(\mathbf{C})$ and $\mathsf{Div}(\mathbf{C})$.

When $\mathbf{C}$ is a regular category (a notion we will define presently), it is possible to construct bicategories $\mathsf{Rel}(\mathbf{C})$, $\mathsf{Rel}^*(\mathbf{C})$ and $\mathsf{Div}(\mathbf{C})$ whose arrows are, respectively, relations, totally defined relations, and divisions between objects of $\mathbf{C}$.

3.1.1 Regular Categories

Fix some category $\mathbf{C}$. Following [10] we define a *strong epimorphism* of $\mathbf{C}$ to be an epimorphism f such that if

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, "i"'] & {} \end{tikzcd}$$

is any commuting diagram with i monomorphic, then there is a (necessarily unique) w, called the *fill-in*, making the following diagram commute.

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d] & {} \arrow[d] \arrow[dl, "w"'] \\ {} \arrow[r, "i"'] & {} \end{tikzcd}$$

We repeat here some useful facts about strong epimorphisms which are taken from [10].

Proposition 3.1 Strong epimorphisms are closed under composition and right division.□

Proposition 3.2 An arrow which is both a monomorphism and strong epimorphism is an isomorphism. □

If f factors as ip where i is a monomorphism and p is a strong epimorphism, then we refer to ip as a *canonical factorization* of f.

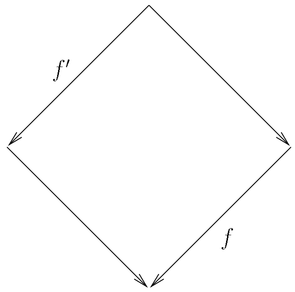
Proposition 3.3 Canonical factorizations are essentially unique. That is to say, if $f = ip = jq$ with i, j monomorphisms and p, q strong epimorphisms, then there is an isomorphism c making the following diagram commute. □

$$\begin{tikzcd} {} \arrow[r, "q"] \arrow[d, "p"'] & {} \arrow[d, "j"] \arrow[dl, "c"'] \\ {} \arrow[r, "i"'] & {} \end{tikzcd}$$

Suppose $f : A \to B$ has canonical factorization ip with $p : A \to C$ and $i : C \to B$, then as a result of Proposition 3.3, we are able to refer to C as the *image* of f.

Finally, following [5], we are able to define a regular category.

Definition 3.4 A *regular category* is a category in which:
(1) Every arrow has a canonical factorization;
(2) All finite limits exist;
(3) Strong epimorphisms are *stable* — i.e. in every pullback diagram as pictured below, f is a strong epimorphism implies that f' is a strong epimorphism.



Some examples of regular categories:

- **Set** — the category of sets and functions. The strong epimorphisms, which are all epimorphisms, are the surjective functions.

- **Mon** — the category of monoids and monoid homomorphisms. The strong epimorphisms are the surjective monoid homomorphisms. These are not, however, all the epimorphisms.

  An example (cited in [1]) of an epimorphism in **Mon** which is not surjective is the canonical inclusion (as multiplicative monoids) of the integers into the rationals.

For the remainder of Section 3.1, $\mathbf{C}$ will denote a regular category.

3.1.2 The bicategory $\mathsf{Rel}(\mathbf{C})$

We define the bicategory $\mathsf{Rel}(\mathbf{C})$, which in the case that $\mathbf{C} = \mathbf{Set}$, has sets as objects and an arrow from the set A to the set B will be a subset of $A \times B$.

In general, following [4], a *relation* $R : A \to B$ in $\mathbf{C}$ is a span from A to B, such that if $S : A \to B$ is any other span, then there is at most one 2-cell in $\mathsf{Span}(\mathbf{C})$ from S to R. It is an easy exercise to see that for any relation $R = (f, R, g)$, the arrow $\langle f, g\rangle : R \to A \times B$ is monic. Conversely, if $i : R \to A \times B$ is monic, then $(\pi_A i, R, \pi_B i)$ is a relation.

Definition 3.5 Every span $S = (f, S, g)$, defines a relation: let $S \xrightarrow{p} Q \xrightarrow{i} A \times B$ be the canonical factorization of $< f, g >: S \to A \times B$. Then $Q = (\pi_A i, Q, \pi_B i)$ is a relation and p is a 2-cell from S to Q. Call Q the *relation defined by S*.

The bicategory $\mathsf{Rel}(\mathbf{C})$ of relations in $\mathbf{C}$ is defined as follows:

Rel–I The objects of $\mathsf{Rel}(\mathbf{C})$ are the objects of $\mathbf{C}$.

Rel–II The arrows of $\mathsf{Rel}(\mathbf{C})$ are the relations in $\mathbf{C}$. The 2-cells are the 2-cells of $\mathsf{Span}(\mathbf{C})$. That is, for any objects A, B of $\mathbf{C}$, the category $\mathsf{Rel}(\mathbf{C})(A, B)$ is the full subcategory of $\mathsf{Span}(\mathbf{C})(A, B)$ whose objects are the relations.

Rel–III The composite of the relations $R : A \to B$ and $S : B \to C$ is obtained by composing as spans and then taking the relation defined by the composite span.

The remaining details of this definition are left to the reader. As one would hope, if $\mathbf{C} = \mathbf{Set}$, then the composite of R and S in $\mathsf{Rel}(\mathbf{C})$ is the set $\{(a, c) \mid (a, b) \in R, (b, c) \in S, \text{ for some } b \in B\}$.

3.1.3 The bicategory $\mathsf{Rel}^*(\mathbf{C})$

The bicategory $\mathsf{Rel}^*(\mathbf{C})$ of 'total relations' or *relational morphisms* is a sub-bicategory of $\mathsf{Rel}(\mathbf{C})$. The objects are those of $\mathbf{C}$ and the category $\mathsf{Rel}^*(\mathbf{C})(A, B)$ is the full subcategory of $\mathsf{Rel}(\mathbf{C})(A, B)$ whose objects are relations (f, R, g) in which the arrow f is a strong epimorphism in $\mathbf{C}$.

One only needs to check that the composite of two such arrows is again an arrow of $\mathsf{Rel}^*(\mathbf{C})$. Consider the diagram below where $< fh', kg' >: R \times_B S \to A \times C$ has the canonical factorization iq. Then the composite relation is $(\pi_A i, Q, \pi_C i) : A \to C$. We need to check that $\pi_A i$ is a strong epi.

$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "h'"'] \arrow[d, "q"] \arrow[dr, "g'"] & & \\
 & R \arrow[dl, "f"'] \arrow[dr, "g"] & Q \arrow[ddl, bend right, "i"] & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C \\
 & A \times C \arrow[ul, "\pi_A"] \arrow[urrr, "\pi_C"'] & & &
\end{tikzcd}$$

Since h is a strong epi and $\mathbf{C}$ is regular, h' is also a strong epi. By construction, $fh' = (\pi_A i)q$. But since q is a strong epi and by the fact (Proposition 3.1) that strong epis are closed under right division, $\pi_A i$ is a strong epimorphism, as required.

By way of example, the arrows of $\mathsf{Rel}^*(\mathbf{Mon})$ are relational morphisms as they are usually defined in the literature (see Section 5 for details). The arrows of $\mathsf{Rel}^*(\mathbf{Set})$ from A to B are relations $R \subseteq A \times B$ such that for each $a \in A$, the set $\{b \mid (a, b) \in R\}$ is nonempty.

3.1.4 The bicategory $\mathsf{Div}(\mathbf{C})$

The bicategory $\mathsf{Div}(\mathbf{C})$ of *divisions* is a sub-bicategory of $\mathsf{Rel}^*(\mathbf{C})$, whose arrows are spans (f, R, g) where f is a strong epimorphism and g is a monomorphism. Composition is defined as for $\mathsf{Rel}(\mathbf{C})$, but it reduces to the composition in $\mathsf{Span}(\mathbf{C})$. To see this, note that in the diagram above, since g is monic, so is g' (see [13, p.72, Ex.5]). Thus, kg' is monic. By construction, $kg' = \pi_C iq$. But since only monos right divide monos, q must also be a monomorphism. By Proposition 3.2 this implies that q is an isomorphism.

In the case that $\mathbf{C} = \mathbf{Mon}$, arrows in $\mathsf{Div}(\mathbf{Mon})$ become monoid divisions as they are usually defined.

4 Representations of Monoids

In this section, a monoid M will be regarded as a category with one object. If m is an arrow of M then write $m \in M$ and say that m is an *element* of M. A *monoid homomorphism* from M to N is then a functor ϕ from M to N.

Fix some category $\mathbf{C}$. A *representation* of the monoid M *over* $\mathbf{C}$ is a functor $\Gamma : M \to \mathbf{C}$. We will sometimes embellish the symbol Γ by writing Γ^M to indicate that Γ is a representation of M and Γ^M_X to indicate that Γ *lands on* X (that is to say, Γ maps the single object of M to X and is therefore a representation of M as endomorphisms of X). If Γ is any representation, then the symbols Γ^M, Γ_X and Γ^M_X will all refer to the representation Γ. If there is only one representation of M as endomorphisms of X in sight, we may simply write M_X, omitting the name of the representation altogether.

If Γ^M_X (resp. M_X) is a representation then for each $m \in M$, $\Gamma(m)$ (resp. $M_X(m)$) is an endomorphism of X. The endomorphism $\Gamma(m)$ will be written $\overline{m}$ if there is no possibility of confusion about which representation of M is used.

The category $\mathbf{Reps}(\mathbf{C})$ is defined to have representations over $\mathbf{C}$ for objects. For $\Gamma_X : M \to \mathbf{C}$ and $\Delta_Y : N \to \mathbf{C}$ objects of $\mathbf{Reps}(\mathbf{C})$, an arrow $\Gamma_X \to \Delta_Y$ is a pair (f_1, η_1), where $f_1 : M \to N$ is a homomorphism and $\eta_1 : X \to Y$ is an arrow of $\mathbf{C}$ which is a natural transformation $\Gamma_X \Rightarrow \Delta_Y \cdot f_1$, i.e. such that for every $m \in M$ the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\eta_1"] \arrow[d, "\Gamma(m)"] & Y \arrow[d, "\Delta(f_1(m))"] \\ X \arrow[r, "\eta_1"] & Y \end{tikzcd}$$

Let $(f_2, \eta_2) : \Delta_Y \to \Lambda_Z$ be another arrow. Their composite is defined by

$$(f_2, \eta_2) \cdot (f_1, \eta_1) = (f_2 \cdot f_1, \eta_2 \cdot \eta_1);$$

it is easy to verify that $\eta_2 \cdot \eta_1$ is a natural transformation $\Gamma_X \Rightarrow \Lambda_Z \cdot f_2 \cdot f_1$ and that these data define a category. If α is an arrow of $\mathbf{Reps}(\mathbf{C})$ we will write $\alpha_{\mathbf{Mon}}$ for the first element of the pair and $\alpha_{\mathbf{C}}$ for the second.

The following proposition is easy to verify.

Proposition 4.1 Let (f, η) be an arrow of $\mathbf{Reps}(\mathbf{C})$. Then (f, η) is an isomorpism if and only if f is an isomorphism of $\mathbf{Mon}$ and η is an isomorphism of $\mathbf{C}$. The inverse of an isomorphism (f, η) is (f^{-1}, η^{-1}). $\square$

If Γ^M is a monoid representation, we can define a congruence $\sim_\Gamma$ on M by $m \sim_\Gamma n$ if $\overline{m} = \overline{n}$. A *concrete monoid* over $\mathbf{C}$ is a monoid representation $\Gamma : M \to \mathbf{C}$ for which $\sim_\Gamma$ is trivial, or equivalently, it is a monoid representation over $\mathbf{C}$ which is faithful.

Remark 4.2 As in the motivating example, the most important representing category is $\mathbf{Set}$, for the objects of $\mathbf{Reps}(\mathbf{Set})$ are the transformation representations of monoids. Let M be some abstract monoid. Then the subcategory of $\mathbf{Reps}(\mathbf{Set})$ whose objects are representations of M and whose arrows all have $\mathbf{Mon}$ component 1_M is known as the category of M-sets.

Let A be a category. In [1] a faithful functor $\Gamma : A \to \mathbf{C}$ is called a *concrete category* over $\mathbf{C}$. Let $\Delta : B \to \mathbf{C}$ be another concrete category. A *concrete functor* $f : \Gamma \to \Delta$ between two concrete categories over $\mathbf{C}$ is defined to be a functor $f : A \to B$ such that $\Delta f = \Gamma$. When A and B are monoids, a concrete functor is simply an arrow of $\mathbf{Reps}(\mathbf{C})$ whose $\mathbf{C}$ component is the identity.

Such an arrow of $\mathbf{Reps}(\mathbf{C})$ in which the $\mathbf{C}$ component is the identity is often called a *strict* morphism from Γ to Δ, as opposed to the general arrow of $\mathbf{Reps}(\mathbf{C})$ which is called a *lax* morphism. Let Γ and Δ be objects of $\mathbf{Reps}(\mathbf{C})$. Pictured below are (from left to right) a lax and a strict morphism in $\mathbf{Reps}(\mathbf{C})$.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\Gamma"'] & \arrow[d, phantom, "\Nearrow\eta"] & B \arrow[dl, "\Delta"] \\ & \mathbf{C} & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\Gamma"'] & \arrow[d, phantom, "="] & B \arrow[dl, "\Delta"] \\ & \mathbf{C} & \end{tikzcd}$$

Lemma 4.3 Let

$$\begin{tikzcd} \Gamma_X^M \arrow[dr, "\alpha"'] & & \Delta_Y^N \arrow[dl, "\beta"] \\ & \Lambda_Z^L & \end{tikzcd} \tag{1}$$

be a diagram in $\mathbf{Reps}(\mathbf{C})$ and let the diagrams

$$\begin{tikzcd} & P \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_N"] & \\ M \arrow[dr, "\alpha_{\mathbf{Mon}}"'] & & N \arrow[dl, "\beta_{\mathbf{Mon}}"] \\ & L & \end{tikzcd} \qquad \begin{tikzcd} & W \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[dr, "\alpha_{\mathbf{C}}"'] & & Y \arrow[dl, "\beta_{\mathbf{C}}"] \\ & Z & \end{tikzcd}$$

be pullbacks in **Mon** and **C** respectively. Then there is a monoid representation $\Psi_W : P \to \mathbf{C}$ with arrows π_Γ and π_Δ which is the pullback of Diagram (1) in $\mathbf{Reps}(\mathbf{C})$.

Proof. To define an action of P on W, let $p \in P$ and define $\gamma_X^p : W \to X$ by $\gamma_X^p = \Gamma(\pi_M(p))\pi_X$ and similarly define $\gamma_Y^p = \Delta(\pi_N(p))\pi_Y : W \to Y$. Now

$$\begin{aligned} \alpha_{\mathbf{C}}\gamma_X^p &= \alpha_{\mathbf{C}}\Gamma(\pi_M(p))\pi_X \\ &= \Lambda(\alpha_{\mathbf{Mon}}\pi_M(p))\alpha_{\mathbf{C}}\pi_X \\ &= \Lambda(\beta_{\mathbf{Mon}}\pi_N(p))\beta_{\mathbf{C}}\pi_Y \\ &= \beta_{\mathbf{C}}\Delta(\pi_N(p))\pi_Y \\ &= \beta_{\mathbf{C}}\gamma_Y^p \end{aligned}$$

so that by the universal property of pullbacks, there is a unique arrow $\gamma^p : W \to W$

making the following diagram commute.

$$
\begin{tikzcd}
& W \arrow[dl, "\gamma^p_X"'] \arrow[d, dashed, "\gamma^p"'] \arrow[dr, "\gamma^p_Y"] & \\
X \arrow[dr, "\alpha_{\mathbf{C}}"'] & W \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \arrow[dl, "\beta_{\mathbf{C}}"] \\
& Z &
\end{tikzcd}
\tag{2}
$$

We now show that setting $\Psi(p) = \gamma^p : W \to W$ defines a representation $\Psi_W : P \to \mathbf{C}$.

Firstly, notice that if $p = 1$ then $\gamma^p = 1_W$ since it is the unique arrow making Diagram (2) commute. Let $p, q \in P$. Then we only need to show that $\gamma^p\gamma^q$ makes Diagram (2) for pq commute, for then, by uniqueness, $\gamma^p\gamma^q = \gamma^{pq}$ as required. First notice that

$$
\begin{aligned}
\gamma^p_X\gamma^q &= \Gamma(\pi_M(p))\pi_X\gamma^q \\
&= \Gamma(\pi_M(p))\Gamma(\pi_M(q))\pi_X \\
&= \Gamma(\pi_M(pq))\pi_X \\
&= \gamma^{pq}_X
\end{aligned}
$$

Similarly, $\gamma^p_Y\gamma^q = \gamma^{pq}_Y$. But $\gamma^p_X = \pi_X\gamma^p$ and $\gamma^p_Y = \pi_Y\gamma^p$ so that

$$\pi_X\gamma^p\gamma^q = \gamma^{pq}_X$$

and

$$\pi_Y\gamma^p\gamma^q = \gamma^{pq}_Y$$

and by uniqueness, $\gamma^p\gamma^q = \gamma^{pq}$ as required.

Having defined a representation Ψ we proceed to show that it is a pullback of Diagram (1) in $\mathbf{Reps}(\mathbf{C})$. By definition of Ψ, $\pi_\Gamma := (\pi_M, \pi_X)$ and $\pi_\Delta := (\pi_N, \pi_Y)$ are both arrows of $\mathbf{Reps}(\mathbf{C})$ and it is immediate that $\alpha\pi_\Gamma = \beta\pi_\Delta$. It only remains to show that Ψ has the universal property. Suppose that the diagram

$$
\begin{tikzcd}
& \Omega^Q_V \arrow[dl, "\delta"'] \arrow[dr, "\epsilon"] & \\
\Gamma \arrow[dr, "\alpha"'] & & \Delta \arrow[dl, "\beta"] \\
& \Lambda &
\end{tikzcd}
$$

commutes. Then there are corresponding commuting diagrams in **Mon** and in **C**. Thus, there are arrows $\xi_{\mathbf{Mon}} : Q \to P$ and $\xi_{\mathbf{C}} : V \to W$ by the universal properties of P and W. We certainly have that if $(\xi_{\mathbf{Mon}}, \xi_{\mathbf{C}})$ is an arrow of $\mathbf{Reps}(\mathbf{C})$, then it is the unique arrow of $\mathbf{Reps}(\mathbf{C})$ which makes the diagram commute, so it only remains to show that, for each $q \in Q$,

$$A_q := \Psi(\xi_{\mathbf{Mon}}(q))\xi_{\mathbf{C}} : V \to W$$

and

$$B_q := \xi_{\mathbf{C}}\Omega(q) : V \to W$$

are equal. It is sufficient, by the universal property of W in $\mathbf{C}$ to show that $\pi_X A_q = \pi_X B_q : V \to X$ and $\pi_Y A_q = \pi_Y B_q : V \to Y$, for then both A_q and B_q are the unique arrow making the following diagram commute.

$$\begin{tikzcd}
 & V \arrow[dl, "\pi_X A_q = \pi_X B_q"'] \arrow[d, dashed] \arrow[dr, "\pi_Y A_q = \pi_Y B_q"] & \\
X \arrow[dr, "\alpha_{\mathbf{C}}"'] & W \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y \arrow[dl, "\beta_{\mathbf{C}}"] \\
 & Z &
\end{tikzcd} \tag{3}$$

Now

$$\begin{aligned}
\pi_X A_q &= \pi_X \Psi(\xi_{\mathbf{Mon}}(q))\xi_{\mathbf{C}} \\
&= \Gamma(\pi_M(\xi_{\mathbf{Mon}}(q)))\pi_X \xi_{\mathbf{C}} \\
&= \Gamma(\delta_{\mathbf{Mon}}(q))\delta_{\mathbf{C}} \\
&= \delta_{\mathbf{C}}\Omega(q) \\
&= \pi_X \xi_{\mathbf{C}}\Omega(q) \\
&= \pi_X B_q
\end{aligned}$$

and similarly, $\pi_Y A_q = \pi_Y B_q$ as required. $\square$

If $\mathbf{C}$ has an initial object, denote it by 0 and if it has a terminal object, denote it by 1. Note that we denote the trivial monoid by the bold **1** to avoid confusion. Then the following is immediate.

Proposition 4.4 If $\mathbf{C}$ has an initial object, then the functor $\mathbf{1} \to \mathbf{C}$ landing on the initial object of $\mathbf{C}$ is the initial object of $\mathbf{Reps}(\mathbf{C})$. $\square$

Proposition 4.5 If $\mathbf{C}$ has a terminal object, then the functor $\mathbf{1} \to \mathbf{C}$ landing on the terminal object of $\mathbf{C}$ is the terminal object of $\mathbf{Reps}(\mathbf{C})$. $\Box$

Proposition 4.5 and Lemma 4.3 yield

Theorem 4.6 If $\mathbf{C}$ is finite complete, then so is $\mathbf{Reps}(\mathbf{C})$. $\Box$

The following trivialities are often useful.

Lemma 4.7 Let M_X be an object of $\mathbf{Reps}(\mathbf{C})$.
(i) If $f : N \to M$ is an arrow of $\mathbf{Mon}$ then $(f, !) : N_0 \to M_X$ is an arrow of $\mathbf{Reps}(\mathbf{C})$, where $! : 0 \to X$ is the unique arrow in $\mathbf{C}$ and N_0 is the unique functor so denoted.
(ii) If $\gamma : Z \to X$ is an arrow of $\mathbf{C}$ then $(\iota, \gamma) : \mathbf{1}_Z \to M_X$ is an arrow of $\mathbf{Reps}(\mathbf{C})$ where $\iota : \mathbf{1} \to M$ is the unique arrow in $\mathbf{Mon}$, and $\mathbf{1}_Z$ is the unique functor so denoted. $\Box$

Next we identify the monomorphisms and strong epimorphisms of $\mathbf{Reps}(\mathbf{C})$ when we place certain conditions on $\mathbf{C}$.

Proposition 4.8 Let $\mathbf{C}$ be a category with an initial object. An arrow (f, α) of $\mathbf{Reps}(\mathbf{C})$ is monic if and only if f and α are monic in $\mathbf{Mon}$ and $\mathbf{C}$ respectively.

Proof. If f is monic in $\mathbf{Mon}$ and α is monic in $\mathbf{C}$ then (f, α) is certainly monic in $\mathbf{Reps}(\mathbf{C})$.

Suppose $(f, \alpha) : M_X \to N_Y$ is monic, and $g, h : L \to M$ are arrows of $\mathbf{Mon}$. Let $(g, !), (h, !) : L_0 \to M_X$ be the arrows of $\mathbf{Reps}(\mathbf{C})$ given by Lemma 4.7 (i). If $f \cdot h = f \cdot g$ then

$$\begin{aligned}(f, \alpha) \cdot (g, !) &= (f \cdot g, !) \\ &= (f \cdot h, !) \\ &= (f, \alpha) \cdot (h, !)\end{aligned}$$

so that $g = h$ as required.

In a similar way, suppose $\gamma, \beta : Z \to X$ are arrows of $\mathbf{C}$. Let $(\iota, \gamma), (\iota, \beta) : \mathbf{1}_Z \to M_X$ be the arrows given by Lemma 4.7 (ii). As above, if $\alpha \cdot \gamma = \alpha \cdot \beta$ then

$$\begin{aligned}(f, \alpha) \cdot (\iota, \gamma) &= (\iota, \alpha \cdot \gamma) \\ &= (\iota, \alpha \cdot \beta) \\ &= (f, \alpha) \cdot (\iota, \beta),\end{aligned}$$

giving the result that $\gamma = \beta$. $\Box$

Proposition 4.9 Let $\mathbf{C}$ be a regular category with initial object. An arrow (f, α) of $\mathbf{Reps}(\mathbf{C})$ is a strong epimorphism if and only if f and α are strong epis in $\mathbf{Mon}$ and $\mathbf{C}$ respectively.

Remark 4.10 The statement of the proposition may be strengthened as it it is enough to assume that **C** has canonical factorizations and initial and terminal objects.

Proof. Let $\phi = (f, \alpha)$ be an arrow $A_X \to B_Y$ in $\mathbf{Reps}(\mathbf{C})$ with f and α both strong epimorphisms. Suppose that the following diagram commutes in $\mathbf{Reps}(\mathbf{C})$ where ι is monic.

$$\begin{tikzcd} A_X \arrow[r, "\phi"] \arrow[d] & B_Y \arrow[d] \\ C_Z \arrow[r, "\iota"'] & D_W \end{tikzcd}$$

Then since **C** has an initial object, Proposition 4.8 gives us that $\iota_{\mathbf{Mon}}$ and $\iota_{\mathbf{C}}$ are monic, so that there are commuting diagrams in **Mon** and **C** as follows.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \arrow[dl, "g"'] \\ C \arrow[r, "\iota_{\mathbf{Mon}}"'] & D \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d] \arrow[dl, "\beta"'] \\ Z \arrow[r, "\iota_{\mathbf{C}}"'] & W \end{tikzcd}$$

It is routine to verify that (g, β) is an arrow of $\mathbf{Reps}(\mathbf{C})$ from B_Y to C_Z. Thus ϕ is a strong epimorphism.

Conversely, let $\phi = (f, \alpha)$ be a strong epimorphism. We will show that both f and α are strong epimorphisms.

Suppose the following diagram commutes in **Mon** with i a monomorphism.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "i"'] & D \end{tikzcd}$$

In order to find an arrow $B \to C$ making the diagram commute, we construct a corresponding diagram in $\mathbf{Reps}(\mathbf{C})$. In the following diagram C_1 and D_1 are the unique representations of C and D as endomorphisms of the terminal object of **C**, $\gamma = (g, !)$, $\delta = (h, !)$ and $\iota = (i, !)$.

$$\begin{tikzcd} A_X \arrow[r, "\phi"] \arrow[d, "\gamma"'] & B_Y \arrow[d, "\delta"] \\ C_1 \arrow[r, "\iota"'] & D_1 \end{tikzcd}$$

Since ϕ is a strong epimorphism, there is a unique arrow from B_Y to C_1 whose **Mon** component is the required arrow.

We have shown that the **Mon** component of a strong epi in **Reps**(**C**) is a strong epi. For the **C** component, suppose the following diagram commutes in **C** with ι a monomorphism.

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & Y \arrow[d, "\beta"] \\ Z \arrow[r, "\iota"'] & W \end{tikzcd}$$

Since **C** is regular, we may write the canonical factorization of α as $X \xrightarrow{\eta} V \xrightarrow{\mu} Y$.

Let $b \in B$. Since f has been shown to be surjective, there is an $a \in A$ with $f(a) = b$. Define δ_b to be the fill-in making the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "A_X(a)"'] & V \arrow[r, "\mu"] \arrow[d, "\delta_b"] & Y \arrow[d, "B_Y(b)"] \\ X \arrow[r, "\eta"'] & V \arrow[r, "\mu"'] & Y \end{tikzcd}$$

It is easy to see that putting $B_V(b) = \delta_b$ defines a representation of B as endomorphisms of V. In **Reps**(**C**) we therefore have arrows $(f, \eta) : A_X \to B_V$ and $(1_B, \mu) : B_V \to B_Y$ with $\phi = (1_B, \mu)(f, \eta)$.

To show that α is a strong epi, it is sufficient to show that μ is an isomorphism. Since (f, α) and (f, η) are both strong epis and by Proposition 3.1, $(1_B, \mu)$ is a strong epi. Thus $(1_B, \mu)$ is a strong epi and a mono, and therefore an iso. By Proposition 4.1, this implies that μ is an isomorphism, as required. $\Box$

We will see in the next section that the following theorem is a powerful tool for constructing bicategories of relations and divisions from **Reps**(**C**).

Theorem 4.11 If $\mathbf{C}$ is a regular category with initial object, then so is $\mathbf{Reps}(\mathbf{C})$.

Proof. Let $\mathbf{C}$ be a regular category with initial object. That $\mathbf{Reps}(\mathbf{C})$ has an initial object is simply Proposition 4.4. We proceed by showing that $\mathbf{Reps}(\mathbf{C})$ satisfies the three axioms of a regular category.
(1) Let $(f, \alpha) : A_X \to B_Y$ be an arrow of $\mathbf{Reps}(\mathbf{C})$, let f have canonical factorization $A \xrightarrow{p} C \xrightarrow{i} B$ and let α have canonical factorization $X \xrightarrow{\eta} Z \xrightarrow{\iota} Y$. We construct an object C_Z of $\mathbf{Reps}(\mathbf{C})$ such that (p, η) is an arrow from A_X to C_Z and (i, ι) is an arrow from C_Z to B_Y. Then by Propositions 4.8 and 4.9, (i, ι) is a mono and (p, η) is a strong epi so that we have a canonical factorization of (f, α). Let $c \in C$. Since f is a strong epi (surjection in $\mathbf{Mon}$) there is some $a \in A$ with $f(a) = c$. Let γ_c be the fill-in making the following daigram commute.

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, "A_X(a)"'] & Z \arrow[r, "\iota"] \arrow[d, "\gamma_c"] & Y \arrow[d, "B_Y(i(c))"] \\
X \arrow[r, "\eta"'] & Z \arrow[r, "\iota"'] & Y
\end{tikzcd}$$

Setting $C_Z(c) = \gamma_c$ defines the required object of $\mathbf{Reps}(\mathbf{C})$, and verifying that (p, η) and (i, ι) are arrows is routine.
(2) That $\mathbf{Reps}(\mathbf{C})$ is finite complete is the statement of Theorem 4.6.
(3) Let ϕ be a strong epimorphism. By Proposition 4.9, its $\mathbf{Mon}$ and $\mathbf{C}$ components are both strong epis. If $\cdot \to \cdot \xleftarrow{\phi} \cdot$ is any diagram in $\mathbf{Reps}(\mathbf{C})$, then Lemma 4.3, implies that there is a pullback diagram in $\mathbf{Reps}(\mathbf{C})$ as in the following diagram where ϕ' is a strong epimorphism in both the $\mathbf{Mon}$ and $\mathbf{C}$ components, and hence a strong epimorphism in $\mathbf{Reps}(\mathbf{C})$.

$$\begin{tikzcd}
 & \cdot \arrow[dl, "\phi'"'] \arrow[dr] & \\
\cdot \arrow[dr] & & \cdot \arrow[dl, "\phi"] \\
 & \cdot &
\end{tikzcd}$$

If there is another pullback diagram in $\mathbf{Reps}(\mathbf{C})$ in which ϕ pulls back to ϕ'', then there is an isomorphism ϵ such that $\phi'' = \phi'\epsilon$. Since strong epimorphisms are closed under composition with isomorphisms, ϕ'' is a strong epimorphism, proving that strong epimorphisms of $\mathbf{Reps}(\mathbf{C})$ are stable. $\square$

5 Old Definitions from New

The standard definitions of relation, relational morphism and division of sets, abstract monoids and transformation representations of monoids may be exhibited as special cases of the constructions of the earlier sections.

A *relation of sets* $f : X \to Y$ is a function from X to the power set 2^Y of Y. The composite of two relations $f : X \to Y$ and $g : Y \to Z$ is the relation $g \circ f$ where $g \circ f(x) = \{z \mid z \in g(y) \text{ for some } y \in f(x)\}$. The following definitions are taken directly from [16]. A relation $f : X \to Y$ is :
(a) *fully defined* if $f(x) \neq \emptyset$ for all $x \in X$;
(b) *injective* if $x \neq x'$ implies $f(x) \cap f(x') = \emptyset$; and
(c) *surjective* if $f(X) = Y$.
The *graph* of f is denoted $\sharp f = \{(x, y) \mid y \in f(x)\}$.

For this section only, regard a monoid not as a one object category, but as a set with an associative binary operation, satisfying the usual axioms. Let M and N be monoids. A *relation $\phi : M \to N$ of monoids* is a relation of sets whose graph $\sharp\phi$ is a submonoid of $M \times N$. A *relational morphism* of monoids is a fully defined relation of monoids. This includes the usual morphism of monoids. A *division* of monoids is an injective relational morphism of monoids. The reader may easily verify that:
(a) a relation of sets is precisely an arrow of $\mathsf{Rel}(\mathbf{Set})$;
(b) a relation of monoids is precisely an arrow of $\mathsf{Rel}(\mathbf{Mon})$;
(c) a relational morphism of monoids is precisely an arrow of $\mathsf{Rel}^*(\mathbf{Mon})$; and
(d) a division of monoids is precisely an arrow of $\mathsf{Div}(\mathbf{Mon})$.
It is also easy to see that composition of the relations in (a), (b), (c) and (d) correspond to horizontal composition in the bicategories $\mathsf{Rel}(\mathbf{Set})$, $\mathsf{Rel}(\mathbf{Mon})$, $\mathsf{Rel}^*(\mathbf{Mon})$ and $\mathsf{Div}(\mathbf{Mon})$ respectively.

There are corresponding notions of morphism, relational morphism and division when discussing transformation representations of monoids. These standard definitions are taken from Nehaniv's recent paper [14]. Let M_X and N_Y be transformation representations of monoids (i.e. objects of $\mathbf{Reps}(\mathbf{Set})$). A *morphism* $M_X \to N_Y$ is defined in [14] as a pair (f, η) where $f : M \to N$ is a monoid homomorphism, and $\eta : X \to Y$ is a function, such that for each $x \in X$ and each $m \in M$, $\overline{f(m)}(\eta(x)) = \eta(\overline{m}(x))$. It is immediate that this is precisely an arrow of $\mathbf{Reps}(\mathbf{Set})$.

A *relational morphism* from M_X to N_Y is a pair (f, η) where f is a totally defined monoid relation, η is a totally defined relation of sets and for all $x \in X$ and all $m \in M$,

$$y \in \eta(x), n \in f(m) \Rightarrow \overline{n}(y) \in \eta(\overline{m}(x)).$$

As noted in Theorem 4.11, we may speak of the bicategory $\mathsf{Rel}^*(\mathbf{Reps}(\mathbf{Set}))$ because $\mathbf{Set}$, and therefore $\mathbf{Reps}(\mathbf{Set})$, is regular with initial object. It is now routine, though tedious, to see that a relational morphism as defined in [14] is precisely an arrow of $\mathsf{Rel}^*(\mathbf{Reps}(\mathbf{Set}))$.

A *division* of monoid representations is a relational morphism in which both the set relation and the monoid relation are injective. Again, a division of monoid relations is simply an arrow of $\mathsf{Div}(\mathbf{Reps}(\mathbf{Set}))$.

6 The Bicategory of Filters

In this section we construct the bicategory $\mathsf{Filt}(\mathbf{C})$ whose arrows are filters — an example of this construction was given in the introduction. This bicategory is essentially the sub-bicategory of $\mathsf{Div}(\mathbf{Reps}(\mathbf{C}))$ whose objects are the concrete monoids over $\mathbf{C}$ and whose arrows are divisions whose strong epic leg are strict morphisms of $\mathbf{Reps}(\mathbf{C})$ (see Remark 4.2).

The bicategory $\mathsf{Filt}(\mathbf{C})$ has much to recommend it: the arrows are more comprehensible than an arbitrary division in $\mathsf{Div}(\mathbf{Reps}(\mathbf{C}))$ (see the example in the introduction); there is a simple description of the hom categories (see Theorem 7.3) and; the Poincaré category $\mathbf{Conc}(\mathbf{C})$ of $\mathsf{Filt}(\mathbf{C})$ has arrows which may be regarded as arrows of $\mathbf{C}$ making computation extremely easy (see Definition 7.4). In fact, the bicategory $\mathsf{Filt}(\mathbf{C})$ is so much simpler than $\mathsf{Div}(\mathbf{Reps}(\mathbf{C}))$ that there is no need for $\mathbf{C}$ to be regular.

Fix a category $\mathbf{C}$ with initial and terminal objects. Define *the bicategory* $\mathsf{Filt}(\mathbf{C})$ *of filters* as follows:

Filt–I The objects of $\mathsf{Filt}(\mathbf{C})$ are the concrete monoids over $\mathbf{C}$, i.e. those objects of $\mathbf{Reps}(\mathbf{C})$ which are faithful.

Filt–II In a similar way to $\mathsf{Div}(\mathbf{Reps}(\mathbf{C}))$, an arrow of $\mathsf{Filt}(\mathbf{C})$ from A_X to B_Y is an object S_X of $\mathbf{Reps}(\mathbf{C})$ together with a mono $i : S_X \to B_Y$ and a strong epi $f : S_X \to A_X$, such that $f_{\mathbf{C}} = 1_X$ (i.e. the strong epi is strict). We shall often abuse notation by referring to the arrow (f, S_X, i) simply as S. Just as in $\mathsf{Div}(\mathbf{Reps}(\mathbf{C}))$, a 2–cell from (f, S_X, i) to (g, T_X, j) is an arrow $\alpha : S_X \to T_X$ such that $g\alpha = f$ and $j\alpha = i$. The following useful facts about the category $\mathsf{Filt}(\mathbf{C})(A_X, B_Y)$ are immediate.

Proposition 6.1 In the category $\mathsf{Filt}(\mathbf{C})(A_X, B_Y)$:
(i) the $\mathbf{C}$ component of every arrow is the identity;
(ii) the $\mathbf{Mon}$ component of every arrow is monic; and
(iii) every diagram commutes — that is to say, $\mathsf{Filt}(\mathbf{C})(A_X, B_Y)$ is a *preorder*. □

Remark 6.2 As usual, we will sometimes identify this preorder with the corresponding poset, writing $S \leq T$ if there is a 2–cell $S \Rightarrow T$.

Filt–III The composition functor is defined as in the Span construction, and in the same manner we must make a choice of pullbacks as part of the definition. For every diagram $X \xrightarrow{f} Y \xleftarrow{1_Y} Y$ in $\mathbf{C}$, the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "1_Y"] \\
 & Y &
\end{tikzcd} \tag{4}$$

is a pullback. Let D be a diagram of the form $\cdot \rightarrow \cdot \leftarrow \cdot$ in $\mathbf{Reps}(\mathbf{C})$ whose $\mathbf{C}$ component is of the form $X \xrightarrow{f} Y \xleftarrow{1_Y} Y$. Then by Lemma 4.3 and the completeness of $\mathbf{Mon}$, we may choose a pullback for D whose $\mathbf{C}$ component is of the form shown in Diagram 4.

Let $A_X \leftarrow S \rightarrow B_Y$ and $B_Y \leftarrow T \rightarrow C_Z$ be arrows of $\mathsf{Filt}(\mathbf{C})$. By the above choice of pullbacks, their composite is given by

$$\begin{tikzcd}
A_X & & B_Y & & C_Z \\
 & S_X \arrow[ul] \arrow[ur, "i"'] & & T_Y \arrow[ul] \arrow[ur, "j"'] & \\
 & & (S \times_B T)_X \arrow[ul, "(\pi_S, 1_X)"] \arrow[ur, "(\pi_T, i_{\mathbf{C}})"'] & &
\end{tikzcd}$$

where $(\pi_T, i_{\mathbf{C}})$ is mono (since monos pull back to monos) and $(\pi_S, 1_X)$ is a strong epi (since $\mathbf{Mon}$ is regular).

Write the composite of S and T as $T \circ S$. To show that composition is a functor we need to show that if $\alpha : S \Rightarrow S'$, $\alpha' : S' \Rightarrow S''$, $\beta : T \Rightarrow T'$ and $\beta' : T' \Rightarrow T''$ are 2-cells, then $(\alpha'\alpha) \circ (\beta'\beta)$ and $(\alpha' \circ \beta')(\alpha \circ \beta)$ are equal 2–cells from $T \circ S$ to $T'' \circ S''$. But this is immediate since $\mathsf{Filt}(\mathbf{C})(A_X, C_Z)$ is a preorder and therefore any two parallel arrows are equal.

Filt–IV The identity arrow of $\mathsf{Filt}(\mathbf{C})(A_X, A_X)$ is simply $(1_{A_X}, A_X, 1_{A_X})$ and the identity 2–cell is 1_{A_X}.

Filt–V Let (S, T, U) be an object of $\mathsf{Filt}(\mathbf{C})(A, B) \times \mathsf{Filt}(\mathbf{C})(B, C) \times \mathsf{Filt}(\mathbf{C})(C, D)$. To define the associativity isomorphism, we need to find a family of 2-cell isomorphisms from $(U \circ T) \circ S$ to $U \circ (T \circ S)$, which is natural in

(S, T, U). Since $\mathsf{Filt}(\mathbf{C})(A, D)$ is a preorder, naturality is trivial, so it is enough to find an isomorphism from $(U \circ T) \circ S$ to $U \circ (T \circ S)$. Again, since $\mathsf{Filt}(\mathbf{C})(A, D)$ is a preorder, it is enough to find any 2–cells from $(U \circ T) \circ S$ to $U \circ (T \circ S)$ and from $U \circ (T \circ S)$ to $(U \circ T) \circ S$.

$$\begin{tikzcd}
 & & (U \circ T) \circ S \arrow[rrr, dotted, "\gamma"] \arrow[d, dotted, "\delta"'] \arrow[drr, "\pi_{U \circ T}"] \arrow[ddl, "\tilde{\pi}_S"'] & & & U \circ (T \circ S) \arrow[dlll, "\pi_{T \circ S}"'] \arrow[dd, "\tilde{\pi}_U"] & \\
 & & T \circ S \arrow[dl, "\pi_S"'] \arrow[dr] \arrow[rr, phantom, "\mathbf{F}"] & & U \circ T \arrow[dl] \arrow[dr, "\pi_U"] & & \\
 & S \arrow[dl, "f_S"'] \arrow[dr] & & T \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\mathbf{G}"] & & U \arrow[dl] \arrow[dr, "i_U"] & \\
A & & B & & C & & D
\end{tikzcd}$$

To find the 2–cell from $(U \circ T) \circ S$ to $U \circ (T \circ S)$ observe that the diagram

$$\begin{tikzcd}
 & (U \circ T) \circ S \arrow[ddl] \arrow[dr] & \\
 & & U \circ T \arrow[d] \\
S \arrow[dr] & & T \arrow[dl] \\
 & B &
\end{tikzcd}$$

commutes, so by the universal property of $(T \circ S)$ there is a unique arrow $\delta : (U \circ T) \circ S \to T \circ S$ of $\mathbf{Reps}(\mathbf{C})$. Since diagrams F and G both commute, the diagram

$$\begin{tikzcd}
 & (U \circ T) \circ S \arrow[ddl] \arrow[dr] & \\
 & & U \circ T \arrow[d] \\
T \circ S \arrow[d] & & U \arrow[ddl] \\
T \arrow[dr] & & \\
 & C &
\end{tikzcd}$$

commutes. By the universal property of $U \circ (T \circ S)$ there is an arrow $\gamma : (U \circ T) \circ S \to U \circ (T \circ S)$ in $\mathbf{Reps}(\mathbf{C})$. To see that γ is a 2–cell simply notice that

$$\begin{aligned}(f_S \pi_S \pi_{T \circ S})\gamma &= f_S \pi_S \delta \\ &= f_S \tilde{\pi}_S\end{aligned}$$

and

$$i_U \tilde{\pi}_U \gamma = i_U \pi_U \pi_{U \circ T}$$

as required.

Since this argument in no way depended upon any fact about the particular spans involved, a symmetric argument shows that there is another 2–cell from $U \circ (T \circ S)$ to $(U \circ T) \circ S$, and we are done.

Filt–VI In the same manner as the definition above of the associativity isomorphism, it is enough to show that for any objects A and B of $\mathsf{Filt}(\mathbf{C})$, and any arrow $S : A \to B$, there is a 2–cell isomorphism from $1_A \circ S$ to S.

Consider the following diagram.

$$\begin{tikzcd}
A_X & & A_X & & B_Y \\
& A_X \arrow[ul] \arrow[ur, "1_{A_X}"] & & S_X \arrow[ul] \arrow[ur, "i"'] & \\
& & (A \times_A S)_X \arrow[ul] \arrow[ur, "l"'] & &
\end{tikzcd}$$

The arrow l is clearly a 2–cell from $1_A \circ S$ to S. Since isos in $\mathbf{Mon}$ are monic strong epis, the arrow l is an iso in the $\mathbf{Mon}$ component (and the identity in the $\mathbf{C}$ component). Proposition 4.1 gives us that l is an isomorphism. The right identity is defined in a similar way.

Filt–VII The coherence conditions hold trivially by the fact that each of the categories $\mathsf{Filt}(\mathbf{C})(A, B)$ is a preorder.

Definition 6.3 If $\phi = (f, C_X, i)$ is an arrow $A_X \to B_Y$ in $\mathsf{Filt}(\mathbf{C})$ then write $\chi(\phi)$ for the injection $i_{\mathbf{C}} : X \to Y$.

7 The Category of Concrete Monoids

The Poincaré category of the bicategory $\mathsf{Filt}(\mathbf{C})$ is the category which we will refer to as the *category of concrete monoids over* $\mathbf{C}$, denoted $\mathbf{Conc}(\mathbf{C})$.

For the rest of this section we work towards a characterization of the connected components of $\mathsf{Filt}(\mathbf{C})(A_X, B_Y)$ in order to understand the arrows of $\mathbf{Conc}(\mathbf{C})$. We shall use lower case Greek letters to denote arrows of $\mathsf{Filt}(\mathbf{C})$ in order to avoid confusion with objects of $\mathbf{Mon}$.

Lemma 7.1 If ϕ and ψ are objects of the same connected component of $\mathsf{Filt}(\mathbf{C})(A_X, B_Y)$ then $\chi(\phi) = \chi(\psi)$.

Proof. Let $\phi = (f, C_X, i)$ and $\psi = (g, D_X, j)$ and suppose there is an arrow (2-cell) $\alpha : \phi \to \psi$. Now $\chi(\phi) = i_{\mathbf{C}}$. But

$$\begin{aligned} i_{\mathbf{C}} &= \alpha_{\mathbf{C}} j_{\mathbf{C}} \\ &= j_{\mathbf{C}} \end{aligned}$$

since $\alpha_{\mathbf{C}}$ is forced to be the identity as noted in Proposition 6.1(i). The result now follows by a simple induction. $\square$

The following proposition is now easy to verify :

Proposition 7.2 The construction of Definition 6.3 defines a functor $\chi : \mathbf{Conc}(\mathbf{C}) \to \mathbf{C}$. $\square$

Note that χ likewise defines a functor from the classifying category of $\mathsf{Filt}(\mathbf{C})$ to $\mathbf{C}$.

For any category $\mathbf{C}$ and any object c of $\mathbf{C}$, define $\mathbf{Sub}_{\mathbf{C}}(c)$ to be the full subcategory of the comma category $\mathbf{C}/c$ whose objects are the monic arrows into c.

Theorem 7.3 Let ϕ and ψ be objects of $\mathsf{Filt}(\mathbf{C})(A_X, B_Y)$. The following are equivalent:
(1) ϕ and ψ are in the same connected component of the category $\mathsf{Filt}(\mathbf{C})(A_X, B_Y)$;
(2) ϕ and ψ have an upper bound in the poset $\mathsf{Filt}(\mathbf{C})(A_X, B_Y)$;
(3) ϕ and ψ have a least upper bound in the poset $\mathsf{Filt}(\mathbf{C})(A_X, B_Y)$;
(4) $\chi(\phi) = \chi(\psi)$.

Proof. It is immediate that (3) $\Rightarrow$ (2) $\Rightarrow$ (1), and by Lemma 7.1 (1) $\Rightarrow$ (4). It only remains to prove that (4) $\Rightarrow$ (3).

Let $\phi = (f, C_X, i)$ and $\psi = (g, D_X, j)$ be such that $\eta = \chi(\phi) = \chi(\psi) : X \to Y$. We proceed in three steps:
(1) We construct an object (E_X, q) of $\mathbf{Sub}_{\mathbf{Reps}(\mathbf{C})}(B_Y)$.
(2) Define $p_{\mathbf{Mon}} : E \to A$ in $\mathbf{Mon}$ making $(p_{\mathbf{Mon}}, 1_X) : E_X \to A_X$ an arrow of $\mathbf{Reps}(\mathbf{C})$ and $\gamma = ((p_{\mathbf{Mon}}, 1_X), E_X, q) : A_X \to B_Y$ an arrow of $\mathbf{Conc}$. Set $p =$

$(p_{\mathbf{Mon}}, 1_X)$ in $\mathbf{Reps}(\mathbf{C})$.
(3) Show that (p, E_X, q) is the least upper bound of ϕ and ψ.

Step 1. We have the diagram

$$C \xrightarrow{i_{\mathbf{Mon}}} B \xleftarrow{j_{\mathbf{Mon}}} D$$

in **Mon**. Let $(E, q_{\mathbf{Mon}})$ be the coproduct of $(C, i_{\mathbf{Mon}})$ and $(D, j_{\mathbf{Mon}})$ in $\mathbf{Sub}_{\mathbf{Mon}}(B)$ — that is, the join of C and D — with coprojections α and β. The situation in **Mon** is shown in Figure 1.

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[dr, "i"'] & E \arrow[d, "q"] & D \arrow[l, "\beta"'] \arrow[dl, "j"] \\ & B & \end{tikzcd}$$

Figure 1: The join of C and D in **Mon** constructed as the coproduct of $(C, i_{\mathbf{Mon}})$ and $(D, j_{\mathbf{Mon}})$ in $\mathbf{Sub}_{\mathbf{Mon}}(B)$.

Define $E_X(e)$ as follows. If $e = \alpha(c)$, for some $c \in C$ then define $E_X(e) = C_X(c)$. If $e = \beta(d)$, for some $d \in D$ then define $E_X(e) = D_X(d)$. This is a well-defined function on $\alpha(C) \cup \beta(D)$ since if $\alpha(c) = \beta(d)$ then

$$\begin{aligned} \eta C_X(c) &= B_Y(i_{\mathbf{Mon}}(c))\eta \\ &= B_Y(q_{\mathbf{Mon}}\alpha(c))\eta \\ &= B_Y(q_{\mathbf{Mon}}\beta(d))\eta \\ &= B_Y(j_{\mathbf{Mon}}(d))\eta \\ &= \eta D_X(d) \end{aligned}$$

but since η is monic this implies that $C_X(c) = D_X(d)$. Otherwise, e can be expressed as a word in elements of $\alpha(C)$ and $\beta(D)$. Define E_X by homomorphic extension. Suppose that

$$\alpha(u_1)\beta(v_1)\ldots\alpha(u_k)\beta(v_k) = \alpha(a_1)\beta(b_1)\ldots\alpha(a_m)\beta(b_m)$$

in E. To show that E_X is a well-defined functor, it is enough to show that

$$E_X(\alpha(u_1)\beta(v_1)\ldots\alpha(u_k)\beta(v_k)) = E_X(\alpha(a_1)\beta(b_1)\ldots\alpha(a_m)\beta(b_m)).$$

Now

$$\eta E_X(\alpha(u_1)\beta(v_1)\ldots\alpha(u_k)\beta(v_k)) \;=\; \eta C_X(u_1)D_X(v_1)\ldots C_X(u_k)D_X(v_k)$$

$$
\begin{aligned}
&= B_Y(i_{\mathbf{Mon}}(u_1))\eta D_X(v_1)\dots C_X(u_k)D_X(v_k)\\
&= \dots\\
&= B_Y(i_{\mathbf{Mon}}(u_1)j_{\mathbf{Mon}}(v_1)\dots i_{\mathbf{Mon}}(u_k)j_{\mathbf{Mon}}(v_k))\eta
\end{aligned}
$$

while

$$\eta E_X(\alpha(a_1)\beta(b_1)\dots\alpha(a_m)\beta(b_m)) = B_Y(i_{\mathbf{Mon}}(a_1)j_{\mathbf{Mon}}(b_1)\dots i_{\mathbf{Mon}}(a_m)j_{\mathbf{Mon}}(b_m))\eta.$$

Since

$$\alpha(u_1)\beta(v_1)\dots\alpha(u_k)\beta(v_k) = \alpha(a_1)\beta(b_1)\dots\alpha(a_m)\beta(b_m)$$

applying $q_{\mathbf{Mon}}$ on both sides gives

$$i_{\mathbf{Mon}}(u_1)j_{\mathbf{Mon}}(v_1)\dots i_{\mathbf{Mon}}(u_k)j_{\mathbf{Mon}}(v_k) = i_{\mathbf{Mon}}(a_1)j_{\mathbf{Mon}}(b_1)\dots i_{\mathbf{Mon}}(a_m)j_{\mathbf{Mon}}(b_m).$$

Therefore

$$\eta E_X(\alpha(u_1)\beta(v_1)\dots\alpha(u_k)\beta(v_k)) = \eta E_X(\alpha(a_1)\beta(b_1)\dots\alpha(a_m)\beta(b_m))$$

and by the fact that η is monic

$$E_X(\alpha(u_1)\beta(v_1)\dots\alpha(u_k)\beta(v_k)) = E_X(\alpha(a_1)\beta(b_1)\dots\alpha(a_m)\beta(b_m)).$$

We have therefore constructed a well-defined functor E_X.

To see that $(q_{\mathbf{Mon}}, \eta)$ is an arrow $E_X \to B_Y$ of $\mathbf{Reps}(\mathbf{C})$, let $e = \alpha(a_1)\dots\beta(b_m) \in E$ and simply notice that

$$
\begin{aligned}
\eta E_X(e) &= \eta C_X(a_1)\dots D_X(b_m)\\
&= B_Y(i_{\mathbf{Mon}}(a_1))\dots B_Y(j_{\mathbf{Mon}}(b_m))\eta\\
&= B_Y(q_{\mathbf{Mon}}(\alpha(a_1)\dots\beta(b_m)))\eta\\
&= B_Y(q_{\mathbf{Mon}}(e))\eta
\end{aligned}
$$

as required. Henceforth, q will refer to $(q_{\mathbf{Mon}}, \eta)$.

Step 2. Here we will omit the subscript **Mon** from the names of some arrows as the context makes clear the fact that we are speaking of an arrow of **Mon**. Define $p(\alpha(c)) = f(c)$ and $p(\beta(d)) = g(d)$ and extend homomorphically. We need to show that this is well-defined. Set $w_1 = \alpha(u_1)\beta(v_1)\dots\alpha(u_k)\beta(v_k)$ and $w_2 = \alpha(a_1)\beta(b_1)\dots\alpha(a_m)\beta(b_m)$. Suppose $w_1 = w_2$ in E, then we need to show that $p(w_1) = p(w_2)$. By faithfulness of A_X it is sufficient to show that $A_X(p(w_1)) = A_X(p(w_2))$. By definition

$$
\begin{aligned}
A_X(p(w_1)) &= A_X(f(u_1)g(v_1)\dots g(v_k))\\
&= C_X(u_1)\dots D_X(v_k)
\end{aligned}
$$

and similarly, $A_X(p(w_2)) = C_X(a_1)\dots D_X(b_m)$. Then

$$\begin{aligned} \eta A_X(p(w_1)) &= \eta C_X(u_1)\dots D_X(v_k) \\ &= B_Y(i(u_1)j(v_1)\dots j(v_k))\eta \end{aligned}$$

and likewise $\eta A_X(p(w_2)) = B_Y(i(u_1)j(v_1)\dots j(v_k))\eta$. Thus if $A_X(p(w_1)) \neq A_X(p(w_2))$, we must have that $\eta A_X(p(w_1)) \neq \eta A_X(p(w_2))$ since η is monic and so

$$B_Y(i(u_1)j(v_1)\dots j(v_k))\eta \neq B_Y(i(a_1)j(b_1)\dots j(b_m))\eta,$$

whence $i(u_1)j(v_1)\dots j(v_k) \neq i(a_1)j(b_1)\dots j(b_m)$. That is to say, $A_X(p(w_1)) \neq A_X(p(w_2))$ implies that

$$q(\alpha(u_1)\beta(v_1)\dots\alpha(u_k)\beta(v_k)) \neq q(\alpha(a_1)\beta(b_1)\dots\alpha(a_m)\beta(b_m))$$

which contradicts the assumption that $w_1 = w_2$. Thus p is a well-defined homomorphism of E onto A (it is onto since both f and g are onto).

To see that $(p_{\mathbf{Mon}}, 1_X)$ is an arrow $E_X \to A_X$ of $\mathbf{Reps(C)}$,

$$\begin{aligned} 1_X E_X(\alpha(a_1)\dots\beta(b_k)) &= C_X(a_1)\dots D_X(b_k) \\ &= 1_X A_X(f(a_1)\dots g(b_k)) \\ &= A_X(p(\alpha(a_1)\dots\beta(b_k))1_X \end{aligned}$$

as required. Henceforth p will refer to $(p_{\mathbf{Mon}}, 1_X)$.

Step 3. It is clear that in $\mathbf{Reps(C)}$, $p\alpha = f$ and $p\beta = g$, and by definition of α, β that $q\alpha = i$ and $q\beta = j$. Thus $\gamma = (p, E_X, q)$ is certainly an upper bound of ϕ and ψ.

To see that γ is the least upper bound of ϕ and ψ, suppose $\delta = (h, H_X, k)$ is another arrow $A_X \to B_Y$ and $\alpha' : \psi \Rightarrow \delta$ and $\beta' : \phi \Rightarrow \delta$ are 2-cells. By Lemma 7.1 $k_{\mathbf{C}} = \eta$. By the universal property of $(E, q_{\mathbf{Mon}})$ in $\mathbf{Sub_{Mon}}(B)$ there is a unique arrow $\xi : E \to H$ in $\mathbf{Mon}$ such that $\xi\alpha_{\mathbf{Mon}} = \alpha'_{\mathbf{Mon}}$ and $\xi\beta_{\mathbf{Mon}} = \beta'_{\mathbf{Mon}}$. The pair $(\xi, 1_X)$ is certainly an arrow of $\mathbf{Reps(C)}$ since

$$\begin{aligned} 1_X E_X(\alpha(a_1)\dots\beta(b_m)) &= 1_X C_X(a_1)\dots D_X(b_m) \\ &= H_X(\alpha'(a_1))\dots H_X(\beta'(b_m))1_X \\ &= H_X(\xi(\alpha(a_1)\dots\beta(b_m))1_X. \end{aligned}$$

as required.

To see that γ is the least upper bound, it only remains to show that $(\xi, 1_X)$ is a 2-cell $\gamma \Rightarrow \delta$. As ξ is an arrow from $(H, k_{\mathbf{Mon}})$ to $(E, q_{\mathbf{Mon}})$ in $\mathbf{Sub_{Mon}}(B)$, $k_{\mathbf{Mon}}\xi = q_{\mathbf{Mon}}$. Thus $k(\xi, 1_X) = q$, so we only need to show that $h(\xi, 1_X) = p$. This is certainly true in the $\mathbf{C}$ component, so we prove it for the $\mathbf{Mon}$ component. Leaving off subscripts for brevity, we wish to prove that $h\xi = p$. In $\mathbf{Mon}$, $h\xi\alpha = h\alpha' = f = p\alpha$ and similarly, $h\xi\beta = p\beta$. Thus for any element $\alpha(a_1)\ldots\beta(b_m) \in E$, we have $h\xi(\alpha(a_1)\ldots\beta(b_m)) = p(\alpha(a_1)\ldots\beta(b_m))$ as required. $\square$

As a result of Theorem 7.3 we are able to give an equivalent definition of the category of concrete monoids over $\mathbf{C}$.

Definition 7.4 The category $\mathbf{Conc}(\mathbf{C})$ of concrete monoids over $\mathbf{C}$ has concrete monoids as objects, and an arrow from A_X to B_Y is a monomorphism $i : X \to Y$ in $\mathbf{C}$ such that there is an arrow $\phi : A_X \to B_Y$ of $\mathsf{Filt}(\mathbf{C})$ with $\chi(\phi) = i$. Composition is the same as in $\mathbf{C}$.

As an immediate consequence of this observation we have

Corollary 7.5 All arrows of $\mathbf{Conc}(\mathbf{C})$ are monic. $\square$

Finally, we give a theorem relating functors between representing categories and functors between representations in them.

Theorem 7.6 [Change of Base] Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and $F : \mathbf{C} \to \mathbf{D}$ a functor. Then F induces a functor $F_{\mathbf{Reps}} : \mathbf{Reps}(\mathbf{C}) \to \mathbf{Reps}(\mathbf{D})$, which is faithful if F is faithful. If F is faithful and preserves monos, then it also induces a faithful functor $F_{\mathbf{Conc}} : \mathbf{Conc}(\mathbf{C}) \to \mathbf{Conc}(\mathbf{D})$.

Proof. Let $\Phi_X : M \to \mathbf{C}$ be an object of $\mathbf{Reps}(\mathbf{C})$, and $(f, \eta) : \Phi_X \to \Psi_Y$ an arrow of $\mathbf{Reps}(\mathbf{C})$. Define $F_{\mathbf{Reps}}(\Phi) = F\Phi : M \to \mathbf{D}$. Then the $F\Phi$ maps the object of M to $F(X)$ in $\mathbf{D}$. Write $F\Phi_X = \tilde{\Phi}_{FX}$. Define $F_{\mathbf{Reps}}(f, \eta) = (f, F\eta)$. To see that this is a valid arrow $\tilde{\Phi}_{FX} \to \tilde{\Psi}_{FY}$ in $\mathbf{Reps}(\mathbf{D})$ we need to show that for each $m \in M$, $\tilde{\Psi}_{FY}(f(m))F\eta = F\eta\tilde{\Phi}_{FX}(m)$. Since (f, η) is an arrow of $\mathbf{Reps}(\mathbf{C})$ we already have that $\Psi_Y(f(m))\eta = \eta\Phi_X(m)$, whence

$$\begin{aligned}
F\eta\tilde{\Phi}_{FX}(m) &= F\eta(F\Phi_X)(m) \\
&= F\eta F(\Phi_X(m)) \\
&= F(\eta\Phi_X(m)) \\
&= F(\Psi_Y(f(m))\eta) \\
&= F(\Psi_Y(f(m)))F\eta \\
&= \tilde{\Psi}_{FY}F\eta
\end{aligned}$$

proving that $F_{\mathbf{Reps}}(f, \eta) = (f, F\eta)$ is an arrow of $\mathbf{Reps}(\mathbf{D})$. The functoriality of this definition is immediate.

Suppose now that F is faithful and mono preserving. Then if $\Phi_X : M \to \mathbf{C}$ is an object of $\mathbf{Conc}(\mathbf{C})$ then $F\Phi_X$ is certainly a faithful functor $M \to \mathbf{D}$ so define $F_{\mathbf{Conc}}(\Phi_X) = F\Phi$. Let $\alpha : \Phi_X \to \Psi_Y$ be an arrow of $\mathbf{Conc}(\mathbf{C})$ with representative

$((f, 1_X), \Gamma_X, (i_{\mathbf{Mon}}, i_{\mathbf{C}}))$ in $\mathsf{Filt}(\mathbf{C})$. Then $((f, F1_X), F\Gamma_X, (i_{\mathbf{Mon}}, Fi_{\mathbf{C}}))$ is certainly an arrow $F\Phi \to F\Psi$ in $\mathsf{Filt}(\mathbf{D})$. Define $F_{\mathbf{Conc}}(\alpha)$ to be the Poincaré class of this arrow. Equivalently, by Theorem 7.3 and Definition 7.4, $F_{\mathbf{Conc}(\mathbf{C})} = F(i_{\mathbf{C}})$ and functoriality and faithfulness are immediate. $\square$

Part II – Applications

8 Concrete Matrix Monoids

Let K be a field. Then $\mathbf{Mat_K}$ is the category whose objects are the natural numbers $0, 1, 2, \ldots$ and arrows $m \to n$ are the $n \times m$ matrices over K.

In this section we exhibit an arrow in the category $\mathbf{Conc}(\mathbf{Mat_K})$. A concrete monoid A_m over $\mathbf{Mat_K}$ is a set of $m \times m$ matrices (including the identity) which is closed under matrix multiplication. Suppose there is an abstract submonoid Q of A whose elements are represented by matrices of the form

$$\left(\begin{array}{c|c} n \times n & * \\ \hline 0 & p \times p \end{array}\right)$$

for some $n + p = m$. It is well known that multiplication of two matrices of this form is given by

$$\left(\begin{array}{c|c} B & * \\ \hline 0 & C \end{array}\right) \cdot \left(\begin{array}{c|c} B' & * \\ \hline 0 & C' \end{array}\right) = \left(\begin{array}{c|c} BB' & * \\ \hline 0 & CC' \end{array}\right).$$

This observation yields concrete monoids B_n and C_p where B_n is the set of top left $n \times n$ corners of matrices in $\{A_m(q) \mid q \in Q\}$ and C_p is the set of bottom right $p \times p$ corners of matrices in $\{A_m(q) \mid q \in Q\}$. There are arrows $B_n \to A_m$ and $C_p \to A_m$ in $\mathsf{Filt}(\mathbf{Mat_K})$ as follows. Let Q_n be the (not necessarily faithful) representation of Q by mapping q to the top left $n \times n$ corner of $A_m(q)$. Let $i_{\mathbf{Mon}} : Q \to A$ be the natural inclusion, and

$$i_{\mathbf{Mat_K}} = \left(\begin{array}{c} I_{n \times n} \\ \hline 0_{p \times n} \end{array}\right).$$

Let $f_{\mathbf{Mon}} : Q \to B$ simply map each $q \in Q$ to the element of B which is represented by the matrix $Q_n(q)$. The reader may easily verify that putting $f_{\mathbf{Mat_K}} = I_{n \times n}$ makes (f, Q_n, i) into an arrow $B_n \to A_m$ of $\mathsf{Filt}(\mathbf{Mat_K})$, and hence $i_{\mathbf{Mat_K}}$ is an arrow $B_n \to A_m$ in $\mathbf{Conc}(\mathbf{Mat_K})$. Similarly

$$i_{\mathbf{Mat_K}} = \left(\begin{array}{c} 0_{n \times p} \\ \hline I_{p \times p} \end{array}\right)$$

is an arrow from C_p to A_m in $\mathbf{Conc}(\mathbf{Mat_K})$.

(Note: There is a well known faithful functor from $\mathbf{Mat_K}$ *to the category* $\mathbf{Vect_K}$ *of vector spaces over* K*. Theorem 7.6 therefore allows us to regard the above arrows as arrows of* $\mathbf{Conc}(\mathbf{Vect_K})$*.)*

9 Transformation Monoids

Questions about monoid representations over **Set** motivated the development of this work where the category **Conc**(**Set**) was discovered as the codomain of the functor which corresponds to the Catalan construction of [15]. Though it is the motivating example, the Catalan construction is not discussed here. Rather, we focus on two issues of more central importance to Semigroup Theory : the Cayley representation and the subgroup structure of a transformation monoid.

9.1 The Cayley Construction for Set

The well known Cayley construction can be viewed as a functor $Cay : \mathbf{Mon} \to \mathbf{Reps}(\mathbf{Set})$. The Cayley construction gives, for any monoid M a representation $Cay(M) : M \to \mathbf{Set}$ where the object of M is mapped to the set $|M|$ - the set of elements of M. If $f : M \to N$ is a monoid homomorphism, then denote by $|f|$ the function from the set $|M|$ to the set $|N|$ which is induced by f. Then it is easy to verify that $(f, |f|)$ is an arrow $Cay(M) \to Cay(N)$ in **Reps**(**Set**), and that setting $Cay(f) = (f, |f|)$ makes Cay into a functor.

9.2 Subgroups of Transformation Monoids

When studying semigroups, Green's relations are an important source of structural information (see [9]). In particular, Green's $\mathcal{H}$ relation identifies the maximal subgroups to be among the basic building blocks of a semigroup. Here we show that in a transformation monoid, the subgroups are permutation groups which can be found by filtering the original transformation monoid.

Let M be a monoid, and let S be a subsemigroup of M, not necessarily containing the identity. We say that S is a *group* subsemigroup of M if there is an idempotent $e \in S$, such that for each $s \in S$, there is a unique $t \in S$ with $st = ts = e$.

Theorem 9.1 Let M_X be a transformation monoid and S an (abstract) group subsemigroup of M. Then there is an arrow

$$S_Y \to M_X$$

in **Conc**(**Set**), where S_Y is some faithful permutation group representation of S.

Proof. Let $e \in S$ be the identity of the group and put $Y = Im(\overline{e}) \subseteq X$. Then S acts on Y since for all $g \in S$, $Im(\overline{g}) = Im(\overline{eg}) \subseteq Im(\overline{e}) = Y$. In fact, we will show that the representation of S as transformations of Y is a faithful permutation representation. Since e is idempotent, it acts identically on Y. If $g \in S$, then $\overline{g} : Y \to Y$ is surjective, since for each $y \in Y$, $y = \overline{e}(y) = \overline{g\overline{g^{-1}}}(y)$. Similarly, $\overline{g}$ is injective on Y since its action is invertible.

It only remains to show that the representation of S as permutations of Y is faithful. Suppose g_1 and g_2 are such that $\overline{g_1}(y) = \overline{g_2}(y)$ for all $y \in Y$. Then $\overline{g_1 e} = \overline{g_2 e}$ since $Im(\overline{e}) = Y$. But since M_X is a faithful representation, this implies that $g_1 = g_1 e = g_2 e = g_2$ as required.

We now give an arrow in $\mathsf{Filt}(\mathbf{Set})$ from S_Y to M_X. Let S^1 be the monoid formed by adding a new, disjoint identity element 1 to S, and let S^1_Y be the monoid representation of S^1 as endomorphisms of Y induced by the action of S on Y. Define $i_{\mathbf{Mon}} : S^1 \to M$ to be the unique monoid morphism which restricts to the natural inclusion of S into M. Let $i_{\mathbf{Set}} : Y \to X$ also be the natural inclusion. Let $f_{\mathbf{Mon}} : S^1 \to S$ act identically on S and map 1 to e, and set $f_{\mathbf{Set}} = 1_Y$. Then it is easy to see that $i : S^1_Y \to M_X$, and $f : S^1_Y \to S_Y$ are arrows of $\mathbf{Reps}(\mathbf{Set})$, making (f, S^1_Y, i) the required division. Thus, $i_{\mathbf{Set}}$ is an arrow from $S_Y \to M_X$ in $\mathbf{Conc}(\mathbf{Set})$. $\square$

10 The Theory of Automata

A nice result of the developments in Part I of this paper is that we are able to make functorial the so called 'action monoid' construction in Automaton Theory in a more satisfying way than has been done before.

To give an account of this new development we begin in Section 10.1 by recapitulating the elements of the Theory of Automata. The new construction of the action monoid functor is then given in Section 10.2. To the author's knowledge, the only other functorial approach to the action monoid construction is given in [6]. We sketch this approach in Section 10.3 to allow the reader to draw their own conclusions about the relative merits of the two approaches.

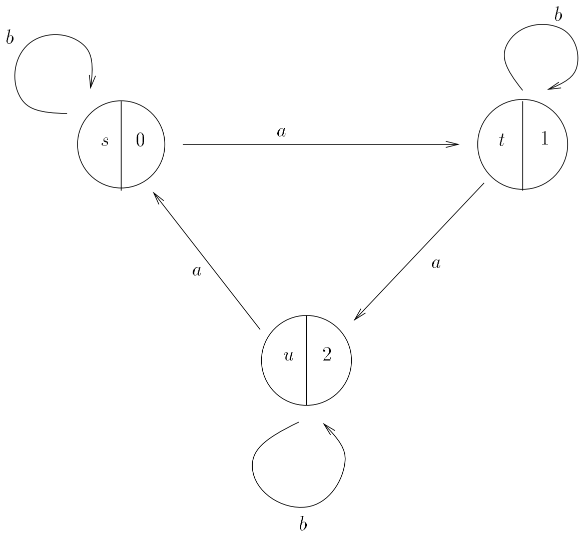
10.1 Introduction to the Theory of Automata

In [6] Ehrig gives a categorical treatment of $\mathbf{C}$-automata whose inputs, outputs and states are objects of any closed monoidal category $\mathbf{C}$. For completeness, we give a brief account of this theory, but, in order to avoid alienating the reader and to simplify the exposition, we restrict our attention to the case $\mathbf{C} = \mathbf{Set}$. It should, however, be borne in mind that all of the developments in this section hold for any closed monoidal category. The interested reader may refer to [8] for a similar exposition where the automata have the additional (but for our purposes, unnecessary) feature of an initial state.

Fix sets X and Y, referred to as the *input* and *output* alphabets respectively. An (X, Y) *automaton* (or 'automaton' for short) is a pair (Φ_S, λ), where $\Phi_S : X^* \to \mathbf{Set}$ is a monoid representation landing on some set S, and $\lambda : S \to Y$ is any function. The set S is referred to as the set of *states*, and λ is called the *output function*. This corresponds to the well established notion of a *Moore automaton* without initial state.

Remark 10.1 If S and X are finite sets, an automaton may be represented by a directed graph with vertex set S, each vertex s labelled by $\lambda(s)$ of Y and for each $x \in X$ an edge labelled x from s to $\Phi_S(x)(s)$. So for each word $w = x_1 \dots x_n$ of X^*, and each $s \in S$ there is a path $s \xrightarrow{x_n} \overline{x_n}(s) \xrightarrow{x_{n-1}} \cdots \xrightarrow{x_1} \overline{w}(s)$. If, in addition, Y is the set $\{0, 1\}$, then our definition corresponds to the familiar notion of a finite state acceptor, where the state s is terminal if and only if $\lambda(s) = 1$.

For example, consider the automaton below with input alphabet $X = \{a, b\}$, output alphabet $Y = \{0, 1, 2\}$ and state set $S = \{s, t, u\}$.



The representation of X^* as endomorphisms of S which this describes is given as follows: the single object $\cdot$ of the category X^* is mapped to S, the letter a is mapped to the 3-cycle $(s\ t\ u)$ and b is mapped to 1_S.

Let $A = (\Phi_S, \lambda)$ and $B = (\Psi_T, \rho)$ be automata. A *morphism* $\phi : A \to B$ is a function $\phi : S \to T$ such that $(1_{X^*}, \phi) : \Phi_S \to \Psi_T$ is an arrow of **Reps**(**Set**) and $\lambda = \rho\phi$. If ϕ is onto, then we say that B is a *quotient* of A.

Definition 10.2 The category of (X, Y)-automata and their morphisms will be denoted **Aut**(X, Y).

Let A be an automaton as above, and let $s \in S$ be a state of A. Define a function $f_s : X^* \to Y$ by $f_s(w) = \lambda(\Phi_S(w)(s))$. Call f_s the *behaviour of* A *at* s. As a heuristic, f_s is the 'function computed by A when it is started in state s'. In the example above, f_s is the number of times (modulo 3) the letter a occurs in the input. The set $E(A) = \{f_s \mid s \in S\}$ is called the *external behaviour of* A and we say that A *realizes* the behaviour $E(A)$.

Let f be any function $X^* \to Y$ and let $u \in X^*$. Define the *left shift* of f by u to be the function

$$\begin{aligned} \sigma_u f : X^* &\longrightarrow Y \\ w &\longmapsto f(wu). \end{aligned}$$

Denote by Y^{X^*} the set of all functions $X^* \to Y$. The operator σ is clearly a left action of X^* on Y^{X^*} since $\sigma_u \sigma_v = \sigma_{uv}$.

It is easy to see that for any automaton A, the external behaviour $E(A)$ is closed under left shift — if s is a state of A, and $x \in X$ then $\sigma_x(f_s) = f_{\overline{x}(s)}$. In general we define a *behaviour* to be a subset $P \subseteq Y^{X^*}$ which is closed under left shift by words of X^*. In the sequel we will see that every behaviour is realized by some automaton.

Definition 10.3 The category $\mathbf{Beh}(X, Y)$ has behaviours as objects with an arrow $P \to Q$ if $P \subseteq Q$. One can think of an arrow $P \to Q$ as the canonical inclusion function.

Let A and B be automata as above. The construction of $E(A)$ from A is made functorial by defining, for any arrow $\phi : A \to B$, the arrow $E(\phi) : E(A) \to E(B)$ by $f_s \mapsto f_{\phi(s)}$. To show that $E(\phi)$ is a well-defined arrow of $\mathbf{Beh}(X, Y)$ we merely show that f_s and $f_{\phi(s)}$ describe the same function $X^* \to Y$. Let $w \in X^*$ then

$$\begin{aligned} f_s(w) &= \lambda(\Phi_S(w)(s)) \\ &= \rho\phi(\Phi_S(w)(s)) \\ &= \rho\Psi_T(w)(\phi(s)) \\ &= f_{\phi(s)}(w) \end{aligned}$$

as required. It is now immediate that E is functorial and it is referred to as the *external behaviour functor* $\mathbf{Aut}(X, Y) \to \mathbf{Beh}(X, Y)$.

Let P be a behaviour. Then define an automaton $N(P)$, the *minimal realization*, of P to have state set P, and action of X^* on P given by left shift. Define the output function λ by $\lambda(p) = p(1)$ for each $p \in P$. Suppose P and Q are behaviours with $P \subseteq Q$. Let the canonical inclusion function be $i : P \to Q$. Then define $N(i) : N(P) \to N(Q)$ by $p \mapsto i(p)$. That $N(i)$ is a well-defined morphism of automata and that N is a functor $\mathbf{Beh}(X, Y) \to \mathbf{Aut}(X, Y)$ are both easy to verify.

The following theorem is proved in [6] and [8].

Theorem 10.4 The external behaviour functor E is left adjoint to the minimal realization N. $\square$

Remark 10.5 Let $A = (\Phi_S, \lambda)$ be an automaton. The unit of the adjunction $\eta : 1_{\mathbf{Aut}(X,Y)} \to NE$ is given by

$$\begin{aligned} \eta_A : A &\longrightarrow NE(A) \\ s &\longmapsto f_s. \end{aligned}$$

This definition makes each arrow η_A onto so that $NE(A)$ is a quotient of A. The reader familiar with the Theory of Automata, will recognize η_A as the Nerode quotient and $NE(A)$ as the minimal automaton of A. (That is, the minimal automaton with behaviour $E(A)$.)

Theorem 10.6 EN is the identity functor of $\mathbf{Beh}(X, Y)$. Consequently, if P is any behaviour, $N(P)$ has behaviour P.

Proof. Since $\mathbf{Beh}(X, Y)$ is just a poset, it is sufficient to show that EN is the identity on objects. Let P be a behaviour. Then $N(P)$ has state set P, with action of X^* on P given by left shift and output function γ which maps p to $p(1)$. Then $EN(P)$ is the set $\{f_p \mid p \in P\}$ where $f_p(w) = \gamma(\sigma_w(p)) = \sigma_w(p)(1) = p(w)$ proving that $EN(P) = P$. $\square$

We are now able to explain why N is called the minimal realization functor. Theorem 10.6 shows that $N(P)$ is an automaton which realizes the behaviour P. If A is any automaton with the behaviour P, then $N(P) = NE(A)$ which is a quotient of A, by Remark 10.5. Thus, if we partially order the automata with behaviour P by $A \succ B$ if B is a quotient of A, then $N(P)$ is the minimum element of this poset.

Definition 10.7 Denote by $\mathbf{Aut}_{\mathrm{min}}(X, Y)$ the full subcategory of $\mathbf{Aut}(X, Y)$ whose objects are minimal realizations of behaviours.

Theorem 10.8 The minimal realization functor N is full and faithful, allowing us to regard $\mathbf{Beh}(X, Y)$ as isomorphic to $\mathbf{Aut}_{\mathrm{min}}(X, Y)$. Since the inclusion N has a left adjoint, $\mathbf{Aut}_{\mathrm{min}}(X, Y)$ is a reflective subcategory of $\mathbf{Aut}(X, Y)$ (see [13, p.89]).

Proof. The faithfulness of N follows immediately from the definition. To see that N is full, suppose $\phi : N(P) \to N(Q)$ is an arrow of $\mathbf{Aut}(X, Y)$ for behaviours P and Q. Then by the definition of E above, for each state s of $N(P)$, the behaviour f_s of $N(P)$ at s and the behaviour $f_{\phi(s)}$ of $N(Q)$ at $\phi(s)$ are equal. But a simple calculation shows that the behaviour of $N(P)$ at s is s itself. Thus $\phi(s) = f_{\phi(s)} = f_s = s$, showing that ϕ is the canonical inclusion of the state set of $N(P)$ into the state set of $N(Q)$. That is, ϕ is also an arrow of $\mathbf{Beh}(X, Y)$. $\square$

Corollary 10.9 Every arrow of $\mathbf{Aut}_{\mathrm{min}}(X, Y)$ is injective as a function on state sets.

10.2 The Action Monoid construction

Now, to the construction of the action monoid as a functor $\Omega : \mathbf{Aut}_{\mathrm{min}}(X, Y) \to \mathbf{Conc}(\mathbf{Set})$.

Recall that the action monoid $\Omega(A)$ of an automaton $A = (\Phi_S, \lambda)$ is the concrete monoid of endomorphisms of the set S consisting of elements $\{\Phi_S(w) \mid w \in X^*\}$. In the example above, the action monoid would be the cyclic group generated by the 3-cycle $(s\ t\ u)$.

Let $A = (\Phi_S, \lambda)$ and $B = (\Psi_T, \rho)$ be minimal automata, and $\phi : A \to B$ an automaton morphism. Then $\Omega(A)$ is a faithful representation of the monoid $M =$

$X^*/\sim_\Phi$ landing on the object S of **Set**. The action of $\Omega(A)$ is given by $\overline{[w]_\Phi} = \Phi(w)$ where $[w]_\Phi$ is the $\sim_\Phi$ class of w. (Refer to Section 4 for the definition of $\sim_\Phi$). Let N be the monoid $X^*/\sim_\Psi$. Then $\Omega(B)$ is the corresponding faithful representation of N by endomorphisms of T.

Then since every arrow of $\mathbf{Aut}_{\mathrm{min}}$ is injective, S embeds in T, so that $\sim_\Phi \supseteq \sim_\Psi$. Define the representation $N_S : N \to \mathbf{Set}$ by the action $N_S([w]_\Psi)(s) = \Phi_S(w)(s)$. Then N_S embeds in $\Omega(B)$ by $(1_N, \phi)$. The quotient map $\psi : N \to Q$ defined by $[w]_\Psi \mapsto [w]_\Phi$ gives an strong epimorphism $(\psi, 1_S) : N_S \to \Omega(A)$ in $\mathbf{Reps}(\mathbf{Set})$. Then the triple $((\psi, 1_S), N_S, (1_N, \phi))$ is an arrow from $\Omega(A)$ to $\Omega(B)$ in $\mathsf{Filt}(\mathbf{Set})$.

Since $\chi((\psi, 1_S), N_S, (1_N, \phi)) = \phi$ we may now define $\Omega(\phi) = \phi : \Omega(A) \to \Omega(B)$ in $\mathbf{Conc}(\mathbf{Set})$. This completes the definition of the action monoid functor $\Omega : \mathbf{Aut}_{\mathrm{min}}(X, Y) \to \mathbf{Conc}(\mathbf{Set})$.

Remark 10.10 The reader will note that Ω may be composed with NE to give a functor from the category $\mathbf{Aut}(X, Y)$ to $\mathbf{Conc}(\mathbf{Set})$ rather than from $\mathbf{Aut}_{\mathrm{min}}(X, Y)$. However, for any (X, Y) automaton A this functor computes the action monoid of its minimal automaton $NE(A)$.

10.3 Ehrig's Action Monoid Functor

In [6, §5.6], Ehrig defines a functor from a category $\mathbf{Aut}(S)$ of automata to a category of transformation monoids, whose object function is precisely the action monoid of the automaton (cf. Remark 10.10).

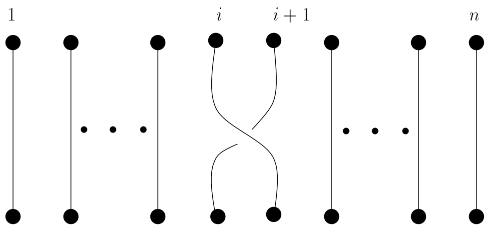
Fix some set S. An S-*automaton* is simply an object $\Phi_S : X^* \to \mathbf{Set}$ of $\mathbf{Reps}(\mathbf{Set})$ landing on S, where X is any set. In other words, an S-automaton can have any input set but it has state set S. Let $B = \Psi_S : X'^* \to \mathbf{Set}$ be another S-automaton. An S-automaton *morphism* from Φ to Ψ is a homomorphism $\alpha : X^* \to X'^*$ such that for each $w \in X^*$, $\Phi(w) = \Psi\alpha(w)$. Therefore the action monoid of Φ is contained in the action monoid of Ψ. Notice that this definition of morphism is just a strict morphism of $\mathbf{Reps}(\mathbf{Set})$. Denote by $\mathbf{Aut}(S)$ the category of S-automata and their morphisms.

The monoid of all endomorphisms of the set S will be denoted $\mathit{End}(S)$. Let $\mathbf{End}(S)$ be the poset of submonoids of $\mathit{End}(S)$ regarded as a category where arrows are inclusions. It is now obvious by the definition of a morphism in $\mathbf{Aut}(S)$, that mapping every A in $\mathbf{Aut}(S)$ to its action monoid in $\mathbf{End}(S)$ describes a functor.

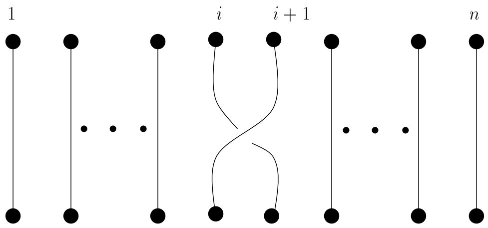
11 An Introduction to Vines

In this section, we give an informal introduction to Lavers' Theory of Vines [12], and from this basis, develop the category **Vine** of vines and investigate the category of concrete vine monoids. The reader is assumed to be familiar with Artin's Theory of Braids [2]. The category **Vine** is not regular (since it is not finite complete) and so provides us with an example of a category from which we can construct **Conc**(**Vine**) but not Div(**Reps**(**Vine**)).

In the following discussion $[n]$ will refer to the set $\{1, 2, \ldots, n\}$, with the usual total order. Fix $n > 0$. An n-braid is a set of n arcs in $\boldsymbol{R}^3$ from the n initial nodes $I(n) = \{(i, 0, 1) \mid i \in [n]\}$ to the n terminal nodes $T(n) = \{(i, 0, 0) \mid i \in [n]\}$. Every braid may be regarded as a product of the two generators pictured below.

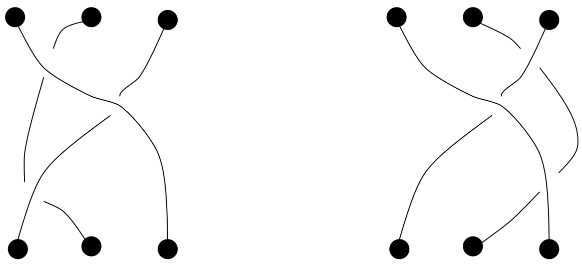


σ_i

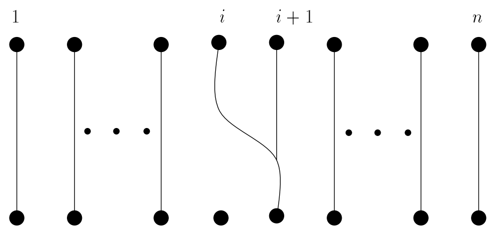


σ_i^{-1}

Two braids are regarded as equivalent if they can be deformed, one into the other, by a homotopy as in the following diagram.

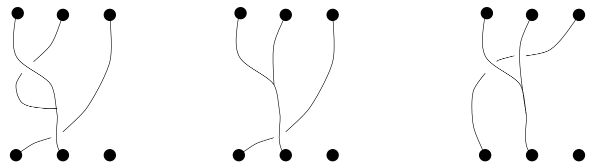


Rather than give a formal definition for vines, we rely on the reader's intuition about braids and some illuminating examples. A *vine* from $I(n)$ to $T(n)$ may be written as a product of the generators σ_i and σ_i^{-1} as pictured above, together with the generator g_i shown below.

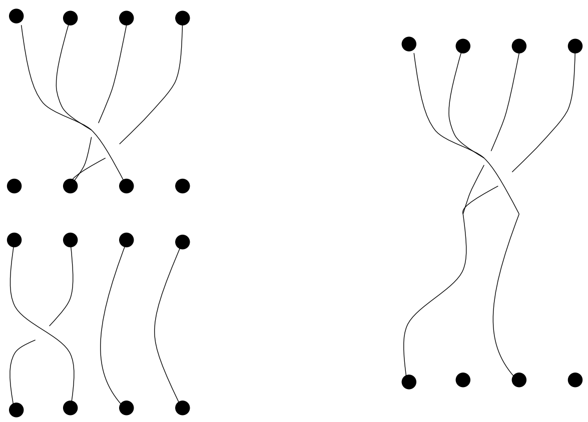


g_i

As a result, the strings of a vine may merge, but not separate. Among the homotopy equivalence relations we have $\sigma_i g_i = \sigma_i^{-1} g_i = g_i$, meaning that strings can twist and untwist about a join. Some examples of equivalent vines follow.



(Note: The word 'vine' may be taken to mean either the set of arcs or, the homotopy equivalence class of the set of arcs, depending on the context.) As with braids, composition is given by concatenation and shrinkage, and is a well-defined and associative operation on homotopy equivalence classes of vines. Note that if a string in the concatenation is not connected to any initial node, it is homotopically shrunk to a point. An example of composition of vines follows.



The set of equivalence classes of vines is a monoid with the same identity element as the braid group — a string connects the ith initial node to the ith terminal node with no intertwining or joining of the strings. Defining relations for the monoid of all vines on n nodes are given in [12].

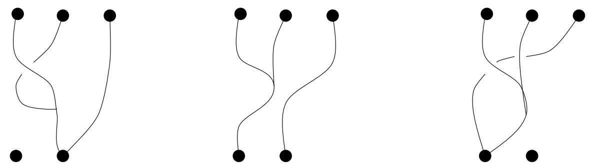
Lavers' vines always go from n intial nodes to n terminal nodes. We make here the straightforward generalization that a vine may go from $I(n)$ to $T(m)$ where n is not necessarily equal to m — it consists of n strings connected to each of the n initial nodes, each string ending at a node in $T(m)$. This will be known as an $n \to m$

vine. Composition of vines is then only allowed when the number of terminal nodes of the first vine is equal to the number of initial nodes of the second. This generalization permits us to define the category **Vine** whose objects are the natural numbers $0, 1, 2, \ldots$ and whose arrows from n to m are precisely the (homotopy equivalence classes of) $n \to m$ vines.

Definition 11.1 If v is an $n \to m$ vine, let $\overline{v}$ denote the map $[n] \to [m]$ that maps i to k if the ith string meets the kth terminal node $(k, 0, 0) \in T(m)$. The map $\overline{v}$ will be called the *function associated with* v.

Proposition 11.2 There is a functor **Vine** $\to$ **Set** which takes n to $[n]$ and the $n \to m$-vine v to $\overline{v}$. $\square$

A *simple vine* is one which is homotopic to a vine in which the strings never intertwine. Some simple vines are pictured below.



A *braid* is a vine which has no joins, that is to say, a vine v such that $\overline{v}$ is injective.

Proposition 11.3 The simple $n \to n$-vines form a monoid isomorphic to the monoid of order-preserving transformations of $\{1, 2, \ldots, n\}$. The isomorphism is given by mapping v to $\overline{v}$, the function associated with v. $\square$

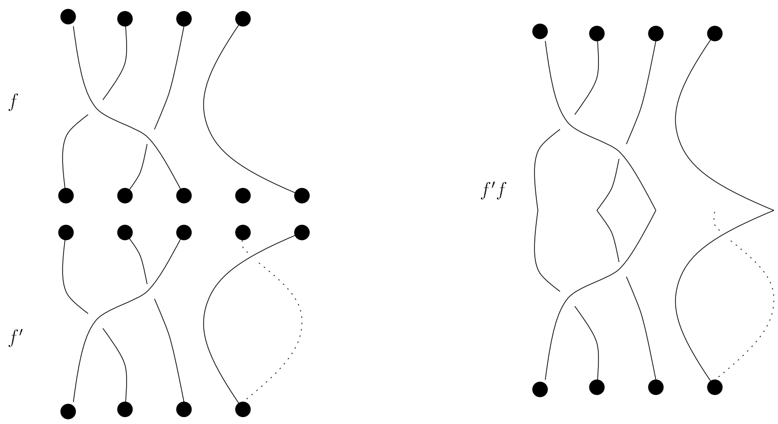
The following theorem is a direct consequence of [12, Proposition 7].

Theorem 11.4 Every vine $u : m \to n$ can be written $u = vb$ where b is a $m \to m$-braid and $v : m \to n$ is a simple vine. $\square$

Lemma 11.5 Every braid $n \to p, n > 0$ has a left inverse.

Proof. Let $f : n \to p$ be a braid. Consider the picture obtained by reflection of f in the $x - y$ plane. This may not be a vine since there is not necessarily an arc emanating from every node in the top plane of the reflection. Construct a vine by adding arcs emanating from top nodes which are not already at the top of an arc. The bottom nodes at which they terminate may be chosen arbitrarily. Call the vine so obtained f'. Then it is clear that $f'f = 1_n$. $\square$

An example of the construction of Lemma 11.5 is pictured below.



Theorem 11.6 Braids are the monic arrows of **Vine**.

Proof. By Lemma 11.5 we see that braids $n \to p, n > 0$ are certainly monic. Further, since there are is only one arrow into 0, every braid $0 \to p$ is monic, thus all braids are monic. Conversely, suppose the vine $u : m \to n$ is not a braid. Write (by Theorem 11.4) $u = vb$ where b is a $m \to m$-braid and $v : m \to n$ is a simple vine. Let b^{-1} be the (group) inverse of b. Since vb has joins, there must be consecutive nodes indexed by $i, i+1$ in $I(m)$ such that $\overline{v}(i) = \overline{v}(i+1)$. Let $s : 2 \to m$ be a braid which connects $1 \in I(2)$ to $i \in T(m)$ and $2 \in I(2)$ to $i+1 \in T(m)$. Let $t : 2 \to m$ be a braid which connects $1 \in I(2)$ to $i+1 \in T(m)$ and $2 \in I(2)$ to $i \in T(m)$. Then it is clear that

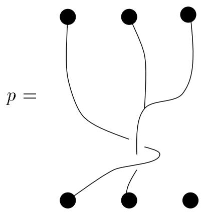
$$\begin{aligned} ub^{-1}s &= vbb^{-1}s \\ &= vs \\ &= vt \\ &= ub^{-1}t. \end{aligned}$$

But since b^{-1} is monic, $b^{-1}s \neq b^{-1}t$, proving that u is not monic. $\Box$

The category **Vine** has initial object 0 and terminal object 1. Thus we may construct the category **Conc**(**Vine**). An object of **Conc**(**Vine**) is called a *vine monoid.*

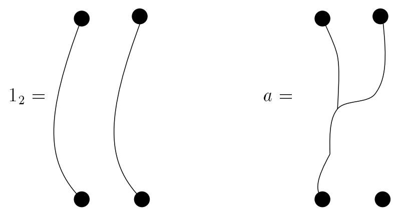
11.1 An arrow of Conc(Vine).

Consider the vine monoid M_3 on 3 strings generated by p as in the following diagram.



The single generator p of M_3.

This is clearly infinite, and does not contain the two element monoid N_2 (shown below) as a submonoid.



The vine monoid N_2 with elements $\{1, a\}$.

However N_2 may certainly be found as a divisor by the division $((\natural, 1_2), M_2, (1_M, \eta))$ where $M_2(p)$ is defined to be the 2-braid a, η is $\sigma_1^2 : 2 \to 3$ (shown in Figure 2), and $\natural : M \to N$ is the natural quotient. To see that $(1_M, \eta)$ is an arrow of **Reps**(**Vine**) simply notice that $\eta M_2(p) = M_3(p)\eta$ as in Figure 2.

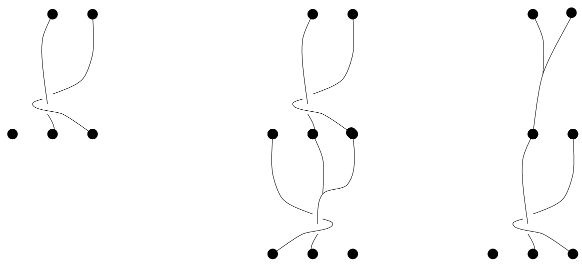


Figure 2: From left to right : η, $M_3(p)\eta$ and $\eta M_2(p)$.

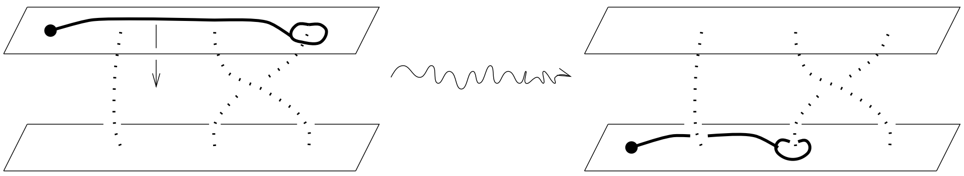
11.2 From Vine to FreeGrp

Let $\{x_1, x_2, \ldots\}$ be an infinite alphabet. Denote by F_n the free group on the generators $\{x_1, \ldots x_n\}$. Let **FreeGrp** be the category whose objects are $F_n, n \geq 0$, and whose arrows are all group homomorphisms between them. The trivial group is free of rank 0, so that both **FreeGrp** and **FreeGrp**^{op} have it for an initial and terminal object. We can therefore speak of the category **Conc**(**FreeGrp**^{op}).

Firstly, it is easy to see that

Proposition 11.7 The epis of **FreeGrp** are precisely the onto maps.□

Artin's famous representation theorem states that every $n \to n$-braid defines an automorphism on the free group of rank n. This proceeds by identifying the free group with generators $\{x_1, \ldots, x_n\}$ with the homotopy group of the plane with a puncture at each of the coordinates of $I(n)$. Then the braid acts on the fundamental group as in the following diagram.



Via the observation that this action can be reversed, we formulate the following representation theorem for vines. It is the first part of [12, Theorem 6.2].

Theorem 11.8 The monoid V_n of all $n \to n$ vines has a faithful contravariant representation as a monoid of endomorphisms of a free group F_n of rank n. The representation is induced by a mapping $\zeta : V_n \to End(F_n)$ defined by

$$\zeta(s_i) : x_j \longmapsto \begin{cases} x_j & \text{if } j \notin \{i, i+1\} \\ x_i x_{i+1} x_i^{-1} & \text{if } j = i \\ x_i & \text{if } j = i+1 \end{cases}$$

and

$$\zeta(g_i) : x_j \longmapsto \begin{cases} x_j & \text{if } j \notin \{i, i+1\} \\ 1 & \text{if } j = i \\ x_i x_{i+1} & j = i+1 \end{cases} \quad .$$

□

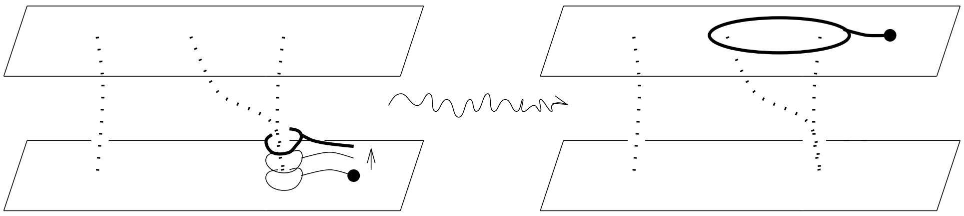
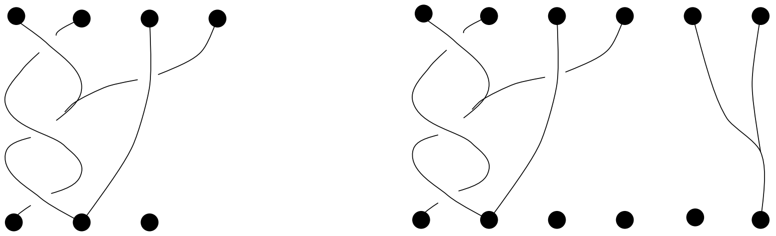


Figure 3: The contravariant action of a vine on a generator of the fundamental group of the 3-punctured plane.

This representation theorem is now used to define a functor Γ : **Vine** $\to$ **FreeGrp**^{op} which is faithful and preserves monos, and hence induces a faithful functor **Conc**(**Vine**) $\to$ **Conc**(**FreeGrp**^{op}).

Let $m < q$ be positive integers. Define $p_{q,m} : F_q \to F_m$ to be the homomorphism which acts identically on the generator x_i if $i \leq m$ and maps $x_i \mapsto 1$ otherwise. Let $i_{m,q} : F_m \to F_q$ be the natural injection.

Let $v : m \to n$ be a vine and let q be greater than both m and n. Define $\gamma_v^q : q \to q$ to be the vine which is the same as v from the first m starting nodes to the first n terminal nodes but which has a simple vine from the starting nodes numbered $m+1$ through q, to the q^{th} terminal node, as in the following diagram.



The vines $v : 4 \to 3$ and $\gamma_v^6 : 6 \to 6$.

When there can be no confusion as to the value of q, we will often write γ_v instead of γ_v^q.

Definition 11.9 For each object n of **Vine** define $\Gamma(n)$ to be F_n in **FreeGrp**. Let $v : m \to n$ be a vine, and let q be any integer greater than both m and n. Then $\Gamma(v)$ is the homomorphism $p_{q,m}\zeta(\gamma_v^q)i_{n,q} : F_n \to F_m$.

This definition is independent of q and corresponds to the action described by Figure 4.

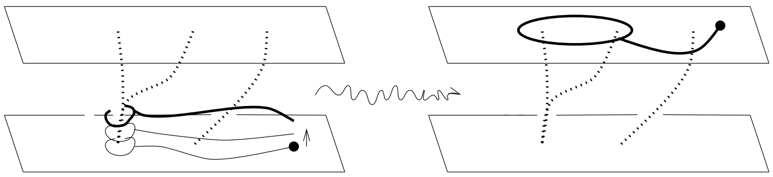


Figure 4: The geometric interpretation of the action of a vine $v : 3 \to 2$ as a homomorphism $\Gamma(v) : F_2 \to F_3$.

The following proposition is clear.

Proposition 11.10 Let $k \xrightarrow{u} m \xrightarrow{v} n$ be a diagram in **Vine**, and let q be greater than k, m and n. Then $\gamma_v^q \gamma_u^q = \gamma_{vu}^q$. $\square$

Lemma 11.11 Γ is a functor **Vine** $\to$ **FreeGrp**^{op}.

Proof. It is clear that $\Gamma(1_n) = 1_{F_n}$. We just need to show that for each diagram $k \xrightarrow{u} m \xrightarrow{v} n$ in **Vine**, $\Gamma(vu) = \Gamma(u)\Gamma(v)$ in **FreeGrp**. Fix $q > \max\{k, m, n\}$. By definition $\Gamma(vu) = p_{q,k}\zeta(\gamma_{vu})i_{n,q}$, while $\Gamma(u)\Gamma(v) = p_{q,k}\zeta(\gamma_u)i_{m,q}p_{q,m}\zeta(\gamma_v)i_{n,q}$. Now the image of $\zeta(\gamma_v)i_{n,q}$ lies in the subgroup of F_q generated by $\langle x_1, \ldots, x_m\rangle$ upon which $p_{q,m}$ acts identically, so that $\Gamma(u)\Gamma(v) = p_{q,k}\zeta(\gamma_u)\zeta(\gamma_v)i_{n,q}$.

By Theorem 11.8, ζ is a contravariant monoid homomorphism on objects so that $\zeta(\gamma_u)\zeta(\gamma_v) = \zeta(\gamma_v\gamma_u)$. But by Proposition 11.10 $\gamma_v\gamma_u = \gamma_{vu}$ so that $\Gamma(u)\Gamma(v) = p_{q,k}\zeta(\gamma_{vu})i_{n,q} = \Gamma(vu)$ as required.$\square$

Lemma 11.12 The functor Γ is faithful.

Proof. Let $u, v : m \to n$ be vines, and let q be greater than both m and n. For the remainder of this proof p will denote $p_{q,m}$ and i will denote $i_{n,q}$. Suppose $\Gamma(u) = \Gamma(v)$. Then for each $s \in F_n$ and hence all generators x_j of F_n, $p\zeta(\gamma_v)i(x_j) = p\zeta(\gamma_u)i(x_j)$, whence, for each generator x_j of F_q with $j \leq n$,

$$p\zeta(\gamma_v)(x_j) \quad = \quad p\zeta(\gamma_u)(x_j). \tag{5}$$

If $\zeta(\gamma_v)$ were not equal to $\zeta(\gamma_u)$, then by the construction of γ they would only differ on some generators $x_i, i \leq n$. In combination with equation (5) this yields $p\zeta(\gamma_v)(x_j) = p\zeta(\gamma_u)(x_j)$ for all $j \leq q$.

Suppose $\zeta(\gamma_v) \neq \zeta(\gamma_u)$, then for some $x_i, i \leq n$, $\zeta(\gamma_v)(x_i) \neq \zeta(\gamma_u)(x_i)$ while both $\zeta(\gamma_v)(x_i)$ and $\zeta(\gamma_u)(x_i)$ are elements of the subgroup $\langle x_1, \ldots, x_m \rangle$ of F_q, by the construction of γ. But p is injective on this subgroup, therefore $\zeta(\gamma_v)(x_i) = \zeta(\gamma_u)(x_i)$ for all generators x_i of F_q. By Theorem 11.8, this implies that $\gamma_v = \gamma_u$ and hence, by the construction of γ, $v = u$. □

Lemma 11.13 If v is a monic vine then $\Gamma(v)$ is a surjective homomorphism, and therefore a monic arrow of $\mathbf{FreeGrp}^{op}$.

Proof. Since a monic vine is a braid (b, say), any generating loop x_i at the top may be "pushed down" via Artin's representation to get a loop σ at the bottom. When σ is acted upon by $\Gamma(b)$ we get x_i back again. Thus, every generating loop is in the image of $\Gamma(b)$ as required.□

The preceeding three lemmas in combination with Theorem 7.6 show that

Theorem 11.14 The functor Γ induces a faithful functor $\mathbf{Conc}(\mathbf{Vine}) \to \mathbf{Conc}(\mathbf{FreeGrp}^{op})$.

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